Chapter 2

Introduction for students

2.1. This is a textbook about propositional logics. A more detailed overview of the contents can be found in 1.7.

This chapter, and to a lesser extent Chapter 5, are intended as an overview of logic. *Caution:*

- These introductory chapters are not intended to be mathematically precise. Arguments sketched in these chapters should not be viewed as proofs, and will not be used as ingredients in the rigorous proofs developed later in the book.
- These introductory chapters are not typical of the style of the whole book. Formal logic, which begins in Chapter 6, is much more computational. The student who is undecided about registering for or dropping the course should look at some of the later chapters.

Who should study logic?

2.2. The subjects of logic, algebra, and computer science are enough alike that a previous course in any one of these three subjects may aid the student in pursuing the other two — partly because of specific results, but more because of a familiarity with the general style of thinking.

Regardless of previous background, however, any liberal arts undergraduate student might wish to take at least one course in logic, for the subject has great philosophical significance: It tells us something about our place in the universe. It tells us how we think — or more precisely, how we think we think. Mathematics contains eternal truths about number and shape; mathematical logic contains truths about the nature of truth itself.

The grandeur of logic's history (sketched later in this chapter) and the power and beauty of its ideas can be appreciated by any student in a first course on logic, but only a few students will greatly enjoy its computations. It is those few who may choose to study beyond the first course (i.e., beyond this book) and become logicians. I would expect that most students taking a first course in mathematical logic are simply liberal arts students fulfilling a mathematics requirement and seeking a sampling of our culture. Logic is an important part of that culture, and I will try to present it in a way that does not require all the background of a mathematician.

2.3. Logic and set theory are sometimes called the *foundations* of mathematics, because they are used as a basis for other branches of mathematics; those other branches are then called *ordinary mathematics*.

- Logic is a foundation because the logician studies the kinds of *reasoning* used in the individual steps of a proof. An ordinary mathematical proof may be long and complicated, involving many steps, but those steps involve only a few different kinds of reasoning. The logician studies the nature of those few kinds of reasoning.
- Set theory (also studied briefly in this book) is a foundation for the *objects* that mathematicians reason about. For instance, 3 and $\sqrt{2}$ are members of the *set* of real numbers. Likewise, $\pi_1 \rightarrow \pi_2$ and $\pi_2 \wedge \neg\, \pi_1$ are members of the *set* of all formulas in the formal language studied in this book.

Logic and set theory are fundamental but not central in mathematics. Knowing a little about the foundations may make you a better mathematician, but knowing a lot about the foundations will only make you a better logician or set theorist.

That disclaimer surprises some people. It might be explained by this analogy: If you want to write The Great American Novel, you'll need to know English well; one or two grammar books might help. But studying dozens of grammar books will make you a grammarian, not a novelist. Knowing the definitions of "verb" and "noun" is rather different from being able to use verbs and nouns effectively. Reading many novels might be better training for a novelist, and writing many novels — or rewriting one novel many times — may be the best training. Likewise, to become proficient in the kinds of informal proofs that make up "ordinary" mathematics, you probably should read and/or write more of "ordinary" mathematics.

For that purpose, I would particularly recommend a course in general topology, also known as "point set topology." We will use topology as a tool in our development of formal logic, so a very brief introduction to topology is given in Chapter 4 of this book, but mathematics majors should take one or more courses on the subject.

General topology has the advantage that its results are usually formulated in sentences rather than equations or other mathematical symbols. Also, it has few prerequisites — e.g., it does not require algebra, geometry, or calculus.

General topology is one of the subjects best suited for the *Moore method*. That is a teaching method in which the students are supplied with just definitions and the statements of theorems, and the students must discover and write up the proofs. In my opinion, that is the best way to learn to do proofs.

2.4. A few students — particularly young adults — may begin to study logic in the naive hope of becoming better organized in their personal lives, or of becoming more "logical" people, less prone to making rash decisions based on transient emotions. I must inform those students that, unfortunately, a knowledge of mathematical logic is *not* likely to help with such matters. Those students might derive greater understanding of their lives by visiting a psychotherapist, or by reading biochemists' accounts of the hormones that storm through humans during adolescence.

Logic was misrepresented by the popular television series *Star Trek*. The unemotional characters Spock (half human, half Vulcan) and Data (an android) claimed to be motivated solely by logic. But that claim was belied by their behavior. Logic can only be used as an aid in acting on one's values, not in choosing those values. Spock's and Data's very positive values, such as respect for other sentient beings and loyalty to their crew mates, must have been determined by extralogical means. Indeed, a few episodes of the program pitted Spock or Data against some "logical" villains — evil Vulcans, evil androids — who exhibited just as much rationality in carrying out quite different values. (See the related remarks about logic and ethics, in 2.19.d.)

Logicians are as likely as anyone to behave illogically in their personal lives. Kurt Gödel (1906–1978) was arguably the greatest logician in history, but he was burdened with severe psychiatric problems for which he was sometimes hospitalized. In the end, he believed he was being poisoned; refusing to eat, he starved to death. In the decades since Gödel's time, medical science has made great progress, and many afflictions of the mind can now be treated successfully with medication or by other means. Any reader who feels emotionally troubled is urged to seek help from a physician, not from this logic book.

2.5. *Why should we formalize logic?* What are the advantages of writing $\wedge$ instead of "and," or writing $\exists$ instead of "there exists"? These symbols are seldom used in other branches of mathematics — e.g., you won't often find them in a research paper on differential equations.

One reason is if we're interested in logic itself. Pursuit of that subject will lead us to logical expressions such as

$$\Big\{(\forall x)\,\big[W(x) \to R(x)\big]\Big\} \vee \Big\{(\forall x)\,\big[R(x) \to W(x)\big]\Big\},$$

much more complicated than any arising from merely abbreviating the reasoning of, say, differential equations. Clearly, without symbols an expression like this would be difficult to analyze.

But even students who do not intend to become logicians would benefit from using these symbols for at least a while. Basic properties of the symbols, such as the difference between $\forall x\, \exists y$ and $\exists y\, \forall x$ are easier to understand if presented symbolically.

Moreover, even if the symbols do not show up explicitly elsewhere, the underlying concepts are implicit in some other definitions (particularly in mathematics). For instance, the expression

"$\limsup_{t\to\infty} f(t)$" might appear somewhere in a paper on differential equations. One way to define limsup is as follows: It is a number with the property that, for numbers r,

$$\limsup_{t\to\infty} f(t) \le r \quad\Longleftrightarrow\quad (\forall\varepsilon > 0)(\exists u \in \mathbb{R})(\forall t > u)\ f(t) < r+\varepsilon.$$

(Other definitions of limsup say essentially the same thing, but may differ in appearance.) Researchers in differential equations generally think of limsup in terms of its familiar properties rather than in terms of this definition, but they first must learn it in terms of its definition — either in words or symbolically. In effect, there are $\forall$'s and $\exists$'s in a paper on differential equations, but they are hidden inside the limsup's and other tools of that subject.

2.6. *Teaching students how to think* — how to analyze, how to question, etc. — is sometimes cited as one of the goals of a mathematics course, especially a mathematical logic course. But that justification for logic may be erroneous. Though "how to think" and mathematical logic do have some overlap, they are two different subjects. Techniques of thinking — e.g., look for a similar but more familiar problem, think about conditions that would hold if the problem were already solved, draw a diagram — can be found, not in logic books, but in a book such as Polya's *How to Solve It*.

Though logic does formalize some thinking techniques, the use of those techniques and the study of the formalizations occur on very different levels.

- The lower level consists of equations, formulas, and computational techniques. For instance, *CNL* shows that the "proof by contradiction" formula $(A \to B) \to ((A \to \overline{B}) \to \overline{A})$ is a theorem in classical, relevant, and constructive logics, and shows how that formula is used in the derivations of other formulas. Practice should make the student adept at computations of this sort.
- The higher level consists of paragraphs of reasoning *about*

those symbolic formulas. Section 5.35 discusses proof by contradiction in informal terms, and the technique is applied in paragraph-style arguments in 3.70, 7.7, 7.14, 22.16, 27.6, 29.6, 29.14, and a few other parts of the book. All of the higher-level reasoning in this book is in an informal version of classical logic (see discussion of metalogic in 2.17). However, this book is not written for use with the Moore method — the exercises in this book are mostly computational and do not require much creativity.

2.7. Granted that you have decided to study logic, is *this* book the right one for you? This book is unconventional, and does not follow the standard syllabus for an introductory course on logic. It may be unsuitable if you need a *conventional* logic course as a prerequisite for some other, more advanced course. The differences between this book and a conventional treatment are discussed in 1.1, 1.10, and at the end of 2.36.

Formalism and certification

2.8. To describe what logic is, we might begin by contrasting formal and informal proofs. A *formal proof* is a purely computational derivation using abstract symbols such as $\vee$, $\wedge$, $\neg$, $\rightarrow$, $\vdash$, $\forall$, $\exists$, $\Rightarrow$, $=$, $\in$, $\subseteq$. All steps must be justified; nothing is taken for granted. Completely formal proofs are seldom used by anyone except logicians. *Informal* reasoning may lack some of those qualifications.

2.9. Here is an example. *Informal set theory* is presented without proofs, as a collection of "observed facts" about sets. We shall review informal set theory in Chapter 3, before we get started on formal logic; we shall use it as a tool in our development of formal logic.

In contrast, *formal set theory* (or *axiomatic set theory*) can only be developed *after* logic. It is a rigorous mathematical theory in which all assertions are proved. In formal set theory, the symbols $\in$ ("is a member of") and $\subseteq$ ("is a subset of") are stripped of their traditional intuitive meanings, and given only the meanings determined by consciously chosen axioms. Indeed, $\subseteq$ is not taken as a primitive symbol at all; it is defined in terms of $\in$. Here is its definition: $x \subseteq y$ is an abbreviation for

$$(\forall z) \quad [\,(z \in x) \to (z \in y)\,].$$

That is, in words: For each z, if z is a member of x, then z is a member of y.

In informal set theory, a "set" is understood to mean "a collection of objects." Thus it is *obvious* that

(a) there exists a set with no elements, which we may call the *empty set*; and

(b) any collection of sets has a union.

But both of those "obvious" assertions presuppose some notion of what a "set" is, and assertion (b) also presupposes an understanding of "union." That understanding is based on the usual meanings of words in English, which may be biased or imprecise in ways that we have not yet noticed.

Axiomatic set theory, in contrast, rejects English as a reliable source of understanding. After all, English was not designed by mathematicians, and may conceal inherent erroneous assumptions. (An excellent example of that is Russell's Paradox, discussed in 3.11.) Axiomatic set theory takes the attitude that we do *not* already know what a "set" is, or what "$\in$" represents. In fact, in axiomatic set theory we don't even use the word "set," and the meaning of the symbol "$\in$" is specified only by the axioms that we choose to adopt. For the existence of empty sets and unions, we adopt axioms something like these:

(a) $(\exists x)\,(\forall y) \quad \neg\,(y \in x)$.

(b) $(\forall x)\,(\exists y)\,(\forall z) \quad [\,(z \in y) \leftrightarrow (\exists w\,(z{\in}w) \wedge (w{\in}x))\,]$.

That is, in words,

(a) there exists a set x with the property that for every set y, it is not true that y is a member of x; and

(b) for each set x there exists a set y with the property that the members of y are the same as the members of members of x.

The issue is not whether empty sets and unions "really exist," but rather, what consequences can be proved about some abstract objects from an abstract system of axioms, consisting of (a) and (b) and a few more such axioms *and nothing else* — i.e., with no "understanding" based on English or any other kind of common knowledge.

Because our natural language is English, we cannot avoid thinking in terms of phrases such as "really exist," but such phrases may be misleading. The "real world" may include three apples or three airplanes, but the abstract concept of the number 3 itself does not exist as a physical entity anywhere in our real world. It exists only in our minds.

If some sets "really" do exist in some sense, perhaps they are not described accurately by our axioms. We can't be sure about that. But at least we can investigate the properties of any objects that *do* satisfy our axioms. We find it convenient to call those objects "sets" because we believe that those objects correspond closely to our intuitive notion of "sets" — but that belief is not really a part of our mathematical system.

Our successes with commerce and technology show that we are in agreement about many abstract notions — e.g., the concept of "3" in my mind is essentially the same as the concept of "3" in your mind. That kind of agreement is also present for many higher concepts of mathematics, but not all of them. Our mental universes may differ on more complicated and more abstract objects. See 3.10 and 3.33 for examples of this.

2.10. *Formalism* is the style or philosophy or program that formalizes each part of mathematics, to make it more clear and less prone to error. *Logicism* is a more extreme philosophy that advocates reducing all of mathematics to consequences of logic.

The most extensive formalization ever carried out was *Principia Mathematica*, written by logicists Whitehead and Russell near the beginning of the 20th century. That work, three volumes totaling nearly 2000 pages, reduced all the fundamentals of mathematics to logical symbols. Comments in English appeared occasionally in the book but were understood to be outside the formal work. For instance, a comment on page 362 points out that the main idea of $1+1=2$ has just been proved!

In principle, all of known mathematics can be formulated in terms of the symbols and axioms. But in everyday practice, most ordinary mathematicians do not completely formalize their work; to do so would be highly impractical. Even partial formalization of a two-page paper on differential equations would turn it into a 50-page paper. For analogy, imagine a cake recipe written by a nuclear physicist, describing the locations and quantities of the electrons, protons, etc., that are included in the butter, the sugar, etc.

Mathematicians generally do not formalize their work completely, and so they refer to their presentation as "informal." However, this word should not be construed as "careless" or "sloppy" or "vague." Even when they are informal, mathematicians do check that their work is *formalizable* — i.e., that they have stated their definitions and theorems with enough precision and clarity that any competent mathematician reading the work *could* expand it to a complete formalization if so desired. Formalizability is a requirement for mathematical publications in refereed research journals; formalizability gives mathematics its unique ironclad certainty.

2.11. Complete formalization *is* routinely carried out by computer programmers. Unlike humans, a computer cannot read between the lines; every nuance of intended meaning must be spelled out explicitly. Any computer language, such as Pascal or C++, has a very small vocabulary, much like the language of formal logic studied in this book. But even a small vocabulary can express complicated ideas if the expression is long enough; that is the case both in logic and in computer programs.

In recent years some researchers in artificial intelligence have begun carrying out complete formalizations of *mathematics* —

they have begun "teaching" mathematics to computers, in excruciating detail. A chief reason for this work is to learn more about intelligence — i.e., to see how sentient beings think, or how sentient beings *could* think. However, these experiments also have some interesting consequences for mathematics. The computer programs are able to check the correctness of proofs, and sometimes the computers even produce new proofs. In a few cases, computer programs have discovered new proofs that were shorter or simpler than any previously known proofs of the same theorems.

Aside from complete formalizations, computers have greatly extended the range of feasible proofs. Some proofs involve long computations, beyond the endurance of mathematicians armed only with pencils, but not beyond the reach of electronic computers. The most famous example of this is the *Four Color Theorem*, which had been a conjecture for over a century: Four colors suffice to color any planar map so that no two adjacent regions have the same color. This was finally proved in 1976 by Appel and Haken. They wrote a computer program that verified 1936 different cases into which the problem could be classified.

In 1994 another computer-based proof of the Four Color Theorem, using only 633 cases, was published by Robertson, Sanders, Seymour, and Thomas. The search for shorter and simpler proofs continues. There are a few mathematicians who will not be satisfied until/unless a proof is produced that can actually be read and verified by human beings, unaided by computers. Whether the theorem is actually *true* is a separate question from what kind of proof is acceptable, and what "true" or "acceptable" actually mean. Those are questions of philosophy, not of mathematics.

2.12. One of the chief attractions of mathematics is its ironclad *certainty*, unique among the fields of human knowledge. (See Kline [1980] for a fascinating treatment of the history of certainty.) Mathematics can be certain only because it is an artificial and finite system, like a game of chess: All the ingredients in a mathematical problem have been put there by the mathematicians who formulated the problem. In contrast, a problem in physics or chemistry may involve ingredients that have not yet been detected by our imperfect instruments; we can never be sure that our observations have taken everything into account.

The absolute certainty in the mental world of pure mathematics is intellectually gratifying, but it disappears as soon as we try to apply our results in the physical world. As Albert Einstein said,

> As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

Despite its uncertainty, however, applied mathematics has been extraordinarily successful. Automobiles, television, the lunar landing, etc., are not mere figments of imagination; they are real accomplishments that used mathematics.

This book starts from the uncertain worlds of psychology and philosophy, but only for motivation and discovery, not to justify any of our reasoning. The rigorous proofs developed in later chapters will not depend on that uncertain motivation.

Ultimately this is a book of *pure* mathematics. Conceivably our results could be turned to some real-world application, but that would be far beyond the expertise of the author, so no such attempt will be made in this volume.

2.13. Some beginning students confuse these two aspects of mathematics:

(*discovery*)	How did you find that proof?
(*certification*)	How do you know that proof is correct?

Those are very different questions. This book is chiefly concerned with the latter question, but we must understand the distinction between the two questions.

2.14. Before a mathematician can write up a proof, he or she must first discover the ideas that will be used as ingredients in that proof, and any ideas that might be related. During the discovery phase, all conclusions are uncertain, but all methods are permitted. Certain heuristic slogans and mantras may be followed — e.g., “First look for a similar, simpler problem that we *do* know how to solve.”

Correct reasoning is not necessary or even highly important. For instance, one common discovery process is to *work backwards* from what the mathematician is trying to prove, even though this is known to be a faulty method of reasoning. (It is discussed further starting in 5.36.)

Another common method is *trial-and-error*: Try something, and if it doesn’t work, try something else. This method is more suitable for some problems than others. For instance, on a problem that only has eight conceivable answers, it might not take long to try all eight. The method may be less suitable for a problem that has infinitely many plausible answers. But even there we may gain something from trial-and-error: When a trial fails, we may look at *how* it fails, and so our next guess may be closer to the target. One might call this “enlightened trial-and-error”; it does increase the discoverer’s understanding after a while.

The discovery process may involve experimenting with numerous examples, searching through books and articles, talking to oneself and/or to one’s colleagues, and scribbling rough ideas on hamburger sacks or tavern napkins. There are many false starts, wrong turns, and dead ends. Some mathematicians will tell you that they get their best inspirations during long evening walks or during a morning shower. They may also tell you that the inspiration seems to come suddenly and effortlessly, but only after weeks of fruitless pacing and muttering. Though it is not meant literally, mathematicians often speak of “banging one’s head against a wall for a few weeks or months.”

The discovery phase is an essential part of the mathematician’s work; indeed, the researcher devotes far more time to discovery than to writing up the results. But discovery may be more a matter of psychology than of mathematics. It is personal, idiosyncratic, and hard to explain. Consequently, few mathematicians make any attempt to explain it.

Likewise, this book will make little attempt to explain the discovery process; this book will be chiefly concerned with the certification process of mathematical logic. To solve this book's exercises, we recommend to the student the methods of working backward and enlightened trial-and-error. Students with a knowledge of computers are urged to use those as well; see 1.20.

The need for discovery may not be evident to students in precollege or beginning college math courses, since those courses mainly demand mechanical computations similar to examples given in the textbook. Higher-level math courses require more discovery, more original and creative thought.

Actually, for some kinds of problems a mechanical, computational procedure *is* known to be available, though explaining that procedure may be difficult. For other problems it is known that there *cannot* be a mechanical procedure. For still other problems it is not yet known whether a mechanical procedure is possible. The subject of such procedures is *computability theory*. But that theory is far too advanced for this introductory course.

2.15. After discovery, the mathematician must start over from scratch, and rewrite the ideas into a legitimate, formalizable proof following rigid rules, to *certify* the conclusions. Each step in the proof must be justified. "Findings" from the discovery phase can be used as suggestions, but not as justifications.

The certification process is more careful than the discovery process, and so it may reveal gaps that were formerly overlooked; if so, then one must go back to the discovery phase. Indeed, I am describing discovery and certification as two separate processes, to emphasize the differences between them; but in actual practice the two processes interact with each other. The researcher hops back and forth between discovery and certification; efforts on one yield new insights on the other.

In the presentation of a proof, any comment about intuition or about the earlier discovery of the ideas is a mere aside; it is not considered to be part of the actual proof. Such comments are optional, and they are so different in style from the proof itself that they generally seem out of place. Skilled writers may sometimes find a way to work such comments into the proof, in hopes of assisting the reader if the proof is hard to understand. But it is more commonplace to omit them altogether.

Indeed, it is quite common to rewrite the proof so that it is more orderly, even though this may further remove any traces of the discovery process. (An example of this is sketched in 5.39.) Consequently, some students get the idea that there is little or no discovery process — i.e., that the polished proof simply sprang forth from the mathematician's mind, fully formed. These same students, when unable to get an answer quickly, may get the idea that they are incapable of doing mathematics. Some of these students need nothing more than to be told that they need to continue "banging their heads against the the walls" for a bit longer — that it is perfectly normal for the ideas to take a while to come.

2.16. Overall, what procedures are used in reasoning? For an analogy, here are two ways behavior may be prescribed in religion or law:

(a) A few *permitted* activities ("you may do this, you may do that") are listed. Whatever is not expressly allowed by that list is forbidden.

(b) Or, a few *prohibited* activities ("thou shalt nots") are listed. Whatever is not explicitly forbidden by that list is permitted.

Real life is a mixture of these two options — e.g., you *must* pay taxes; you must *not* kill other people. But we may separate the two options when trying to understand how behavior is prescribed.

Theologians and politicians have argued over these options for centuries — at least as far back as Tertullian, an early Christian philosopher (c. 150–222 A.D.). The arguments persist. For instance, pianos are not explicitly mentioned in the bible; should they be permitted or prohibited in churches? Different church denominations have chosen different answers to this question. (See Shelley [1987].)

In the *teaching* of mathematics, we discuss both

- *permitted* techniques, that can and should be used, and

- common errors, based on unjustified techniques — i.e., *prohibited* techniques.

Consequently, it may take some students a while to understand that mathematical *certification* is based on *permissions only* — or more precisely, a modified version of permissions:

(a′) We are given an explicit list of axioms (permitted statements), and an explicit list of rules for deriving further results from those axioms (permitted methods). Any result that is not among those axioms, and that cannot be derived from those axioms via those rules, is prohibited.

Language and levels

2.17. *Levels of formality.* To study logic mathematically, we must reason about reasoning. But isn't that circular? Well, yes and no.

Any work of logic involves at least two levels of language and reasoning; the beginner should take care not confuse these:

a. The *inner system*, or *object system*, or *formal system*, or *lower system*, is the formalized theory that is being studied and discussed. Its language is the *object language*. It uses symbols such as

$$(,\quad),\quad \wedge,\quad \neg,\quad \forall,\quad \rightarrow,\quad \pi_1,\quad \pi_2,\quad \pi_3,\quad \ldots$$

that are specified explicitly; no other symbols may be used. Also specified explicitly are the grammatical rules and the axioms. For instance, in a typical grammar,

$$\pi_1 \rightarrow (\pi_1 \vee \pi_2) \text{ is a formula;} \qquad \pi_1)(\rightarrow \pi_1\pi_2\vee \text{ is not.}$$

(Precise rules for forming formulas will be given in 6.7.) Each inner system is fixed and unchanging (though we may study different inner systems on different days). The inner system is entirely artificial, like a language of computers or of robots.

The individual components of that language are simpler and less diverse than those of a natural language such as English. However, by combining a very large number of these simple components, we can build very complicated ideas.

b. The *outer system*, or *metasystem*, or *higher system*, is the system in which the discussion is carried out. The reasoning methods of that system form the *metalogic* or *metamathematics*. The only absolute criteria for correctness of a metamathematical proof are that the argument must be clear and convincing. The *metalanguage* may be an informal language, such as the slightly modified version of English that mathematicians commonly use when speaking among themselves. ("Informal" should not be construed as "sloppy" or "imprecise"; see 2.10.) For instance,

assume $\mathcal{L}$ is a language with
only countably many symbols

is a sentence *in* English *about* a formal language $\mathcal{L}$; here English is the informal metalanguage.

c. (Strictly speaking, we also have an intermediate level between the formal and informal languages, which we might call the "level of systematic abbreviations." It includes symbols such as $\vdash$, $\vDash$, $\Rightarrow$, and $A, B, C, \ldots$, which will be discussed starting in 2.23.)

Throughout this book, we shall use classical logic for our metalogic. That is, whenever we assert that one statement about some logic *implies* another statement about some logic, the "implies" between those two statements is the "implies" of classical logic. For instance, Mints's Admissibility Rule for Constructive Logic can be stated as

$$\vdash \overline{S} \to (Q \vee R) \qquad \Rightarrow \qquad \vdash (\overline{S} \to Q) \vee (\overline{S} \to R);$$

this rule is proved in 27.13. Here the symbols "$\to$" are implications in the object system, which in this case happens to be constructive logic. The symbols "$\vdash$" mean "is a theorem" — in this case, "is a theorem of constructive logic." The symbol

$\Rightarrow$ is in the metasystem, and represents implication in *classical* logic. The whole line can be read as

$$\begin{array}{ll} \textit{if} & \overline{S} \to (Q \vee R) \quad \text{is a theorem of constructive logic,} \\ \textit{then} & (\overline{S} \to Q) \vee (\overline{S} \to R) \quad \text{is too.} \end{array}$$

The words "if" and "then," shown in italic for emphasis here, are the classical implication ($\Rightarrow$) of our metalogic.[1]

The metalogic is classical not only in this book, but in most of the literature of logic. However, that is merely a convenient convention, not a necessity. A few mathematicians prefer to base their outer system on more restrictive rules, such as those of the relevantist or the constructivist (see 2.41 and 2.42). Such an approach requires more time and greater sophistication; it will not be attempted here.

Returning to the question we asked at the beginning of this section: Is our reasoning circular? Yes, to some extent it is, and unavoidably so. Our outer system is classical logic, and our inner system is one of several logics — perhaps classical. How can we justify using classical logic in our study of classical logic?

Well, actually it's two different kinds of classical logic.

- Our inner system is some formal logic — perhaps classical — which we study without any restriction on the complexity of the formulas.
- Our outer system is an informal and simple version of classical logic; we use just fundamental inference rules that we feel fairly confident about.

It must be admitted that this "confidence" stems from common sense and some experience with mathematics. We assume a background and viewpoint that might not be shared by all readers.

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[1] Actually, even that last explanation of Mints's rule is an abbreviation. In later chapters we shall see that $\overline{S} \to (Q \vee R)$ is not a formula, but a formula scheme, and it is *not* a theorem scheme of constructive logic as stated. A full statement of Mints's rule is the following:

Suppose that $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are some formula schemes. Suppose that some particular substitution of formulas for the metavariables appearing in $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ makes $\overline{\mathcal{Z}} \to (\mathcal{X} \vee \mathcal{Y})$ into a theorem. Then the same substitution also makes $(\overline{\mathcal{Z}} \to X) \vee (\overline{\mathcal{Z}} \to Y)$ into a theorem.

There is no real bottom point of complete ignorance from which we can begin.

An analogous situation is that of the student in high school who takes a course in English.[2] The student already must know *some* English in order to understand the teacher; the course is intended to extend and refine that knowledge. Likewise, it is presumed that the audience of this book already has some experience with reasoning in everyday nonmathematical situations; this book is intended to extend and refine that knowledge of reasoning.

2.18. To reduce confusion, this book will use the terms *theorem* and *tautology* for truths in the inner system, and *principle* and *rule* for truths on a higher level. However, we caution the reader that this distinction is not followed in most other books and papers on logic. In most of the literature, the words "principle," "theorem," and "tautology" are used almost interchangeably (and outside of logic, the word "rule" too); the word "theorem" is used most often for all these purposes. In particular, what I have called the Deduction Principle, Glivenko's Principle, and Gödel's Incompleteness Principles are known as "theorems" in most of the literature. One must recognize from the context whether inner or outer is being discussed.

I will use the term *corollary* for "an easy consequence," in all settings — inner and outer, semantic and syntactic. See also 13.16.

2.19. *Adding more symbols to the language.* One major way of classifying logics is by what kinds of symbols they involve.

a. This book is chiefly concerned with *propositional logic*, also known as *sentential logic*. That logic generally deals with $\vee$ (or), $\wedge$ (and), $\neg$ (not), $\rightarrow$ (implies), and propositional variable symbols such as $\pi_1, \pi_2, \pi_3, \ldots$. A typical formula in propositional logic is $(\pi_1 \vee \pi_2) \rightarrow \pi_3$. (The symbols $\leftrightarrow$, $\circ$, $\rightsquigarrow$, &, $\perp$ are also on this level but are used less often in this book.)

b. *Predicate logic* adds some symbols for "individual variables" $x, y, \ldots$ to the language, as well as the *quantifier* symbols $\forall$ and $\exists$. Propositions $\pi_1, \pi_2, \ldots$ may be functions of those variables. For instance, $\forall x\ \exists y\ \pi_2(x, y)$ says that "for each x there exists at least one y (which may depend on x) such that

[2] Replace "English" with whatever is your language, of course.

$\pi_2(x, y)$ holds." This book includes a brief study of quantifiers in informal classical logic, starting in 5.41.

c. An *applied logic*, or *first-order theory*, adds some *nonlogical symbols*. For instance,

$$= \text{ (equals)}, \qquad + \text{ (plus)}, \qquad \times \text{ (times)}$$

are symbols, not of logic, but of arithmetic, another subject. To obtain the theory of arithmetic we add to predicate logic those nonlogical symbols, plus a few axioms governing the use of those symbols. A typical theorem about the set $\mathbb{Z}$ of integers (but not about the positive integers $\mathbb{N}$) is

$$\forall x\, \forall y\, \exists z \;\; x + z = y,$$

which says that for each choice of x and y there exists at least one z such that $x + z = y$. (In other words, we are able to subtract.) Another important applied logic is axiomatic set theory, described in 2.9. Applied logics will not be studied in this book.

d. *Modal logic* (also not studied in this book) adds some modal operators to one of the logics described above. Here are some examples of modal operators:

In this logic	the symbol "$\Box$" means	the symbol "$\Diamond$" means
alethic	it is necessary that	it is possible that
deontic	it is obligatory that	it is permitted that
epistemic	it is known that	it is not known to be false that

(See Copeland [1996] and references therein.) These modal operators occur in dual pairs, satisfying $\neg(\Diamond x) = \Box(\neg x)$. For instance,

it is not permitted that we kill people

says the same thing as

it is obligatory that we do not kill people.

Each modal logic has many variants. For instance, different ethical systems can be represented by the deontic logics stemming from different axiom systems. We cannot prove that one ethical system is "more correct" or "more logical" or "better" than another, but in principle we should be able to calculate the different consequences of those different ethical systems.

Semantics and syntactics

2.20. The logician studies the form and the meanings of language separately from each other. For a nonmathematical example of this separation, consider Lewis Carroll's nonsense poem *Jabberwocky*, which begins

'Twas brillig, and the slithy toves
Did gyre and gimble in the wabe;
All mimsy were the borogoves,
And the mome raths outgrabe.

We don't know exactly what "slithy" or "tove" means, but we're fairly certain that "slithy" is an adjective and "tove" is a noun — i.e., that "slithy" specifies some particular *kind* of "tove" — because the verse seems to follow the grammatical rules of ordinary English.

In much the same fashion, in mathematical logic we can study the grammar of formulas separately from the meanings of those formulas. For instance, the expression "$(A \vee B) \to C$" conforms to grammatical rules commonly used in logic, while the expression "$A)(\vee \to BC$" does not. All the logics studied in this book use the same grammatical rules, which we will study in Chapter 6.

2.21. *Two faces of logic.* Logic can be classified as either semantic or syntactic.[3] We will study these separately at first, and then together.

---

[3] Van Dalen refers to these as the "profane" and "sacred" sides of logic. That description may be intended humorously but has some truth to it.

Semantics is the concrete or applied approach, which initially may be more easily understood. We study examples, and investigate which statements are true as computational facts. A typical computation is

$$\llbracket \pi_1 \vee \neg\, \pi_1 \rrbracket \;=\; \llbracket \pi_1 \rrbracket \varovee \Big(\ominus \llbracket \pi_1 \rrbracket \Big) \;=\; \max \Big\{ \llbracket \pi_1 \rrbracket,\, 1 - \llbracket \pi_1 \rrbracket \Big\} \;\in\; \Sigma_+,$$

which would be read as follows: The expression $\pi_1 \vee \neg\, \pi_1$ is an uninterpreted string of symbols. We are interested in finding its value, $\llbracket \pi_1 \vee \neg\, \pi_1 \rrbracket$, in some valuation, in some particular interpretation. We first expand that expression to $\llbracket \pi_1 \rrbracket \varovee \ominus \llbracket \pi_1 \rrbracket$ by replacing each uninterpreted connective symbol, $\vee$ or $\neg$, with an interpretation of that symbol. Different logics give different meanings to those symbols; in the fuzzy interpretation which we are taking as an example, disjunction is the maximum of two numbers, and negation is subtraction from 1. Finally, we evaluate those numbers, and determine whether the resulting value is a member of Σ_+, the set of true values. The formula $\pi_1 \vee \neg\, \pi_1$ always evaluates to "true" in the two-valued interpretation and in some other interpretations; we write that fact as $\vDash \pi_1 \vee \neg\, \pi_1$. It evaluates sometimes to true and sometimes to false, in other interpretations, such as the fuzzy interpretation; we write that fact as $\nvDash \pi_1 \vee \neg\, \pi_1$. Semantic logic can also prove deeper results; for instance, it is shown in 9.12 that if some two formulas A and B satisfy $\vDash A \to B$ in the crystal interpretation, then the formulas A and B must have at least one π_j in common — they cannot be unrelated formulas like $\pi_1 \to \pi_6$ and $\pi_2 \vee \neg\pi_2$.

Syntactics is the abstract or theoretical approach. In syntactic logics, we do not have a set Σ_+ of true values or a set Σ_- of false values; we do not evaluate a formula to semantic values at all. Instead we start from some axioms (assumed formulas) and assumed inference rules, and investigate what other formulas and inference rules can be proved from the given ones. As a typical example, we shall now present a proof of the *Third Contrapositive Law*, $\vdash (\overline{A} \to B) \to (\overline{B} \to A)$, though the notation used here is less concise than that used later in the book. Of course, the proof makes use of results that are established prior to the Third Contrapositive Law.

#	formula	justification
(1)	$(\overline{A} \to B) \to (\overline{B} \to \overline{\overline{A}})$	This is a specialization of the Second Contrapositive Law, which is proved earlier.
(2)	$\overline{\overline{A}} \to A$	This is an axiom (an assumed formula).
(3)	$(\overline{B} \to \overline{\overline{A}}) \to (\overline{B} \to A)$	This follows by applying the detachmental corollary of $\to$-Prefixing to step (2).
(4)	$(\overline{A} \to B) \to (\overline{B} \to A)$	This follows by applying Transitivity, proved earlier, to steps (1) and (3).

2.22. *Example (optional): the real numbers.* Though the theory of $\mathbb{R}$ will not be studied in this book, we sketch the definition of $\mathbb{R}$ briefly here, because it makes a good example of the relation between semantics and syntactics.

In lower-level mathematics courses, the real numbers are presented as infinite decimal expansions or as points on a line. Those presentations are concrete and appeal to the intuition, and they will suffice for the needs of this book as well, but they do not really suffice to support proofs of deeper theorems about the real number system. A freshman college calculus book proves some of its theorems but leaves many unproved, stating that their proofs can be found in more advanced courses; those proofs actually require a more rigorous definition. For instance:

> *Intermediate Value Theorem.* If f is continuous on $[a, b]$ and $f(a) < m < f(b)$, then there is at least one number c in (a, b) such that $f(c) = m$.
>
> *Maximum Value Theorem.* If f is continuous on $[a, b]$, then there is at least one number c in $[a, b]$ with $f(c) = \max\{f(t) : a \leq t \leq b\}$.

These results are evident from pictures (using the "points on a line" explanation of the real numbers), but the pictures do not really constitute a rigorous proof, and the proof is omitted from such books. The rigorous proof *must* be omitted, because it requires notions that cannot be seen in pictures and that are too advanced to be discussed in any detail in freshman calculus books. The proof requires the completeness of the reals (discussed in 2.27), or equivalently the *least upper bound property*: Every subset of $\mathbb{R}$ that has an upper bound in $\mathbb{R}$, also has a *least* upper bound in $\mathbb{R}$. Some calculus books mention that property, but they don't follow through with the proofs.

A more advanced course, usually called "Introduction to Real Analysis," is taken by some mathematics majors in their senior year of college. It investigates the real number system more carefully. It lists the axioms for a *complete ordered field*; these include the least upper bound property and rules such as $x + y = y + x$. The axioms give us a *syntactic* system.

Some introductory textbooks on real analysis simply *assume* that the real numbers satisfy those axioms, but that really is evading part of the question. What is this thing that satisfies those axioms? How do we know that there *is* a thing satisfying those axioms? What are the "real numbers," really? More mature mathematicians take a slightly longer route to the real numbers:

One can prove that the infinite decimal expansions satisfy these axioms and thus form a complete ordered field. However, it is much easier to prove that those axioms are satisfied by *Dedekind cuts of rationals* or by Cantor's *equivalence classes of Cauchy sequences of rationals*; these are two "constructions" of the real numbers from the rational numbers. Decimal expansions, Dedekind cuts, and equivalence classes of Cauchy sequences are three different *semantic* systems that satisfy the axioms for a complete ordered field.

In a still more advanced course on algebra, one can prove that any two complete ordered fields are *isomorphic*.[4] For instance, each infinite decimal expansion corresponds to one Dedekind cut, or to one equivalence class of Cauchy sequences. Thus it doesn't really matter *which* complete ordered field we use. All such fields have the same algebraic properties, and thus they are just different manifestations or representations or relabelings of the same algebraic system.

Finally, one can *define* the real number system to *be* that algebraic system. Once we've done all this, we can put aside the semantic examples (decimal expansions, Dedekind cuts, equivalence classes), and just concentrate on the syntactic axioms. This may seem excessively abstract, but ultimately it focuses on the useful part. For developing abstract theorems such as the Intermediate Value Theorem and the Maximum Value Theorem, we are more concerned with what the real numbers do than with what they look like. In the case of real numbers, the axioms (syntactics) give us a more useful description or explanation than do the examples (semantics).

Actually, the usual axioms for the real numbers do not make a "first-order theory." Most of the axioms, such as $x+y = y+x$, are about numbers; but the least upper bound property is an axiom about numbers and about

[4] The meaning of the term "isomorphic" varies from one part of mathematics to another. In general, we say that two mathematical objects are isomorphic if one is just a relabeling of the other — i.e., if there exists a mapping between them that preserves all structures currently of interest. The precise definition varies because different mathematicians may find different structures to be of interest. For instance, the rational numbers and the integers are isomorphic if we're just interested in cardinality, but not if we're interested in multiplication and division.

sets of numbers. It is not possible to describe the real numbers as a first-order theory. One might expect that a simpler system, such as the integers, can be fully explained by a first-order theory, but Gödel showed that even that is not possible. See 2.34.

2.23. *Formulas* are expressions like $\pi_1 \to (\pi_2 \wedge \pi_1)$ or $\pi_1 \vee \neg\pi_1$. The Greek letters $\pi_1, \pi_2, \ldots$ are propositional variable symbols. A formula such as (for instance) $\pi_1 \vee \neg\pi_1$ may be "valid" in a couple of different ways:

- Semantics is concerned with values and meanings of formulas. For instance, the expression $\vDash \pi_1 \vee \neg\, \pi_1$ means "the formula $\pi_1 \vee \neg\, \pi_1$ is a *tautology* — i.e., it is an 'always-true' formula; it is true in all the valuations (examples) in the interpretation that we're currently considering."
- In a syntactic logic, $\vdash \pi_1 \vee \neg\pi_1$ means "the formula $\pi_1 \vee \neg\, \pi_1$ is a *theorem* — i.e., it is a *provable* formula; it is derivable from the assumptions of the inference system that we're currently considering." (Derivations are explained in greater detail starting in 12.5.) If the formula *is* one of those assumptions, we would also say it is an *axiom*. (Thus, each axiom is a theorem; it has a one-line derivation.)

Formula *schemes* are expressions like $A \to (B \vee A)$; the letters are metavariables. Each formula scheme represents infinitely many formulas, because each metavariable can be replaced by any formula. For instance, the formula scheme $A \vee B$ includes as instances the formulas

$$\pi_1 \vee \pi_2, \qquad (\pi_2 \to \pi_3) \vee (\pi_1 \wedge \neg\pi_5), \qquad \pi_1 \vee (\pi_2 \vee \pi_3)$$

and infinitely many others. Those individual formulas are part of the object system, but the formula scheme $A \vee B$ is at a slightly higher level of conceptualization. Our reasoning will lead us to draw conclusions about $A \vee B$, but this might be best understood as a way of summarizing conclusions drawn about the individual formulas. See 6.27 for further discussion.

A *tautology scheme* or *theorem scheme* is a formula scheme whose instances are all tautologies or theorems, respectively.

2.24. At a higher level, an *inference rule* is an expression involving two or more formulas, related by a turnstile symbol ($\vdash$ or $\vDash$). Again, inference rules come in two flavors:

- In a semantic logic, $\{A \to B,\ B \to C\} \ \vDash \ A \to C$ means that (regardless of whether the formulas involved are tautologies) in each valuation where the hypotheses $A \to B$ and $B \to C$ are true, the conclusion $A \to C$ is also true. See 7.10 for further discussion of such inference rules.
- In syntactic logic, $\{A{\to}B,\ B{\to}C\} \ \vdash \ A{\to}C$ means that (regardless of whether the formulas involved are theorems) from the hypotheses $A \to B$ and $B \to C$, we can derive the conclusion $A \to C$. See 12.5 for further discussion of derivations.

In this book, any inference rule must have at least one hypothesis — i.e., there must be at least one formula in the set on the left of the turnstile. That is part of this book's definition of "inference rule." (Permitting inference rules with no hypotheses — i.e., classifying tautologies and theorems as special cases of "inference rules" — would make sense, but would complicate some parts of our exposition.)

2.25. In addition to tautologies/theorems and inference rules, we will also study a few *higher principles*, such as admissibility rules. Here are three different levels of consequences:

Tautology $\vDash A{\to}B$	The formula $A \to B$ is always true.
Inference $A \vDash B$	Whenever A is true, then B is true.
Admissibility $\vDash A \ \Rightarrow \ \vDash B$	If some substitution of formulas for metavariables makes A an always-true formula, then the same substitution makes B always-true.

Beginners grasp the semantic notions of "always true" and "whenever true" with little difficulty, but are more challenged by the syntactic analogues, which are more abstract:

Theorem $\vdash A{\to}B$	The formula $A \to B$ can be derived from nothing (i.e., from just the logic's assumptions).

Inference $A \vdash B$	From formula A we can derive formula B.
Admissibility $\vdash A \;\Rightarrow\; \vdash B$	If some substitution of formulas for metavariables makes A into a formula that can be derived from nothing, then the same substitution also makes B derivable from nothing.

These three notions can easily be confused with one another by students, and even by experienced logicians who are new to pluralism. The distinction is not strongly supported by our language; it requires phrasing that is sometimes awkward. That may stem from the fact that in classical logic, the three notions turn out to be equivalent, and so they are used interchangeably in much of the literature. But in some other logics they are not equivalent. Moreover, even in classical logic the proof of equivalence (given in 22.5 and 29.15) is far from trivial; we must not treat the notions as equivalent until after we have proceeded through that proof.

The best way to alleviate the beginner's confusion is to look at several examples in different logics. We will do that in later chapters; following is a brief preview.

- In most logics of interest, $\vdash A \to B$ implies $A \vdash B$. Thus we can go from an implication theorem to an inference rule; we can move A leftward past the turnstile. This is essentially just *detachment* (13.2.a), an assumed inference rule in all the logics considered in this book.

But what about moving A to the *right* past the turnstile? That is the subject of three deduction principles[5] listed below:

- *Constructive version.* $A \vdash B \;\Rightarrow\; \vdash A \to B$. For this to hold, constructive implication is both necessary and sufficient; we prove that in 22.5 and 22.6.
- *Łukasiewicz version.* $A \vdash B \;\Rightarrow\; \vdash A \to (A \to B)$ in the three-valued Łukasiewicz logic; see 24.26.
- *Relevant version.* $A \vdash B \;\Rightarrow\; \vdash \overline{A} \vee B$ in the comparative

[5] Called "Theorems" in most of the literature. See 2.18.

and Sugihara interpretations and in relevant logic; see 8.30.c and 23.6.

All three of these principles are valid in classical logic. Indeed, they coalesce into one principle, since the three formulas $A \to B$, $A \to (A \to B)$, and $\overline{A} \vee B$ are equivalent in classical logic.

2.26. *Relating the two faces.* After we study some syntactic systems and semantic systems separately, we will study the interaction between them. In particular, by *completeness* we will mean a pairing of some syntactic system with some semantic system such that $\vdash A \Leftrightarrow \vDash A$. That is, these two sets of formulas are equal:

$$\left\{ \begin{array}{c} \text{theorems of the} \\ \text{syntactic system} \end{array} \right\} = \left\{ \begin{array}{c} \text{tautologies of the} \\ \text{semantic system} \end{array} \right\}.$$

In effect, the abstract axioms "explain" the concrete interpretation, and we obtain a complete analysis of the set of truths:

> any statement can be proved (P) by an abstract derivation or disproved (D) by a concrete counterexample.

That dichotomy is illustrated by the table below, which considers five syntactic logics (classical, Wajsberg, constructive, RM, or Abelian) and the corresponding five semantic logics (two-valued, Łukasiewicz, topological, Sugihara, comparative). For

syntactics	class. 25.1	Waj. 24.14	constr. 22.1	RM 23.13	Abel. 26.1
semantics	2-val. 8.2	Łuk. 8.17	topol. 10.1	Sugi. 8.38	comp. 8.28
complete?	29.12	29.20	29.29	(*23.11*)	(*26.8*)
$A \vee \neg A$	P	D	D	P	P
$(\neg\neg A) \to A$	P	P	D	P	P
$(A \to \neg A) \to \neg A$	P	D	P	P	D
$A \to (B \to A)$	P	P	P	D	D
$(A \wedge \neg A) \to B$	P	D	P	D	D
$((A \to B) \to B) \to A$	D	D	D	D	P

the first three of these five pairings, completeness is proved in this book. The last two pairings are also complete, but their completeness proofs are too advanced for this book, and are merely mentioned in the sections indicated by the italicized and parenthesized numbers.

A much longer list of completeness pairings is given in the next table below; again, italicized reference numbers indicate discussion in lieu of proof.

syntactics		*semantics*		*complete?*
classical	25.1	two-valued	8.2	29.12
classical	25.1	powerset	9.3	11.11
classical	25.1	six-valued	9.4	11.12
constructive	22.1	all topologies	10.1	29.29
constructive	22.1	finite topologies	10.1	29.29
constructive	22.1	no finite functional		22.16
constructive	22.1	$\mathbb{R}$ topology	10.1	(*29.29*)
Dummett	22.18	upper sets	4.6.h	(*22.18*)
relevant	23.1	Church chain	9.13	sound
relevant	23.1	Ch. diamond	9.14	sound
relevant	23.1	no finite functional		23.11.a
Brady	23.11.b	crystal	9.7	(*23.11.b*)
RM	23.13	Sugihara	8.38	(*23.11.c*)
RM	23.13	no finite functional		23.11.a
implications		worlds	28.1	28.13
Wajsberg	24.14	Łukasiewicz	8.16	29.20
Rose-Rosser	24.1	Zadeh	8.16	(*24.2*)
Abelian	26.1	comparative	8.28	(*26.8*)
not finite		Dziobiak		(*29.31*)
not findable		arithmetic		(*2.34*)

We caution the student that the pairs in this introductory book were selected in part for their simplicity, and so are atypical; "most" logics are more complicated. In particular, relevant logic (23.1) is one of the most important and interesting logics studied in this book, but its characterizing semantics are algebraic structures too complicated for us even to describe in this book. (Those semantics can be found in Anderson and Belnap [1975] and Dunn [1986].) The logic RM (relevant plus mingle) is of interest because it has some of the flavor of relevant logic and yet has a very simple semantics, but the proof of that completeness pairing is too advanced for this book to do more than state it; see 23.11.c.

Advanced books and papers on logic often use the terms "theorem" and "tautology" interchangeably, because they assume that the reader is already familiar with some completeness pairing. We may follow that practice in these introductory/preview chapters; but we will cease that practice when we begin formal logic in Chapter 6, and we will not resume it until after we prove completeness near the end of the book.

Remarks. More precisely, the pairing $\vdash A \Leftrightarrow \vDash A$ is called *weak completeness* in some of the literature. *Strong completeness* says that the syntactic and semantic systems have the same inference rules; that is, $\mathcal{S} \vdash A \Leftrightarrow \mathcal{S} \vDash A$ for any formula A and set of formulas $\mathcal{S}$. We will consider both kinds of completeness; see particularly 21.1.

2.27. The word "completeness" has many different meanings in math. Generally, it means "nothing is missing — all the holes have been filled in"; but different parts of mathematics deal with different kinds of "holes."

For instance, the rational number system has a hole where $\sqrt{2}$ "ought to be," because the rational numbers

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad 1.41421, \quad 1.414213, \quad \ldots$$

get closer together but do not converge to a rational number. If we fill in all such holes, we get the real number system, which is the completion of the rationals.

Even within logic, the word "completeness" has a few different meanings. Throughout this book, it will usually have the meaning sketched in 2.26, but some other meanings are indicated in 5.15, 8.14, and 29.14.

2.28. The two halves of completeness have their own names:

> *Adequacy* means {tautologies} $\subseteq$ {theorems}. That is, our axiomatic method of reasoning is *adequate* for proving all the true statements; no truths are missing.
>
> *Soundness* means {theorems} $\subseteq$ {tautologies}. That is, our method of reasoning is *sound*; it proves *only* true statements.

In most pairings of interest, soundness is much easier to establish. Consequently, soundness is sometimes glossed over, and some mathematicians use the terms "completeness" and "adequacy" interchangeably.

Historical perspective

2.29. Prior to the 19th century, mathematics was mostly empirical. It was viewed as a collection of precise observations about the physical universe. Most mathematical problems arose from physics; in fact, there was no separation between math and physics. Every question had a unique correct answer, though not all the answers had yet been found. Proof was a helpful method for organizing facts and reducing the likelihood of errors, but each physical fact remained true by itself regardless of any proof. This pre-19th-century viewpoint still persists in many textbooks, because textbooks do not change rapidly, and because a more sophisticated viewpoint may require higher learning.

2.30. Prior to the 19th century, Euclidean geometry was seen as the best known description of physical space. Some non-Euclidean axioms for geometry were also studied, but not taken seriously; they were viewed as works of fiction. Indeed, most early investigations of non-Euclidean axioms were carried out with the intention of proving those axioms *wrong*: Mathematicians hoped to prove that Euclid's parallel postulate was a consequence of Euclid's other axioms, by showing that a denial of the parallel postulate would lead to a contradiction. All such attempts were unsuccessful — the denial of the parallel postulate merely led to bizarre conclusions, not to outright contradictions — though sometimes errors temporarily led mathematicians to believe that they had succeeded in producing a contradiction.

However, in 1868 Eugenio Beltrami published a paper showing that some of these geometries are not just fictions, but skewed views of "reality" — they are *satisfied* by suitably peculiar *interpretations* of Euclidean geometry. For instance, in "double elliptic geometry," any two straight lines in the plane must meet.

This axiom is satisfied if we interpret "plane" to mean the surface of a sphere and interpret "straight line" to mean a great circle (i.e., a circle whose diameter equals the diameter of the sphere).

By such peculiar interpretations, mathematicians were able to prove that certain non-Euclidean geometries were *consistent* — i.e., free of internal contradiction — and thus were legitimate, respectable mathematical systems, not mere hallucinations. These strange interpretations had an important consequence for conventional (nonstrange) Euclidean geometry as well: We conclude that the Euclidean parallel postulate can *not* be proved from the other Euclidean axioms.[6]

After a while, mathematicians and physicists realized that we don't actually know whether the geometry of our physical universe is Euclidean, or is better described by one of the non-Euclidean geometries. This may be best understood by analogy with the two-dimensional case. Our planet's surface appears flat, but it is revealed to be spherical if we use delicate measuring instruments and sophisticated calculations. Analogously, is our three-dimensional space "flat" or slightly "curved"? And is it curved positively (like a sphere) or negatively (like a horse saddle)? Astronomers today, using radio-telescopes to study "dark matter" and the residual traces of the original big bang, may be close to settling those questions.

2.31. *The formalist revolution.* Around the beginning of the 20th century, because of Beltrami's paper and similar works, mathematicians began to change their ideas about what is mathematics and what is truth. They came to see that their symbols can have different interpretations and different meanings, and consequently there can be multiple truths. Though some branches of math (e.g., differential equations) continued their close connections with physical reality, mathematics as a whole has been freed from that restraint, and elevated to the realm

[6]These consistency results are actually *relative*, not *absolute*. They show that *if* Euclidean geometry is free of contradictions, *then* certain non-Euclidean geometries are also free of contradictions, and the parallel postulate cannot be proved from the other Euclidean axioms.

of pure thought.[7] Ironically, most mathematicians — even those who work regularly with multiple truths — generally retain a Platonist attitude: They see their work as a human investigation of some sort of objective "reality" which, though not in our physical universe, nevertheless somehow "exists" independently of that human investigation.

In principle, any set of statements could be used as the axioms for a mathematical system, though in practice some axiom systems might be preferable to others. Here are some of the criteria that we might wish an axiom system to meet, though generally we do not insist on all of these criteria. The system should be:

Meaningful. The system should have some uses, or represent something, or make some sort of "sense." This criterion is admittedly subjective. Some mathematicians feel that if an idea is sufficiently fundamental — i.e., if it reveals basic notions of mathematics itself — then they may pursue it without regard to applications, because some applications will probably become evident later — even as much as a century or two later. This justification-after-the-theory has indeed occurred in a few cases.

Adequate (or "complete"). The axioms should be sufficient in number so that we can prove all the truths there are about whatever mathematical object(s) we're studying. For instance, if we're studying the natural number system, can we list as axioms enough properties of those numbers so that all the other properties become provable?

Sound. On the other hand, we shouldn't have too many axioms. The axioms should not be capable of proving any *false* statements about the mathematical object(s) we're studying.

Independent. In another sense, we shouldn't have too many axioms: They should not be redundant; we shouldn't include any axioms that could be proved using other axioms. This criterion affects our efficiency and our aesthetics, but it does not really affect the "correctness" of the system; repetitions of axioms are tolerable. Throughout most of the axiom systems studied in this book, we will *not* concern ourselves about independence. This

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[7] Or reduced to a mere game of marks on paper, in the view of less optimistic mathematicians.

book is primarily concerned with comparing different logics, but an axiom that works adequately in studying several logics is not necessarily optimal for the study of any of them.

Consistent. This is still another sense in which we shouldn't have too many axioms: Some mathematicians require that the axioms do not lead to a contradiction. In some kinds of logic, a contradiction can be used to prove *anything*, and so any one contradiction would make the notion of "truth" trivial and worthless. In paraconsistent logics, however, a contradiction does not necessarily destroy the entire system; see discussion in 5.16.

Recursive. This means, roughly, that we have some algorithm that, after finitely many steps, will tell us whether or not a given formula is among our axioms. (Nearly every logic considered in this book is defined by finitely many axiom schemes, and so it is recursive.)

2.32. Over the last few centuries, mathematics has grown, and the confidence in mathematical certainty has also grown. During the 16th–19th centuries, that growth of certainty was part of a wider philosophical movement known as the Enlightenment or the Age of Reason. Superstitions about physical phenomena were replaced by rational and scientific explanations; people gained confidence in the power of human intellect; traditions were questioned; divine-right monarchies were replaced by democracies. Isaac Newton (1643–1727) used calculus to explain the motions of the celestial bodies. Gottfried Wilhelm von Leibniz (1646–1716) wrote of his hopes for a universal mathematical language that could be used to settle all disputes, replacing warfare with computation.

The confidence in mathematics was shown at its peak in David Hilbert's famous speech[8] in 1900. Here is an excerpt from the beginning of that speech:

[8]Hilbert (1862–1943) was the leading mathematician of his time. In his address to the International Congress of Mathematicians in 1900, he described 23 well-chosen unsolved problems. Attempts to solve these famous "Hilbert problems" shaped a significant part of mathematics during the 20th century. A few of the problems still remain unsolved.

> This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no "unknowable."

Hilbert was the leader of the formalists. He advocated working to put all of mathematics on a firm axiomatic foundation; this plan is now known as *Hilbert's program*. He and other mathematicians were encouraged by some successes in the 1920s, and particularly by Gödel's proof of completeness of first-order logic in 1930.

2.33. In 1931, however, Gödel proved his Incompleteness Principles (discussed below), thereby demolishing Hilbert's program: Gödel proved that, in fact, mathematics does have some "unknowable" propositions. At nearly the same time, Heisenberg formulated his Uncertainty Principle of particle physics. Evidently, there are inherent limitations on what we can know, in any field of investigation. However, these discoveries mark only a boundary, not a refutation, of the Enlightenment. Despite its limitations, reason remains a powerful tool.

2.34. *Gödel's First Incompleteness Principle* is a result of applied (first-order) logic, not propositional logic, so its detailed explanation is beyond the scope of this book; but we will describe it briefly:

> It is not possible to specify a set of axioms that specify precisely the truths of $\mathbb{N}$.

By "the truths of $\mathbb{N}$" we mean facts about the positive integers, including advanced facts about prime numbers.

A subtlety must be pointed out here: "not possible to specify" is not the same thing as "there does not exist." For instance, for our set of axioms, we could simply use the set of *all* true statements about $\mathbb{N}$. That set exists, but we can't actually *find* it. A set of axioms that we can "specify" means a recursive set — a set that is finite, or an infinite set of axioms, for which membership or nonmembership can be determined by some explicit algorithm.

The system $\mathbb{N}$ is particularly important because most parts of mathematics can be expressed in countable languages — i.e., only a sequence of symbols is needed. Consequently, most statements of mathematics can be encoded as statements about $\mathbb{N}$. (Indeed, statements could be encoded in terms of sequences of 0's and 1's; that is the language used inside modern electronic computers.) If we could axiomatize $\mathbb{N}$, then we could axiomatize most of mathematics.

After Gödel's work, other examples of incompleteness were found — i.e., mathematical theories which, if viewed as semantic systems, could not be suitably axiomatized. In the late 20th century, Gregory Chaitin showed that *most* of mathematics cannot be suitably axiomatized — that mathematics is riddled with holes. As Chaitin himself put it,

> most of mathematics is true for no particular reason.

Thus, a reasonable goal for mathematicians is not to explain everything (Hilbert's program, discussed in 2.32), but just to find some parts of mathematics that *can* be explained and that have some usefulness or other interest.

2.35. *Gödel's Second Incompleteness Principle* was concerned with *consistency*, mentioned in 2.31 as a minimal requirement for an axiom system. Hilbert hoped to prove that each of the main axiom systems of mathematics was consistent. This was in fact accomplished for some elementary axiom systems.

Any proofs about a logical "inner" system depend on the use of other mathematics in the "outer" system (see terminology in 2.17), but Hilbert hoped that such proofs could be arranged so that the outer mathematics used in the proof was more elementary than the inner system whose consistency was being proved. Thus we could "bootstrap" our way up, from systems so simple that they were obviously consistent, to more complicated systems. Gödel himself is credited with one of the most successful of these bootstrapping operations: In 1931 he proved the consistency of first-order logic, assuming only much more elementary and more obvious systems.

But Gödel's Second Incompleteness Principle showed limitations in this bootstrapping process: It cannot be applied to some axiomatic systems a bit higher than first-order logic. For some of those systems, consistency can only be proved by assuming the consistency of other systems that are at least as complicated and as subject to doubt. Thus an absolute proof of consistency is not possible.

One of the axiomatic systems to which Gödel's incompleteness result is applicable is set theory, the "foundation" of the objects of modern mathematics. Set theory is *empirically* consistent: It is now over a century old; if it had any contradictions, probably we would have found one by now. But we're still not certain of that, and Gödel has made it clear that we never will be. Even a mathematician must accept some things on faith or learn to live with uncertainty.

2.36. Though Gödel demonstrated the impossibility of Hilbert's *program*, Hilbert's *style* of formalism and formalizability continued to spread throughout mathematics, and it remains dominant to this day. One of the last branches of mathematics to embrace the formalist revolution was logic itself. Just as "geometry" meant "Euclidean geometry" until near the end of the 19th century, so too logic was dominated by a classical-only viewpoint until late in the 20th century.

Early in the 20th century, a few pioneering mathematicians began to investigate nonclassical logics. The resulting proofs have the same ironclad certainty as any other mathematical results, but the investigators themselves were motivated by philosophical beliefs. Many of them were not pluralists (espousing many logics), but were enamored of one particular nonclassical logic, and advocated it as "the one true logic," the one correct approach to thinking. Fighting against the majority view made them controversial figures.

One particularly interesting example was constructivism, introduced briefly in 2.42 and in greater detail in later chapters. Errett Bishop's book, *Foundations of Constructive Analysis* [1967], was not actually a book on logic — in fact, it barely mentioned formal logic — but it was nevertheless a radical breakthrough affecting logic. The book presented a substantial portion of analysis (a branch of mathematics) at the level of an advanced undergraduate or beginning graduate course, covering fairly standard material but in a constructivist style. This had never been done before, nor had previous mathematicians even believed it could be done.

Most of Bishop's book was just mathematics — objective and indisputable. However, Bishop explained his unconventional viewpoint in a preface titled "A Constructivist Manifesto." This told not only *how* his methods of reasoning worked (an objective, mathematical matter) but also *why* he felt these methods were preferable (a more subjective matter). Here is one of the more colorful passages:

> Mathematics belongs to man, not to God. We are not interested in properties of the positive integers that have no descriptive meaning for finite man. When a man proves a positive integer to exist, he should show how to find it. If God has mathematics of his own that needs to be done, let him do it himself.

(See the footnote in 29.2 for an example of God's mathematics.)

During the last decades of the 20th century, logic finally accepted the formalist revolution. Mathematical logic became a respectable subject in its own right, no longer dependent on philosophy or even particularly concerned about it. Researchers saw different logics merely as different types of abstract algebraic structures. (For instance, the rules in 7.5.c(ii) show that a functional valuation $[\![\ ]\!]$ is nothing other than a homomorphism from the algebra of formulas to the algebra of semantic values.) With this philosophy-free view, it was inevitable that research logicians would become pluralists; the many-logic viewpoint is now taken for granted in the research literature of logic. But that replacement of philosophy with algebra also made the research literature less accessible to beginners. Textbooks lagged behind, continuing to emphasize classical logic.

So far, only a few textbooks have been written with the pluralist approach; this book is one of them. Though pluralistic in content, this book is old-fashioned in style: It is intended for beginners, so we use philosophy for motivation and we attempt to keep the algebra to a minimum.

Pluralism

2.37. How can there be different kinds of logics? Isn't logic just common sense? No, it's not that simple. Different situations may call for different kinds of reasoning. As we already indicated in 2.26, different logics have different truths (i.e., theorems or tautologies). They also have different inference rules and higher-level principles, as indicated in the table below. We may understand these principles better if we study the contrasts between the different logics; that is one of the chief strategies of this book. Some of our main logics are briefly introduced in the next few pages.

property $\rightarrow$ logic $\downarrow$	Excluded middle or a variant of it	Deduction principle: If $A \vdash B \ldots$	If $\vdash A \rightarrow B$ with A, B unrelated	Explosion of some sort
relevant	$\vdash A \vee \overline{A}$	$\vdash \overline{A} \vee B$	can't be	none
integer-valued	$\vdash A \vee \overline{A}$	$\vdash \overline{A} \vee B$	$\vdash \overline{A}$ and $\vdash B$	$\vdash (A \wedge \overline{A}) \rightarrow (B \vee \overline{B})$
Wajsberg	$\vdash A \vee \overline{A} \vee \widetilde{A}$	$\vdash A \rightarrow (A \rightarrow B)$	no consequences	$A \wedge \overline{A} \vdash B$
constructive	$(\vdash A \vee B) \Rightarrow (\vdash A$ or $\vdash B)$	$\vdash A \rightarrow B$	$\vdash \overline{A}$ or $\vdash B$	$\vdash (A \wedge \overline{A}) \rightarrow B$
classical	$\vdash A \vee \overline{A}$	all of the above	$\vdash \overline{A}$ or $\vdash B$	$\vdash (A \wedge \overline{A}) \rightarrow B$

2.38. So-called *classical logic* is the logic developed by Frege, Russell, Gödel, and others. Among commonly used logics, it is computationally the simplest, and it is adequate for the needs of most mathematicians. In fact, it is the only logic with which most mathematicians (other than logicians) are familiar. The main ideas of propositional classical logic can be summarized by the simple true/false table given here; a similar table can

inputs		outputs			
P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$
0	0	1	0	0	1
0	1	1	0	1	1
1	0	0	0	1	0
1	1	0	1	1	1

be found as early as the works of Philo of Megara, one of the Greek Stoics (approximately 200 B.C.). For compatibility with other parts of this book, we shall abbreviate 0 = "false" and 1 = "true."

We emphasize that $P \rightarrow Q$ is considered to be true (1) in all cases except the case where P is true (1) and Q is false (0). Some noteworthy corollaries are

(a) a false statement implies anything;

(b) anything implies a true statement; and

(c) $P \rightarrow Q$ is true precisely when $P \leq Q$.

Exercise. Extend the table, giving output columns for the three formulas $Q \rightarrow P$, $(P \rightarrow Q) \vee (Q \rightarrow P)$, and $(P \rightarrow Q) \rightarrow (Q \rightarrow P)$.

Remarks. The "noteworthy corollary" (c) will generalize to most of the nonclassical logics in this book, but (a) and (b) may fail in logics that have several kinds of "true" or several kinds of "false." An example is examined in detail in 2.40.

2.39. Aristotle objected to the Megarian approach. He pointed out that if every statement — even a statement about a future event — is already either "true" or "false" (perhaps without our knowing which), then the future is predetermined. For instance, one of these statements is true, and the other is false:

- There will be a sea-battle tomorrow.
- There will not be a sea-battle tomorrow.

Łukasiewicz (1878–1956; pronounced woo-kah-sheay-vitch) extended this objection; if the future is predetermined then we have no free will. So in the 1920s he began studying *multivalued logics*. His earliest work involved only three semantic values — true (1), false (0), and indeterminate (1/2) — but later he considered *quantitative logics* with a whole range of values, e.g., all the numbers in the interval $[0, 1]$. A statement such as

most old men are *mostly* bald

is *mostly* true. We can make this precise by assigning numbers between 0 and 1 to "old," "bald," and the three *most*'s.

Closely related to Łukasiewicz's logics are the fuzzy logics studied by L. A. Zadeh in the 1960s. Zadeh observed that mechanical sensory devices such as thermostats cannot supply information with perfect precision, and even high precision is prohibitively expensive. In the real world, data are unavoidably imprecise. Thus, the circuits using that information must be designed to make the best precise use of imprecise data. Fuzzy logics are now used in the design of control circuitry in dishwashers, clothes dryers, automobile cruise controls, and other devices.

The term "fuzzy" may be unfortunate, for it has picked up a rather negative connotation in our society in recent years; a person who uses "fuzzy thinking" is a person who does not think clearly. But the two notions are nearly opposites: Fuzzy thinking is unnecessarily imprecise reasoning, while fuzzy logic is precise reasoning about imprecise information.

Fuzzy logic is investigated further in 8.16–8.26, Chapter 24, and 29.16–29.20.

2.40. Aristotle also mentioned a type of *comparative logic* that is incompatible with classical logic. He wrote[9]

(a) if there are two things both more desirable than something, the one which is more desirable to a greater

[9] This can be found in *Topics* III, 3, 118b2–3 of Aristotle [1984]. The analysis given here is modified from Casari [1989]; our development of this logic later in this book is based largely on Meyer and Slaney [1989, 2002].

degree is more desirable than the one more desirable to a less degree.

That's admittedly a bit hard to parse, but an example may be easier. Suppose that I'm serving coffee, tea, and punch. Then

(a′) if "the coffee is hotter than the punch" is more true than "the tea is hotter than the punch," then the coffee is hotter than the tea.

How can we restate this mathematically?

Assume that I've picked some particular temperature, and I call anything above that temperature "hot." This "hot" cutoff is a sort of absolute mark on our temperature scale, but it is a sliding absolute: We only use it temporarily, and our reasoning ultimately will not depend on where we put this cutoff point. Our reasoning will depend only on the differences in temperatures — i.e., how the coffee, tea, and punch stand *relative* to each other. (This explanation may justify the use of the word "relativity" in 20.2.)

Now, let us abbreviate the three propositions:

$$\begin{aligned} C &= \text{“the coffee is hot,”} \\ T &= \text{“the tea is hot,”} \\ P &= \text{“the punch is hot.”} \end{aligned}$$

As in 2.38(c), we understand that a less true statement implies a more true statement. For instance, saying that

(b) the coffee is hotter than the punch

is the same as saying that, no matter what we have chosen for the cutoff temperature in our definition of "hot,"

(b′) if the punch has a high enough temperature to be called "hot," then the coffee also has a high enough temperature to be called "hot,"

or, in other words,

(b″) $P \to C$.

Similarly, "the tea is hotter than the punch" can be restated as $P \to T$. Putting these statements together, since a less true statement implies a more true statement, "$P \to C$ is more true than $P \to T$" can be restated as $(P \to T) \to (P \to C)$. Finally, condition (a′) can be restated as

$$\text{(a}''\text{)} \quad \big[(P \to T) \to (P \to C)\big] \to (T \to C).$$

This "prefix cancellation" formula is always true in *comparative logic*, a mathematical logic developed further in 8.28–8.37 and Chapters 20 and 26.

But the prefix cancellation formula is not always true in classical logic — i.e., classical logic cannot be used to compare beverages in the fashion indicated above. For instance, suppose that the tea is hot, and the punch and coffee are not. Then $T = 1$ and $P = C = 0$. In two-valued classical logic, this yields

$$\underbrace{\underbrace{\Big[\underbrace{(P \to T)}_{1} \to \underbrace{(P \to C)}_{1}\Big]}_{1\ (*)} \to \underbrace{(T \to C)}_{0}}_{0}$$

so prefix cancellation is false. Evidently, classical logic is not a good method for analyzing statements such as (a′).

How does classical logic go astray? The error first appears at the step marked by the asterisk $(*)$ in the evaluation above. The statements $P \to T$ and $P \to C$ are both true, so classical logic calls $(P \to T) \to (P \to C)$ true. But $P \to T$ is *more* true than $P \to C$, so comparative logic calls $(P \to T) \to (P \to C)$ false.

2.41. In classical logic, $P \to Q$ is true whenever at least one of $\overline{P}$ or Q is true — regardless of how P and Q are related. Thus,

if the earth is square then today is Tuesday

is true to a classical logician (since the earth is *not* square); but it is nonsense to anyone else. *Relevant logic* is designed to avoid implications between unrelated clauses.

Most mathematicians believe that they are using classical logic in their everyday work, but that is only because they are unfamiliar with relevant logic. Here is an example:

> *A new Pythagorean theorem.* If $\lim_{\theta\to 0}\theta\csc\theta = 1$, then the sides of a right triangle satisfy $a^2 + b^2 = c^2$.

A theorem of this type would not be accepted for publication in any research journal. The editor or referee might respond that "one of the hypotheses of the theorem has not been used," or might use much stronger language in rejecting the paper. A superfluous hypothesis does not make the theorem false (classically), but it does make the theorem unacceptably weak; to even *think* of publishing such a theorem would be in very poor taste. Most mathematicians, not being familiar with relevant logic, do not realize that it describes much of their "good taste."

Relevant logic is investigated further in 5.29, 8.31, 8.43, 8.44, 9.12, and Chapters 23 and 28.

2.42. In the classical viewpoint, mathematics is a collection of statements; but to constructivists,[10] mathematics is a collection of procedures or constructions. The rules for combining procedures are slightly different from the rules for combining statements, so they require a different logic.

For instance, in classical logic, $A \vee \neg A$ is always true (as evidenced by the truth table in 2.38); either a thing is or it is not. That is known as the *Law of the Excluded Middle.* Thus, most mathematicians would agree that

> Goldbach's conjecture is true or the negation of Goldbach's conjecture is true.

But constructivists feel that it is meaningless to talk about the truthfulness of Goldbach's conjecture separately from the proof of Goldbach's conjecture. They point out that it is *not* presently true that

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[10]The student is cautioned not to confuse these similar-sounding words: converse (5.25), contrapositive (5.27), contradiction (5.35), contraction (15.4.c), and constructive (2.42 and 22.1).

> we know how to prove Goldbach's conjecture or we know how to disprove Goldbach's conjecture.

Thus the formula $A \vee \neg A$ does not always represent a construction, and so it is not a theorem of *constructive* logic.

Constructivism (or the lack of it) is illustrated further by Jarden's example (2.44–2.46), by the notion of inhabited sets (3.10), and by the Axiom of Choice (3.33). Constructive logic is investigated further in Chapters 10, 22, and 27–29.

2.43. *Goldbach's Conjecture*, mentioned in the preceding example, is a question of number theory, not logic; but as an unsolved problem it is useful for illustrating certain ideas of logic. It involves the *prime numbers* (2, 3, 5, 7, 11, 13, ...) — that is, the integers greater than 1 that are evenly divisible by no positive integers other than themselves and one. In 1742 Goldbach observed that

$$4 = 2+2, \quad 6 = 3+3, \quad 8 = 3+5, \quad 10 = 3+7, \quad 12 = 5+7, \quad \ldots \quad .$$

He conjectured that this sequence continues — i.e., that every even number greater than 2 can be written as the sum of two primes in at least one way. In a quarter of a millennium, Goldbach's Conjecture has not yet been proved or disproved, despite the fact that it is fairly simple to state.

Jarden's example (optional)

2.44. *Prerequisites for Jarden's proof*

a. The number $\sqrt{2}$ is irrational. *Proof.* Assume (for contradiction) that $\sqrt{2}$ is rational; say $\sqrt{2} = p/q$ where p and q are positive integers. Choose p as small as possible — i.e., let p be the smallest positive integer whose square is equal to twice a square. Then $p^2 = 2q^2$, hence $p > q$. Also then, p^2 is even, hence p is even, hence $p = 2r$ for some positive integer r. But then $q^2 = 2r^2$. Thus q's square is twice a square — i.e., q is smaller than p but satisfies the condition for which p was supposed to be smallest.

b. The number $\log_2 9$ is irrational. *Proof.* Assume that $\log_2 9 = p/q$ where p and q are positive integers. Then algebra yields $9^q = 2^p$. But the left side of that equation is a product of 3's and is not divisible by 2; the right side is a product of 2's and is not divisible by 3.

2.45. *Jarden's theorem.* There exist positive, irrational numbers p and q such that q^p is rational.

Jarden's nonconstructive proof. Consider two cases:

(a) If ${\sqrt{2}}^{\sqrt{2}}$ is rational, use $p = q = \sqrt{2}$.

(b) If ${\sqrt{2}}^{\sqrt{2}}$ is irrational, use $p = \sqrt{2}$ and $q = {\sqrt{2}}^{\sqrt{2}}$. Then a bit of algebra shows that q^p is a rational number. (*Exercise.* Find that number.)

In either case we have demonstrated the existence of p and q.

2.46. *Discussion of Jarden's proof.* The proof in the preceding paragraph is peculiar: After you read the proof, you may feel convinced that the desired numbers p and q exist, but you still don't know what they are. (More precisely, you don't know what q is.) We have not *constructed* the pair (p, q).

This peculiarity stems from the fact that we know ${\sqrt{2}}^{\sqrt{2}}$ is rational or irrational, but we don't know which. Take A to be the statement that ${\sqrt{2}}^{\sqrt{2}}$ is rational; then Jarden's proof relies on the fact that (classically) we know $A \vee \overline{A}$, the Law of the Excluded Middle, even though we do not know A and we do not know $\overline{A}$.

It is only Jarden's proof, not his theorem, that is nonconstructive. Some assertions (such as 3.33) are inherently nonconstructive, but for other theorems such as Jarden's a nonconstructive proof can be replaced by a (often longer) constructive proof. For instance, it can actually be shown that ${\sqrt{2}}^{\sqrt{2}}$ is irrational, using the Gel'fond-Schneider Theorem (a very advanced theorem whose proof will not be given here); hence we can use case (b) of Jarden's proof. Or, for a more elementary proof, use $p = \log_2 9$ and $q = \sqrt{2}$ it is easy to show that those numbers are irrational (see 2.44) and a bit of freshman algebra yields the value of q^p, which turns out to be a rational number. (*Exercise.* Find that number.)