CHAPTER ONE

Mathematical Preliminaries

The underlying theory for geophysics, planetary physics, and space physics requires a solid understanding of many of the methods of mathematical physics as well as a set of specialized topics that are integral to the diverse array of real-world problems that we seek to understand. This chapter will review some essential mathematical concepts and notations that are commonly employed and will be exploited throughout this book. We will begin with a review of vector analysis focusing on indicial notation, including the Kronecker δ and Levi-Civita ϵ permutation symbol, and vector operators. Cylindrical and spherical geometry are ubiquitous in geophysics and space physics, as are the theorems of Gauss, Green, and Stokes. Accordingly, we will derive some of the essential vector analysis results in Cartesian geometry in these curvilinear coordinate systems. We will proceed to explore how vectors transform in space and the role of rotation and matrix representations, and then go on to introduce tensors, eigenvalues, and eigenvectors. The solution of the (linear) partial differential equations of mathematical physics is commonly used in geophysics, and we will present some materials here that we will exploit later in the development of Green's functions. In particular, we will close this chapter by introducing the ramp, Heaviside, and Dirac δ functions. As in all of our remaining chapters, we will provide a set of problems and cite references that present more detailed investigations of these topics.

1.1 Vectors, Indicial Notation, and Vector Operators

This book primarily will pursue the kinds of geophysical problems that emerge from scalar and vector quantities. While mention will be made of tensor operations, our primary focus will be upon vector problems in three dimensions that form the basis of geophysics. Scalars and vectors may be regarded as tensors of a specific rank. *Scalar* quantities, such as density and temperatures, are *zero-rank* or *zero-order* tensors. *Vector* quantities such as velocities have an associated direction as well as a magnitude. Vectors are *first-rank* tensors and are usually designated by boldface lower-case letters. *Second-rank tensors*, or simply *tensors*, such as the stress tensor are a special case of square matrices. Matrices are generally denoted by boldface, uppercase letters, while tensors are generally denoted by boldface, uppercase, sans-serif letters (Goldstein et al., 2002). For example, $\boldsymbol{M}$ would designate a matrix while $\boldsymbol{\mathsf{T}}$ would designate a tensor. [There are other notations, e.g., Kusse and Westwig (2006), that employ overbars for vectors and double overbars for tensors.] Substantial simplification of notational issues emerges upon adopting *indicial* notation.

In lieu of x, y, and z in describing the Cartesian components for position, we will employ x_1, x_2, and x_3. Similarly, we will denote by $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ the mutually orthogonal *unit vectors* that are in the direction of the x_1, x_2, and x_3 axes. (Historically, the use of $\boldsymbol{e}$ emerged in Germany where the letter "e" stood for the word *Einheit*, which translates as "unit.") The indicial notation implies that any repeated index is summed, generally from 1 through 3. This is the *Einstein summation convention*.

It is sufficient to denote a vector $\boldsymbol{v}$, such as the velocity, by its three components (v_1, v_2, v_3). We note that $\boldsymbol{v}$ can be represented vectorially by its component terms, namely,

$$\boldsymbol{v} = \sum_{i=1}^{3} v_i \hat{\boldsymbol{e}}_i = v_i \hat{\boldsymbol{e}}_i. \tag{1.1}$$

Suppose $\boldsymbol{\mathsf{T}}$ is a tensor with components T_{ij}. Then,

$$\boldsymbol{\mathsf{T}} = \sum_{i=1,j=1}^{3} T_{ij} \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j = T_{ij} \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j. \tag{1.2}$$

We now introduce the *inner product*, also known as a scalar product or dot product, according to the convention

$$\boldsymbol{u} \cdot \boldsymbol{v} \equiv u_i v_i. \tag{1.3}$$

Moreover, we define u and v to be the lengths of $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively, according to

$$u \equiv \sqrt{u_i u_i} = |\boldsymbol{u}|; \qquad v \equiv \sqrt{v_i v_i} = |\boldsymbol{v}|; \tag{1.4}$$

we can identify an angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ that we define according to

$$\boldsymbol{u} \cdot \boldsymbol{v} \equiv uv \cos\theta, \tag{1.5}$$

which corresponds directly to our geometric intuition.

We now introduce the Kronecker δ according to

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.6}$$

The Kronecker δ is the indicial realization of the identity matrix. It follows, then, that

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}, \tag{1.7}$$

and that

$$\delta_{ii} = 3. \tag{1.8}$$

This is equivalent to saying that the *trace*, that is, the sum of the diagonal elements, of the identity matrix is 3. An important consequence of Eq. (1.7) is that

$$\delta_{ij}\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i. \tag{1.9}$$

A special example of these results is that we can now derive the general scalar product relation (1.3), namely,

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i\hat{\boldsymbol{e}}_i \cdot v_j\hat{\boldsymbol{e}}_j = u_i v_j \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = u_i v_j \delta_{ij} = u_i v_i, \tag{1.10}$$

by applying Eq. (1.7).

We introduce the Levi-Civita or permutation symbol ϵ_{ijk} in order to address the *vector product* or *cross product*. In particular, we define it according to

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i\,j\,k \text{ are an even permutation of } 1\,2\,3, \\ -1 & \text{if } i\,j\,k \text{ are an odd permutation of } 1\,2\,3, \\ 0 & \text{if any two of } i, j, k \text{ are the same.} \end{cases} \tag{1.11}$$

We note that ϵ_{ijk} changes sign if any two of its indices are interchanged. For example, if the 1 and 3 are interchanged, then the sequence 1 2 3 becomes 3 2 1. Accordingly, we define the cross product $\boldsymbol{u} \times \boldsymbol{v}$ according to its ith component, namely,

$$(\boldsymbol{u} \times \boldsymbol{v})_i \equiv \epsilon_{ijk} u_j v_k, \tag{1.12}$$

or, equivalently,

$$\boldsymbol{u} \times \boldsymbol{v} = (\boldsymbol{u} \times \boldsymbol{v})_i \hat{\boldsymbol{e}}_i = \epsilon_{ijk}\hat{\boldsymbol{e}}_i u_j v_k = -(\boldsymbol{v} \times \boldsymbol{u}). \tag{1.13}$$

It is observed that this structure is closely connected to the definition of the determinant of a 3×3 matrix, which emerges from expressing the scalar triple product

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \epsilon_{ijk} u_i v_j w_k, \tag{1.14}$$

and, by virtue of the cyclic permutivity of the Levi-Civita symbol, demonstrates that

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}). \tag{1.15}$$

The right-hand side of Eq. (1.14) is the determinant of a matrix whose rows correspond to $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

Indicial notation facilitates the calculation of quantities such as the vector triple cross product

$$\begin{aligned}\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) &= \boldsymbol{u} \times \epsilon_{ijk} \hat{\boldsymbol{e}}_i v_j w_k = \epsilon_{lmi} \hat{\boldsymbol{e}}_l u_m \epsilon_{ijk} v_j w_k \\ &= (\epsilon_{ilm} \epsilon_{ijk}) \hat{\boldsymbol{e}}_l u_m v_j w_k. \end{aligned} \tag{1.16}$$

It is necessary to deal first with the $\epsilon_{ilm}\epsilon_{ijk}$ term. Observe, as we sum over the i index, that contributions can emerge only if $l \neq m$ and $j \neq k$. If these conditions both hold, then we get a contribution of 1 if $l = j$ and $m = k$, and a contribution of -1 if $l = k$ and $m = j$. Hence, it follows that

$$\epsilon_{ilm} \epsilon_{ijk} = \delta_{\ell j} \delta_{mk} - \delta_{lk} \delta_{mj}. \tag{1.17}$$

Returning to (1.16), we obtain

$$\begin{aligned}\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) &= (\delta_{lj} \delta_{mk} - \delta_{lk} \delta_{mj}) \hat{\boldsymbol{e}}_l u_m v_j w_k \\ &= \hat{\boldsymbol{e}}_l v_l u_m w_m - \hat{\boldsymbol{e}}_l w_l u_m v_m \\ &= \boldsymbol{v}(\boldsymbol{u} \cdot \boldsymbol{w}) - \boldsymbol{w}(\boldsymbol{u} \cdot \boldsymbol{v}), \end{aligned} \tag{1.18}$$

thereby reproducing a familiar, albeit otherwise cumbersome to derive, algebraic identity. Finally, if we replace the role of $\boldsymbol{u}$ in the triple scalar product (1.18) by $\boldsymbol{v} \times \boldsymbol{w}$, it immediately follows that

$$\begin{aligned}(\boldsymbol{v} \times \boldsymbol{w}) \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= |\boldsymbol{v} \times \boldsymbol{w}|^2 = \epsilon_{ijk} v_j w_k \epsilon_{ilm} v_l w_m \\ &= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) v_j w_k v_l w_m \\ &= v^2 w^2 - (\boldsymbol{v} \cdot \boldsymbol{w})^2 = v^2 w^2 \sin^2 \theta, \end{aligned} \tag{1.19}$$

where we have made use of the definition for the angle θ given in (1.5).

The Kronecker δ and Levi-Civita ϵ permutation symbols simplify the calculation of many other vector identities, including those with respect to *derivative* operators. We define ∂_i according to

$$\partial_i \equiv \frac{\partial}{\partial x_i}, \tag{1.20}$$

and employ it to define the gradient operator $\boldsymbol{\nabla}$, which is itself a vector:

$$\boldsymbol{\nabla} = \partial_i \hat{\boldsymbol{e}}_i. \tag{1.21}$$

Another notational shortcut is to employ a subscript of “, i” to denote a derivative with respect to x_i; importantly, a comma “,” is employed together with the subscript to designate differentiation. Hence, if f is a scalar function of $\boldsymbol{x}$, we write

$$\frac{\partial f}{\partial x_i} = \partial_i f = f_{,i}; \tag{1.22}$$

but if $\boldsymbol{g}$ is a vector function of $\boldsymbol{x}$, then we write

$$\frac{\partial g_i}{\partial x_j} = \partial_j g_i = g_{i,j}. \tag{1.23}$$

Higher derivatives may be expressed using this shorthand as well, for example,

$$\frac{\partial^2 g_i}{\partial x_j \partial x_k} = g_{i,jk}. \tag{1.24}$$

Then, the usual divergence and curl operators become

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \partial_i u_i = u_{i,i} \tag{1.25}$$

and

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \epsilon_{ijk} \hat{\boldsymbol{e}}_i \partial_j u_k = \epsilon_{ijk} \hat{\boldsymbol{e}}_i u_{k,j}. \tag{1.26}$$

Our derivations will employ Cartesian coordinates, primarily, since curvilinear coordinates, such as cylindrical and spherical coordinates, introduce a complication insofar as the unit vectors defining the associated directions change. However, once we have obtained the fundamental equations, curvilinear coordinates can be especially helpful in solving problems since they help capture the essential geometry of the Earth.

1.2 Cylindrical and Spherical Geometry

Two other coordinate systems are widely employed in geophysics, namely, cylindrical coordinates and spherical coordinates. As we indicated earlier, our starting point will always be the fundamental equations that we derived using Cartesian coordinates and then we will convert to coordinates that are more "natural" for solving the problem at hand. Let us begin in two dimensions with polar coordinates (r, θ) and review some fundamental results.

As usual, we relate our polar and Cartesian coordinates according to

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned} \tag{1.27}$$

which can be inverted according to

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan(y/x). \end{aligned} \tag{1.28}$$

Unit vectors in the new coordinates can be expressed

$$\begin{aligned} \hat{\boldsymbol{r}} &= \quad \cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{y}}. \end{aligned} \tag{1.29}$$

We recall how to obtain the various differential operations, such as the gradient, divergence, and curl, by using the chain rule of multivariable calculus. Suppose that f is a scalar function of x and y, and we wish to transform its Cartesian derivatives into derivatives with respect to polar coordinates. From the chain rule, it follows that

$$\frac{\partial f}{\partial x} = \left.\frac{\partial r}{\partial x}\right|_y \left.\frac{\partial f}{\partial r}\right|_\theta + \left.\frac{\partial \theta}{\partial x}\right|_y \left.\frac{\partial f}{\partial \theta}\right|_r = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}, \tag{1.30}$$

where the vertical bar followed by a subscript designates the variable or variables that are held fixed. In like fashion, we can derive

$$\frac{\partial f}{\partial y} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}. \tag{1.31}$$

Finally, we can obtain the *Laplacian* of a scalar quantity in two dimensions, ∇^2, defined according to

$$\nabla^2 f \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}. \tag{1.32}$$

Integrals in two dimensions require the transformation of differential area elements from $\mathrm{d}x\,\mathrm{d}y$ to $r\,\mathrm{d}\theta\,\mathrm{d}r$. Therefore, the integral of f over some area A can be expressed equivalently as

$$\begin{aligned}\int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_A f(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_A f(\boldsymbol{x})\,\mathrm{d}A \\ &= \int_A f(\boldsymbol{x})\,\mathrm{d}^2x = \int_A f(r,\theta)r\,\mathrm{d}r\,\mathrm{d}\theta, \quad (1.33)\end{aligned}$$

where the areas of integration are kept the same and integration over two variables is implicit.

We now move on to review three-dimensional geometry wherein polar coordinates become either cylindrical or spherical polar coordinates. We begin with cylindrical coordinates, which now introduce the third or z dimension. Accordingly, we observe that the Laplacian becomes

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}, \quad (1.34)$$

where we assume that r is measured in the x-y plane, that is, it is not the radial distance from the origin to the point in question. Suppose, as before, that $\boldsymbol{g}$ is a vector function and we wish to obtain its divergence and curl. We will designate its components in the cylindrical coordinate system (r,θ,z) by (g_r,g_θ,g_z). These can be calculated directly by taking projections of $(f_1,f_2,f_3) \equiv (f_x,f_y,f_z)$ onto the (r,θ,z) directions. The z direction requires no elaboration. However, we note that

$$\begin{aligned} g_r &= \cos\theta g_x + \sin\theta g_y \\ g_\theta &= -\sin\theta g_x + \cos\theta g_y \end{aligned} \quad (1.35)$$

and

$$\begin{aligned} g_x &= \cos\theta g_r - \sin\theta g_\theta \\ g_y &= \sin\theta g_r + \cos\theta g_\theta. \end{aligned} \quad (1.36)$$

With these results in hand, we can show that the divergence of $\boldsymbol{g}$ becomes

$$\nabla\cdot\boldsymbol{g} = \frac{1}{r}\frac{\partial}{\partial r}(rg_r) + \frac{1}{r}\frac{\partial g_\theta}{\partial\theta} + \frac{\partial g_z}{\partial z}, \quad (1.37)$$

where $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$ are unit vectors in the associated directions. Similarly, we write the curl of $\boldsymbol{g}$ as

$$\nabla \times \boldsymbol{g} = \frac{1}{r}\left\{\left[\frac{\partial g_z}{\partial \theta} - \frac{\partial (r g_\theta)}{\partial z}\right]\hat{\boldsymbol{r}} + \left[\frac{\partial g_r}{\partial z} - \frac{\partial g_z}{\partial r}\right] r\hat{\boldsymbol{\theta}} + \left[\frac{\partial (r g_\theta)}{\partial r} - \frac{\partial g_r}{\partial \theta}\right]\hat{\boldsymbol{z}}\right\}. \tag{1.38}$$

Finally, the integral over some volume V of a scalar function f can be written equivalently as

$$\begin{aligned}\int_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_V f(x_1,x_2,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ &= \int_V f(\boldsymbol{x})\,\mathrm{d}V = \int_V f(\boldsymbol{x})\,\mathrm{d}^3x \\ &= \int_V f(r,\theta,z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z. \end{aligned} \tag{1.39}$$

This concludes our summary of cylindrical coordinates.

We now adopt spherical coordinates (Figure 1.1) according to

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta, \end{aligned} \tag{1.40}$$

which can be inverted according to

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arccos(z/r) = \arccos\left(z/\sqrt{x^2+y^2+z^2}\right) \\ \varphi &= \arctan(y/x). \end{aligned} \tag{1.41}$$

Without elaboration, we list here some essential results.

1. Unit vector relationships, from which g_r, g_θ, and g_φ can also be extracted:

$$\begin{aligned} \hat{\boldsymbol{r}} &= \sin\theta\cos\varphi\,\hat{\boldsymbol{x}} + \sin\theta\sin\varphi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\varphi\,\hat{\boldsymbol{x}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\,\hat{\boldsymbol{x}} + \cos\varphi\,\hat{\boldsymbol{y}}. \end{aligned} \tag{1.42}$$

   We note, as a check, that all three of these unit vectors are of unit length and are mutually orthogonal.

2. Gradient of a scalar f:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}. \tag{1.43}$$

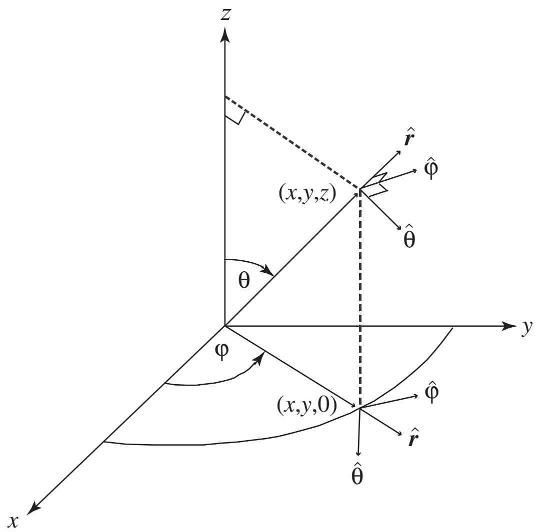

Figure 1.1. Spherical coordinates.

3. Laplacian of a scalar f:

$$\nabla^2 f = \frac{1}{r^2\sin\theta}\left\{\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\varphi^2}\right\}. \quad (1.44)$$

Note that the Laplacian of vector quantities will differ from the above due to the dependence of the projected components on the coordinates.

4. Divergence of a vector $\boldsymbol{g}$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial(r^2 g_r)}{\partial r} + r\frac{\partial(\sin\theta\, g_\theta)}{\partial\theta} + r\frac{\partial g_\varphi}{\partial\varphi}\right]. \quad (1.45)$$

5. Curl of a vector $\boldsymbol{g}$:

$$\boldsymbol{\nabla}\times\boldsymbol{g} = \frac{1}{r^2\sin\theta}\left\{\left[\frac{\partial(r\sin\theta\, g_\varphi)}{\partial\theta} - \frac{\partial(r g_\theta)}{\partial\varphi}\right]\hat{\boldsymbol{r}} + \left[\frac{\partial g_r}{\partial\varphi} - \frac{\partial(r\sin\theta\, g_\varphi)}{\partial r}\right]r\hat{\boldsymbol{\theta}} + \left[\frac{\partial(r g_\theta)}{\partial r} - \frac{\partial g_r}{\partial\theta}\right]r\sin\theta\,\hat{\boldsymbol{\varphi}}\right\}. \quad (1.46)$$

6. Volume integral of a scalar f:

$$\int_V f(\boldsymbol{x})\,\mathrm{d}^3x = \int_V f(r,\theta,\varphi)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi. \quad (1.47)$$

We will now review some of the integral relations involving vector quantities.

1.3 Theorems of Gauss, Green, and Stokes

We wish to present some familiar results from integral calculus. We will not provide proofs but will present a brief sketch as to how they can be obtained. In Figure 1.2, we depict the relevant geometry.

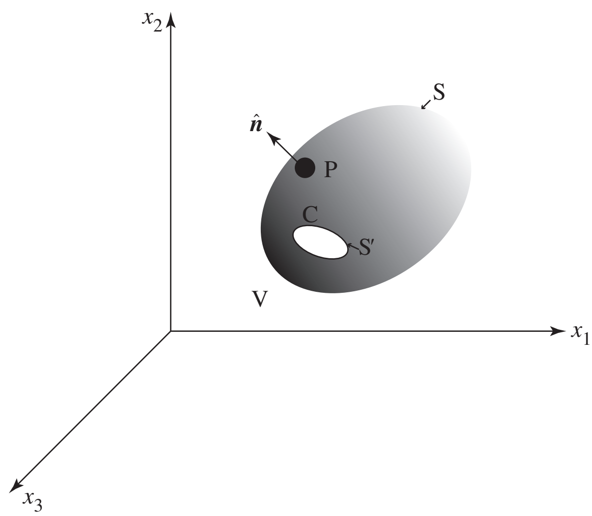

Figure 1.2. Geometry of volume and surface.

We denote by V the volume under consideration, and S denotes the surface of that volume. We identify a point P on the surface of that volume, and show by an arrow the unit vector $\hat{\boldsymbol{n}}$ emerging out from that surface. Finally, we draw a closed curve C on that surface that contains a surface area S′. We denote by $\boldsymbol{g}$ a vector function and by f and h two different scalar functions. We assume that f, $\boldsymbol{g}$, and h all go to zero as our distance from the origin goes to infinity. As before, we denote surface and volume elements by d^2x and d^3x, respectively.

Gauss's theorem can be expressed by

$$\int_{\mathrm{V}} \boldsymbol{\nabla} \cdot \boldsymbol{g}\, \mathrm{d}^3x = \int_{\mathrm{S}} \boldsymbol{g} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}^2x. \tag{1.48}$$

This result can be proved by subdividing the volume V into a set of cubes, going to the limit that the sides of the cubes become vanishingly small. It is simple to show by a direct integration that Gauss's theorem holds for each cube. When we amalgamate all of the cubes, the contributions emerging from the surfaces that are in common cancel, leaving only the contribution from the external enveloping surface.

Green's theorem (Morse and Feshbach, 1999) is generally expressed as

$$\begin{aligned}\int_V \nabla \cdot (f\nabla h - h\nabla f)\, \mathrm{d}^3x &= \int_S (f\nabla h - h\nabla f) \cdot \hat{\boldsymbol{n}}\, \mathrm{d}^2x \\ &= \int_V (f\nabla^2 h - h\nabla^2 f)\, \mathrm{d}^3x. \end{aligned} \tag{1.49}$$

Other textbooks, for example, Greenberg (1998), refer to this as one of Green's identities. The second integral is a direct application of Gauss's theorem. To obtain the third integral, we applied the $\nabla\cdot$ operator on the product of the two terms and eliminated the common $\nabla f \cdot \nabla h$ term, expressing $\nabla \cdot \nabla$ as the Laplacian ∇^2.

Stokes's theorem allows us to relate the integral of the curl of a vector acting upon the surface S' that is enclosed by a curve C to the integral of that vector projected onto and along that curve C. It emerges in electromagnetic theory and has the form

$$\int_{S'} (\nabla \times \boldsymbol{g}) \cdot \hat{\boldsymbol{n}}\, \mathrm{d}^2x = \oint_C \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{\ell}, \tag{1.50}$$

where $\oint$ denotes an integral around a closed curve, in this case C, and $\mathrm{d}\boldsymbol{\ell}$ is a differential line element that resides on C. As in the case of Gauss's theorem, we can prove Stokes's theorem by subdividing the area enclosed by the curve into a set of squares whose sides will ultimately be taken to be vanishingly small. Stokes's theorem can readily be proven on a square, and the lines in common among the squares cancel when calculating the contribution from all squares in the limit.

1.4 Rotation and Matrix Representation

We have already explored one form of vector rotation, namely, the conversion of Cartesian coordinates into spherical geometry where the unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ were replaced by or "rotated" into $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$. We wish to obtain a general expression for converting coordinates from a system of axes $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ to new coordinate axes $\hat{\boldsymbol{e}}'_1$, $\hat{\boldsymbol{e}}'_2$, and $\hat{\boldsymbol{e}}'_3$. (We assume that you have had an introductory linear algebra course as an undergraduate, including the solution of linear equations via Gaussian elimination and introduction to the eigenvalue problem.) The visual approach to coordinate conversions as well as rotation is typically presented diagrammatically in two dimensions, but becomes rather cumbersome and confusing in three. Matrix algebra provides a simple way of clarifying this problem. We recognize that the new coordinate axes $\hat{\boldsymbol{e}}'_i$ for $i = 1, 2, 3$ should be expressible as a linear combination of the $\hat{\boldsymbol{e}}_j$ coordinate axes. Suppose that $\boldsymbol{A}$ is a 3×3 matrix with components A_{ij} so that we can write

$$\hat{\boldsymbol{e}}'_i = \sum_{j=1}^{3} A_{ij} \hat{\boldsymbol{e}}_j = A_{ij} \hat{\boldsymbol{e}}_j \tag{1.51}$$

for $i = 1, 2$, and 3, returning to indicial notation, since this is a general expression for a linear combination of the original coordinate axes. Accordingly, we take the dot product of this expression with $\hat{\boldsymbol{e}}_k$, for $i, k = 1, \ldots, 3$ and observe that

$$\hat{\boldsymbol{e}}_k \cdot \hat{\boldsymbol{e}}'_i = \hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_k = \sum_{j=1}^{3} A_{ij} \hat{\boldsymbol{e}}_k \cdot \hat{\boldsymbol{e}}_j = A_{ik} \tag{1.52}$$

by virtue of the Kronecker δ identity (1.7). A useful mnemonic device for remembering this result is that A_{ik} is the matrix that connects the kth axis from the original coordinate to the ith axis in the new coordinate system. Similarly, we introduce another matrix $\boldsymbol{B}$ so that we can write the inverse operation, going from the new coordinates back to the old, namely,

$$\hat{\boldsymbol{e}}_i = \sum_{i=1}^{3} B_{ij} \hat{\boldsymbol{e}}'_j = B_{ij} \hat{\boldsymbol{e}}'_j, \tag{1.53}$$

returning to indicial notation, and directly obtain that

$$B_{ik} = \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}'_k. \tag{1.54}$$

We observe that

$$B_{ik} = A_{ki}, \tag{1.55}$$

establishing that the matrix $\boldsymbol{B} = \tilde{\boldsymbol{A}}$ where the tilde ~ designates the *transpose* of the matrix. Hence, it immediately follows that

$$\tilde{\boldsymbol{A}} \boldsymbol{A} = \boldsymbol{A} \tilde{\boldsymbol{A}} = \boldsymbol{I}, \tag{1.56}$$

where $\boldsymbol{I}$ designates the identity matrix so that its ij components are δ_{ij}. For self-evident reasons, we refer to both $\boldsymbol{A}$ and $\boldsymbol{B}$ as *rotation matrices*.

The conversion of the coordinates of a point from one coordinate system to another, which is a rotated version (without translation) of the original, can also be regarded as a rotation in the opposing sense of the point undergoing coordinate conversion. For example, suppose our coordinate conversion corresponded to a positive 45° rotation of the x-y axes to a new x'-y' set of axes, leaving the z-axis unchanged. That would have the same effect as a rotation of negative 45° in the x-y plane of the point under consideration.

Returning momentarily to the expression $A_{ij} = \hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_j$, it follows that each of the matrix components is a *direction cosine*, the inner product of the original jth unit vector with the new ith unit vector. Had we attempted to do this using a visual construction, for example, as performed by Goldstein et al. (2002), we would have observed the same relationships, albeit in a geometrically complicated form. One final feature of rotation matrices that is often important is that the three rows of a rotation matrix, if we were to regard them as three vectors, are mutually orthogonal and of unit length. A similar observation can be made for the three columns of a rotation matrix.

Suppose now that we have a vector $\boldsymbol{v}$ that we wish to express in both coordinate systems utilizing (1.1) and indicial notation, namely,

$$\boldsymbol{v} = v_i \hat{\boldsymbol{e}}_i \equiv \boldsymbol{v}' = v'_j \hat{\boldsymbol{e}}'_j. \tag{1.57}$$

Taking the inner product of $\hat{\boldsymbol{e}}'_k$ with this expression, we observe that

$$v'_k = v_j \hat{\boldsymbol{e}}'_k \cdot \hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}'_k \cdot \hat{\boldsymbol{e}}_j v_j = A_{kj} v_j. \tag{1.58}$$

Therefore, we note that the Cartesian coordinate representation in the new reference (rotated) frame transforms in the same way as the coordinate axes transform. For more insight into the nature of rotations, consult Goldstein et al. (2002) or Newman (2012).

Before proceeding, it is useful to describe two further aspects of matrix structure. Any matrix $\boldsymbol{A}$ can be expressed as the sum of a symmetric and an antisymmetric matrix, namely,

$$\boldsymbol{A} = \frac{\boldsymbol{A} + \tilde{\boldsymbol{A}}}{2} + \frac{\boldsymbol{A} - \tilde{\boldsymbol{A}}}{2}, \tag{1.59}$$

where $\tilde{A}$ designates the transpose of the matrix. Rotation matrices have a special structure, inasmuch as they possess both a symmetric and an antisymmetric part. Intuitively, we expect that a rotation matrix identifies a special direction corresponding to the axis around which the rotation takes place, and the angle of rotation that is executed around this axis. It takes two variables to identify the direction—think of them as corresponding to a latitude and longitude—and a third variable to identify the magnitude of the rotation. This situation applies whether we are considering a physical rotation of an object or the expression of its orientation in a new coordinate system. Newman (2012) provides a more detailed discussion of this problem. There are other methodologies for describing a rotation, such as through three *Euler angles*. Goldstein et al. (2002) provides a detailed discussion of this approach. The Euler angles are routinely employed in celestial mechanics in describing the orientation of elliptical orbits in the gravitational two-body problem, for example, for cometary orbits (Roy, 2005).

We consider an x-y plane established by the orbit of Jupiter around the Sun, with the x-direction corresponding to the direction of Jupiter's perihelion; we wish to establish the orientation of the comet's elliptical orbit with respect to this *invariable plane*. To do this, we undertake a series of rotations of the comet's orbit. We begin with the ellipse being in the x-y plane with the Sun at the origin, and the ellipse's axis initially oriented in the x-direction. We then rotate the ellipse in the x-y plane, that is, around the z-axis, by an angle referred to as the *longitude of the ascending node* Ω. The *line of nodes* identifies the y-axis about which the plane of the orbit is now rotated in the x-z plane to corresponds to its *angle of inclination* i. Finally, we perform a third rotation, this time around the new z-axis in the x-y plane to identify the *longitude of perihelion* ω. For details of this procedure, the reader is encouraged to consult Goldstein et al. (2002), Roy (2005), Taff (1985), and Danby (1988). Importantly, what should be evident is that, by executing three successive rotations around the z, y, and z axes, respectively, it is possible to orient properly any three-dimensional object. There are alternative conventions for the Euler angles that are detailed in Goldstein et al. (2002). We have focused here upon the one commonly employed in planetary science as well as in classical mechanics. Rotation matrices occupy an important role in describing the dynamics of objects and materials in many different environments.

A natural question to ask now is, how do matrices derived from physically based considerations such as a variational or energy principle transform under coordinate axis rotation? This is the defining characteristic that distinguishes tensors from matrices.

1.5 Tensors, Eigenvalues, and Eigenvectors

In this section, we will explore how we can make symmetric 3×3 matrices transform under coordinate rotation in the same way as a vector; in so doing, we refer to the matrix as being a second-rank tensor. In addition, we observe that there exists a special coordinate basis where the action of the tensor upon a vector is equivalent to a scalar multiplication. This simplifies many calculations, and eigenvalue-eigenvector analysis is at the heart of this procedure. We will demonstrate how to solve the eigenvalue problem, and show how the eigenvectors form the basis set for the new coordinate system.

Having observed (1.57) describing how vectors transform, namely,

$$v'_i = A_{ij} v_j = \tilde{B}_{ji} v_j, \tag{1.60}$$

we introduce the concept of a second-rank tensor $\boldsymbol{T}$ as being a matrix $\boldsymbol{T}$ with components $T_{k\ell}$ that transforms similarly, namely,

$$T'_{ij} = A_{ik} A_{j\ell} T_{k\ell} = A_{ik} T_{k\ell} \tilde{A}_{\ell j} = \tilde{B}_{ik} T_{k\ell} B_{\ell j} \tag{1.61}$$

and

$$T_{ij} = B_{ik} B_{j\ell} T'_{k\ell} = B_{ik} T'_{k\ell} \tilde{B}_{\ell j} = \tilde{A}_{ik} T_{k\ell} A_{\ell j}. \tag{1.62}$$

Accordingly, we write

$$\boldsymbol{T}' = \boldsymbol{A} \boldsymbol{T} \tilde{\boldsymbol{A}} \quad \text{or} \quad \boldsymbol{T} = \tilde{\boldsymbol{A}} \boldsymbol{T}' \boldsymbol{A}, \tag{1.63}$$

which is routinely referred to as a *similarity transformation*. (Equivalent expressions are available utilizing $\boldsymbol{B}$.)

Matrix quantities in geophysics are generally real valued and frequently *symmetric*; if $\boldsymbol{A}$ is symmetric, we say that

$$A_{ij} = A_{ji}. \tag{1.64}$$

Tensors are often associated with situations where there are special associated directions. For example, in geophysics and engineering applications, the stress and strain tensors describe how compressional or extensional forces act upon materials resulting in a displacement according to a generalization of Hooke's law. In particular, if $\boldsymbol{T}$ is a tensor and $\boldsymbol{u}$ is a vector, we can find a number λ such that $\boldsymbol{T} \cdot \boldsymbol{u} = \lambda \boldsymbol{u}$. In this situation, we refer to λ as an *eigenvalue* or *characteristic value* of $\boldsymbol{T}$ and $\boldsymbol{u}$ is its associated *eigenvector* or *characteristic vector*. For the examples mentioned, the stress and strain tensors are symmetric, thereby guaranteeing the existence of real eigenvalues and eigenvectors. Moreover, there are special directions where the force associated with a solid or plastic material emerges in a direction orthogonal or normal to a surface (Newman, 2012). Similarly, in rotational kinematics (see, e.g., Goldstein et al., 2002) the moment of inertia tensor has special directions associated with it, often as an outcome of symmetry considerations. Since most geophysics graduate students have already completed an analytical mechanics course, but possibly not continuum mechanics, we shall explore rotational motion as an example of the eigenvalue problem.

Suppose we wish to calculate the kinetic energy K of a rigid solid body rotating around its center of mass with angular velocity $\boldsymbol{\omega}$, where the direction of this vector corresponds to the orientation of the spin axis. It follows, for any point $\boldsymbol{x}$ inside the rotating body, that the corresponding velocity is

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{x} \tag{1.65}$$

and the kinetic energy satisfies

$$K = \frac{1}{2} \int \rho(\boldsymbol{x}) v^2(\boldsymbol{x}) \, \mathrm{d}^3 x. \tag{1.66}$$

Importantly, we note that this is an integral over non-negative quantities and must necessarily yield a positive result. After some brief algebra, we find that

$$\begin{aligned} K &= \frac{1}{2} \int \rho(\boldsymbol{x}) [x_k x_k \omega_i \delta_{ij} \omega_j - \omega_i \omega_j x_i x_j] \, \mathrm{d}^3 x \\ &= \omega_i I_{ij} \omega_j, \end{aligned} \tag{1.67}$$

which we refer to as a *quadratic form* where the components of the moment of inertia tensor I_{ij} satisfy

$$I_{ij} = \int \rho(\boldsymbol{x})[x_k x_k \delta_{ij} - x_i x_j]. \tag{1.68}$$

We immediately note that this tensor is real and symmetric.

Tensor analysis, especially the properties of eigenvalues and eigenvectors, is an important topic generally treated in advanced undergraduate mathematics courses. However, there are some features of eigenvalue analysis for tensorial quantities that are conceptually vital in geophysics, and we sketch here some of their properties. More complete physically motivated treatments can be found in Goldstein et al. (2002) and in Newman (2012), which show that 3×3 symmetric tensors have real-valued eigenvalues and eigenvectors. In this particular case, the eigenvalues must be positive (or possibly zero) since the kinetic energy can never be negative. The eigenvalues, in turn, are the solutions to the cubic (characteristic) polynomial $p(\lambda)$ constructed by solving for the three roots of the equation

$$p(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0. \tag{1.69}$$

(Please note that we are employing I_{ij} to designate the components of the moment of inertia tensor. Many linear algebra textbooks employ the same symbol to describe the components of the identity matrix.) For this physically important class of eigenvalue problems, Cardano's method (Newman, 2012) provides an explicit closed-form solution, and we now sketch the derivation of this method.

Suppose the cubic polynomial can be expressed

$$a\lambda^3 + b\lambda^2 + c\lambda + d = 0, \tag{1.70}$$

where $a \neq 0$. We choose to eliminate the quadratic term by replacing λ with $z + \alpha$, where $\alpha = -b/3a$. We now obtain the equation

$$az^3 + c'z + d' = 0, \tag{1.71}$$

where $c' = c + 2\alpha b + 3\alpha^2 a$ and $d' = d + c\alpha + b\alpha^2 + a\alpha^3$. We now exploit the *triple-angle identity*

$$\cos^3\theta = \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta, \tag{1.72}$$

which can be readily verified by writing $\cos\theta$ in exponential form. We replace z in (1.71) by $y\cos\theta$ and then obtain

$$y\cos\theta\left[\frac{3ay^2}{4}+c'\right]+\left[\frac{ay^3\cos 3\theta}{4}+d'\right]=0. \tag{1.73}$$

We are at liberty to select y so that the first bracketed term disappears, and we then select θ so that the second bracketed term disappears. Note, however, that the solution for θ is *degenerate*; upon finding one solution, we can construct two additional solutions that are also real valued by taking our original solution and adding $2\pi/3$ as well as taking our original solution and subtracting $2\pi/3$. This kind of degeneracy is common in complex analysis and is associated with *branch cuts*, a topic we will review in chapter 3. Remarkably, while often overlooked in most textbooks, real-world, three-dimensional eigenvalue solutions are readily within reach.

Finally, we note that for each of the eigenvalues, we can now explicitly calculate the eigenvectors by solving the associated pair of linear equations and then normalizing the vectors obtained to be of unit length. We designate, after ordering, the eigenvalues represented by I_i according to

$$0\leqslant I_1\leqslant I_2\leqslant I_3. \tag{1.74}$$

(In planetary physics, the eigenvalues are often referred to as $0\leqslant A\leqslant B\leqslant C$.) We now define the normalized eigenvectors associated with the three eigenvalues as $\hat{\boldsymbol{e}}'_\ell$, for $\ell=1$, 2, and 3. If this selection does not yield a right-handed coordinate basis, we typically change the sign of one of the eigenvectors to conform with that convention. In terms of our original coordinate basis, the coordinates of the ℓth eigenvector $\hat{\boldsymbol{u}}_\ell$, which we now equate with $\hat{\boldsymbol{e}}'_\ell$, can be expressed as column vectors

$$\hat{\boldsymbol{u}}_\ell=\begin{pmatrix}\hat{\boldsymbol{e}}_1\cdot\hat{\boldsymbol{e}}'_\ell\\ \hat{\boldsymbol{e}}_2\cdot\hat{\boldsymbol{e}}'_\ell\\ \hat{\boldsymbol{e}}_3\cdot\hat{\boldsymbol{e}}'_\ell\end{pmatrix}. \tag{1.75}$$

Taken together, we recognize that the three column vectors (for $\ell=1,2,3$) establish the rotation matrix $\tilde{\boldsymbol{A}}$, namely,

$$\tilde{\boldsymbol{A}}=\begin{pmatrix}\hat{\boldsymbol{e}}_1\cdot\hat{\boldsymbol{e}}'_1 & \hat{\boldsymbol{e}}_1\cdot\hat{\boldsymbol{e}}'_2 & \hat{\boldsymbol{e}}_1\cdot\hat{\boldsymbol{e}}'_3\\ \hat{\boldsymbol{e}}_2\cdot\hat{\boldsymbol{e}}'_1 & \hat{\boldsymbol{e}}_2\cdot\hat{\boldsymbol{e}}'_2 & \hat{\boldsymbol{e}}_2\cdot\hat{\boldsymbol{e}}'_3\\ \hat{\boldsymbol{e}}_3\cdot\hat{\boldsymbol{e}}'_1 & \hat{\boldsymbol{e}}_3\cdot\hat{\boldsymbol{e}}'_2 & \hat{\boldsymbol{e}}_3\cdot\hat{\boldsymbol{e}}'_3\end{pmatrix}. \tag{1.76}$$

It is now easy to verify that

$$\boldsymbol{T}\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}\Lambda, \tag{1.77}$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix containing the eigenvalues λ_i or, equivalently, I_i, namely,

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{1.78}$$

Multiplying each side of the equation from the right by $\boldsymbol{A}$, we then recover the similarity transform (1.61). In the important special case of stress or strain tensors, which are symmetric, a similar analysis is applicable where their eigenvalues are necessarily real valued, but can have either sign. A similar analysis is applicable to stress and strain tensors (although their eigenvalues need not be non-negative).

1.6 Ramp, Heaviside, and Dirac δ Functions

We now wish to introduce the concept of a *generalized function*, a function on the real line that vanishes essentially everywhere but the origin, where it is singular, and has an integral of 1. This concept was introduced by Paul Dirac and is the continuous analogue of the Kronecker δ_{ij} that we introduced earlier. In particular, we define $\delta(x)$ according to

$$\begin{aligned} \delta(x) &= 0 \quad \text{if } x \neq 0 \\ g(y) &= \int_{-\infty}^{\infty} g(x)\delta(x-y)\,\mathrm{d}x \end{aligned} \tag{1.79}$$

for any $g(x)$. Consider the integral over the δ function and we define the step or *Heaviside* function according to

$$\begin{aligned} H(x) &= \int_{-\infty}^{x} \delta(y)\,\mathrm{d}y \\ &= \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \end{aligned} \tag{1.80}$$

Additionally, we introduce the *ramp function* $R(x) \equiv xH(x)$ and observe that it satisfies

$$R(x) = \int_{-\infty}^{x} H(y)\,\mathrm{d}y. \tag{1.81}$$

With these expressions, we also observe that

$$\delta(x) = \frac{\mathrm{d}H(x)}{\mathrm{d}x}$$
$$H(x) = \frac{\mathrm{d}R(x)}{\mathrm{d}x}$$
$$\delta(x) = \frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2}. \quad (1.82)$$

We plot these functions in Figure 1.3.

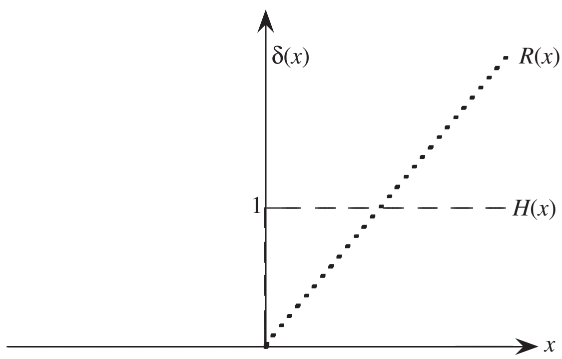

Figure 1.3. Dirac $\delta(x)$, Heaviside $H(x)$, and ramp $R(x)$ functions. Here, the δ function (solid line) vanishes everywhere but the origin, where it is singular. The Heaviside function (dashed line) vanishes for negative values, while the ramp function (dotted line) also vanishes for negative values.

The fundamental aspect present here is that the derivative of a discontinuous function yields a δ function while the second derivative of a function with discontinuous derivative does the same. The quantitative jump in function values or derivatives controls the amplitude of the outcome. We will have much more to say about this topic and its relation to *Green's functions* in later chapters.

Having provided this simple digest of mathematical preliminaries, with an emphasis upon geometry, we turn in the next chapter to a brief discussion of some of the problems expressible via partial differential equations that are central to contemporary geophysics.

1.7 Exercises

1. Prove the following identities using the Kronecker δ and Levi-Civita ϵ permutation symbol identities.

(a) Show that

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = 0.$$

(b) Show that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}.$$

2. Suppose that $\boldsymbol{b}$ is an arbitrary point in 3-space. Let $\mathbb{X}$ be the set of points $\boldsymbol{x}$ such that $(\boldsymbol{x} - \boldsymbol{b}) \cdot \boldsymbol{x} = 0$. Show that this describes the surface of a sphere with center $\frac{1}{2}\boldsymbol{b}$ and radius $\frac{1}{2}b$.
3. Derive using the chain rule the formula for the Laplacian in cylindrical coordinates.
4. Derive using the chain rule the formula for the Laplacian in spherical coordinates.
5. Let $\boldsymbol{\omega}(\boldsymbol{x}, t)$, called the vorticity of a flow, be defined by

$$\boldsymbol{\omega}(\boldsymbol{x}, t) \equiv \boldsymbol{\nabla} \times \boldsymbol{v}(\boldsymbol{x}, t).$$

Suppose that our flow corresponds to solid-body rotation, that is,

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{\Omega} \times \boldsymbol{x},$$

where $\boldsymbol{\Omega}$ is a constant vector that describes the rotation rate of the object. Prove that

$$\boldsymbol{\omega} = 2\boldsymbol{\Omega}.$$

6. Show that the divergence of the velocity field in the previous problem is zero. Therefore, using Gauss's theorem, prove that the flux of material traveling with this velocity through *any* surface is zero, that is, the total amount of fluid in any container satisfying these relationships is conserved. Explain *why* this is the case.
7. Consider a new (primed) set of coordinates defined by the vectors

$$\hat{\boldsymbol{e}}_1' = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}; \qquad \hat{\boldsymbol{e}}_2' = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}; \qquad \hat{\boldsymbol{e}}_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(Coordinate axes of this sort can emerge in the calculation of phase diagrams in geochemistry.) Prove that they are mutually orthogonal and of unit length (orthonormal). Find the rotation matrix that transforms from the original into the new coordinates. Find the inverse transformation for this rotation.

8. Using Cardano's method as described in the text, solve for the roots of the cubic polynomial

$$f(z) = z^3 - 7z^2 + 14z - 8.$$

9. Suppose we have a matrix of the form

$$\begin{pmatrix} \frac{5}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{5}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solve for the eigenvalues of this tensor. Can this be a moment of inertia tensor? Explain why or why not. What are its eigenvectors? Express in similarity form this tensor in terms of its associated diagonal matrix and rotation matrices.