Chapter Two

Classical Electrodynamics

Classical electrodynamics is contained within quantum electrodynamics as a limiting case. It has a domain of validity and applicability to certain problems for which the classical treatment is clearly preferable to the more general quantum-mechanical approach. There is also a close relationship between a number of results in classical radiation theory and corresponding expressions derived in quantum electrodynamics. In fact, sometimes classical formulas have a range of validity greater than that expected on the basis of elementary considerations. The relationship between classical and quantum electrodynamics will be discussed briefly in Chapter 3. The quantum-mechanical formulation relies heavily on the classical theory as a guide, starting with the classical field Hamiltonian.

This chapter will give a purely classical treatment of radiation. It will not attempt to be a complete description of classical electrodynamics, since that general subject is treated well in several textbooks. However, starting from basic principles, a number of useful and general results will be derived. Applications to specific radiative processes will be given in later chapters.

2.1 RETARDED POTENTIALS

2.1.1 Fields, Potentials, and Gauges

In non-covariant form, the four Maxwell equations are

$$\begin{aligned}
\nabla\times\boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t},\\
\nabla\cdot\boldsymbol{E} &= 4\pi\rho,\\
\nabla\cdot\boldsymbol{B} &= 0,\\
\nabla\times\boldsymbol{E} + \frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} &= 0.
\end{aligned} \tag{2.1}$$

These are the "microscopic" Maxwell equations in terms of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. The sources of the fields are the charge and current densities and would include contributions from the polarization and magnetization of the medium. In the "macroscopic" form of Maxwell's equations, only the conduction charge densities and currents appear in the equations, which are now equations for $\boldsymbol{D}$ and $\boldsymbol{H}$ and also involve the dialectric constant and magnetic permeability. The forms (2.1) are more convenient for our purposes.

The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are physical variables in the sense that they are observable, being directly connected to physical quantities like forces. For convenience in the mathematical description of electromagnetic processes, it is useful to introduce *potentials* from which the fields are derived. Since div curl $\boldsymbol{A} = \nabla\cdot(\nabla\times\boldsymbol{A}) = 0$, the third of the equations (2.1) is satisfied automatically if we write

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \tag{2.2}$$

where $\boldsymbol{A}$ is some vector function of $\boldsymbol{r}$ and t. The fourth equation (2.1) is satisfied if we introduce a scalar function $\Phi(\boldsymbol{r}, t)$, such that

$$\boldsymbol{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}, \tag{2.3}$$

since curl grad $\Phi = \nabla\times(\nabla\Phi) = 0$. It is not *necessary* to introduce these functions $\boldsymbol{A}(\boldsymbol{r}, t)$ and $\Phi(\boldsymbol{r}, t)$; rather, it is simply convenient to do so. The vector and scalar potentials are not physical quantities in the sense that they can be measured. In fact, they are not unique, since the same $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are obtained from the potentials if they are replaced by

$$\begin{aligned} \boldsymbol{A} &\to \hat{\boldsymbol{A}} = \boldsymbol{A} + \nabla\Lambda, \\ \Phi &\to \hat{\Phi} = \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}, \end{aligned} \tag{2.4}$$

where Λ is any arbitrary function of $\boldsymbol{r}$ and t.

The equation (2.4) is called a gauge transformation, and the invariance of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ under this transformation is called gauge invariance. Although it introduces subtleties and complications in the general formulation of both classical and quantum electrodynamics, at the same time the gauge invariance can be used to facilitate calculations of electromagnetic phenomena. Because of the freedom of choice in the potentials allowed by the invariance under the transformation (2.4), a subsidiary condition can be imposed on $\boldsymbol{A}$ and Φ. The form of the subsidiary condition establishes the "choice of gauge." For example, if the *Lorentz condition*

$$\nabla\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0 \tag{2.5}$$

is imposed, the inhomogeneous Maxwell equations [first two equations (2.1)] reduce to two equations that are separable in $\boldsymbol{A}$ and Φ:

$$\begin{aligned} \nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} &= -\frac{4\pi}{c}\boldsymbol{j}, \\ \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -4\pi\rho. \end{aligned} \tag{2.6}$$

In terms of $A_\mu = (i\Phi, \boldsymbol{A})$ and $j_\mu = (ic\rho, \boldsymbol{j})$, these equations are manifestly covariant:

$$\Box^2 A_\mu = -(4\pi/c)j_\mu. \tag{2.7}$$

The Lorentz gauge condition is also covariant: $\partial_\mu A_\mu = 0$.

The class of gauges satisfying the condition (2.5) is called the Lorentz gauge; it is also called the covariant gauge. The Lorentz gauge is convenient, because of the covariance property, especially in the general formulation of classical and quantum electrodynamics. However, even within the Lorentz gauge, there is a certain degree of arbitrariness. For example, if the function $\Lambda(\boldsymbol{r}, t)$ satisfies

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0. \tag{2.8}$$

then $\hat{\boldsymbol{A}}$ and $\hat{\Phi}$ satisfy the Lorentz condition if $\boldsymbol{A}$ and Φ do.

It should be noted that the Lorentz condition (2.5) does not have any physical interpretation, although it looks like some kind of conservation equation. Rather, it is to be regarded only as a mathematical subsidiary condition imposed for convenience in the computations. In fact, other types of mathematical subsidiary relations, corresponding to other "gauges," are also convenient. For example, the condition $\nabla\cdot\boldsymbol{A} = 0$ can be imposed; this condition designates the *Coulomb gauge*.[1] It is particularly convenient in certain radiation problems and is sometimes referred to as the radiation gauge (also as the transverse gauge). Its inconvenience lies in the non-covariant nature of the subsidiary relation.

When describing radiation fields, that is, fields corresponding to propagating electromagnetic waves at large distances from their source in empty space where $\rho = 0$ and $\boldsymbol{j} = 0$, the condition

$$\nabla\cdot\boldsymbol{A} = 0 \tag{2.9}$$

can be imposed together with the condition

$$\Phi = 0. \tag{2.10}$$

For general electromagnetic fields it is not possible to impose both conditions (2.9) and (2.10), but the condition (2.10) follows—again, only for radiation fields—if the condition (2.9) is imposed.[2] Actually, the gauge corresponding to the conditions (2.9) and (2.10) can be considered to be within the Lorentz class, since the Lorentz condition (2.5) is satisfied identically. The gauge (2.9, 2.10) is very convenient in its restricted application to radiation fields, since both $\boldsymbol{E}$ and $\boldsymbol{B}$ are derived from the vector potential alone through the simple relations $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ and $\boldsymbol{E} = -(1/c)\partial\boldsymbol{A}/\partial t$. The subsidiary relation (2.9) also simplifies calculations by requiring the fields (and $\boldsymbol{A}$) to be transverse.

2.1.2 Retarded Potentials in the Lorentz Gauge

The field-source (or potential-source) equations (2.6), for Φ and for each component of $\boldsymbol{A}$ are of the form

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi s(\boldsymbol{r}, t). \tag{2.11}$$

The equation is linear in its relationship between the field (potential) $\Psi(\boldsymbol{r}, t)$ and the source $s(\boldsymbol{r}, t)$; therefore, a superposition principle will apply, such that the total field

[1] In this gauge the potential Φ satisfies $\nabla^2\Phi = -4\pi\rho$, which has a solution corresponding to an (instantaneous) Coulomb field. Hence the name *Coulomb gauge*.

[2] This can be seen readily through a substitution of Equation (2.3) into the second of equations (2.1) with $\rho = 0$. The result (2.10) can be considered as a special radiation-field solution of $\nabla^2\Phi = 0$.

will be a result of a summation over contributions from sources at various spatial points. The solution[3] of Equation (2.11) has a very well-known and particular form in terms of a volume integral over the source:

$$\Psi(\boldsymbol{r}, t) = \int \frac{s(\boldsymbol{r'}, t - |\boldsymbol{r} - \boldsymbol{r'}|/c)}{|\boldsymbol{r} - \boldsymbol{r'}|} d^3\boldsymbol{r'}. \tag{2.12}$$

This solution (2.12) exhibits the retardation effect in which the contribution to the field at time t is due to the source characteristics at the time

$$t' = t - |\boldsymbol{r} - \boldsymbol{r'}|/c, \tag{2.13}$$

earlier by an interval $\Delta t = |\boldsymbol{r} - \boldsymbol{r'}|/c$ equal to the time required for propagation at the velocity of light between $\boldsymbol{r'}$ and $\boldsymbol{r}$.

The solution (2.12), in the form of a retarded "Green function," can be derived by various means. The most direct and systematic approach evaluates the Green function in terms of its Fourier transform, with the retardation requirement a consequence of the mathematical analysis. However, a simpler derivation is possible which makes use of a convenient artificial device. We imagine a point charge and current of *variable magnitude*[4] at the origin. The potential $\Phi(\boldsymbol{r}, t)$ due to this charge $q(t)$ will then be a solution of Equation (2.6), which is of the form

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi q(t)\delta(\boldsymbol{r}), \tag{2.14}$$

where $\delta(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)$ is the three-dimensional delta function. From the inherent spherical symmetry in the problem, Φ must be a function of, in addition to the time t, only the magnitude of the radial distance r. Then $\Phi = \Phi(r, t)$ and $\nabla^2\Phi = r^{-2}\partial(r^2\partial\Phi/\partial r)/\partial r$, and if we substitute $\Phi(r, t) = \chi(r, t)/r$, away from the origin χ satisfies

$$\partial^2\chi/\partial r^2 - c^{-2}\partial^2\chi/\partial t^2 = 0. \tag{2.15}$$

This equation has the well-known solution

$$\chi(r, t) = \chi(t \pm r/c), \tag{2.16}$$

and for physical reasons (causality) we choose the minus sign in the argument of χ.

Now we consider the solution to Equation (2.14) in the neighborhood of the origin. As $r \to 0$, the spatial derivatives on the left will be much larger than the time derviative, and Φ will be a solution of the equation

$$\nabla^2\Phi = -4\pi q(t)\delta(\boldsymbol{r}) \qquad (r \to 0). \tag{2.17}$$

The solution to this equation is well known:

$$\Phi(\boldsymbol{r}, t) = q(t)/r \qquad (r \to 0). \tag{2.18}$$

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[3]Here we are referring to the "special" or "particular" solution of the inhomogenous equation (2.11). This represents the field due directly to the source $s(\boldsymbol{r}, t)$. To this solution must be added the so-called "general" solution of the corresponding homogeneous equation. The latter could represent fields not associated with the specific source $s(\boldsymbol{r}, t)$ such as that connected with some external field.

[4]It should be emphasized that this is essentially a mathematical assumption, made for convenience. This is permissable, even though we know that, because of other independent considerations, an isolated charge cannot change its magnitude.

For this solution to match the result (2.16) above for arbitrary t, the function χ must be identified with q itself, so that the solution of equation (2.14) for general r and t is

$$\Phi(\boldsymbol{r}, t) = q(t - r/c)/r. \tag{2.19}$$

For a general charge distribution we can simply add contributions, according to the superposition principle:

$$\Phi(\boldsymbol{r}, t) = \int \frac{\rho(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3\boldsymbol{r}'. \tag{2.20}$$

By similar arguments we obtain the solution to the Equation (2.6) for the vector potential:

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{c} \int \frac{\boldsymbol{j}(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3\boldsymbol{r}'. \tag{2.21}$$

The two solutions (2.20) and (2.21) are represented by Equation (2.12), and we see how the retardation effect comes in. The contribution of the source to the potentials is that swept up by a spherical wave[5] with amplitude proportional to $|\boldsymbol{r} - \boldsymbol{r}'|^{-1}$ converging on the field point $\boldsymbol{r}$ at the radial velocity c. This concept is useful in obtaining (see Section 4) the expressions for the potentials associated with a single moving point charge, i.e., the so-called Liénard-Wiechert potentials. In this case, the potential Φ, for example, is *not* given by q divided by the retarded distance.

2.2 MULTIPOLE EXPANSION OF THE RADIATION FIELD

2.2.1 Vector Potential and Retardation Expansion

The electric and magnetic fields associated with radiation can, as we have seen in the previous section, within the Lorentz gauge, be evaluated in terms of only the vector potential, Φ being set identically to zero. The vector potential at the field point $\boldsymbol{r}$ at time t is determined by the characteristics of the source current density $\boldsymbol{j}$ at the source point $\boldsymbol{r}'$ and the retarded time $t' = t - R/c$, where

$$R = |\boldsymbol{R}| = |\boldsymbol{r} - \boldsymbol{r}'|. \tag{2.22}$$

We can write the result (2.21) in the form

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}, t) &= \frac{1}{c} \int d^3\boldsymbol{r}' \frac{\boldsymbol{j}(\boldsymbol{r}', t - R/c)}{R} \\ &= \frac{1}{c} \iint d^3\boldsymbol{r}' dt' \frac{\boldsymbol{j}(\boldsymbol{r}', t')}{R} \delta(t' - t + R/c). \end{aligned} \tag{2.23}$$

The field point is at a large distance from the source (see Figure 2.1) and if the source has some localization, $r' \ll r$, in

$$R^2 = r^2 - 2\boldsymbol{r} \cdot \boldsymbol{r}' + r'^2, \tag{2.24}$$

[5]This is to be regarded as a "mathematical" rather than a physical wave.

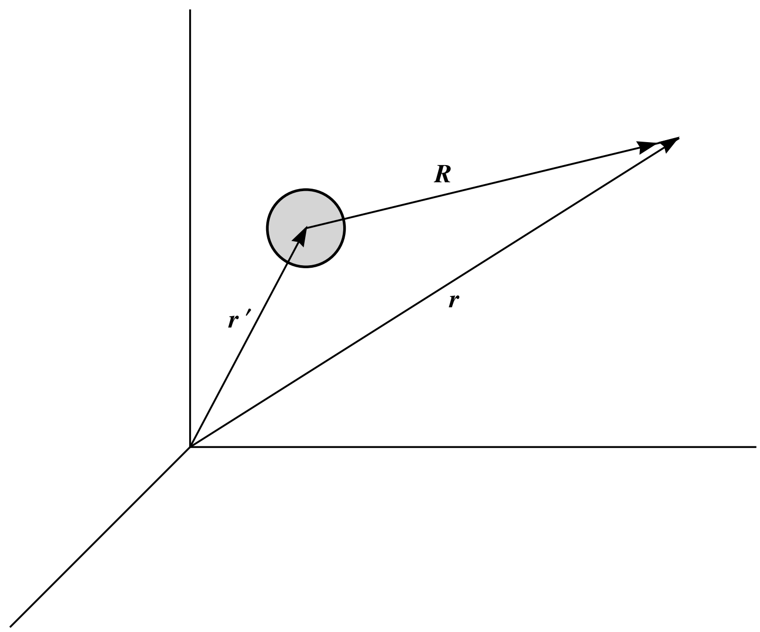

Figure 2.1 Field and source points.

the last term can be neglected. Then

$$R \approx r - \boldsymbol{n} \cdot \boldsymbol{r}', \tag{2.25}$$

where $\boldsymbol{n} = \boldsymbol{r}/r$ is a unit vector in the radial direction. In the current density

$$\boldsymbol{j}(\boldsymbol{r}', t - R/c) \approx \boldsymbol{j}(\boldsymbol{r}', t - r/c + \boldsymbol{n} \cdot \boldsymbol{r}'/c), \tag{2.26}$$

it is convenient to introduce a translated time coordinate

$$t'' = t - r/c, \tag{2.27}$$

which, significantly, does not involve the source coordinate $\boldsymbol{r}'$. Also, the factor $1/R$ in the integrand (2.23) is slowly varying and can be set equal to $1/r$ and taken outside of the integral. We then have, removing the double primes from the translated time coordinate t'',

$$\boldsymbol{A}(\boldsymbol{r}, t) \approx (cr)^{-1} \int d^3\boldsymbol{r}' \boldsymbol{j}(\boldsymbol{r}', t + \boldsymbol{n} \cdot \boldsymbol{r}'/c). \tag{2.28}$$

The retardation term $\boldsymbol{n} \cdot \boldsymbol{r}'/c$ is of the order of the time for propagation at velocity c across the source. If the source motions are non-relativistic, this time is small and the current density can be expanded:

$$\boldsymbol{j}(\boldsymbol{r}', t + \boldsymbol{n} \cdot \boldsymbol{r}'/c) = \boldsymbol{j}(\boldsymbol{r}', t) + \frac{\boldsymbol{n} \cdot \boldsymbol{r}'}{c} \frac{\partial \boldsymbol{j}(\boldsymbol{r}', t)}{\partial t} + \cdots . \tag{2.29}$$

This then yields a *multipole expansion* for the vector potential:

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{cr} \int d^3\boldsymbol{r}' \boldsymbol{j}(\boldsymbol{r}', t) + \frac{1}{c^2 r} \frac{d}{dt} \int d^3\boldsymbol{r}' (\boldsymbol{n} \cdot \boldsymbol{r}') \boldsymbol{j}(\boldsymbol{r}', t) + \cdots . \tag{2.30}$$

An expansion of this type is useful when the source motions are non-relativisitic, for which retardation effects are small. When the source consists of charges in relativistic motion, many higher multipoles contribute to the radiation field and the expansion (2.30) is not useful.

It is convenient to replace the formulation in terms of a continuum distribution of source current density by one in terms of a collection of discrete charges. If we set

$$\boldsymbol{j}(\boldsymbol{r}', t) = \sum_{\alpha} q_\alpha \boldsymbol{v}_\alpha(\boldsymbol{r}')\delta\big(\boldsymbol{r}' - \boldsymbol{r}_\alpha(t)\big), \tag{2.31}$$

where q_α is the charge of particle α and $\boldsymbol{v}_\alpha$ is its velocity, we have

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{cr}\sum_{\alpha} q_\alpha \boldsymbol{v}_\alpha + \frac{1}{c^2 r}\sum_{\alpha} q_\alpha \frac{d}{dt}\big((\boldsymbol{n}\cdot\boldsymbol{r}_\alpha)\boldsymbol{v}_\alpha\big) + \cdots . \tag{2.32}$$

But (leaving off the subscripts) we can rewrite the second term in the expansion (2.32) using

$$\begin{aligned}(\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{v} &= \frac{1}{2}\frac{d}{dt}\big((\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{r}\big) + \frac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{v} - \frac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{r} \\ &= \frac{1}{2}\frac{d}{dt}\big((\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{r}\big) + \frac{1}{2}(\boldsymbol{r}\times\boldsymbol{v})\times\boldsymbol{n}.\end{aligned} \tag{2.33}$$

The vector potential (2.32) can then be written as a sum of (electric) dipole, magnetic dipole, and quadrupole terms:

$$\boldsymbol{A} = \boldsymbol{A}_d + \boldsymbol{A}_m + \boldsymbol{A}_q + \cdots , \tag{2.34}$$

where each term has a time derivative of the corresponding moment. If we define the moments

$$\begin{aligned}\boldsymbol{p} &= \sum_{\alpha} q_\alpha \boldsymbol{r}_\alpha, \\ \boldsymbol{m} &= \sum_{\alpha} q_\alpha \boldsymbol{r}_\alpha \times \boldsymbol{v}_\alpha, \\ \boldsymbol{Q} &= 3\sum_{\alpha} q_\alpha (\boldsymbol{n}\cdot\boldsymbol{r}_\alpha)\boldsymbol{r}_\alpha,\end{aligned} \tag{2.35}$$

the expansion (2.34) can be written

$$\boldsymbol{A} = [\dot{\boldsymbol{p}} + \dot{\boldsymbol{m}} \times \boldsymbol{n} + \ddot{\boldsymbol{Q}}/6c]/cr + \cdots , \tag{2.36}$$

where the dots denote time derivatives.

2.2.2 Multipole Radiated Power

The evaluation of the electric and magnetic fields from the vector potential can be simplified for the case where only the radiation-field components are of interest. These are the fields associated with propagating plane waves and a general superposition of these waves could be represented by a sum over Fourier components;

that is, we could write a corresponding vector potential for a general radiation field in the form[6]

$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_{\boldsymbol{k}} \boldsymbol{a}_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}. \tag{2.37}$$

Here the $\boldsymbol{a}_{\boldsymbol{k}}$ are the Fourier amplitudes corresponding to wave vector $\boldsymbol{k}$ with frequency $\omega = c|\boldsymbol{k}|$. For the gauge condition div $\boldsymbol{A} = 0$ to hold identically for all $\boldsymbol{r}$ and t, for each Fourier component the transversality condition

$$\boldsymbol{a}_{\boldsymbol{k}} \cdot \boldsymbol{k} = 0 \tag{2.38}$$

must hold. We can also make use of another general form for the radiation field. Setting coordinate axes with the x-axis along the direction of propagation in the radial direction away from the source, the components of the vector potential associated with the radiation field will have the general form [see Equation (2.16)], in addition to the $1/r$ factor,[7]

$$\boldsymbol{A} \propto f(x - ct) = f(X) \tag{2.39}$$

where $X = X(x, t) = x - ct$ is the argument of the general function f. The magnetic field $\boldsymbol{B}$ = curl $\boldsymbol{A}$ results from differentiation with respect to x, but it is more convenient to express this in terms of a time derivative. Since $\partial f/\partial x = (\partial f/\partial X)(\partial X/\partial x) = \partial f/\partial X$ and $\partial f/\partial t = (\partial f/\partial X)(\partial X/\partial t) = -c(\partial f/\partial X)$, we can write $\partial f/\partial x = -(1/c)\partial f/\partial t$. In terms of the direction of propagation (away from the source) $\boldsymbol{n} = \boldsymbol{k}/k$, the general magentic field can then be written in terms of the time derivative of the vector potential:

$$\boldsymbol{B} = (1/c)\dot{\boldsymbol{A}}\times\boldsymbol{n}. \tag{2.40}$$

This is a general relation for radiation fields and holds for any gauge, including the one imposed to obtain the retarded potentials (2.20) and (2.21) and their expansions, which we are employing. These potentials do not satisfy the (sometimes convenient) radiation gauge conditions div $\boldsymbol{A} = 0$ and $\Phi = 0$, so $\boldsymbol{E} \neq -(1/c)\dot{\boldsymbol{A}}$; instead, again only for radiation fields,

$$\boldsymbol{E} = \boldsymbol{B}\times\boldsymbol{n}. \tag{2.41}$$

From the convenient relation (2.40), the radiation magnetic field can be computed in terms of the time derivatives of the various moments by means of the retardation expansion (2.36). To the various terms in $\dot{\boldsymbol{A}}$ can be added anything proportional to $\boldsymbol{n}$ without changing the (physical) quantity $\boldsymbol{B}$. It is convenient to, in this manner, replace $\boldsymbol{Q}$ in Equation (2.35) by the expression

$$\boldsymbol{Q} = 3\sum_{\alpha} q_\alpha\big[(\boldsymbol{n}\cdot\boldsymbol{r}_\alpha)\boldsymbol{r}_\alpha - \tfrac{1}{3}\boldsymbol{n}r_\alpha^2\big]. \tag{2.42}$$

The components of this vector can be expressed in terms of a quadrupole tensor through the relation $Q_j = Q_{jk}n_k$ (sum over k), where

$$Q_{jk} = \sum_{\alpha} q_\alpha(3x_jx_k - \delta_{jk}r^2)_\alpha \tag{2.43}$$

[6]Actually, the real part ("Re") of this expression should be taken.

[7]In evaluating the radiation fields $\boldsymbol{E}$ and $\boldsymbol{B}$ from $\boldsymbol{A}$, differentiation of the $1/r$ factor is neglected; the radiation fields, like the potential, have a $1/r$ falloff.

is a symmetric traceless tensor. The magnetic field (2.40) is then

$$\begin{aligned} \boldsymbol{B} &= \frac{1}{c^2 r}\big[\ddot{\boldsymbol{p}} \times \boldsymbol{n} + (\ddot{\boldsymbol{m}} \times \boldsymbol{n}) \times \boldsymbol{n} + \frac{1}{6c}\dddot{\boldsymbol{Q}} \times \boldsymbol{n}\big] \\ &= \boldsymbol{B}_d + \boldsymbol{B}_m + \boldsymbol{B}_q. \end{aligned} \tag{2.44}$$

The radiated power from a system of charges is computed from the Poynting vector

$$\boldsymbol{S} = (c/4\pi)\boldsymbol{E} \times \boldsymbol{B} = (c/4\pi)(\boldsymbol{B} \cdot \boldsymbol{B})\boldsymbol{n}, \tag{2.45}$$

which is the energy flux in the direction $\boldsymbol{n}$. Multiplying by $4\pi r^2$, we get the total radiated power in terms of a directional average (of $\boldsymbol{n}$):

$$P = dW/dt = r^2 c\overline{(\boldsymbol{B} \cdot \boldsymbol{B})}, \tag{2.46}$$

where the bar denotes a directional average. In evaluating this average, the cross terms can be shown to vanish; that is,

$$\overline{\boldsymbol{B}_d \cdot \boldsymbol{B}_m} = \overline{\boldsymbol{B}_d \cdot \boldsymbol{B}_q} = \overline{\boldsymbol{B}_m \cdot \boldsymbol{B}_q} = 0. \tag{2.47}$$

To see this, consider the "parity" of the various components of $\boldsymbol{B}$ (behavior under $\boldsymbol{n} \to -\boldsymbol{n}$). From their definitions, $\boldsymbol{B}_d$ is odd while $\boldsymbol{B}_m$ and $\boldsymbol{B}_q$ are even, and this proves the first two of the relations (2.47). The last one involving $\boldsymbol{B}_m$ and $\boldsymbol{B}_q$ is harder to prove. However, from the vector identity $(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C})$, we have

$$(\dddot{\boldsymbol{Q}} \times \boldsymbol{n}) \cdot \big[(\ddot{\boldsymbol{m}} \times \boldsymbol{n}) \times \boldsymbol{n}\big] = \big[\dddot{\boldsymbol{Q}} \cdot (\ddot{m} \times \boldsymbol{n})\big](\boldsymbol{n} \cdot \boldsymbol{n}) - (\dddot{\boldsymbol{Q}} \cdot \boldsymbol{n})\big[\boldsymbol{n} \cdot (\ddot{\boldsymbol{m}} \times \boldsymbol{n})\big], \tag{2.48}$$

but the last term in brackets on the right is zero. It then remains to prove that the directional average of $\dddot{\boldsymbol{Q}} \cdot (\ddot{\boldsymbol{m}} \times \boldsymbol{n})$ is zero. In terms of the Levi-Civita symbol [see Equation (1.73)], this is

$$\dddot{\boldsymbol{Q}} \cdot (\ddot{\boldsymbol{m}} \times \boldsymbol{n}) = \varepsilon_{jkl}\dddot{Q}_{jn}\ddot{m}_k n_n n_l, \tag{2.49}$$

and since

$$\overline{n_n n_l} = \tfrac{1}{3}\delta_{nl}, \tag{2.50}$$

the directional average is

$$\overline{\dddot{\boldsymbol{Q}} \cdot (\boldsymbol{m} \times \boldsymbol{n})} = \tfrac{1}{3}\varepsilon_{jkl}\dddot{Q}_{jl}\ddot{m}_k. \tag{2.51}$$

But we can relabel the dummy indices j and l, taking half the sum, making use of the symmetric nature of $\dddot{Q}_{jl}$ and the antisymmetric nature of ε_{jkl}, and we see easily that the result (2.51) is identically zero. Thus,

$$P = r^2 c(\overline{\boldsymbol{B}_d^2} + \overline{\boldsymbol{B}_m^2} + \overline{\boldsymbol{B}_q^2}) = P_d + P_m + P_q. \tag{2.52}$$

The terms P_d, P_m, and P_q can be expressed in terms of the individual (time derivatives of) moments through the insertion of the expressions (2.44) and (2.52). In P_d and P_m the angular averages are trivial and we obtain

$$\begin{aligned} P_d &= 2\ddot{\boldsymbol{p}}^2/3c^3, \\ P_m &= 2\ddot{\boldsymbol{m}}^2/3c^3. \end{aligned} \tag{2.53}$$

The quadrupole expression is not so easy to evaluate. Employing for now the notation $q_j = \dddot{Q}_j = \dddot{Q}_{jk}n_k = q_{jk}n_k$, we have

$$(\boldsymbol{q} \times \boldsymbol{n})^2 = \boldsymbol{q}^2 - (\boldsymbol{q} \cdot \boldsymbol{n})^2. \tag{2.54}$$

But

$$\boldsymbol{q}^2 = \boldsymbol{q} \cdot \boldsymbol{q} = q_j q_j = q_{jk} q_{jl} n_k n_l, \tag{2.55}$$

$$(\boldsymbol{q} \cdot \boldsymbol{n})^2 = (q_j n_j)^2 = q_j q_k n_j n_k = q_{jl} q_{kn} n_j n_k n_l n_n. \tag{2.56}$$

The directional average of the product $n_k n_l$ in Equation (2.55) was given in the result (2.50). The average of the four-index product in Equation (2.56) must be symmetric in the indices and expressed in terms of the "fundamental tensor" δ_{jk}. The most general such symmetric function is

$$\overline{n_j n_k n_l n_n} = a(\delta_{jk}\delta_{ln} + \delta_{jl}\delta_{kn} + \delta_{jn}\delta_{kl}), \tag{2.57}$$

where a is a constant. The constant can be evaluated by either contracting on two pairs of indices or by contracting on all four. Doing the latter, letting all four refer to the "z" or "polar" direction for a spherical polar coordinate description, from

$$\overline{n_z^4} = \frac{1}{2}\int_0^{\pi} \cos^4\theta \sin\theta \, d\theta = \frac{1}{5}, \tag{2.58}$$

we obtain $a = 1/15$. The quadrupole power radiated is then, by Equations (2.52), (2.44), (2.54)–(2.58),

$$P_q = \frac{1}{180c^5} \sum_{jk} \dddot{Q}_{jk}^2. \tag{2.59}$$

It should be noted that in P_d, P_m, and P_q the contribution results from a sum over squares of individual moment ($\ddot{p}_j, \ddot{m}_j, \dddot{Q}_{jk}$) contributions. Further, remember that these results refer to the total energy radiated, integrated over photon energies and summed over polarization states.

The multipole expansion is useful, as we have stated earlier, in the limit where the source motions are non-relativistic. In this case, the retardation effects are small and the principal contribution comes from the dipole term. Sometimes, however, because of symmetries in the source characteristics,[8] there is no dipole contribution. Compared with the (electric) dipole term, the magnetic dipole and quadrupole terms are small and of the same order of magnitude. If the characteristic velocities of the source particles are $\sim v$, from the definitions of $\boldsymbol{p}$, $\boldsymbol{m}$, and Q_{jk}, we see that

$$P_m \sim P_q \sim (v^2/c^2) P_d. \tag{2.60}$$

2.3 FOURIER SPECTRA

Depending on the details of the source characteristics, the radiation field contains a spectrum of photon energies or frequencies. If $E(t)$ and $B(t)$ represent the magnitude of the electric and magnetic fields, respectively, associated with a particular

[8]An example of this case—one that will be considered later—is that in which the source consists of two identical particles. In the scattering of two electrons, the lowest-order radiation term is quadrupole.

component[9] of polarization, the corresponding contribution to the energy flux in the magnitude of the Poynting vector in the radial direction away from the source is

$$S(t) = dW/dA\,dt = (c/4\pi)E^2(t) = (c/4\pi)B^2(t). \tag{2.61}$$

The fields can be written in terms of Fourier amplitudes:

$$E(t) = \int E_\omega e^{-i\omega t} d\omega, \quad B(t) = \int B_\omega e^{-i\omega t} d\omega, \tag{2.62}$$

where the Fourier components are determined by the specific time dependence of the fields:

$$E_\omega = (2\pi)^{-1} \int E(t) e^{i\omega t} dt, \quad B_\omega = (2\pi)^{-1} \int B(t) e^{i\omega t} dt. \tag{2.63}$$

If the source motions are periodic, instead of a continuum of frequencies, there would be a sum over a discrete spectrum:

$$E(t) = \sum_k E_k e^{-i\omega_k t}, \quad B(t) = \sum_k B_k e^{-i\omega_k t}. \tag{2.64}$$

Let us take the continuum form (2.62) and consider only the magnetic field, since it is equal in magnitude to the associated electric field (although, of course, the corresponding components are mutually perpendicular). Employing the complex forms (2.61) we should write the Poynting vector as $S(t) = (c/4\pi)|B^2(t)|$. We can also introduce the energy flux per unit frequency, integrated over time:

$$I(\omega) = dW/dA\,d\omega, \tag{2.65}$$

so that we can write for the total energy flow per unit area:

$$dW/dA = \int S(t) dt = \int I(\omega) d\omega, \tag{2.66}$$

with

$$S(t) = (c/4\pi)|B(t)|^2 = (c/4\pi) \iint B^*_{\omega'} B_\omega e^{i(\omega' - \omega)t} d\omega' d\omega. \tag{2.67}$$

When this form is substituted into the first integral on the right side of Equation (2.66) and use is made of the identity

$$\int e^{i(\omega' - \omega)t} dt = 2\pi \delta(\omega' - \omega), \tag{2.68}$$

we have

$$dW/dA = \frac{1}{2} c \int |B_\omega|^2 d\omega. \tag{2.69}$$

In the above expressions, the frequency variable ω extends from $-\infty$ to ∞, but the positive and negative frequencies in the radiation field spectrum are physically equivalent and can be lumped together. Thus, we can write

$$I_\omega \equiv I(|\omega|) = 2I(\omega) = c|E_\omega|^2 = c|B_\omega|^2. \tag{2.70}$$

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[9] It could be one of the mutually perpendicular linear components transverse to the direction of propagation.

If the element of area through which radiation is flowing is at a distance r from the source and subtends a solid angle $d\Omega$, then $dA = r^2 d\Omega$, and the differential energy radiated within frequency $d\omega$ is

$$dW_\omega = I_\omega r^2 d\omega\, d\Omega. \tag{2.71}$$

In this relation, the intensity I_ω will be proportional to $1/r^2$, so that dW_ω is independent of r.

The photon concept can be introduced into our classical formulation by writing

$$dW_\omega = \hbar\omega\, dw_\omega, \tag{2.72}$$

where dw_ω is the *probability* of photon emission within $d\omega$. Then we have

$$dw_\omega = (c/\hbar) r^2 (d\omega/\omega)\big[|E_\omega|^2 \text{ or } |B_\omega|^2\big] d\Omega \tag{2.73}$$

for the differential probability of emitting a photon within frequency $d\omega$ and within solid angle $d\Omega$. As mentioned earlier, E_ω and B_ω refer to the components associated with a particular polarization, and the corresponding probability dw_ω given by Equation (2.73) would then refer to this photon polarization state. If in dw_ω we are not interested in polarization, and want the total dw_ω summed over polarizations, the terms $|E_\omega|^2$ or $|B_\omega|^2$ in Equation (2.73) would be the sum of the squares of their values associated with each polarization.

The expression (2.73), although it contains $\hbar$, is essentially a classical one, since the only quantum mechanics introduced is the photon concept by means of the relation (2.72). For example, we have not provided the necessary quantum mechanics to calculate E_ω or B_ω from the source motions. In this chapter, we employ only classical theory to relate the fields to source characteristics. Nevertheless, the classical theory, together with the subsequent injection of the photon concept by means of the relation (2.72), does provide a description of certain phenomena often considered to be quantum mechanical in nature. One phenomenon is the so-called infrared divergence or infrared "catastrophe," which is the infinite probability of emitting infinitesimally soft photons in charged-partical processes. The simple semi-classical formulation described above is sufficient to yield this effect as well as, in fact, its explanation. We return to discuss this phenomenon in more detail later in this chapter.

Once again, it may be well to emphasize that if we are not interested in photon polarizations, the E_ω or B_ω in the emission probability (2.73) can refer to the (Fourier transform of the) magnitude of the fields. Alternatively, if the total field is expressed in terms of its two mutually perpendicular polarizations ($\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$, with $\boldsymbol{B}_1 \cdot \boldsymbol{B}_2 = 0$; similarly with $\boldsymbol{E}$), the emission probability can be expressed in terms of the sum of the individual polarization components. That is, in the result (2.73) we can write

$$|B_\omega|^2 = |B_{\omega 1}|^2 + |B_{\omega 2}|^2 \tag{2.74}$$

(similarly with $|E_\omega|^2$). For the case of dipole radiation, there is a simple and convenient relation for the angular and spectral distribution of radiation summed over polarizations. By equations (2.36), (2.40), and (2.70), we have

$$(dW/d\omega\, d\Omega)_d = |\ddot{\boldsymbol{p}}_\omega \times \boldsymbol{n}|^2/c^3, \tag{2.75}$$

in terms of the Fourier transform of $\ddot{\boldsymbol{p}}$. When we integrate over angles of emission, the spectrum of energy radiated is

$$(dW/d\omega)_d = (4\pi/c^3)|\ddot{\boldsymbol{p}}_\omega|^2\overline{\sin^2\theta} = (8\pi/3c^3)|\ddot{\boldsymbol{p}}_\omega|^2; \tag{2.76}$$

here θ is the angle between $\ddot{\boldsymbol{p}}_\omega$ and $\boldsymbol{n}$, and the spherical average of $\sin^2\theta$ is 2/3. The result (2.76) can be put in slightly different form, since $\ddot{\boldsymbol{p}}_\omega = -\omega^2\boldsymbol{p}_\omega$, and for a single charge the second time derivative of the dipole moment is directly related to the particle acceleration and thus to the force on it. That is, for a charge q the emission spectrum is directly related to the Fourier transform of its acceleration:

$$(dW/d\omega)_d = (8\pi q^2/3c^3)|\boldsymbol{a}_\omega|^2. \tag{2.77}$$

We shall return to further general developments concerning radiation field spectra and applications to specific processes later in this chapter and in later chapters.

2.4 FIELDS OF A CHARGE IN RELATIVISTIC MOTION

2.4.1 Liénard-Wiechert Potentials

In the convenient Lorentz gauge, the vector and scalar potentials associated with a distribution of charges and currents are given by the expressions (2.20)–(2.23). An important application of these retarded potentials is to the case of a single particle of charge q moving at an arbitrary velocity $\boldsymbol{v}$. As mentioned earlier, for this problem—even in the case of a point particle—the potentials are *not* given by the non-relativistic values ($R = |\boldsymbol{r} - \boldsymbol{r}'|$)

$$\Phi_{\rm NR} \to q/R, \quad \boldsymbol{A}_{\rm NR} \to q\boldsymbol{v}/cR, \tag{2.78}$$

at the retarded position $\boldsymbol{r}'$ and time t'. This is immediately obvious, since the expressions (2.78) do not form the components of a four-vector. Actually, it is not difficult to construct a covariant form for $A_\mu = (i\Phi, \boldsymbol{A})$ that reduces to the non-relativistic limits (2.78) for $v \ll c$. From $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$, the source-to-field point, the four-vector

$$R_\mu = (iR, \boldsymbol{R}) = (ic(t - t'), \boldsymbol{r} - \boldsymbol{r}') \tag{2.79}$$

can be formed [see Equation (2.22)]; this is a null four-vector:

$$R_\mu R_\mu = 0. \tag{2.80}$$

Employing the four-vector velocity $v_\mu = \gamma(ic, \boldsymbol{v})$, a scalar

$$R_\nu v_\nu = -\gamma(Rc - \boldsymbol{R}\cdot\boldsymbol{v}) \tag{2.81}$$

can be constructed that reduces to $-Rc$ in the limit $v \ll c$. The expression

$$A_\mu = -q[v_\mu/R_\nu v_\nu], \tag{2.82}$$

where the brackets are meant to imply the imposition of the retardation condition, satisfies the covariance requirement and reduces to the required limit in a Lorentz frame where the motion is non-relativistic. It is, therefore, a correct general formula valid in a frame where the particle velocity is arbitrarily large.

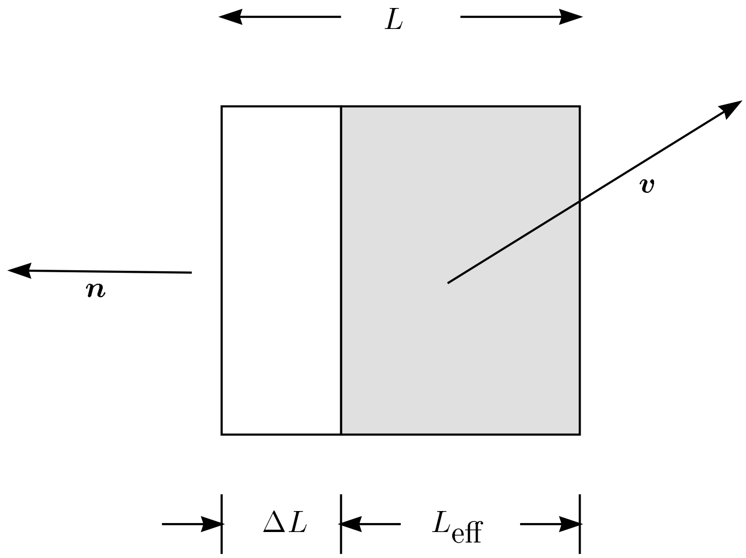

Figure 2.2 “Effective volume” associated with the field of a moving charge.

In terms of the source-to-field point unit vector $\boldsymbol{n} = \boldsymbol{R}/R$, $\boldsymbol{\beta} = \boldsymbol{v}/c$, and

$$\varkappa \equiv 1 - \boldsymbol{n} \cdot \boldsymbol{\beta}, \tag{2.83}$$

the result (2.82) is

$$A_\mu = (i\Phi, \boldsymbol{A}) = [iq/\varkappa R, q\boldsymbol{v}/c\varkappa R]. \tag{2.84}$$

The magnitude of the retardation effect is expressed in the factor $\varkappa^{-1}$. Although we have made use of covariance considerations in deriving this result, this correction factor is not relativistic in nature, the modification being linear in v. The correction is due totally to effects of retardation in the relationship between source motions and field (potential) amplitudes.

The expressions (2.84) are called the *Liénard-Wiechert potentials*, and their derivation as given above, although simple, perhaps obscures the meaning of, in particular, the correction factor $\varkappa^{-1}$. One way of interpreting the factor $q/\varkappa$ is as an “effective charge” (q_{eff}). To see this, refer back to the expressions (2.20)–(2.23) for the potentials that exhibit the manner in which an imaginary spherical wave converging on the field point $\boldsymbol{r}$ at time t sweeps past the charge in motion at the source point $(\boldsymbol{r}', t')$. We assume, for convenience, that the charge has finite spatial extent although, in the end, we can let it approach the point-particle limit. Also for convenience, imagine the field point at the origin, so that $\boldsymbol{n}$ points toward the origin. Then if $\boldsymbol{v}$ is pictured as having a positive component away from the origin, $\boldsymbol{n} \cdot \boldsymbol{v}$ is negative. Now consider the contribution to $\boldsymbol{A}$ and Φ from a cubic (or cylindrical) element of the charge with flat front and back perpendicular to $\boldsymbol{n}$ (see Figure 2.2). During the time Δt that the wave front passes the charge element, the back end of the charge has moved a distance ΔL, so that the *effective* amount of charge that the wave sees is

$$q_{\text{eff}} = q(L_{\text{eff}}/L), \tag{2.85}$$

where[10] L is the length of the element in the radial ($-\boldsymbol{n}$) direction, and

$$L_{\text{eff}} = L + \Delta L \tag{2.86}$$

(remember that ΔL is negative if $\boldsymbol{n}\cdot\boldsymbol{v}$ is negative). But $\Delta L = \boldsymbol{n}\cdot\boldsymbol{v}\Delta t$, $\Delta t = L_{\text{eff}}/c$ and, by the relations (2.85) and (2.86), we have

$$q_{\text{eff}} = q(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})^{-1} = q/\varkappa. \tag{2.87}$$

The spatial extent of the charge can now be made infinitesimally small with no effect on the correction factor $\varkappa^{-1}$.

2.4.2 Charge in Uniform Motion

The fields associated with a charge moving at constant velocity can be computed easily, and the results are of interest in connection with, for example, a topic (Weizsäcker-Williams Method) that is treated later in this chapter. For the case of uniform motion the particle retarded position (P_r) and present position (P_p) are related in a simple way to the field point (P_f), as is indicated in Figure 2.3. In terms of the distance

$$s \equiv \varkappa R = R - \boldsymbol{R}\cdot\boldsymbol{\beta}, \tag{2.88}$$

the vector and scalar potentials are given by

$$\boldsymbol{A} = q\boldsymbol{\beta}/s, \quad \Phi = q/s, \tag{2.89}$$

where, for convenience in the notation, the brackets indicating retardation have been omitted. The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are computed from $\boldsymbol{A}$ and Φ by differentiation with respect to the field-point coordinates. To do this, it is convenient to employ a coordinate system (K_0) with the origin at the instantaneous present position of the particle. Then, by Equations (2.2) and (2.3), we have $E_x = -\partial\Phi/\partial x_0 - c^{-1}\partial A_x/\partial t_0$, etc. However, since the fields are carried by the particle (moving in the x- or x_0-direction), the time derivatives can be computed from

$$\frac{\partial}{\partial t_0} = -v\frac{\partial}{\partial x_0}. \tag{2.90}$$

Also (see Figure 2.3),

$$\begin{aligned} s^2 &= r_0^2 - (R\beta\sin\theta)^2 = r_0^2(1 - \beta^2\sin^2\psi) \\ &= x_0^2 + y_0^2 + z_0^2 - \beta^2(y_0^2 + z_0^2). \end{aligned} \tag{2.91}$$

The vector potential has only an x-component (direction of $\boldsymbol{v}$) and we obtain for the fields:

$$\begin{aligned} \boldsymbol{E} &= qs^{-3}(1 - \beta^2)\boldsymbol{r}_0, \\ \boldsymbol{B} &= \boldsymbol{\beta}\times\boldsymbol{E}. \end{aligned} \tag{2.92}$$

[10] The lengths are, of course, lab-frame values, which are different from those in the particle rest frame.

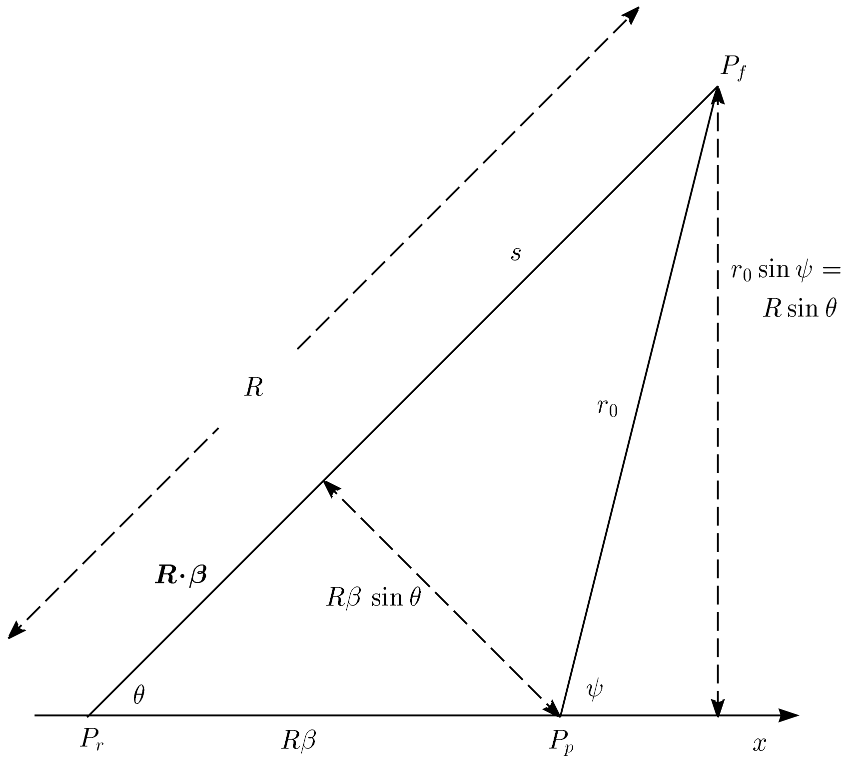

Figure 2.3 Present and retarded positions for a charge in uniform motion.

Note that $\boldsymbol{r}_0$ is the vector from the *present* position of the charge to the field point (rather than from the retarded position). The longitudinal and transverse components of the fields should also be noted, especially in the extreme relativistic limit. The magnitude of the electric field in the transverse direction (perpendicular to the x-axis) can be expressed as a function of ψ and r_0 or as a function of ψ and the "impact" parameter $b = r_0 \sin\psi$:

$$E_t = \frac{q}{r_0^2}\frac{(1-\beta^2)\sin\psi}{(1-\beta^2\sin^2\psi)^{3/2}} = \frac{q}{b^2}\frac{(1-\beta^2)\sin^3\psi}{(1-\beta^2\sin^2\psi)^{3/2}}. \tag{2.93}$$

The longitudinal component is

$$E_l = \frac{q}{r_0^2}\frac{(1-\beta^2)\cos\psi}{(1-\beta^2\sin^2\psi)^{3/2}} = \frac{q}{b^2}\frac{(1-\beta^2)\sin^2\psi\cos\psi}{(1-\beta^2\sin^2\psi)^{3/2}}. \tag{2.94}$$

The magnetic field has only a transverse component, which, at the field point, is perpendicular to both $\boldsymbol{E}_t$ and $\boldsymbol{\beta}$ [see Equation (2.92)]:

$$|\boldsymbol{B}| = \beta E_t. \tag{2.95}$$

The dependence of the fields on ψ provides a description of their time dependence as seen by an observer at the fixed field point P_f as the particle (at P_p) passes. The largest electric field component is the transverse field E_t, which attains a value $E_t \to (q/b^2)(1-\beta^2)^{-1/2}$ as $\psi \to \pi/2$. The longitudinal component E_l does not attain this value and, in fact, changes sign at $\psi = \pi/2$. As $\beta \to 1$, the electric field E_t and its accompanying magnetic field (2.95) become large at $\psi = \pi/2$, and the observer at P_f sees a strong *transverse pulse*. The idea behind the Weizsäcker-Williams method is that this pulse can be regarded as a flux of photons (see Section 8).

2.4.3 Fields of an Accelerated Charge

For a charge in non-uniform motion the calculation of the fields is a more complicated problem. The case of finite acceleration is of great importance, however, because there are now radiation-field components; that is, there are electric and magentic fields that fall off as $1/r$ at a large distance from the charge. The existence of these fields is due to, as in the non-relativistic problem, the effects of retardation. In the computation of the fields from the spatial and time derivatives of the potentials, the retardation results in a certain amount of complexity. This is because, in evaluating the spatial gradients with respect to the field-point coordinate ($\boldsymbol{r}$), it is necessary to include the explicit functional dependence of the potentials on $\boldsymbol{r}$ and dependence that is contained in the retardation condition.

We consider the motion of the source particle to be known; that is, $\boldsymbol{r}' = \boldsymbol{r}'(t')$ is specified, as is its velocity $\boldsymbol{v} = \partial \boldsymbol{r}'/\partial t'$ and acceleration $\dot{\boldsymbol{v}} = \partial \boldsymbol{v}/\partial t'$ (we leave the primes off $\boldsymbol{v}$ asnd $\dot{\boldsymbol{v}}$). The retarded time coordinate t' is the independent variable in the problem. To compute the fields $\boldsymbol{E} = -\nabla \Phi - (1/c)\partial \boldsymbol{A}/\partial t$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ from the Liénard-Wiechert potentials (2.84), it is necessary to express the derivatives with respect to the retarded time t'. This is easily accomplished through consideration of the mathematical statement of the retardation condition, that is, Equation (2.80) or

$$R = \left[(\boldsymbol{r} - \boldsymbol{r}') \cdot (\boldsymbol{r} - \boldsymbol{r}')\right]^{1/2} = c(t - t'). \qquad (2.96)$$

Differentiating with respect to t, we have

$$\frac{\partial R}{\partial t} = c\left(1 - \frac{\partial t'}{\partial t}\right) = \frac{\partial R}{\partial t'}\frac{\partial t'}{\partial t} = -\boldsymbol{n} \cdot \boldsymbol{v}\frac{\partial t'}{\partial t}. \qquad (2.97)$$

Then

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \frac{R}{s}\frac{\partial}{\partial t'} = \frac{1}{\varkappa}\frac{\partial}{\partial t'}. \qquad (2.98)$$

We also need a convenient expression for the gradient operator ∇, which can be written

$$\nabla = \nabla_{\boldsymbol{r}} + \nabla_{t'} = \nabla_{\boldsymbol{r}} + (\nabla t')\partial/\partial t'. \qquad (2.99)$$

Here $\nabla_{\boldsymbol{r}}$ means differentiation with respect to the field-point coordinate $\boldsymbol{r}$, ignoring the dependence contained within the retardation condition, and $\nabla_{t'}$ accounts for the latter contribution. The factor $\nabla t'$ can be found from the retardation condition (2.96), recognizing that varying $\boldsymbol{r}$ implies a variation in t':

$$\begin{aligned} \nabla t' = -\frac{1}{c}\nabla R = &= -\frac{1}{c}\left(\frac{\boldsymbol{R}}{R} + \frac{\partial R}{\partial t'}\nabla t'\right) \\ &= -\frac{1}{c}\left(\frac{\boldsymbol{R}}{R} - \boldsymbol{n} \cdot \boldsymbol{v}\nabla t'\right). \end{aligned} \qquad (2.100)$$

Solving for $\nabla t'$, we have $\nabla t' = -\boldsymbol{R}/sc$, and

$$\nabla = \nabla_{\boldsymbol{r}} - \frac{\boldsymbol{R}}{sc}\frac{\partial}{\partial t'}. \tag{2.101}$$

The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are derived from the potentials $\boldsymbol{A}(= q\boldsymbol{v}/cs)$ and $\Phi(= q/s)$, employing the operators (2.98) and (2.101). Carrying through these operations and rearranging the terms, we find

$$\frac{\boldsymbol{E}}{q} = \frac{\boldsymbol{R}_v}{s^3}(1-\beta^2) + \frac{1}{c^2 s^3}\boldsymbol{R}\times(\boldsymbol{R}_v\times\dot{\boldsymbol{v}}), \tag{2.102}$$

$$\boldsymbol{B} = \boldsymbol{n}\times\boldsymbol{E}, \tag{2.103}$$

where

$$\boldsymbol{R}_v = \boldsymbol{R} - R\boldsymbol{\beta} \tag{2.104}$$

is the "virtual present source-to-field-point vector." That is, if the source (charge) at the retarded position and velocity were to move at constant velocity, it would be at a certain "virtual" position, and $\boldsymbol{R}_v$ is the vector from that position to the field point. Compare this result with that for a charge in uniform motion (see Figure 2.3). Again, the brackets [ ] designating retardation have been left off the terms on the right-hand side.

For considerations of radiation effects, the important term in the field (2.102) is the second, since it falls off as $1/r$ at large distances. It then yields a Poynting vector component $S \propto 1/r^2$ and an energy flow rate $dW/dt = S\,dA \propto d\Omega$ through an element of area $dA = r^2 d\Omega$ and solid angle $d\Omega$. The radiation term is finite only when there is an acceleration of the particle, and it is a consequence of the effects of retardation.

2.5 RADIATION FROM A RELATIVISTIC CHARGE

It is convenient to express the radiation fields in terms of the dimensionless quantities $\boldsymbol{n} = \boldsymbol{R}/R$, $\boldsymbol{\beta} = \boldsymbol{v}/c$, and $\varkappa = 1 - \boldsymbol{n}\cdot\boldsymbol{\beta}$. Then the second term on the right of Equation (2.102) can be written

$$\boldsymbol{E}_{\text{rad}} = (q/cR\varkappa^3)\boldsymbol{n}\times\big((\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\big). \tag{2.105}$$

The Poynting vector at the field point $\boldsymbol{r}$, t is then

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{r},t) &= \boldsymbol{n}(c/4\pi)|\boldsymbol{E}_{\text{rad}}|^2 \\ &= \boldsymbol{n}\frac{q^2}{4\pi c}\left[\frac{\boldsymbol{n}\times\big((\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\big)}{\varkappa^3 R}\right]^2,\end{aligned} \tag{2.106}$$

where the brackets have now been reinserted to emphasize that quantities therein are to be evaluated at the retarded coordinates. That is, for example, $\boldsymbol{\beta}c$ and $\dot{\boldsymbol{\beta}}c$ are the velocity and acceleration at $\boldsymbol{r}'$ and $t' = t - R/c$.

The energy flow rate per unit solid angle $d\Omega = dA/R^2$ at $\boldsymbol{r}$, t would be computed from

$$dW/dt\,d\Omega = R^2 S, \tag{2.107}$$

but this is *not* equal to charge's energy radiation rate. If dE' is the change in the energy of the charged particle as a result of the emission of radiation, we can write $dE' = -dW$. However, the interval of time dt associated with the passage of photons through the element of area dA at the field point $\boldsymbol{r}$ is different from the interval dt' corresponding to the emission of these photons by the charge at $\boldsymbol{r'}$. The relationship between these intervals has already been derived and is given by the result (2.98): $dt'/dt = R/s = 1/\varkappa$. Then the rate of radiation of energy by the charge is

$$-dE'/dt' = (dW/dt)(dt/dt') = \varkappa\, dW/dt. \tag{2.108}$$

The times t and t' both refer to events in the lab frame; that is, the prime does not refer to a different Lorentz frame. The $\varkappa$ factor is associated with retardation effects, being first order in β. Since, also, $d\Omega = d\Omega'$, the angular distribution of emitted radiation is given by

$$-\frac{dE'}{dt'd\Omega'} = \frac{q^2}{4\pi c}\frac{\left[\boldsymbol{n}\times\big((\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\big)\right]^2}{\varkappa^5}. \tag{2.109}$$

Employing the well-known identity for the triple vector product, we can write the quantity in brackets in Equation (2.109) as

$$\begin{aligned}\boldsymbol{n}\times\big((\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\big) &= (\boldsymbol{n}-\boldsymbol{\beta})(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}}) - \dot{\boldsymbol{\beta}}\big(\boldsymbol{n}\cdot(\boldsymbol{n}-\boldsymbol{\beta})\big)\\ &= (\boldsymbol{n}-\boldsymbol{\beta})(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}}) - \varkappa\dot{\boldsymbol{\beta}}.\end{aligned} \tag{2.110}$$

Squaring this expression then yields the squared bracket in Equation (2.109):

$$\left[\boldsymbol{n}\times\big((\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\big)\right]^2 = \varkappa^2\dot{\beta}^2 + 2\varkappa(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}) - (1-\beta^2)(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})^2. \tag{2.111}$$

This relation is convenient for an evaluation of the angular integration over $d\Omega'$ to obtain the total radiative energy loss rate. The form (2.111), together with the factor $\varkappa^{-5}$, describes the angular distribution of the emitted radiation. In the non-relativistic limit, the result is the familiar dipole pattern of the form $\sin^2\vartheta$, where ϑ if the angle between $\boldsymbol{n}$ and $\dot{\boldsymbol{\beta}}$. The result in the extreme relativistic limit ($\beta \to 1$) is more interesting, since in this limit $\varkappa$ can get very small when $\boldsymbol{n}$ is along $\boldsymbol{\beta}$. In this case, the distribution is peaked in the direction of the instantaneous velocity $\boldsymbol{\beta}$; if θ is the angle between $\boldsymbol{n}$ and $\boldsymbol{\beta}$, it is easy to show that the angular width is of the order $\Delta\theta \sim \gamma^{-1} = (1-\beta^2)^{1/2} \ll 1$. The effect is basically kinematic in nature and could also be derived through considerations of transformations between the lab frame and one in which the motion is non-relativistic.

Another characteristic of the angular distribution in the general case should be mentioned. From the form (2.109) we see that the intensity is zero in two directions: when $\boldsymbol{n}-\boldsymbol{\beta}$ is along and opposite to the direction of $\dot{\boldsymbol{\beta}}$.

To compute the total rate of radiation of energy, an integration over $d\Omega'$ (directions of $\boldsymbol{n}$) can be performed. It is convenient to employ coordinate axes in which one axis (say, the y-axis) is instantaneously (at the retarded time) aligned with the velocity $\boldsymbol{\beta}$. Choosing this axis as, in addition, the polar axis of a spherical polar coordinate system describing $\boldsymbol{n} = (\sin\theta\cos\varphi, \cos\theta, \sin\theta\sin\varphi)$, we have $\varkappa = 1 - \beta\cos\theta$. In the terms on the right-hand side of Equation (2.111) the integrations over the azimuthal angle φ are trivial. The subsequent integrations over $d\theta$ are elementary, although the algebra is a little tedious. The result can be written

$$-dE'/dt' = (2q^2/3c)\gamma^6\big[\dot{\beta}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2\big]. \tag{2.112}$$

The derivation of the form (2.112) as outlined above requires a considerable amount of work, especially if we include that necessary to obtain the angular distribution (2.109). However, there is much simpler method of obtaining the energy radiation rate (2.112) that makes use of ideas of covariance. The differential energy radiated can be written $dE_{\text{rad}} = E\,dN$, where E is the photon energy and dN is the differential number of photons[11] emitted. As we have seen in Section 1.2.3, the ratio dt/E associated with photon coordinates is an invariant, where the photons may be moving in any direction. Then the radiated power dE_{rad}/dt must be an invariant or scalar. But we have already derived the form for this scalar in the non-relativistic limit; this is the dipole or lowest-order term (2.53), which is, of course, contained within the rate (2.112) when $\beta \ll 1$. It is always possible to consider a Lorentz frame where the motion is, at a particular time, non-relativistic. In this limit, the radiative energy loss is

$$-dE/dt = (2q^2/3m^2c^3)\dot{p}_j\dot{p}_j, \tag{2.113}$$

with a sum over the three spatial indices j. Since dE/dt (for this particular problem) is an invariant, the exact relativistic expression must be a scalar function of the particle's kinematic quantities. It is easy to guess the appropriate generalization, the validity of which is established by its agreement with the result (2.113) in a Lorentz frame where the motion is non-relativistic. The relativistic generalization is obtained through the replacement

$$\dot{p}_j\dot{p}_j \rightarrow (dp_\mu/d\tau)(dp_\mu/d\tau), \tag{2.114}$$

and it remains to show that the covariant form is identical to the non-covariant expression (2.112). Since $d\tau = dt/\gamma$, and $p_\mu = \gamma mc(i, \boldsymbol{\beta})$, we have

$$(dp_\mu/d\tau)(dp_\mu/d\tau) = m^2c^2\gamma^2\big[(\dot{\gamma}\boldsymbol{\beta} + \gamma\dot{\boldsymbol{\beta}})^2 - \dot{\gamma}^2\big]. \tag{2.115}$$

But $\dot{\gamma} = \gamma^3\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}$, $\dot{\beta}^2 - (\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 = (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2$, and elementary algebraic manipulations yield the form (2.112).

This very simple derivation illustrates again the very considerable power of covariance considerations. For example, the non-relativistic expression (2.113) contains only the dipole contribution, while the relativistic result includes contributions from many multipoles. Yet, the exact formula has been obtained from its limiting form without having to employ the formulation of radiation theory for relativistic particles. The relativistic result (2.112) is, in addition, of great significance in exhibiting important features of the radiation phenomenon. In particular, the factor γ^6 indicates the increased *efficiency* of the radiation phenomenon for relativistic particles. This is especially so for the case where the accelerating force is perpendicular to the particle velocity. The force is related to the acceleration by

$$\boldsymbol{F} = mc\big[\gamma^3(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta} + \gamma\dot{\boldsymbol{\beta}}\big]. \tag{2.116}$$

[11] It is really not necessary to introduce the photon concept here, but it is convenient.

When $\boldsymbol{F}$ is along $\boldsymbol{\beta}$ and $\gamma \gg 1$, $\dot{\boldsymbol{\beta}} \to F/mc\gamma^3$ and the radiated power approaches

$$P_{\parallel} \to (2q^2/3m^2c^3)F^2, \tag{2.117}$$

which is independent of energy. On the other hand, if $\boldsymbol{F}$ is perpendicular to $\boldsymbol{\beta}$, we see from Equation (2.116) that $\dot{\beta} \to F/mc\gamma$, and

$$P_{\perp} \to (2q^2/3m^2c^3)F^2\gamma^2, \tag{2.118}$$

increasing as the square of the energy. Comparison of the two results shows why, for example, in the field of high-energy physics, radiative losses are far more important in synchrotrons than in linear accelerators (linacs). The comparison is relevant for astrophysics, where nature has provided both high-energy particles and magnetic fields, which provide the (perpendicular) deflecting force. The associated "synchrotron radiation" from relativistic electrons in cosmic radio sources is produced, as we see, by a very efficient mechanism.

2.6 RADIATION REACTION

The radiation of energy by an accelerated charge has an effect on the motion of the particle that can be described in terms of a radiation reaction force. The form of the expression for this force can be established through considerations of energy conservation. However, a more revealing derivation of the result involves a calculation of the force from various elements of the charge acting on one another. Because of the effects of retardation, the total force is not zero and there results a total "self-force." Treatment of the phenomenon also involves the evaluation of the charge's "self-energy," and, in this simple classical problem, we can introduce the ideas of "renormalization." A full treatment of the problem is clearly beyond the scope of this book. We only touch on the subject, first in the non-relativistic limit—or in a reference frame where the motion is such that $\beta \ll 1$.

2.6.1 Non-Relativistic Limit

Radiation reaction was considered by Lorentz, who first introduced the electron into the subject of electromagnetic theory. For definiteness in the formulation, our charged particle is referred to as an electron, although the classical description would be the same for any particle. We consider a slowly moving electron of finite size and compute the reaction force that results from the interaction of the radiation field from different parts of the electron acting on other parts (see Figure 2.4). Let de and de' be two elements of charge on the electron, located at $\boldsymbol{r}$ and $\boldsymbol{r'}$, respectively. The reaction force is computed from

$$\boldsymbol{F} = \iint de\, d\boldsymbol{E}, \tag{2.119}$$

where $d\boldsymbol{E}$ is the differential electric field at $\boldsymbol{r}$ due to the charge element de' at the source point $\boldsymbol{r'}$. The double integral is then over the elements de and de'.

It is convenient to evaluate the radiation reaction force in an inertial frame in which the electron is instantaneously at rest. The radiation-reaction force terms are

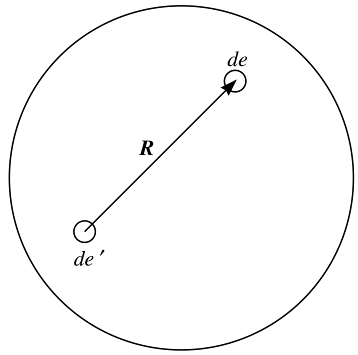

Figure 2.4 Infinitesimal charge elements associated with the same charge distribution.

proportional to $\dot{\boldsymbol{v}}$ and to higher derivatives of the velocity, and these are the same in the lab frame as in the instantaneous rest frame. The integration over de in the double integral (2.119) is instantaneous, but the differential field $d\boldsymbol{E}$ at de is due to the motion of the element de' at the retarded time at which the velocity is not zero. In terms of the source-to-field point distance $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$, $\boldsymbol{n} = \boldsymbol{R}/R$, and $\varkappa = 1 - \boldsymbol{n} \cdot \boldsymbol{\beta}$, the differential field $d\boldsymbol{E}$ due to de' would be given by the expression (2.102):

$$d\boldsymbol{E} = \frac{de'}{R^3\varkappa^3}\left(1 - \frac{v^2(t')}{c^2}\right)\left(\boldsymbol{R} - \frac{R}{c}\boldsymbol{v}(t')\right) + \frac{1}{c^2}\boldsymbol{R} \times \left[\left(\boldsymbol{R} - \frac{R}{c}\boldsymbol{v}(t')\right) \times \dot{\boldsymbol{v}}(t')\right]. \tag{2.120}$$

Note that all quantities on the right are retarded values; that is, $\dot{\boldsymbol{v}}(t') = d\boldsymbol{v}(t')/dt'$, and $\varkappa = 1 - \boldsymbol{n} \cdot \boldsymbol{v}(t')/c$. The retarded time is $t' = t - R/c$, where at the present time the velocity $\boldsymbol{v}(t)$ is zero. We then express the (retarded) velocity and acceleration as expansions in terms of quantities at the time t:

$$\begin{aligned} \boldsymbol{v}(t') &= -(R/c)\dot{\boldsymbol{v}}(t) + \frac{1}{2}(R/c)^2\ddot{\boldsymbol{v}}(t) + \cdots, \\ \dot{\boldsymbol{v}}(t') &= \dot{\boldsymbol{v}}(t) - (R/c)\ddot{\boldsymbol{v}}(t) + \cdots. \end{aligned} \tag{2.121}$$

In the expression for $d\boldsymbol{E}$ we neglect higher-derivative terms ($\dddot{\boldsymbol{v}}$, etc.) and consider only terms linear in $\boldsymbol{v}$, $\dot{\boldsymbol{v}}$, and $\ddot{\boldsymbol{v}}$, neglecting terms in v^2, $v\dot{v}$, etc.; terms having coefficients c^{-p} with $p > 3$ are also ignored. Then

$$\begin{aligned} \varkappa &= 1 + \boldsymbol{R} \cdot \dot{\boldsymbol{v}}/c^2 - (R/2c^3)\boldsymbol{R} \cdot \ddot{\boldsymbol{v}} + \cdots, \\ \varkappa^{-3} &= 1 - 3\boldsymbol{R} \cdot \dot{\boldsymbol{v}}/c^2 + (3R/2c^3)\boldsymbol{R} \cdot \ddot{\boldsymbol{v}} + \cdots, \end{aligned} \tag{2.122}$$

where $\dot{\boldsymbol{v}}$ and $\ddot{\boldsymbol{v}}$ are the values at time t. The differential electric field is, neglecting the higher-order terms in Equation (2.120),

$$d\boldsymbol{E} = de'\left[-\frac{2\boldsymbol{R}(\dot{\boldsymbol{v}} \cdot \boldsymbol{R})}{R^3c^2} + \frac{\boldsymbol{R}(\boldsymbol{R} \cdot \ddot{\boldsymbol{v}})}{2R^2c^3} + \frac{\boldsymbol{R}}{R^3} + \frac{\ddot{\boldsymbol{v}}}{2c^3} + \cdots\right]. \tag{2.123}$$

For component k of the field, summing over dummy index j, the result can be written

$$dE_k = de'\left[-\frac{2R_k\dot{v}_jR_j}{R^3c^2} + \frac{R_kR_j\ddot{v}_j}{2R^2c^3} + \frac{R_k}{R^3} + \frac{\ddot{v}_k}{2c^3} + \cdots\right]. \tag{2.124}$$

The total force on the electron is found from Equation (2.119) by integrating over de' and de. For a spherically symmetric distribution,

$$\begin{aligned}&\iint de\,de' f(R)R_k = 0,\\ &\iint de\,de' f(R)R_kR_j = \frac{1}{3}\delta_{kj}\iint de\,de' f(R)R^2,\end{aligned} \tag{2.125}$$

and we have

$$\boldsymbol{F} = \frac{2e^2}{3c^3}\ddot{\boldsymbol{v}} - \frac{4W_{\mathrm{el}}}{3c^2}\dot{\boldsymbol{v}} + \cdots, \tag{2.126}$$

where

$$W_{\mathrm{el}} = \frac{1}{2}\iint de\,de'/R \tag{2.127}$$

is the electrostatic self-energy. The result can be written

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{rr}} - m_{\mathrm{el}}\dot{\boldsymbol{v}}, \tag{2.128}$$

where the radiation reaction $\boldsymbol{F}_{\mathrm{rr}}$ is given by the first term on the right-hand side of Equation (2.126), and $m_{\mathrm{el}} = 4W_{\mathrm{el}}/3c^2$ is the "electromagentic mass." Note the factor 4/3; it was suggested long ago by Poincaré that there should be non-electromagnetic binding forces providing a "glue" to stabilize the electron, and that the associated binding energy is $\frac{1}{3}W_{\mathrm{el}}$. In fact, if there are external forces acting on the electron, it is appropriate to write the equation of motion in the form

$$m_0\dot{\boldsymbol{v}} - (2e^2/3c^3)\ddot{\boldsymbol{v}} = \boldsymbol{F}_{\mathrm{ext}}, \tag{2.129}$$

where the electromagnetic and mechanical inertia terms are combined into an "observed" mass m_0, which can be determined experimentally. Moreover, with m_0 determined empirically, it is then not necessary to know what percentage of it is electromagnetic in origin. In fact, for a point particle, W_{el} diverges. There are higher terms in $\boldsymbol{F}$ that would result from, for example, the next-order terms in the expansions (2.121), and these would be of the order $F_{\mathrm{rr}}(\dddot{\boldsymbol{v}}/\ddot{\boldsymbol{v}})r_0/c$, where r_0 is the electron radius. For a point particle these terms go to zero, but, of course, W_{el} is infinite as $r_0 \to 0$. This is what classical electron theory in this elementary formulation has to live with. On the other hand, the expression for $\boldsymbol{F}_{\mathrm{rr}}$ is independent of the structure of the electron.

The combining of electromagnetic mass into a total observed mass is an essential part of what is known as "renormalization" in modern quantum electrodynamics. The idea goes back to J. J. Thomson in classical electron theory. It is clear from elementary considerations that quantum mechanics must be brought into the problem. There are still divergences in quantum electrodynamics, although they are not as severe (logarithmic). Nevertheless, quantum electrodynamics does treat the electron as a point particle and these divergences are there, so there is still is a certain amount of, as Feynman used to say, "sweeping the dirt under the rug." We seem to be able to endure these difficulties, however.

There is an extensive literature on electron theory—even on the classical theory and in recent decades. Suggested references are listed at the end of this chapter.[12]

2.6.2 Relativistic Theory: Lorentz-Dirac Equation

The radiation-reaction force is a real effect whose magnitude is increased when charged particles experience rapid changes in their acceleration. It is then of interest to obtain a relativistic generalization of the non-relativistic equation of motion including radiation reaction. If "mass renormalization" has already been carried out, the relativistic generalization of Equation (2.129) should be of the form

$$m_0\, dv_\mu/d\tau = K_\mu + \Gamma_\mu, \tag{2.130}$$

where m_0 is the observed mass, K_μ is some external four-vector force, and Γ_μ is the term due to radiation reaction. The equation of motion (2.130) is sometimes called the Lorentz-Dirac equation and Γ_μ is referred to as the Abraham radiation-reaction four-vector.

It is not hard to obtain the expression for Γ_μ. The most obvious try would be to set it equal to $(2e^2/3c^3)d^2v_\mu/d\tau^2$, whose space component does reduce to non-relativistic form $\boldsymbol{F}_{\text{rr}}$ is that limit. However, such an expression for Γ_μ does not satisfy the relation [see Equation (1.65)]

$$\Gamma_\mu v_\mu = 0, \tag{2.131}$$

which is a conservation equation that all forces must satisfy; that is, in general, $v_\mu d^2v_\mu/d\tau^2$ is not identically zero. The next step would be to try the form[13]

$$\Gamma_\mu = (2e^2/3c^3)(d^2v_\mu/d\tau^2 + Sv_\mu), \tag{2.132}$$

where S is a scalar function. This scalar can be established by imposing the condition (2.131), which leads immediately to the result

$$S = \frac{1}{c^2}v_\mu\frac{d^2v_\mu}{d\tau^2} = \frac{1}{c^2}\left[\frac{d}{d\tau}\left(v_\mu\frac{dv_\mu}{d\tau}\right) - \frac{dv_\mu}{d\tau}\frac{dv_\mu}{d\tau}\right] = -\frac{1}{c^2}a_\mu a_\mu, \tag{2.133}$$

since $v_\mu v_\mu = -c^2 =$ constant, and $a_\mu = dv_\mu/d\tau$. The solutions to and properties of the Lorentz-Dirac equation have been studied extensively (see Footnote 12). We shall not go further into this topic, since it is somewhat outside the principal program of this text.

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[12]A thorough discussion of, in particular, classical electron theory is given in the scholarly book *Classical Charged Particles* by F. Rohrlich (Reading, MA: Addison-Wesley Publ. Co., Inc., 1965). Formulations that can avoid divergence problems are described and an extensive collection of relevant papers are cited, including those published early in the last century.

[13]We do not try an expression with a term proportional to $dv_\mu/d\tau$ because this would be contained within the left-hand side of Equation (2.130).

2.7 SOFT-PHOTON EMISSION

Photons are produced whenever charged particles suffer a change in velocity. There is also photon production accompanying the production of a charged particle, as in, for example, β-decay ($n \to p + e^- + \overline{\nu}_e$ for bound or free neutrons or for protons bound in a nucleus $p \to n + e^+ + \nu_e$).[14] In this case, the electron or positron can be considered to be accelerated instantaneously from rest to a velocity v, and the overall radiative process corresponds to the production of a photon accompanying the other particles in the final state. When the photon has a very low energy, the description of the process is simplified in that usually (but not always) we can derive a photon-emission probability that does not depend on the details of the associated radiationless process. This is because the photons are "soft," and their emission does not disturb the accompanying (radiationless) process that causes the particle acceleration. The formulas for the soft-photon emission probability can be derived by classical electrodynamics, as is done in this section. In Chapter 3, the same formulas will be derived by quantum electrodynamics. By employing the two different approaches, we learn more about the range of validity of the resulting expressions. The formulas are of great value because of their generality, and they allow a convenient calculation of certain important processes like, in particular, bremsstrahlung. Also, although they are restricted to the soft-photon limit, the formulas allow simple estimates for photon-producing processes at general photon energies.

We consider emission by non-relativistic and relativistic particles, and derive some useful expressions by means of the purely classical formulation of the chapter. The formulas will be derived again in the following chapter wherein their range of validity will be discussed further. Chapter 3 will also consider the effects of photon production connected with interactions with particles' intrinsic magnetic moments.

2.7.1 Multipole Formulation

For a system of particles in non-relativistic motion, the total rate of radiation of energy is given by the result (2.52), which can be written

$$dW/dt = \sum_M C_M |M(t)|^2, \tag{2.134}$$

where $M(t)$ is some moment (or, rather, its time derivative), and C_M is a numerical coefficient. The values of C_M and $M(t)$ are given in Equations (2.53) and (2.59) for the first few terms of the multipole expansion (2.134). Further, we can introduce the moment's Fourier transform M_ω by writing

$$M(t) = \int M_\omega e^{-i\omega t} d\omega, \tag{2.135}$$

with

$$M_\omega = (2\pi)^{-1} \int M(t) e^{i\omega t} dt. \tag{2.136}$$

[14] Here the photon "coupling" (see Chapter 3) or emission is associated with the production of the e^- or e^+ rather than with the proton. The e^- and e^+, together with the neutrino, carry away most of the energy and their velocities are much larger than that of the more massive proton.

Then, including only positive frequencies in the emission spectrum $dW/d\omega$ and introducing the photon concept by means of the relation (2.72), we have

$$dw_\omega = (4\pi/\hbar)(d\omega/\omega)\sum_M C_M|M(\omega)|^2 \tag{2.137}$$

for the photon-emission probability (within $d\omega$).

If the individual moment can be written (as it can) as a time derivative

$$M(t) = \dot{\mu}_M \tag{2.138}$$

in terms of some related moment $\mu(t)$, in the soft-photon limit $M_\omega \to \Delta\mu_M/2\pi$, where $\Delta\mu_M$ is the change in μ_M. Then, in this limit, the result (2.137) becomes

$$dw_\omega \to (\pi\hbar)^{-1}(d\omega/\omega)\sum_M C_M|\Delta\mu_M|^2. \tag{2.139}$$

This is a very general formula and certain features of it should be emphasized. First, note that the quantities $\Delta\mu_M$ are independent of ω and so the factor $d\omega/\omega$ exhibits the infrared divergence effect. That is, there is always a factor of this precise form, independent of whether the lowest-order emission is electric or magnetic dipole or quadrupole radiation or from higher multipole radiation. Basically, the fundamental assumption that yields this result is that the system of charges can exist in a continuum of states rather than in a quantized spectrum. This will be true for a system of free particles but not for a system in bound states. It should also be noted how the contributions from the various moments contribute additively with the appropriate coefficients C_M; this is the case when dw_ω represents the probability integrated over angles for the outgoing photon. The coefficients are the same for the x, y, and z components of the dipole contributions and for each contribution from the quadrupole tensor. The probability dw_ω is then determined by the combination of quantities $\Delta\mu_M$.

2.7.2 Dipole Formula

Let us now obtain an important result for soft-photon emission by a single charge ze in non-relativistic motion. In this case, the emission is dipole radiation, and, instead of employing the expressions given above, we refer back to the more general formula (2.73) to exhibit the angular distribution as well. In Equation (2.73) we employ[15] the magnetic-field term with

$$\begin{aligned} B_\omega &= (2\pi)^{-1}\int |\text{curl } \boldsymbol{A}|e^{i\omega t}dt \\ &= (2\pi c)^{-1}\int |\dot{\boldsymbol{A}}\times\boldsymbol{n}|e^{i\omega t}dt. \end{aligned} \tag{2.140}$$

[15]Although we are computing a radiation field, we do not evaluate its intensity from the electric-field magnitude derived from $-\dot{A}/c$; that is, we are not employing the gauge (2.9) and (2.10). This is because we are deriving the fields from the Liénard-Wiechert potentials, which do not satisfy that gauge condition. With the fields derived from the L-W potentials, it is simpler to employ the relation (2.40) for the magnetic component of the radiation field, since no gauge is specified therein. These remarks were also made in Section 2.2 and are relevant to the upcoming Section 2.7.3.

In the low-frequency limit this becomes

$$B_\omega \rightarrow (2\pi c)^{-1}|\Delta \boldsymbol{A} \times \boldsymbol{n}|, \tag{2.141}$$

where $\Delta \boldsymbol{A}$ is the change in the vector potential associated with the charge motion; in the non-relativistic limit, $\Delta \boldsymbol{A}$ approaches $(ze/R)\Delta\boldsymbol{\beta}$, where $c\Delta\boldsymbol{\beta}$ is the velocity change. Equation (2.73) then gives

$$dw_\omega = (\alpha/4\pi^2)(d\omega/\omega)z^2|\Delta\boldsymbol{\beta}\times\boldsymbol{n}|^2 d\Omega \tag{2.142}$$

for the soft-photon emission probability within frequency $d\omega$ and in the direction $\boldsymbol{n}$ within the solid angle $d\Omega$. We see that the result is proportional to the fine-structure constant α and is determined by $\Delta\boldsymbol{\beta}$. Integrating over $d\Omega$ we get the total dw_ω (if we are not interested in the direction of the outgoing soft photon):

$$dw_\omega = (2\alpha/3\pi)z^2(\Delta\boldsymbol{\beta})^2 d\omega/\omega. \tag{2.143}$$

The dipole result (2.142) can be expressed in a form that is even more general. Both results (2.142) and (2.143) correspond to the photon production probability summed over polarization states. However, the dw per polarization state can be seen from the result (2.142) by rewriting the factor $|\Delta\boldsymbol{\beta}\times\boldsymbol{n}|^2$. The two photon (linear) polarization states can be described by two unit vectors $\boldsymbol{\varepsilon}_a$ and $\boldsymbol{\varepsilon}_b$ in mutually perpendicular directions perpendicular to $\boldsymbol{n} = \boldsymbol{k}/k$; that is, if $\boldsymbol{\varepsilon}$ is a general polarization state, $\boldsymbol{\varepsilon}\cdot\boldsymbol{n} = 0$. Then, if $\boldsymbol{q}$ is any vector [see Equation (2.54)],

$$(\boldsymbol{q}\times\boldsymbol{n})^2 = q^2 - (\boldsymbol{q}\cdot\boldsymbol{n})^2 = \sum_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}\cdot\boldsymbol{q})^2. \tag{2.144}$$

Another way of obtaining the result (2.144) makes use of the relation $\boldsymbol{n} = \boldsymbol{\varepsilon}_a\times\boldsymbol{\varepsilon}_b$; since $\boldsymbol{\varepsilon}_a\cdot\boldsymbol{\varepsilon}_b = 0$, the elementary vector identity $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$ yields $\big(\boldsymbol{q}\times(\boldsymbol{\varepsilon}_a\times\boldsymbol{\varepsilon}_b)\big)^2 = (\boldsymbol{q}\cdot\boldsymbol{\varepsilon}_a)^2 + (\boldsymbol{q}\cdot\boldsymbol{\varepsilon}_b)^2$. That is, from the formula (2.142) we can identify the expression for the probability per photon state for emission in the direction $\boldsymbol{n}$ desginated by the element $d\Omega$:

$$dw_\omega = (\alpha/4\pi^2)z^2(\boldsymbol{\varepsilon}\cdot\Delta\boldsymbol{\beta})^2(d\omega/\omega)d\Omega. \tag{2.145}$$

2.7.3 Emission from Relativistic Particles

For the more general case of emission by relativistic particles it is also convenient to employ the result (2.73) with B_ω given by Equation (2.141). Now we use the relativistic expression (2.84) for the vector potential, and for a system of particles, readily obtain the general formula

$$dw_\omega = \alpha|\boldsymbol{\Delta}_\beta|^2(d\omega/\omega)d\Omega \tag{2.146}$$

for the soft-photon-emission probability. Here

$$\boldsymbol{\Delta}_\beta = (2\pi)^{-1}\left[\sum_k \Delta(z\boldsymbol{\beta}/\varkappa)_k\right]\times\boldsymbol{n}, \tag{2.147}$$

the sum being over the charged particles, and the soft-photon-emission probability is determined by the particles' velocities and changes in velocities. Further, we again obtain the result that dw_ω is proportional to $d\omega/\omega$ in this very general case.

Even for the case of a single particle, integration over $d\Omega$ does not yield a simple result for arbitrary initial and final velocities. We give expressions for dw_ω only for two special, but important, cases. First, consider the case where a single particle is accelerated from rest to (or suddenly produced at) velocity $\boldsymbol{\beta}c$. Then

$$dw_\omega = \frac{\alpha}{4\pi^2} z^2 \beta^2 \frac{d\omega}{\omega} \frac{\sin^2\theta}{(1-\beta\cos\theta)^2} d\Omega, \tag{2.148}$$

where θ is the angle between $\boldsymbol{\beta}$ and $\boldsymbol{n}$. The integration over $d\Omega$ $(= 2\pi \sin\theta d\theta)$ is elementary and yields the total probability

$$dw_\omega = \frac{\alpha}{\pi} z^2 \left(\frac{1}{\beta} \ln \frac{1+\beta}{1-\beta} - 2 \right) \frac{d\omega}{\omega}, \tag{2.149}$$

which is consistent with the dipole formula (2.143) when $\beta \ll 1$.

The second important example of emission from relativistic particles for which a simple expression for dw_ω results is the case where a single highly relativistic particle suffers small-angle elastic scattering. Then, since we are considering soft-photon emission, β is close to unity before and after scattering, $\Delta\beta$ is small and $\boldsymbol{\beta} \cdot \Delta\boldsymbol{\beta}$ is negligible. Then, with $\boldsymbol{f}(\boldsymbol{\beta}) = (1 - \boldsymbol{\beta}\cdot\boldsymbol{n})^{-1}\boldsymbol{\beta}$, and $\Delta\boldsymbol{f} = (\partial\boldsymbol{f}/\partial\boldsymbol{\beta})\cdot\Delta\boldsymbol{\beta}$,

$$dw_\omega = (4\pi^2)^{-1} \alpha z^2 (d\omega/\omega) \int (\Delta\boldsymbol{f} \times \boldsymbol{n})^2 d\Omega. \tag{2.150}$$

The evaluation of the integrals[16] of the various terms here is elementary. The algebra is a little tedious, but the result is simple:

$$dw_\omega = (2\alpha/3\pi) z^2 \gamma^2 (\Delta\boldsymbol{\beta})^2 (d\omega/\omega), \tag{2.151}$$

that is, just $\gamma^2 = (1-\beta^2)^{-1}$ times the non-relativistic expression (2.143). It should be emphasized again that this formula holds only in the limit of small-angle scattering and for $\gamma \gg 1$. The formula is useful, however, and will be applied in a later chapter on bremsstrahlung.

Formula (2.151) can be derived in a much simpler way by making use of the non-relativistic expression (2.143). We consider the process in the lab frame (K) where the initial velocity is along, say, the x-axis and in a frame (K') moving in the same direction such that in this frame the particle motion is non-relativistic. Since dw_ω is a probability (a number), it must be an invariant:

$$dw_\omega(\boldsymbol{v}, \Delta\boldsymbol{v}) = dw'_{\omega'}(\boldsymbol{v}', \Delta\boldsymbol{v}'). \tag{2.152}$$

For the right-hand side we can employ the result (2.143). Since $d\omega'/\omega' = d\omega/\omega$, and $\Delta\boldsymbol{v}'$ is in, say, the y-direction, we have only to perform an elementary Lorentz transformation of $\Delta\beta'_y$:

$$\Delta\beta'_y = \Delta\beta_y / \gamma(1 - \beta\beta_x). \tag{2.153}$$

But $\beta_x \to \beta$, so $\Delta\beta'_y \to \gamma\Delta\beta_y$ and the result (2.151) is readily obtained.

Finally, let us rewrite and generalize some of the relativistic expressions given here, and again make use of covariance arguments to show the most general formula can be obtained easily from the corresponding expression in the non-relativistic

[16] They are similar to those involved in the derivation of the result (2.112).

limit. We express the formulas in terms of the photon momentum $\boldsymbol{k}$ rather than ω ($= |\boldsymbol{k}|c$), being part of the four-vector $k_\mu = (ik, \boldsymbol{k})$. Also, we let p_μ and p'_μ be, respectively, the initial and final four-momenta of the charged particle. A photon polarization four-vector $\varepsilon_\mu = (\varepsilon_0, \boldsymbol{\varepsilon})$ is introduced, with a gauge chosen such that $\varepsilon_0 = 0$. Then the invariant[17]

$$k \cdot p \equiv k_\mu p_\mu = -\gamma mck\varkappa \tag{2.154}$$

can be used to write

$$\frac{\boldsymbol{\beta}}{\varkappa} = -k\frac{\boldsymbol{p}}{k \cdot p}. \tag{2.155}$$

The factor $(d\omega/\omega)d\Omega$ is expressed in terms of the invariant $d^3\boldsymbol{k}/k$:

$$\frac{dk}{k}d\Omega = \frac{1}{k^2}\frac{d^3\boldsymbol{k}}{k} \tag{2.156}$$

In the gauge with $\varepsilon_0 = 0$,

$$\varepsilon_\mu p_\mu = \varepsilon \cdot p = \boldsymbol{\varepsilon} \cdot \boldsymbol{p}, \tag{2.157}$$

and the formula (2.146) becomes [see Equation (2.144)]

$$dw_\omega = \frac{z^2\alpha}{4\pi^2}\sum_{\boldsymbol{\varepsilon}}\left|\Delta\left(\frac{\varepsilon \cdot p}{k \cdot p}\right)\right|^2\frac{d^3\boldsymbol{k}}{k}. \tag{2.158}$$

That is, per polarization state, the generalization of the non-relativistic expression (2.145) is

$$dw_\omega = \frac{z^2\alpha}{4\pi^2}\left|\frac{\varepsilon \cdot p}{k \cdot p} - \frac{\varepsilon \cdot p'}{k \cdot p'}\right|^2\frac{d^3\boldsymbol{k}}{k}. \tag{2.159}$$

The expression is manifestly covariant, involving factors that are Lorentz invariants. This is to be expected, since dw_ω should be invariant, as has already been noted [see Equation (2.152)]. Actually, the general expression (2.159) could be obtained *directly* from the formula (2.145) by rewriting the latter in terms of factors that are manifestly covariant and that reduce to the non-relativistic factors in that limit. It is not difficult to do this; the $\boldsymbol{\varepsilon} \cdot \boldsymbol{\beta}$ term is replaced by the invariant $\varepsilon \cdot p$ and the $1/k$ inside the square is replaced by the form $(k \cdot p)^{-1}$ [see Equation (2.155)]. In the next chapter, in Section 3.5, all of the soft-photon formulas derived here will be derived in a quantum-mechanical treatment.

2.8 WEIZSÄCKER-WILLIAMS METHOD

The idea for this method was first introduced by Fermi[18] in 1924 and was developed more fully ten years later by von Weizsäcker and especially by Williams. Sometimes the procedure is referred to, in a more descriptive way, as the "Method of Virtual

[17] For a four-dimensional dot product we do not use boldface symbols, to distinguish it from the three-dimensional case.

[18] *Z. Phys.* **29**, 315 (1924).

Quanta," and, more recently, the designation "Equivalent Photon Method" (e.p.m.) has been used. We stick with the older name, employing, for brevity, simply "W-W."

The W-W method is really quite powerful. It allows the calculation, by very simple means, of certain proceses that otherwise would be extremely difficult to evaluate. Generally, it is the cross section for some process that is computed, and the method allows its evaluation to a relative accuracy $\sim (\ln N)^{-1}$, where N is some large number. The actual application of the method for various specific processes will be deferred to later chapters. Applications will show, for example, how it allows an alternative derivation for a process and how it always provides more insight into the nature of the process. Here the foundations for the development of the procedure will be given and the basic assumptions involved will be discussed, along with the limitations of the method. We also try to indicate basically why it works so well in general, although this will only be fully clarified later when the specific applications are outlined.

2.8.1 Fields of a Moving Charge

In the W-W method the effects of the fields on a moving charge q on a target system T are described in terms of an "equivalent" flux of photons. The charge is incident on T at an impact parameter b with a velocity $\boldsymbol{v}$, and to describe the fields it is convenient to introduce two reference frames K and K' with x- and x'-axes oriented along $\boldsymbol{v}$, with q moving at the origin of K' (see Figure 2.5). It is assumed that the charge is moving fast enough that the interaction with T does not cause an appreciable deviation from a straight line path. The system T is located at a distance b along the y-axis of K, and experiences fields from q that are time variable. The equivalent flux of photons incident on T then has a frequency spectrum that is determined by the details of this time dependence.

In the frame K' there is only an electric field with components at T equal to $\boldsymbol{E}' = (q/r'^3)(x'_1, b, 0)$; here x'_1 is the coordinate of T in K', and $r'^2 = b^2 + x'^2_1$. With K and K' coinciding at $t' = 0$, $x'_1 = -vt'$. What are needed are the fields in K at the point $(0, b, 0)$ where the target is located. The time coordinate t at this point is gotten from t' by the elementary Lorentz transformation $t' = \gamma(t - vx_1/c^2) = \gamma t$ (since $x_1 = 0$), so that we can write $x'_1 = -\gamma vt$. The electromagnetic fields in K are found from the tensor transformation [see Equations (1.28), (1.73)]

$$F'_{\rho\lambda} = a_{\rho\mu} a_{\lambda\nu} F_{\mu\nu}, \tag{2.160}$$

where

$$F_{\mu\nu} = \begin{pmatrix} 0 & iE_1 & iE_2 & iE_3 \\ -iE_1 & 0 & B_3 & -B_2 \\ -iE_2 & -B_3 & 0 & B_1 \\ -iE_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{2.161}$$

is the electromagnetic field tensor. As in this problem, when the relative motion is along the x- and x'-axes, the transformation coefficients are given by Equations (1.23)–(1.25). Then, the transformations (2.152) for individual components

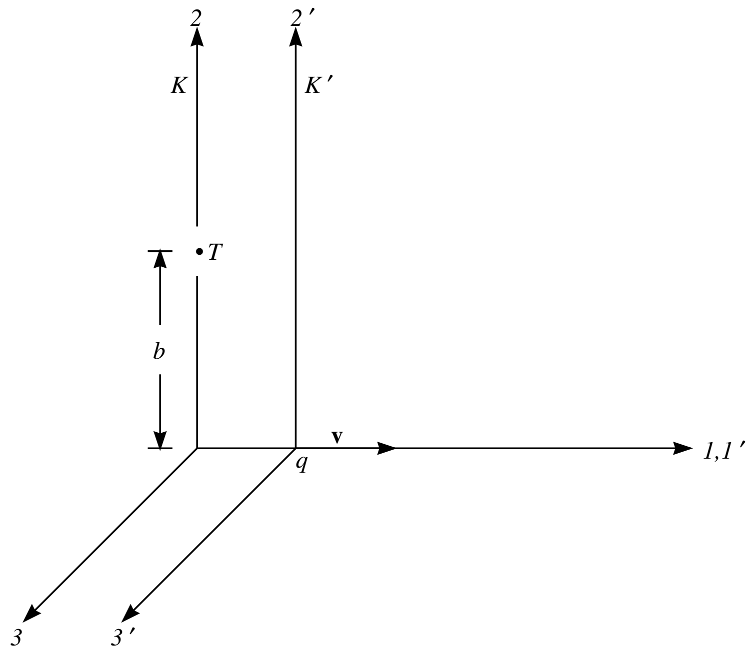

Figure 2.5 Moving-charge and "target" reference frames.

are found to be

$$\begin{aligned} E_1' &= E_1, \\ E_2' &= \gamma(E_2 - \beta B_3), \\ E_3' &= \gamma(E_3 + \beta B_2), \\ B_1' &= B_1, \\ B_2' &= \gamma(B_2 + \beta E_3), \\ B_3' &= \gamma(B_3 - \beta E_2). \end{aligned} \tag{2.162}$$

Actually, in our problem we have to go from the frame K' to K, but those transformations are the same as the ones (2.162) with the sign of β changed and with primes transferred to the unprimed fields. Of course, there are only the fields E_1' and E_2' in K'. The only fields in K are then

$$\begin{aligned} E_1 &= E_1' = -\frac{q\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \\ E_2 &= \gamma E_2' = \frac{q\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \\ B_3 &= \beta\gamma E_2' = \beta E_2, \end{aligned} \tag{2.163}$$

now expressed in terms of the K-frame variable t. These are the fields experienced by the target system.

2.8.2 Equivalent Photon Fluxes

If the target system consists of charges in non-relativistic motion,[19] magnetic forces are negligible and the principal perturbation is from the electric fields E_1 and E_2. Then, if there were, in addition to E_1 and E_2, two fictitious[20] magnetic fields $(B_3)_a = E_2$ and $(B_3)_b = -E_1$, the perturbation would be essentially the same. However, now the perturbation can be described in terms of two radiation pulses with Poynting vectors $S_1 = (c/4\pi)E_2^2$ and $S_2 = (c/4\pi)E_1^2$ striking the target in two directions. These pulses can be considered to consist of photons of linear polarization, with a spectrum of frequencies determined by the time dependence of E_2 and E_1. In terms of the Fourier transforms of these fields, the spectral distributions of the equivalent photon fluxes will be [see Equation (2.70)]

$$\begin{aligned} dJ_1/d\omega &= dN_1/dA\,d\omega = (c/\hbar\omega)|E_{2\omega}|^2, \\ dJ_2/d\omega &= dN_2/dA\,d\omega = (c/\hbar\omega)|E_{1\omega}|^2. \end{aligned} \tag{2.164}$$

Specifically, the Fourier amplitudes are given by the integrals

$$E_{2\omega} = \frac{q}{2\pi bv}\int_{-\infty}^{\infty}\frac{e^{ia\xi}}{(1+\xi^2)^{3/2}}d\xi, \tag{2.165}$$

$$E_{1\omega} = -\frac{q}{2\pi bv\gamma}\int_{-\infty}^{\infty}\frac{\xi e^{ia\xi}}{(1+\xi^2)^{3/2}}d\xi, \tag{2.166}$$

where the obvious variable change $\xi = \gamma vt/b$ has been made, and the dimensionless parameter a is given by

$$a = \omega b/\gamma v. \tag{2.167}$$

All of the frequency dependence is contained in a. Moreover, the main contributions to the integrals (2.165) and (2.166) come from $|\xi| \lesssim 1$, so that for $a \gg 1$ the integrands are oscillatory and the integrals are small. The characteristic frequency of the equivalent photon fluxes is then

$$\omega_c \sim \gamma b/v, \tag{2.168}$$

and this is also the effective maximum frequency of both of the distributions (2.164). The integrals (2.165) and (2.166) are actually representations of modified Bessel functions, so that they can be evaluated from tables for any value of the parameter a. However, only the asymptotic forms will be important for our considerations, both as to the foundations of the method as well as for most (but not all) the applications.

It is the asymptotic form at low frequency (small a) that is most relevant. The Fourier amplitudes in this limit may be obtained through consideration of the integrals (2.165) and (2.166) for $a \ll 1$. The first integral is very simple and we have

$$E_{2\omega} \to q/\pi bv. \tag{2.169}$$

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[19] It will be shown below that this assumption can be relaxed, so that the W-W method can be employed when there are relativistic motions in the target system.

[20] Actually, as $\beta \to 1$, the field $B_3 = E_2$ really *is* present [see Equations (2.163)], so that in this limit $(B_3)_a$ is not fictitious.

In the second integral the asymptotic form can be obtained in good approximation by setting $e^{ia\xi} = 1 + ia\xi + \cdots$ and by applying effective cutoffs to the integral, giving the approximate result[21]

$$2\int_1^{1/a} d\xi/\xi = 2\ln(1/a), \tag{2.170}$$

$$E_{1\omega} \to -(iqa/\pi bv\gamma)\ln(1/a). \tag{2.171}$$

The important ratio is then

$$|E_{1\omega}|^2/|E_{2\omega}|^2 \to \left[(a/\gamma)\ln(1/a)\right]^2, \tag{2.172}$$

which is very small in the limit $a \ll 1$, so that in the limit of very soft "virtual" photons, $dJ_1/d\omega \propto \omega^{-1}$ while $dJ_2/d\omega \propto \omega$. It is because of this result and, therefore, in applications (treated later) that the main contribution to the evaluation of cross sections by the W-W method comes from these virtual photons, and that the pulse J_1 always gives the principal result, J_2 being unimportant. The method always gives a final formula involving a logarithmic factor whose argument (N) is very large. Since N is evaluated to an accuracy up to an undetermined multiplying constant of order unity, use of the asymptotic forms given above is sufficient in the general formulation. That is, it is really unnecessary to introduce the precise forms for the distributions (2.164) in terms of modified Bessel functions. In the end, the asymptotic forms are always taken and the argument N is not precisely determined.

It is, however, appropriate to exhibit here a general formula for the cross section $d\sigma$ for a process evaluated by means of the W-W method. If we describe the process in terms of the interaction of the equivalent virtual photons accompanying the fast charge, we can write

$$d\sigma = dN\,d\sigma_v \tag{2.173}$$

where dN is the differential number of virtual photons and $d\sigma_v$ is their interaction cross section. For example, in an important application of the W-W method, bremsstrahlung can be considered as Compton scattering of the virtual photons of the Coulomb field of the scattering center by the incoming fast particle. In this case, $d\sigma_v$ would be the Compton cross section and $d\sigma$ the bremsstrahlung cross section. The differential dN is obtained by multiplying the virtual photon flux $dJ_1/d\omega$ by $d\omega$ and the differential area $2\pi b\,db$ associated with charged particles incident on the target at impact parameters within db. This summary over azimuthal angles means that $d\sigma_v$ should be the cross section for unpolarized (virtual) photons. Employing the asymptotic form (2.169) for $E_{2\omega}$, we then have, for $q = ze$, the result

$$dN = (2\alpha/\pi)(z/\beta)^2(d\omega/\omega)(db/b), \tag{2.174}$$

where α is, again, the fine structure constant and $\beta = v/c$. There will, in the end, be an integration over db, yielding a factor $\ln(b_{\max}/b_{\min})$ and, depending on the

[21] The exact asymptotic form of this integral is obtained by replacing $\ln(1/a)$ by $\ln(2/\Gamma_E a) = \ln(1.123/a)$, where $\ln \Gamma_E = 0.5772$ is Euler's constant. An outline of an elementary derivation of this more precise result, without resort to general identities on Bessel funcitons, may be found in R. J. Gould, *Am. J. Phys.* **38**, 189 (1970).

process considered, there may be an integration over $d\omega$ (or a transformation of $d\omega$ into, say, the differential energy of the particle produced in the process).

Finally, we might note another reason for neglecting the effects of the pulse associated with J_2, at least for the case of the highly relativistic incident charge. This is exhibited in the factor $1/\gamma$ multiplying the integral (2.166) and means that, for $\gamma \gg 1$, the flux $dJ_2/d\omega$ is small compared with $dJ_1/d\omega$ for *all* frequencies of virtual photons. Since, for $\gamma \gg 1$, the flux $dJ_2/d\omega$ is truly fictitious for a relativistic target (since there is no magnetic field to accompany E_1), and since magnetic interactions may be important, the equivalent-photon description for this "transverse" pulse has its limitations. On the other hand, as has been remarked earlier, the field E_2 really does have a magnetic counterpart (B_3) of equal magnitude as $\beta \to 1$, so that the "longitudinal" pulse J_1 does have a valid equivalent-photon description, even when the target is relativistic and magnetic forces are important.

2.9 ABSORPTION AND STIMULATED EMISSION

A radiation field, incident on a system of charges, can, itself, provide the perturbation to cause the system to undergo a transition. This "external" radiation field can thereby induce a transition resulting in the production of a new photon (stimulated emission) or a transition in which the system absorbs energy from the radiation field, removing a photon from it (absorption). The two processes, stimulated emission and absorption, are a result of the same interaction or perturbation and can be considered as just the time reverse of one another. This is indicated pictorially in Figure 2.6 where a photon beam (wavy lines) is shown incident on the charge system s. The beam is considered to have a definite direction specified by, say, a solid angle element $d\Omega$, and we consider a class of its photons having a particular polarization and a frequency within $d\omega$. Thus, the photon states of the beam are completely specified and $\overline{n}$ is the associated photon occupation number. Also, we consider a transition in s between two specified states ("1" and "2"). There can occur, in addition to the processes induced by the beam, spontaneous emission, in general in any direction; Figure 2.6 indicates the case of emission in the beam direction.[22]

A most important result is the relation between the rates for stimulated emission and absorption and that for spontaneous emission. The relation is extremely simple and is sometimes referred to in terms of the "Einstein A and B coefficients," although that old-fashioned terminology and notation will not be employed here. Also, it should be noted that, to derive the fundamental result, the only quantum-mechanical concept that will be introduced is that of the photon. That is, it is not necessary to make use of the detailed formalism of quantum field theory. This fact has dictated the inclusion of the topic in a chapter on classical radiation theory rather than in the following one on quantum electrodynamics.

[22] Stimulated emission, being the exact reverse of absorption, always takes place in the direction of the incident radiation beam.

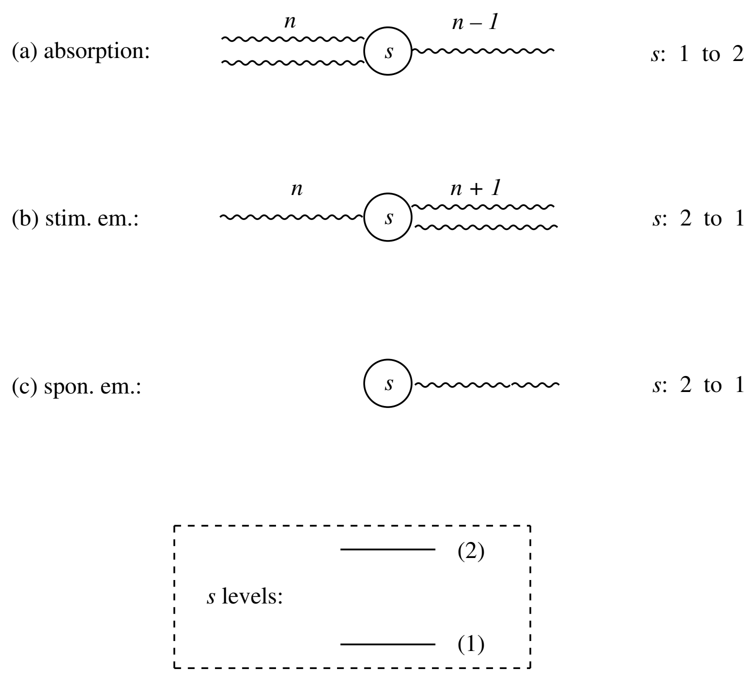

Figure 2.6 Representations of absorption, stimulated emission, and spontaneous emission.

2.9.1 Relation to Spontaneous Emission

Consider again the three processes indicated in Figure 2.6 with the photon state completely specified in terms of its polarization, frequency (or energy), and direction of motion. As a result of the interaction with the photon beam, the system s undergoes transitions between states 1 and 2. The system can also undergo spontaneous radiative transitions that can deposit photons into the beam. The rates for the absorption (a) and stimulated-emission (b) processes will be proportional to the occupation number ($\overline{n}$) of the photon beam and to the numbers N_1 and N_2 of the system in the lower and upper state, respectively. That is,

$$\begin{aligned} R_{\mathrm{a}} &= R_{1\to 2} = R_{\mathrm{abs}} \propto \overline{n} N_1, \\ R_{\mathrm{b}} &= R_{2\to 1} = R_{\mathrm{stim}} \propto \overline{n} N_2. \end{aligned} \tag{2.175}$$

The proportionality constants (w_{abs} and w_{stim}) in the rates (2.175) must be identical, since the two processes are just the time reverse of one another. The spontaneous rate will, of course, be proportional to N_2:

$$R_{\mathrm{c}} = R_{\mathrm{spon}} \propto N_2, \tag{2.176}$$

and we designate the associated proportionality constant as w_{spon}. The rate constants w_{abs} ($= w_{\mathrm{stim}}$) and w_{spon} are determined by the characteristics of the charge system s; that is, they are "atomic" parameters. A convenient way of determining the relationship between them is to consider the case where the system s is in thermal equilibrium with the surrounding photon gas. Then, balancing the $1 \to 2$ and $2 \to 1$ rates involving photons produced or absorbed within the beam direction and with particular polarization, we have

$$\overline{n} N_1 w_{\text{abs}} = \overline{n} N_2 w_{\text{stim}} + N_2 w_{\text{spon}}. \tag{2.177}$$

With[23]

$$\begin{aligned} \overline{n} &= (e^{\hbar\omega/kT} - 1)^{-1}, \\ N_2/N_1 &= e^{-\hbar\omega/kT}, \end{aligned} \tag{2.178}$$

and the identity $w_{\text{abs}} = w_{\text{stim}}$, we obtain the simple and fundamental result

$$w_{\text{spon}} = w_{\text{abs}} = w_{\text{stim}} \equiv w. \tag{2.179}$$

In other words, there is only *one* (w) fundamental radiative transition constant. The coefficient w will, of course, be different for each type of system and transition—and, in general, the w's are difficult to compute—but the identity (2.179) must hold for each given transition. This identity corresponds to the relation between the "Einstein A and B coefficients."

2.9.2 General Multiphoton Formula

Because of the result (2.179), for any process for which there is a finite rate coefficient w for the spontaneous production of a photon in some polarization state and in some direction of motion and with frequency ω, the total rate will be given by

$$R_{\text{spon}} + R_{\text{stim}} = w N_2 (1 + \overline{n}). \tag{2.180}$$

The factor $1 + \overline{n}$ simply corrects for simulated emission. The result (2.180) holds for any background photon gas (not just for that of a blackbody, of course), and $\overline{n}$ is the occupation number of the gas causing the stimulated process. Again, we must remember that stimulated emission takes place only in the direction of the (incident) photons that cause it. Note, further, that $\overline{n}$ is the occupation number for the characteristics of the outgoing photon. The process is stimulated only by these same kind of photons.

There are some radiative processes that take place by the spontaneous production of two photons. For example, the neutral pion decays spontaneously into two γ-rays: $\pi^0 \to \gamma_1 + \gamma_2$. The total decay rate for photons emitted in specific directions is obtained from the spontaneous rate by multiplying by $(1 + \overline{n}_1)(1 + \overline{n}_2)$, where $\overline{n}_1$ and $\overline{n}_2$ are the occupation numbers of the surrounding photon gas at the energies $\hbar\omega_1$ and $\hbar\omega_2$ (for a pion at rest these energies are equal) and corresponding directions and polarizations. This corrects for stimulated emission. That precisely the factor $(1 + \overline{n}_1)(1 + \overline{n}_2)$ is required can be seen if we consider the pion to be in thermal equilibrium with a (blackbody) photon gas. Then in a steady state condition

$$N(\pi^0) w (1 + \overline{n}_1)(1 + \overline{n}_2) = N(2\gamma) w \overline{n}_1 \overline{n}_2 \tag{2.181}$$

[23] The blackbody occupation number [first of Equation (2.178)] is a result of (only) the assumption that the photon is a massless boson.

where w is the transition rate. The "2γ" two-photon state of the π^0 will have relative numbers $N(2\gamma)/N(\pi^0) = \exp[(\hbar\omega_1 + \hbar\omega_2)/kT]$ and $1+\overline{n} = \overline{n}e^{\hbar\omega/kT}$ for a blackbody distribution. Thus, we see that the condition (2.181) is satisfied.

Another example of a two-photon radiative process is the decay of the $2s$ state of a hydrogenic system:

$$a_{2s} \to a_{1s} + \gamma_1 + \gamma_2, \tag{2.182}$$

with $\hbar\omega_1 + \hbar\omega_2 = \frac{3}{4}Z^2\text{Ry}$. Corrected for stimulated emission, the total decay rate is obtained from the spontaneous rate by multiplying by $(1+\overline{n}_1)(1+\overline{n}_2)$; there is an integration over photon energies and directions. Again, we see that this factor is needed to satisfy the detailed balance relation

$$N_{2s}w(1+\overline{n}_1)(1+\overline{n}_2) = N_{1s}w\overline{n}_1\overline{n}_2, \tag{2.183}$$

with $N_{2s}/N_{1s} = \exp[-(\hbar\omega_1 + \hbar\omega_2)/kT]$ and $\overline{n}_1$ and $\overline{n}_2$ the blackbody occupation numbers.

The generalization to the case of a process in which there are any number of photons in the final state is clear. If $R_{\text{spon}}(\omega_1, \omega_2, \ldots)$ is the spontaneous rate, the rate corrected for stimulation by an external radiation field is

$$R = R_{\text{spon}}(\omega_1, \omega_2, \ldots)(1+\overline{n}_1)(1+\overline{n}_2)\cdots . \tag{2.184}$$

2.9.3 Stimulated Scattering

In addition to stimulated photon-emission process, there can be stimulation of photon scattering by an external radiation field. The rate is enhanced by a factor $1+\overline{n'}$, where $\overline{n'}$ is the occupation number of the photon gas for the state (energy, polarization, direction of motion) of the scattered photon. The scattering can be by a free electron (Compton scattering) or by an atomic or molecular system. For a system with internal degrees of freedom the scattering can either leave the system unchanged (Rayleigh scattering) or cause an excitation in the system at the same time (Raman scattering). If, in the scattering, the photon frequency changes from ω to ω' (and its polarization and direction changes) and the scattering system changes from s to s', we can see the necessity of precisely the factor $1+\overline{n'}$ to correct for stimulated scattering by considering detailed balance. With the scattering system in equilibrium with a surrounding blackbody radiation field, the condition

$$\overline{n}Nw_{\text{sc}}(1+\overline{n'}) = \overline{n'}N'w_{\text{sc}}(1+\overline{n}) \tag{2.185}$$

is satisfied identically with $\overline{n}$ and $\overline{n'}$ the blackbody occupation numbers and $N/N' = \exp[-\hbar(\omega-\omega')/kT]$. Of course, the stimulation correction $1+\overline{n'}$ (or $1+\overline{n}$) applies whether the external radiation field is or is not in thermal equilibrium with the scattering system.

Bibliographical Notes

For general classical electrodynamics and classical radiation theory the three books by Jackson, Panofsky and Phillips, and Landau and Lifshitz are excellent. These texts, listed as References 4, 5, and 6 at the end of Chapter 1, will be referred to here, for brevity, as J, P^2, and L^2. Each is especially good on certain topics covered in this chapter, as we indicate below. Another scholarly work is that of Rohrlich (R), mentioned in Footnote 12. Also useful is the book by W. Heitler, *The Quantum Theory of Radiation*, 3rd ed., Oxford, UK: Oxford University Press, 1954, referred to as H; the first chapter of Heitler's book is on classical electrodynamics. For the various topics in this chapter, the suggested references are as follows:

Gauge invariance: H.
Retarded potentials: L^2, J, P^2.
Multipole expansion: L^2.
Fourier spectra: L^2.
Fields of relativistic charge: P^2.
Radiation from relativistic charges: L^2, J, P^2.
Radiation reaction: P^2, R, H.
Soft-photon emission: L^2, J, P^2.
Weizsäcker-Williams method: J.
Absorption and stimulated emission (quantum mechanical approach): H.