In cases for which $\langle d^2 \rangle$ is much larger than $\langle d \rangle$, (4.11) is approximately

$$\ell = \frac{\log[n] + \log\left[\langle d^2 \rangle\right] - 2\log[\langle d \rangle]}{\log\left[\langle d^2 \rangle\right] - \log[\langle d \rangle]} = \frac{\log\left[n/\langle d \rangle\right]}{\log\left[\langle d^2 \rangle / \langle d \rangle\right]} + 1, \tag{4.12}$$

although (4.12) should be treated with caution, as cycles are ignored, and when we are above the threshold for a giant component to exist (e.g., when $\langle d^2 \rangle$ is much larger than $2\langle d \rangle$), then there can be nontrivial clustering for some degree sequences.

If we examine (4.11) for the case of a Poisson random network, then $\langle d^2 \rangle - \langle d \rangle = \langle d \rangle^2$, and then

$$\ell = \frac{\log\left((n-1)\frac{\langle d \rangle - 1}{\langle d \rangle} + 1\right)}{\log(\langle d \rangle)}.$$

When $\langle d \rangle$ is substantially greater than 1, this is roughly $\log(n)/\log(\langle d \rangle)$, which is very similar to the result for the regular tree example. If p is held constant, then as n increases, ℓ decreases and converges to 1 from above. In that case we would estimate the diameter to be 2. In fact, it can be shown that for a constant p, this crude approximation is right on the mark: the diameter of a large random graph with constant p is 2 with a probability tending to 1 (see Corollary 10.11 in Bollobás [86]). Next let us consider the case in which the average degree is not exploding but instead is held constant so that $p(n-1) = \langle d \rangle > 1$. Then our estimate for diameter is on the order of $\log(n)/\log(\langle d \rangle)$. Here the estimate is not as accurate.[30] Applying this estimate to the network generated in Figure 1.6, where $n = 50$, $p = .08$, and the average degree is roughly $\langle d \rangle = 4$, yields an estimated diameter of 2.8. While the calculation is only order of magnitude, this value is not far off for the largest component in Figure 1.6.

Developing accurate estimates for diameters, even for such completely random networks, turns out to be a formidable task that has been an active area of study in graph theory for the past four decades.[31] Nevertheless, the above approximations reflect the fact that the diameter of a random network is likely to be "small" in the sense that it is significantly smaller than the number of nodes, and one can work with specific models to develop accurate estimates.

4.3 ▪ An Application: Contagion and Diffusion

To develop a feel for how some of the derivations from random networks might be useful, consider the following application. There is a society of n individuals. One of them is initially infected with a contagious virus (possibly even a computer virus). Let the network of interactions in the society be described by a Poisson random network with link probability p.

30. In this range of p, the network generally has a giant component but is most likely not completely connected.

31. See Chapter 10 in Bollobás [86] for a report on some of the results and references to the literature.