2.0 CLASSICAL MECHANICS

2.1 FUNDAMENTAL TECHNIQUES

2.1.1 The Virial Theorem

The equation of motion of a system can be written in the form

$$\vec{F}_i - \dot{\vec{p}}_i = 0. \tag{2.1}$$

We are interested in the quantity

$$G = \sum_i \vec{p}_i \cdot \vec{r}_i . \tag{2.2}$$

Differentiating this expression gives

$$\frac{dG}{dt} = \sum_i \dot{\vec{r}}_i \cdot \vec{p}_i + \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i = \sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i + \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i . \tag{2.3}$$

From Equation 2.1 this reduces to

$$\frac{dG}{dt} = 2T + \sum_i \vec{F}_i \cdot \vec{r}_i . \tag{2.4}$$

Averaging over a period of time τ, we obtain

$$\frac{1}{\tau}\int_0^{\tau} \frac{dG}{dt}\, dt = \frac{1}{\tau}\left[G(\tau) - G(0)\right] = \langle 2T \rangle + \left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle . \tag{2.5}$$

If the motion is periodic, with τ = period, then then left hand side of Equation 2.5 is zero and we see that

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle . \tag{2.6}$$

2.1.2 D'Alembert's Principle

From Equation 2.1 it follows that the virtual work done by this system is also zero,

$$\sum_i \left(\vec{F}_i - \dot{\vec{p}}_i\right)\cdot \delta\vec{r}_i = 0. \tag{2.7}$$

The total force will be a combination of externally applied forces and internal constraints,

$$\vec{F}_i = \vec{F}_i'^{a} + \vec{f}_i, \tag{2.8}$$

so that Equation 2.8 reduces to

$$\sum_i \left(\vec{F}_i'^{a} - \dot{\vec{p}}_i\right)\cdot \delta\vec{r}_i - \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0. \tag{2.9}$$

If we restrict our attention to rigid bodies and other systems for which the forces of constraint do no work then we conclude that the condition for equilibrium of a system is given by D'Alemberts principle which states

$$\sum_i \left(\vec{F}_i^{a} - \dot{\vec{p}}_i\right)\cdot \delta\vec{r}_i = 0. \tag{2.10}$$

2.1.3 Lagrange's Equation

If $\vec{r}_i$ is a function of independent variables q_i, then

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j. \tag{2.11}$$

Dropping the superscript "*a*" for convenience, the first term from Equation 2.10 is

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_{i,j} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j, \tag{2.12}$$

where Q_j is the generalized force. The second term in Equation 2.10 is

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j. \qquad (2.13)$$

By definition

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left(\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right)\right)\right), \qquad (2.14)$$

and

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \frac{\partial \vec{v}_i}{\partial q_j}. \qquad (2.15)$$

Similarly

$$\vec{v}_i = \sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t} \qquad (2.16)$$

so it follows that

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}. \qquad (2.17)$$

Using Equations 2.15 and 2.17 in Equation 2.14 we find that

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left(\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right). \qquad (2.18)$$

Equations 2.12 and 2.18 are combined to give

$$\sum_j \left(\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right)\right) - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) - Q_j \right)\delta q_j = 0, \quad (2.19)$$

or

$$\sum_j \left(\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) - Q_j \right) \delta q_j = 0. \qquad (2.20)$$

Consequently,

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) = Q_j. \qquad (2.21)$$

If $F_i = -\nabla V_i$, then

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}. \qquad (2.22)$$

Equation 2.21 is equivalent to

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial}{\partial q_j}(T - V) \right) = 0. \qquad (2.23)$$

We define $L=T–V$ to be the Lagrangian. If V is not a function of time then

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right) = 0, \qquad (2.24)$$

which is known as Lagrange's equation, can be used to determine the equations of motion. We also define

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \qquad (2.25)$$

to be the canonical, or conjugate, momentum. If the Lagrangian does not contain a given coordinate q_j then the coordinate is said to be cyclic or ignorable.

2.1.3.1 The Two Body Central Force Problem

As an example of the application of Lagrange's equation, consider a system of two mass points m_1 and m_2 subject to an interaction potential *V*, where *V* is any function of the vector between the particles. The kinetic energy is

$$T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 . \tag{2.26}$$

The kinetic energy can also be written as the kinetic energy of the center of mass plus the kinetic energy about the center of mass. We define $\vec{R}$ = position of the center of mass, and $\vec{r} = \vec{r}_1 - \vec{r}_2$ = vector between m_1 and m_2. The kinetic energy of the center of mass is given by

$$T_{cm} = \frac{1}{2}(m_1 + m_2)\dot{R}^2 . \tag{2.27}$$

Relative to the center of mass, the position of m_1 and m_2 are given by

$$\vec{r}_1^{\,'} = \frac{-m_2}{m_1 + m_2}\vec{r} , \tag{2.28}$$

and

$$\vec{r}_2^{\,'} = \frac{m_1}{m_1 + m_2}\vec{r} . \tag{2.29}$$

Therefore, the kinetic energy about the center of mass is given by

$$T' = \frac{1}{2} m_1 \dot{r}_1'^2 + \frac{1}{2} m_2 \dot{r}_2'^2 = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{r}^2 . \tag{2.30}$$

Consequently, the Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{r}^2 - V(r, \dot{r}, \ldots) . \tag{2.31}$$

We see immediately that because the potential is only a function of the vector between the particles the conjugate momentum of *R*, (the momentum of the center of mass), is constant. That is, the motion of the center of mass has no effect on the motion about the center of mass. This also implies that there will be no out of plane motion.

2.1.3.2 The Inverse Square Law of Forces

When V is a function of r only, as is the case for gravitational or electrostatic forces, Equation 2.31 may be expressed in polar coordinates as

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r). \tag{2.32}$$

Note that we have chosen to ignore the term describing the motion of the center of mass since it has no effect on other parameters and we have introduced the definition

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \tag{2.33}$$

where μ is termed the reduced mass. The equations of motion are found from Lagrange's equation, (Equation 2.24). For the variable $q = \theta$ we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = \frac{d}{dt}(\mu r^2 \dot{\theta}) = 0, \tag{2.34}$$

while for $q = r$ we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0. \tag{2.35}$$

For many problems of interest, such as orbital mechanics, $m_1 >> m_2$ and $\mu \to m_2$. The physical consequence is that the smaller particle, m_2, is subjected to the largest perturbation in its motion. From this point forward we will follow the usual convention and replace μ with the symbol m with the understanding that it refers to the motion of the smaller of m_1 and m_2 about the center of mass.

Equation 2.34 is the statement of conservation of angular momentum. That is,

$$l = mr^2\dot{\theta}, \tag{2.36}$$

is a constant. Equation 2.36 can be rewritten in the form

$$l dt = mr^2 d\theta, \tag{2.37}$$

which implies

$$\frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}, \tag{2.38}$$

and

$$\frac{d^2}{dt^2} = \frac{l}{mr^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{d}{d\theta}\right). \tag{2.39}$$

The area swept out by a moving body is given by

$$A = \frac{1}{2}r(r\theta). \tag{2.40}$$

It follows that

$$\frac{dA}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}\left[\frac{1}{2}r(r\theta)\right] = \frac{l}{2}. \tag{2.41}$$

Because angular momentum is conserved *dA/dt* is also constant. This is Kepler's Second Law which states that the planets sweep out equal areas in equal times.

From the definition of *l*, equation 2.35 becomes

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{\partial V}{\partial r}, \tag{2.42}$$

or, because *V* is only a function of *r*,

$$m\ddot{r} = -\frac{d}{dr}\left(V + \frac{1}{2}\frac{l^2}{mr^2}\right). \tag{2.43}$$

The particle moves in an effective potential given by

$$V_{eff} = \left(V + \frac{l^2}{2mr^2}\right). \tag{2.44}$$

Equation 2.43 reduces to

$$m\ddot{r} = -\frac{dV_{eff}}{dr}, \tag{2.45}$$

thus

$$m\dot{r}\ddot{r} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{dr}{dt}\frac{dV_{eff}}{dr} = -\frac{dV_{eff}}{dt}. \tag{2.46}$$

Consequently,

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right) = 0, \tag{2.47}$$

which is the statement that energy is conserved. From this equation, we also find that

$$\dot{r} = \pm\sqrt{\left(\frac{2}{m}\left(E - V_{eff}\right)\right)} = \pm\sqrt{\left(\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)\right)}, \tag{2.48}$$

or simply

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}. \tag{2.49}$$

Substituting the relation between dt and $d\theta$, Equation 2.37, and we find that

$$d\theta = \frac{1}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}}\frac{dr}{r^2}. \tag{2.50}$$

Consider the case when the potential is of the form $V = -\frac{k}{r} = -ku$. Equation 2.50 becomes

$$d\theta = \frac{-1}{\sqrt{\frac{2mE}{l^2} + \frac{2mk}{l^2}u - u^2}}du. \tag{2.51}$$

Integrating this expression gives

$$\theta_f - \theta_i = \arcsin\frac{-2u + \frac{2mk}{l^2}}{\sqrt{\frac{8mE}{l^2} + \left(\frac{2mk}{l^2}\right)^2}} = \arcsin\frac{\left(1 - \frac{l^2}{mk}\frac{1}{r}\right)}{\sqrt{\frac{2El^2}{mk^2} + 1}}. \tag{2.52}$$

Inverting this expression gives

$$\frac{1}{r} = \frac{mk}{l^2}\left\{1 - \left[\sqrt{1 + \frac{2El^2}{mk^2}}\right]\sin\left(\theta_f - \theta_i\right)\right\}. \tag{2.53}$$

This is usually written in the form

$$\frac{1}{r} = \frac{mk}{l^2}\left\{1 - e\sin\left(\theta_f - \theta_i\right)\right\}, \tag{2.54}$$

where

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}. \tag{2.55}$$

This is the equation for a conic section having several classes of solutions as shown below.

1.) If $e > 1$, and $E > 0$, the orbit is a hyperbola.

2.) If $e = 1$, and $E = 0$, the orbit is a parabola.

3.) If $e < 1$, and $E < 0$, the orbit is an ellipse.

4.) If $e = 0$, and $E = -\frac{mk^2}{2l^2}$, the orbit is a circle.

2.2 VARIATIONAL TECHNIQUES

2.2.1 The Calculus of Variations

Consider a function $f(y, \dot{y}, x)$ defined on a path $y=y(x)$ between x_1 and x_2 where $\dot{y} = dy/dx$. We wish to find a particular path $y(x)$ such that the integral

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)dx, \tag{2.56}$$

has a stationary value relative to paths differing infinitesimally from the correct function $y(x)$. Since J must have a stationary value for the correct path relative to any neighboring path, the variation must be zero relative to some particular set of neighboring paths. Such a set of paths can be denoted by

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x), \tag{2.57}$$

where $y(x, 0)$ is the correct path and $\eta(x_1) - \eta(x_2) = 0$. Explicitly,

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x, \alpha), \dot{y}(x, \alpha), x)dx. \tag{2.58}$$

A necessary condition for a stationary point is

$$\left(\frac{dJ}{d\alpha}\right)\bigg|_{\alpha=0} = 0. \tag{2.59}$$

From Equation 2.58

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} \right) dx. \tag{2.60}$$

It is easily seen that

$$\frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} = \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x \partial \alpha}, \tag{2.61}$$

so that

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x \partial \alpha} dx = \left. \frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha} \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}} \right) \frac{\partial y}{\partial \alpha} dx. \tag{2.62}$$

From the boundary conditions, the first term on the right hand side vanishes and Equation 2.60 reduces to

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} \right) \frac{\partial y}{\partial \alpha} dx. \tag{2.63}$$

The fundamental lemma of the calculus of variations states that if

$$\int_{x_1}^{x_2} M(x)\eta(x) dx = 0, \tag{2.64}$$

for all $\eta(x)$ continuous through the second derivative then $M(x)$ must be identically zero on the interval. Thus J is stationary only if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}} \right) = 0. \tag{2.65}$$

If f is a function of many independent variables then

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_i} \right) = 0. \tag{2.66}$$

2.2.2 Hamilton's Principle

Hamilton's principle states that the motion, in configuration space, of a system where all non-constraining forces are derivable from a generalized scalar potential that may be a function of coordinates, velocities, and time is such that the integral

$$I = \int_{t_1}^{t_2} L dt \, , \tag{2.67}$$

has a stationary value for the correct path of the motion. That is,

$$\delta I = \delta \int_{t_1}^{t_2} L dt = 0 \, . \tag{2.68}$$

The integral I is termed the action, and Hamilton's principle states that the variation in I is zero. In other words, the action is minimized. By comparison with Equation 2.65 it follows that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \tag{2.69}$$

which is Lagrange's equation.

2.2.3 Lagrange Multipliers

D'Alembert's principle, and the resulting form of Lagrange's equation, assume no constraint forces. Consider a treatment when the equations of constraint can be put in the form

$$\sum_k a_{lk} dq_k + a_{lt} = 0 \, , \tag{2.70}$$

where the a_{lk} and a_{lt}'s may be functions of a, t. For virtual displacements it follows that

$$\sum_k a_{lk} \delta q_k = 0 \ . \tag{2.71}$$

If this is true, then it must also follow that

$$\lambda_l \sum_k a_{lk} \delta q_k = 0 \ , \tag{2.72}$$

where the λ_l are undetermined coefficients called Lagrange multipliers. From Equations 2.63, 2.68, and 2.69 it is seen that Hamilton's principle is equivalent to

$$\int \sum_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k = 0. \tag{2.73}$$

By the same process, Equation 2.72 is equivalent to

$$\int \sum_{k,l} \lambda_l a_{lk} \delta q_k dt = 0. \tag{2.74}$$

We combine these two relations to obtain

$$\int \sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} + \sum_l \lambda_l a_{lk} \right) \delta q_k dt = 0. \tag{2.75}$$

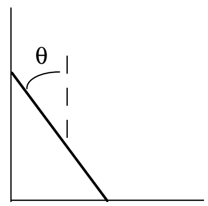
The δq_k 's are not necessarily independent, but because the values of the λ_l's are undetermined we may choose them such that

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \sum_l \lambda_l a_{lk} \ . \tag{2.76}$$

Thes equations, together with Equation 2.70, can be used to determine the equations of motions for systems with constraining forces.

Example 2.1

Consider the case of a ladder of length L that is inclined against a frictionless wall and floor as shown at right. Find the equations of motion.



The position of the center of mass of the ladder, and its orientation, can be described with the variables x, y, θ. The motion of the ladder is constrained by the wall and floor. We have the two constraints

$$x = \frac{L}{2}\sin\theta, \tag{2.77}$$

and

$$y = \frac{L}{2}\cos\theta. \tag{2.78}$$

From Equation 2.60 it follows that these give the constraining relations

$$\lambda_1\left[dx - \frac{L}{2}\cos\theta d\theta\right] = 0, \tag{2.79}$$

and

$$\lambda_2\left[dy + \frac{L}{2}\sin\theta d\theta\right] = 0, \tag{2.80}$$

respectively. By inspection, the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\theta}^2, \tag{2.81}$$

where $I = \frac{1}{12}mL^2$. Similarly, the potential energy is

$$V = mgy, \tag{2.82}$$

so that the Lagrangian is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{mL^2}{24}\dot{\theta}^2 - mgy. \tag{2.83}$$

From Equations 2.76, 2.79, and 2.80 the equations of motion are

$$m\ddot{x} = \lambda_1, \tag{2.84}$$

$$m\ddot{y} + mg = \lambda_2, \tag{2.85}$$

and

$$\frac{mL^2}{24}\ddot{\theta} = -\lambda_1\left(\frac{L}{2}\cos\theta\right) + \lambda_2\left(\frac{L}{2}\sin\theta\right), \tag{2.86}$$

respectively. From Equations 2.77 and 2.78 we see that

$$\ddot{x} = -\frac{L}{2}\sin\theta\dot{\theta}^2 + \frac{L}{2}\cos\theta\ddot{\theta}, \tag{2.87}$$

and

$$\ddot{y} = \frac{L}{2}\cos\theta\dot{\theta}^2 - \frac{L}{2}\sin\theta\ddot{\theta}. \tag{2.88}$$

From Equations 2.84 and 2.87 we see that

$$\lambda_1 = m\left(-\frac{L}{2}\sin\theta\dot{\theta}^2 + \frac{L}{2}\cos\theta\ddot{\theta}\right), \tag{2.89}$$

while from Equation 2.85 and 2.88 we see that

$$\lambda_2 = m\left(\frac{L}{2}\cos\theta\dot{\theta}^2 - \frac{L}{2}\sin\theta\ddot{\theta} + g\right). \tag{2.90}$$

When these are combined with Equation 2.86 and simplified we obtain

$$\frac{1}{6}\ddot{\theta} = \left(-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}\right)\left(\cos\theta\right) - \left(\cos\theta\dot{\theta}^2 - \sin\theta\ddot{\theta} + g\right)\left(\sin\theta\right), \tag{2.91}$$

which is equivalent to

$$-\frac{5}{6}\ddot{\theta} = -2\sin\theta\cos\theta\dot{\theta}^2 - g\sin\theta. \tag{2.92}$$

2.2 RIGID BODY MOTION

2.2.1 Rotations

A rigid body in space needs 6 independent generalized coordinates to specify its configuration. For example, 3 coordinates are needed to specify the location of the center of mass relative to some external axes and 3 other coordinates are needed to specify the orientation of the body relative to a coordinate system parallel to the external axes. The orientation is specified by stating the direction cosine of the body axes relative to the external axes. That is, if the prime denotes body axes then

$$\hat{i}' = \left(\hat{i}'\cdot\hat{i}\right)\hat{i} + \left(\hat{i}'\cdot\hat{j}\right)\hat{j} + \left(\hat{i}'\cdot\hat{k}\right)\hat{k}, \tag{2.93}$$

or

$$\hat{i}' = \alpha_1\hat{i} + \alpha_2\hat{j} + \alpha_3\hat{k}, \tag{2.94}$$

or

$$\hat{i}' = \cos\left(\hat{i}',\hat{i}\right)\hat{i} + \cos\left(\hat{i}',\hat{j}\right)\hat{j} + \cos\left(\hat{i}',\hat{k}\right)\hat{k}. \tag{2.95}$$

Similarly,

$$\vec{x}' = \vec{\vec{R}}\vec{x}, \tag{2.96}$$

where

$$\vec{\vec{R}} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}. \tag{2.97}$$

Because

$$\|\vec{x}'\| = \|\vec{x}\|, \tag{2.98}$$

we have

$$\vec{R}\vec{R}^t = \vec{I}, \tag{2.99}$$

or

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1, \tag{2.100}$$

for l = 1, 2, 3. That is,

$$x_i' = a_{ij} x_j, \tag{2.101}$$

and

$$x_i' x_i' = a_{ij} a_{ik} x_j x_k = x_i x_i. \tag{2.102}$$

Therefore $\alpha_{ij}\alpha_{ik} = 1$, if $j = k$, $\alpha_{ij}\alpha_{ik} = 0$ otherwise. In two dimensions

$$\vec{R} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{2.103}$$

2.2.1.1 General Properties of Rotations

If $\vec{G} = \vec{A}\vec{F}$ and we transform to a new coordinate system then $\vec{B}\vec{G} = \vec{B}\vec{A}\vec{F} = \vec{B}\vec{A}\vec{B}^{-1}\vec{B}\vec{F}$ we say that $\vec{B}\vec{A}\vec{B}^{-1}$ is the form of $\vec{A}$ in the new coordinate system. $\vec{A}' = \vec{B}\vec{A}\vec{B}^{-1}$ defines a similarity transformation. In some coordinate system

$$\vec{A}' = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.104}$$

so that $Tr\vec{A} = 1 + 2\cos\phi$. This property holds true in all coordinate systems.

Example 2.2

Find the axis of rotation and compute the angle of rotation for

$$\vec{R} = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{6}}{4} & \frac{1}{4} \\ -\frac{\sqrt{6}}{4} & \frac{1}{2} & \frac{\sqrt{6}}{4} \\ \frac{1}{4} & -\frac{\sqrt{6}}{4} & \frac{3}{4} \end{pmatrix}.$$

It can easily be verified that $\vec{R}\vec{R}^T = \vec{I}$, so $\vec{R}$ satisfies the requirements of a rotation matrix. The axis of rotation can be defined as the direction that any vector which remains unchanged by $\vec{R}$ points. That is, if $\vec{x} = \vec{R}\vec{x}$, then $\vec{x}$ is the axis of rotation. We must solve

$$\begin{pmatrix} \frac{3}{4} & \frac{\sqrt{6}}{4} & \frac{1}{4} \\ -\frac{\sqrt{6}}{4} & \frac{1}{2} & \frac{\sqrt{6}}{4} \\ \frac{1}{4} & -\frac{\sqrt{6}}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}, \tag{2.105}$$

which gives the equations

$$3x_1 + \sqrt{6}x_2 + x_3 = 4x_1, \tag{2.106}$$

$$-\sqrt{6}x_1 + 2x_2 + \sqrt{6}x_3 = 4x_2, \tag{2.107}$$

$$x_1 - \sqrt{6}x_2 + 3x_3 = 4x_3. \tag{2.108}$$

These three equations can be solved to show that $x_1 = x_3$ and $x_2 = 0$. The normalized axis of rotation is therefore $\frac{1}{\sqrt{2}}(1,0,1)$. By examination, Tr $\vec{R}$ = 1 + 2 cos θ = 2, so that θ = 60º.

2.2.2.2 The Euler Angles

Rather than specify the 9 independent elements of the rotation matrix we may describe the orientation in terms of 3 Euler angles. For example, rotate the initial system of axes, *xyz*, by an angle ψ counterclockwise about *z*. This defines the ξηζ axes. Next rotate about the ξ axis by an angle θ in the counterclockwise direction. This defines the ξ'η'ζ' axes. Finally rotate counterclockwise by an angle φ about the ζ' axes. This defines the ξ'ψ'ζ' axes. In matrix form

$$\vec{x}' = \vec{A}\vec{x}, \tag{2.109}$$

where

$$\vec{A} = \vec{B}\vec{C}\vec{D}, \tag{2.110}$$

and

$$\vec{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.111}$$

$$\vec{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \tag{2.112}$$

$$\vec{D} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.113}$$

2.2.2.3 The Cayley-Klein Parameters

Consider a general linear transformation in 2 dimensional space

$$\vec{u}' = \alpha\vec{u} + \beta\vec{v}, \tag{2.114}$$

and

$$\vec{v}' = \gamma\vec{u} + \delta\vec{v}, \qquad (2.115)$$

where the transformation matrix is

$$\vec{Q} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \qquad (2.116)$$

Note that α, β, γ, δ may be complex. If we require $\vec{Q}\vec{Q}^t = \vec{I}$ and $|\vec{Q}| = +1$ we find that β = −γ* and δ = α*, that is

$$\vec{Q} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}. \qquad (2.117)$$

Consider a matrix of the form

$$\vec{P} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}, \qquad (2.118)$$

such that

$$\vec{P}' = \vec{Q}\,\vec{P}\,\vec{Q}^t \cdot \qquad (2.119)$$

The hermitian property and the trace of a matrix are unaffected by similarity transformations. Consequently, $\vec{P}'$ is of the form

$$\vec{P}' = \begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix}. \qquad (2.120)$$

If we let $x_+ = x + iy$ and $x_- = x - iy$, then

$$\vec{P}' = \begin{pmatrix} z' & x_-' \\ x_+' & z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z & x_- \\ x_+ & -z \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}. \qquad (2.121)$$

In this way, we may define a 9 element rotation matrix in terms of 4 Cayley-Klein parameters.

2.2.2 The Rate of Change of a Vector

The rate of change of a vector $\vec{r}$ as seen by an observer in the body system of axes will differ from the corresponding change as seen by an observer fixed in space. If the body axes are rotating with angular velocity ω the general solution is

$$\left(\frac{d\vec{r}}{dt}\right)_{space} = \left(\frac{d\vec{r}}{dt}\right)_{body} + \vec{\omega}\times\vec{r}\,. \qquad (2.122)$$

We have

$$\vec{v}_s = \vec{v}_b + \vec{\omega}\times\vec{r}\,, \qquad (2.123)$$

and a successive application of Equation 2.122 gives

$$\vec{a}_s = \vec{a}_b + 2(\vec{\omega}\times\vec{v}_b) + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{d\vec{\omega}}{dt}\times\vec{r}\,. \qquad (2.124)$$

If the angular velocity of the body is constant Newton's law is

$$\vec{F} = m\vec{a}_s = m\vec{a}_b + 2m(\vec{\omega}\times\vec{v}_b) + m\vec{\omega}\times(\vec{\omega}\times\vec{r})\,. \qquad (2.125)$$

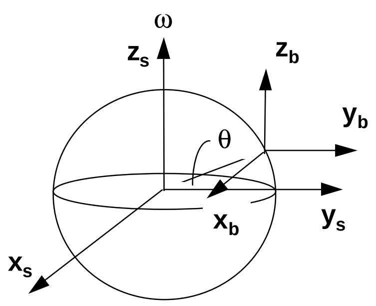
To an observer in the rotating system it appears as though the particle is moving under an effective force

$$\vec{F}_{eff} = m\vec{a}_b = \vec{F} - 2m(\vec{\omega}\times\vec{v}_b) - m\vec{\omega}\times(\vec{\omega}\times\vec{r})\,. \qquad (2.126)$$

The second term on the right hand side is called the Coriolis force and the last term on the right hand side is called the centrifugal force.

Example 2.3

Show that if a particle is thrown up vertically with initial speed v_o and reached a height h, that it will experiment a Coriolis deflection that is opposite in direction, and four times greater in magnitude, than the deflection it would experience if it were dropped at rest from the same maximum height.



The rate of rotation of the Earth, ω, is identical for each scenario. The difference is the initial velocity and position of the particle. From equation 2.126, the Coriolis force is $2(\vec{\omega}\times\vec{v}_b)$. Defining the coordinate axes as shown, we have $\omega_x = \omega_y = 0$ and $\omega_z = \omega$, while $v_x = 0$, $v_y = v\cos\theta$, and $v_z = v\sin\theta$.
As a result,

$$-2(\vec{\omega}\times\vec{v}_b) = +2(\omega v\cos\theta)\hat{i} + 2(0)\hat{j} + 2(0)\hat{k}, \tag{2.127}$$

As a result, the acceleration due to the Coriolis force is simply

$$\ddot{x} = 2\omega v\cos\theta. \tag{2.128}$$

A body falling under the influence of gravity satisfies the condition

$$v = v(0) - gt, \tag{2.129}$$

where $v(0)$ is defined as the velocity at time $t = 0$. Substituting this into Equation 2.128 and integrating gives

$$\dot{x} = 2\omega\cos\theta\left[v(0)t - \frac{1}{2}gt^2\right]. \tag{2.130}$$

Integrating a second time gives

$$x = 2\omega\cos\theta\left[\frac{1}{2}v(0)t^2 - \frac{1}{6}gt^3\right], \tag{2.131}$$

To solve the problem at hand, we let t be the time it takes for the particle to complete the trip. The position of the particle at any time is given by

$$h(t) = h(0) + v(0)t - \frac{1}{2}gt^2, \qquad (2.132)$$

In one case, the particle starts at rest, $v(0) = 0$, at height h. From the definition of the problem, it can be seen that the time required to complete the drop to the ground is

$$t_1 = \sqrt{\frac{2h}{g}} = \frac{v_o}{g}, \qquad (2.133)$$

Consequently, the deflection when dropped from rest at height h is

$$x_1 = 2\omega\cos\theta\left[-\frac{1}{6}g\left(\frac{v_o}{g}\right)^3\right] = -\left(\frac{1}{3}\right)\left(\frac{v_o}{g}\right)^3 g\omega\cos\theta. \qquad (2.134)$$

If instead, we let t_2 be the time it takes to fall from height h, we have $v_i = v_o$ and $t_2 = 2t_1$. The deflection is

$$x_2 = 2\omega\cos\theta\left[\frac{1}{2}v_o\left(\frac{2v_o}{g}\right)^2 - \frac{1}{6}g\left(\frac{2v_o}{g}\right)^3\right] = +\left(\frac{4}{3}\right)\frac{v_o^3}{g^2}\omega\cos\theta. \qquad (2.135)$$

Thus, the deflection when thrown from the ground up is opposite in direction, and four times greater in magnitude, than then deflection when dropped from rest.

2.2.3 The Rigid Body Equations of Motion

We have previously seen that the total kinetic energy of a rigid body may be expressed as the sum of the kinetic energy of the entire body as if concentrated at the center of mass, plus the kinetic energy of the motion about the center of mass. If a rigid body moves with one point stationary, the angular momentum about the center of mass is

$$\vec{L} = m_i \, (\vec{r}_i \times \vec{v}_i). \tag{2.136}$$

Since r_i is a fixed vector in the body

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i. \tag{2.137}$$

Thus

$$\vec{L} = m_i (\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)) = m_i (\vec{\omega} r_i^2 - \vec{r}_i \, (\vec{r}_i \cdot \vec{\omega})). \tag{2.138}$$

The x-component is

$$L_x = \omega_x m_i (r_i^2 - x_i^2) - \omega_y m_i x_i y_i - \omega_z m_i x_i z_i, \tag{2.139}$$

or

$$L_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z. \tag{2.140}$$

We define

$$I_{jk} = \int_v \rho(\vec{r})(r^2 \delta_{jk} - x_j x_k) dV, \tag{2.141}$$

to be the inertia tensor so that

$$\vec{L} = \vec{I} \, \vec{\omega}. \tag{2.142}$$

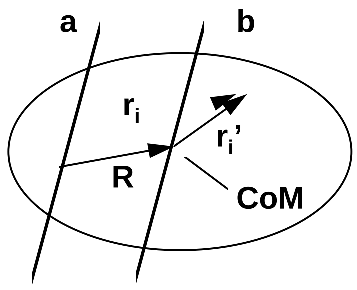
The kinetic energy of motion about a point is

$$T = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i), \tag{2.143}$$

or

$$T = \frac{\vec{\omega} \cdot \vec{L}}{2} = \frac{1}{2} \vec{\omega} \cdot \vec{I} \cdot \vec{\omega}. \tag{2.144}$$

The moment of inertia, $\vec{I}$, about some given axis is related to the moment about a parallel axis through the center of mass. Let the vector from the origin to the center of mass be $\vec{R}$ and let the radii vectors from the origin and the center of mass to the i^{th} particle be $\vec{r}_i$ and $\vec{r}_i\,'$ respectively. That is



$$\vec{r}_i = \vec{R} + \vec{r}_i\,'. \qquad (2.145)$$

The moment of inertial about axis *a* is

$$I_a = m_i(\vec{r}_i \times \hat{n})^2 = m_i\,[(\vec{R} + \vec{r}_i\,') \times \hat{n}]^2, \qquad (2.146)$$

or

$$I_a = M(\vec{R} \times \hat{n})^2 + m_i\,(\vec{r}_i\,' + \hat{n})^2 + 2m_i(\vec{R} \times \hat{n})(\vec{r}_i\,' \times \hat{n}). \qquad (2.147)$$

The last term is $2(\vec{R} \times \hat{n}) \cdot (\hat{n} \times m_i \vec{r}_i\,')$, but $\sum_i m_i r_i\,' = 0$ and $\vec{I}_b = m_i(\vec{r}_i\,' \times \hat{n})^2$ so

$$\vec{I}_a = \vec{I}_b + M(\vec{R} \times \hat{n})^2. \qquad (2.148)$$

2.2.3.1 The Euler Equations and Torque-Free Motion

By definition

$$\left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_b + \vec{\omega} \times \vec{L} = \vec{N}. \qquad (2.149)$$

In the body system, using $L_i = I_{ii}\,\omega_i$, we find

$$I_i \frac{d\omega_i}{dt} + \varepsilon_{ijk}\,\omega_j \omega_k I_k = N_i. \qquad (2.150)$$

The Euler equations are

$$I_1\dot{\omega}_1 - \omega_2\omega_3\ (I_2 - I_3) = N_1, \tag{2.151}$$

$$I_2\dot{\omega}_2 - \omega_3\omega_1\ (I_3 - I_1) = N_2, \tag{2.152}$$

$$I_3\dot{\omega}_3 - \omega_1\omega_2\ (I_1 - I_2) = N_3. \tag{2.153}$$

The principle axes of a system are those where the tensor $\bar{I}$ is diagonal. If this is the case,

$$T = \frac{1}{2}\frac{L_1^2}{I_1} + \frac{1}{2}\frac{L_2^2}{I_2} + \frac{1}{2}\frac{L_3^2}{I_3}. \tag{2.154}$$

Because T is constant, the relation defines an ellipsoid fixed in the body axes. Because angular momentum is conserved $\vec{L}$ must be on a fixed sphere defined by

$$L^2 = L_1^2 + L_2^2 + L_3^2. \tag{2.155}$$

For the given initial conditions, kinetic energy and angular momentum, the path of $\vec{L}$ is constrained to be the intersection of the sphere, L^2, and the ellipsoid, T. If we have a body symmetrical about the z axis so that $I_1 = I_2$ the Euler equations are, in the absence of torque's,

$$I_1\dot{\omega}_1 = (I_1 - I_3)\ \omega_3\ \omega_1, \tag{2.156}$$

$$I_2\dot{\omega}_2 = -(I_1 - I_3)\ \omega_3\ \omega_1, \tag{2.157}$$

$$I_3\dot{\omega}_3 = 0. \tag{2.158}$$

We have ω_3 = constant, and

$$\dot{\omega}_1 = -\Omega\omega_2,\ \ \dot{\omega}_2 = \Omega\omega_1, \tag{2.159}$$

where

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3. \tag{2.160}$$

Thus,

$$\ddot{\omega}_1 = -\Omega\omega_1, \qquad (2.161)$$

which has solutions $\omega_1 = A\cos\Omega t$ and $\omega_2 = A\sin\Omega t$. Hence, the total angular velocity is constant in magnitude but precesses with frequency Ω about the ζ-axis. We may solve for A and ω_3 by noting that

$$T = \frac{1}{2}I_1A^2 + \frac{1}{2}I_3\omega_3^2, \qquad (2.162)$$

and

$$L^2 = I_1^2A^2 + I_3^2\omega_3^2. \qquad (2.163)$$

2.2.3.2 The Heavy Symmetrical Top

Consider a heavy top with a symmetry axis taken to be the z-axis of the coordinate system fixed in the body as shown. The 3 Euler angles are: θ = inclination of the z-axis from the vertical, ϕ = azimuth of the top about the vertical, and ψ = rotation angle of the top about its own z-axis. The kinetic energy is

$$T = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2, \qquad (2.164)$$

or

$$T = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2. \qquad (2.165)$$

The potential energy is

$$V = mgl\cos\theta, \qquad (2.166)$$

so that the Lagrangian is given by

$$L = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos\theta)^2 - mgl \cos\theta. \qquad (2.167)$$

We have

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \dot{\phi} \cos\theta) = I_3 \omega_3 = I_1 a, \qquad (2.168)$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_1 \sin^2 \theta + I_3 \cos^2 \theta)\dot{\phi} + I_3 \dot{\psi} \cos\theta = I_1 b, \qquad (2.169)$$

and

$$E = T + V = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} I_3^2 \omega_3^2 + Mgl \cos\theta. \qquad (2.170)$$

Combining these first two expressions gives

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2 \theta}, \qquad (2.171)$$

and

$$\dot{\psi} = \frac{I_1 a}{I_3} - \cos\theta \frac{b - a\cos\theta}{\sin^2 \theta}. \qquad (2.172)$$

2.3 OSCILLATORY MOTION

2.3.1 Oscillations

Consider a system that is subjected to a potential that is only a function of coordinates. If we expand the coordinates q_i about their equilibrium position according to

$$q_i = q_{io} + \eta_i, \qquad (2.173)$$

and then perform a Taylor series expansion on the potential we obtain

$$V(q_i) = V(q_{io}) + \left(\frac{\partial V}{\partial q_i}\right)\bigg|_o \eta_i + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)\bigg|_o \eta_i \eta_j + \ldots \tag{2.174}$$

The first term on the right hand side may be redefined to be zero by shifting the zero potential to be the equilibrium value. Similarly, if the generalized forces are zero the second term is also zero. As a result, the potential can be approximated by the matrix relation

$$V(q_i) \approx \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)\bigg|_o \eta_i \eta_j . \tag{2.175}$$

Similarly, we can define

$$T = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} m_{ij} \dot{\eta}_i \dot{\eta}_j = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j , \tag{2.176}$$

so that the Lagrangian is

$$L = \frac{1}{2}\left(T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j\right). \tag{2.177}$$

The equations of motion are given by

$$T_{ij}\, \ddot{\eta}_j + V_{ij}\, \eta_j = 0 . \tag{2.178}$$

If we try a solution of the form $\eta_i = C a_i \exp^{-i\omega t}$ we find that

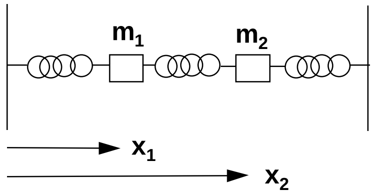
$$V_{ij}\, a_j - \omega^2\, T_{ij}\, a_j = 0 , \tag{2.179}$$

which is equivalent to

$$\left| V - \omega^2 T \right| = 0 . \tag{2.180}$$

Example 2.4

Two particles move in one dimension at the junction of three springs as shown. The springs all have unstretched length a and force constants as shown. Find the frequencies of the normal modes of oscillation.



We define the coordinates x_1 and x_2 to describe the displacement of the two blocks, relative to the left attachment point. In these coordinates

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2. \qquad (2.181)$$

Expanding about equilibrium, we define $x_1 = a + \eta_1$, $x_2 = 2a + \eta_2$. Equation 2.181 is equivalent to

$$T = \frac{1}{2} m \dot{\eta}_1^2 + \frac{1}{2} m \dot{\eta}_2^2 = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j, \qquad (2.182)$$

where

$$T_{ij} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}. \qquad (2.183)$$

Similarly,

$$V = \frac{1}{2} k (x_1 - a)^2 + \frac{1}{2} 3k [x_2 - x_1 - a]^2 + \frac{1}{2} k (2a - x_2)^2, \qquad (2.184)$$

or

$$V = \frac{1}{2} V_{ij} \eta_i \eta_j, \qquad (2.185)$$

with

$$V_{ij} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right) \Bigg|_a . \qquad (2.186)$$

By examination

$$\frac{\partial V}{\partial x_1} = k(x_1 - a) - 3k(x_2 - x_1 - a), \tag{2.187}$$

$$\frac{\partial^2 V}{\partial x_1^2} = k + 3k = 4k, \tag{2.188}$$

and

$$\frac{\partial V}{\partial x_2} = 3k(x_2 - x_1 - a) - k(2a - x_2), \tag{2.189}$$

$$\frac{\partial^2 V}{\partial x_2^2} = 3k + k = 4k. \tag{2.190}$$

Likewise, the cross terms are

$$\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial^2 V}{\partial x_2 \partial x_1} = -3k, \tag{2.191}$$

so that

$$V_{ij} = \begin{pmatrix} 4k & -3k \\ -3k & 4k \end{pmatrix}. \tag{2.192}$$

The secular equation is

$$\left|V - \omega^2 T\right| = \begin{vmatrix} 4k - \omega^2 m & -3k \\ -3k & 4k - \omega^2 m \end{vmatrix} = 0, \tag{2.193}$$

which reduces to

$$\left(4k - \omega^2 m\right)^2 - 9k = m^2\omega^4 - 8km\omega^2 + 7k^2 = 0. \tag{2.194}$$

From the quadratic equation, this has solutions

$$\omega_1 = \sqrt{\frac{k}{m}}, \tag{2.195}$$

and

$$\omega_2 = \sqrt{\frac{7k}{m}}. \tag{2.196}$$

2.4 HAMILTON'S EQUATIONS

2.4.1 Legendre Transformations and Hamilton's Equations of Motion

Consider a function f of two variables such that

$$df = udx + vdy, \tag{2.197}$$

where $u = \frac{\partial f}{\partial x}$ and $v = \frac{\partial f}{\partial y}$. We wish to change from the variables x, y to the variables u, y. We define a function

$$g = f - ux. \tag{2.198}$$

We have

$$dg = df - dux - udx = vdy - xdu, \tag{2.199}$$

so that $x = -\frac{\partial g}{\partial u}$ and $v = \frac{\partial g}{\partial y}$. This is an example of a Legendre transformation. It is frequently used in thermodynamics.

Recall that Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \tag{2.200}$$

where $L = L(q_i, \dot{q}_i, t)$, and the conjugate momenta are

$$p_i = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i}. \tag{2.201}$$

We can transform from the variables $(q_i, \dot{q}_i, t)$ to the variables (q_i, p_i, t) by the use of

$$H(q_i, p_i, t) = \dot{q}_i p_i - L(q_i, \dot{q}_i, t), \tag{2.202}$$

where H is called the Hamiltonian. We have

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt. \tag{2.203}$$

However, from the definition

$$dH = p_i d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt, \tag{2.204}$$

or simply

$$dH = \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt. \tag{2.205}$$

Comparing the two expressions for dH we obtain Hamilton's equations

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i, \tag{2.206}$$

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \tag{2.207}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{2.208}$$

If the equations defining the generalized coordinates don't depend on time explicitly, and if the forces are derivable from a conservative potential V, then

$$H = T + V = E. \tag{2.209}$$

If we define a column matrix $\vec{\eta}$ with 2n elements such that $\eta_i = q_i$, $\eta_{i+n} = p_i$, for $i \le n$, then

$$\left(\frac{\partial H}{\partial \eta_i}\right) = \frac{\partial H}{\partial q_i}, \tag{2.210}$$

and

$$\left(\frac{\partial H}{\partial \vec{\eta}_{i+n}}\right) = \frac{\partial H}{\partial p_i}. \tag{2.211}$$

If we define $\bar{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then Hamilton's equations may be written in the form

$$\dot{\vec{\eta}} = \bar{J}\frac{\partial H}{\partial \vec{\eta}}. \tag{2.212}$$

This is called symplectic notation.

Example 2.5

The Lagrangian for a simple spring is given by

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

Find the Hamiltonian and the equations of motion using the Hamiltonian formulation. Identify any conserved quantities.

From the definition of p_i we have

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}. \tag{2.213}$$

From the definition of the Hamiltonian, equation 2.227, we see that

$$H = \dot{x}(m\dot{x}) - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2, \tag{2.214}$$

or simply

$$H = \frac{1}{2m} p_x^2 + \frac{1}{2} kx^2 . \qquad (2.215)$$

From Hamilton's equations we have

$$\frac{\partial H}{\partial q_i} = kx = -\dot{p}_x , \qquad (2.216)$$

$$\frac{\partial H}{\partial p_i} = \frac{p_x}{m} = \dot{x} , \qquad (2.217)$$

$$\frac{\partial H}{\partial t} = 0 . \qquad (2.218)$$

The first equation is the equation of motion in one dimension,

$$m\ddot{x} + kx = 0 , \qquad (2.219)$$

the second equation is the definition of momentum, and the last equation is the statement of conservation of energy.

2.5.2 Canonical Transformations

If a generalized coordinate q_i has constant conjugate momenta it is said to be cyclic. If this is the case, then p_i = 0, which tells us that the Hamiltonian is independent of that p_i. If all coordinates q_i are cyclic the conjugate momenta can be defined by $p_i = \alpha_i$. Consequently,

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i , \qquad (2.220)$$

or

$$q_i = \omega_i t + \beta_i . \qquad (2.221)$$

A problem is often easier to solve if we can find a system where the number of cyclic coordinates is maximum. How do we transform to this set of coordinates? We need a new set of coordinates Q_i, P_i, where $Q_i = Q_i(q_i, p_i, t)$, $P_i = P_i(q_i, p_i, t)$. We require Q_i, P_i to be canonical coordinates. Therefore, some function $K = K(Q_i, P_i, t)$ exists such that

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \tag{2.222}$$

and

$$\dot{P}_i = \frac{\partial K}{\partial Q_i}. \tag{2.223}$$

If Q_i, P_i are canonical coordinates they must satisfy a modified Hamilton's principle that can be put in the form

$$\delta \int_{t_1}^{t_2} \left(P_i \dot{Q}_i - K(Q_i, P_i, t) \right) dt = 0, \tag{2.224}$$

because the old coordinates satisfy

$$\delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q_i, p_i, t) \right) dt = 0. \tag{2.225}$$

Both requirements can be satisfied if we require a relation

$$\lambda\left(p_i \dot{q}_i - H \right) = P_i \dot{Q}_i - K + \frac{dF}{dt}, \tag{2.226}$$

where λ = constant and F is any function of the phase space coordinates continuous through the second derivative. λ is related to a scale transformation. If $\lambda = 1$ the relation defines a canonical transformation. The function F is termed the generating function. It may be a function of q_i, p_i, Q_i, P_i, t and defines the transformation.

2.5.3 Symplectic Transformations and Poisson Brackets

Recall that Hamilton's equations can be written in the from

$$\dot{\vec{\eta}} = \vec{J}\frac{\partial H}{\partial \vec{\eta}}. \tag{2.227}$$

If we have a canonical transformation from $\eta \to \xi = \xi(\eta)$, then

$$\dot{\xi}_i = \frac{\partial \xi_i}{\partial \eta_j}\dot{\eta}_j. \tag{2.228}$$

In matrix form,

$$\dot{\vec{\xi}} = \vec{M}\dot{\vec{\eta}}, \tag{2.229}$$

where $M_{ij} = \frac{\partial \xi_i}{\partial \eta_j}$. We have

$$\dot{\vec{\xi}} = \vec{M}\vec{J}\frac{\partial H}{\partial \vec{\eta}}, \tag{2.230}$$

also

$$\frac{\partial H}{\partial \eta_i} = \frac{\partial H}{\partial \xi_j}\frac{\partial \xi_j}{\partial \eta_i}, \tag{2.231}$$

and

$$\frac{\partial H}{\partial \vec{\eta}} = \tilde{\vec{M}}\frac{\partial H}{\partial \vec{\xi}}. \tag{2.232}$$

Consequently,

$$\dot{\vec{\xi}} = \vec{M}\vec{J}\tilde{\vec{M}}\frac{\partial H}{\partial \vec{\xi}} = \vec{J}\frac{\partial H}{\partial \vec{\xi}}. \tag{2.233}$$

Therefore, a transformation is canonical if

$$\vec{M}\vec{J}\tilde{\vec{M}} = \vec{J}. \tag{2.234}$$

The Poisson bracket of a function is defined by

$$[u,v]_{PB} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}. \tag{2.235}$$

We have

$$[\vec{\eta},\vec{\eta}]_{PB} = \tilde{J}, \tag{2.236}$$

and

$$[\vec{\xi},\vec{\xi}]_{PB} = \frac{\partial \tilde{\vec{\xi}}}{\partial \vec{\eta}}\tilde{J}\frac{\partial \vec{\xi}}{\partial \vec{\eta}} = \tilde{\tilde{M}}\tilde{J}\tilde{M} = \tilde{J}. \tag{2.237}$$

In other words, the fundamental Poisson brackets are invariant under canonical transformations. We also have

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t}, \tag{2.238}$$

or

$$\frac{du}{dt} = [u,H]_{PB} + \frac{\partial u}{\partial t}. \tag{2.239}$$

Similarly,

$$\dot{q}_i = [q_i, H]_{PB}, \ \dot{p}_i = [p_i, H]_{PB}. \tag{2.240}$$

2.6 CONTINUOUS SYSTEMS

2.6.1 The Transition from a Discrete to a Continuous System

Consider an infinitely long elastic rod that can undergo small longitudinal vibrations. We approximate this by an infinite chain of equal mass points a distance a apart and connected by uniform massless springs having force constraints k. The kinetic energy is

$$T = \frac{1}{2}\sum_i m\dot{\eta}_i^2 , \qquad (2.241)$$

where m is the mass of each particle and η_i is the location of the ith particle. The potential energy is

$$V = \frac{1}{2}\sum_i k(\eta_{i+1} - \eta_i)^2 . \qquad (2.242)$$

We have

$$L = \frac{1}{2}\sum_i \left[m\dot{\eta}_i^2 - k(\eta_{i+1} - \eta_i)^2 \right], \qquad (2.243)$$

or

$$L = \frac{1}{2}\sum_i a\left[\frac{m}{a}\dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1} - \eta_i}{a} \right)^2 \right]. \qquad (2.244)$$

The equations of motion are

$$\frac{m}{a}\ddot{\eta}_i - ka\left(\frac{\eta_{i+1} - \eta_i}{a^2} \right) + ka\left(\frac{\eta_i - \eta_{i-1}}{a^2} \right) = 0 . \qquad (2.245)$$

We have $m/a = \mu$ = mass/unit length. Hooke's law states that the extension of a rod/unit length is proportional to the force, i.e.

$$F = Y\xi , \qquad (2.246)$$

where $\xi = \left(\frac{\eta_{i+1} - \eta_i}{a} \right)$ is the extension/unit length. The force necessary to stretch the string by an amount x is

$$F = k(\eta_{i+1} - \eta_i) = ka\left(\frac{\eta_{i+1} - \eta_i}{a} \right). \qquad (2.247)$$

Consequently, $ka = Y$ = Young's modulus. Note that

$$\frac{\eta_{i+1}-\eta_i}{a}=\frac{\eta(x+a)-\eta(x)}{a}\rightarrow\frac{d\eta}{dx}, \qquad (2.248)$$

as $a \rightarrow 0$. Also as $a \rightarrow 0$ the summation over the particles becomes an integral and Equation 2.244 becomes

$$L=\frac{1}{2}\int\left(\mu\dot{\eta}^2-Y\left(\frac{d\eta}{dx}\right)^2\right)dx. \qquad (2.249)$$

The equation of motion is

$$\mu\frac{d^2\eta}{dt^2}-Y\frac{d^2\eta}{dx^2}=0. \qquad (2.250)$$

This is a wave equation with wave propagation velocity $v=\sqrt{\frac{Y}{\mu}}$. Equation 2.249 is said to define a Lagrangian density

$$\tilde{L}=\mu\left(\frac{d\eta}{dt}\right)^2-Y\left(\frac{d\eta}{dx}\right)^2. \qquad (2.251)$$

2.6.2 The Lagrangian Formulation

Consider a Lagrangian density

$$\tilde{L}=\tilde{L}\left(\eta,\frac{d\eta}{dx},\frac{d\eta}{dt},x,t\right). \qquad (2.252)$$

Hamilton's principle is

$$\delta I=\delta\int_1^2\int\tilde{L}\,dx\,dt=0. \qquad (2.253)$$

We choose value of η such that

$$\eta(x,t;\alpha)=\eta(x,t;0)+\alpha\xi(x,t), \qquad (2.254)$$

where $\eta(x,t;0)$ is the function that satisfies Hamilton's principle. We have

$$\delta I = \left(\frac{dI}{d\alpha}\right)_{\alpha=0} = 0, \tag{2.255}$$

where

$$\frac{dI}{d\alpha} = \int_{t_1}^{t_2}\int_{x_1}^{x_2} dxdt\left\{\frac{\partial\tilde{L}}{\partial\eta}\frac{\partial\eta}{\partial\alpha} + \frac{\partial\tilde{L}}{\partial\left(\frac{d\eta}{dx}\right)}\frac{\partial}{\partial\alpha}\left(\frac{d\eta}{dx}\right) + \frac{\partial\tilde{L}}{\partial\left(\frac{d\eta}{dt}\right)}\frac{\partial}{\partial\alpha}\left(\frac{d\eta}{dt}\right)\right\}. \tag{2.256}$$

This expression may be simplified as follows. First,

$$\int_{t_1}^{t_2} dt\left\{\frac{\partial\tilde{L}}{\partial\left(\frac{d\eta}{dt}\right)}\frac{\partial}{\partial\alpha}\left(\frac{d\eta}{dt}\right)\right\} = -\int_{t_1}^{t_2}\frac{d}{dt}\left[\frac{\partial\tilde{L}}{\partial\left(\frac{d\eta}{dt}\right)}\right]\frac{\partial\eta}{\partial\alpha}dt, \tag{2.257}$$

plus a boundary term that goes to zero. Also,

$$\int_{x_1}^{x_2} dx\left\{\frac{\partial\tilde{L}}{\partial\left(\frac{d\eta}{dx}\right)}\frac{\partial}{\partial\alpha}\left(\frac{d\eta}{dx}\right)\right\} = -\int_{x_1}^{x_2}\frac{d}{dx}\left[\frac{\partial\tilde{L}}{\partial\left(\frac{d\eta}{dx}\right)}\right]\frac{\partial\eta}{\partial\alpha}dx, \tag{2.258}$$

plus another boundary term that goes to zero. Therefore,

$$\delta I = \int_{t_1}^{t_2}\int_{x_1}^{x_2} dxdt\left\{\frac{\partial\tilde{L}}{\partial\eta} - \frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\left(\frac{d\eta}{dx}\right)}\right) - \frac{d}{dx}\left(\frac{\partial\tilde{L}}{\partial\left(\frac{d\eta}{dt}\right)}\right)\right\}\left(\frac{\partial\eta}{\partial\alpha}\right)_{\alpha=0} = 0, \tag{2.259}$$

and

$$\frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\left(\frac{d\eta}{dx}\right)}\right)+\frac{d}{dx}\left(\frac{\partial\tilde{L}}{\partial\left(\frac{d\eta}{dt}\right)}\right)-\frac{\partial\tilde{L}}{\partial\eta}=0. \tag{2.260}$$

Similarly for other Lagrangians.

2.6.3 Noether's Theorem

A formal description of the connection between invariance or symmetry properties and conserved quantities is contained in Noether's theorem. We consider transformations where

$$x_\mu \to x_\mu^{'} = x_\mu + \delta x_\mu, \tag{2.261}$$

$$\eta_\rho(x_\mu) \to \eta_\rho^{'}(x_\mu^{'}) = \eta_\rho(x_\mu) + \eta_\rho(\delta x_\mu), \tag{2.262}$$

$$\tilde{L}(\eta_\rho(x_\mu), \eta_{\rho,\nu}(x_\mu), x_\mu) \to \tilde{L}'\left(\eta_\rho^{'}(x_\mu^{'}), \eta_{\rho,\nu}^{'}(x_\mu^{'}), x_\mu^{'}\right). \tag{2.263}$$

We make three assumptions:

1.) 4-space is Euclidean,

2.) The Lagrangian density has the same functional form after transformation,

3.) The magnitude of the action integral is invariant under the transformation.

From assumptions 2, 3 we have

$$\int_{\Omega'} \tilde{L}(\eta_\rho^{'}, \eta_{\rho,\nu}^{'}, x_\mu^{'})dx_\mu^{'} - \int_{\Omega} \tilde{L}(\eta_\rho, \eta_{\rho,\nu}, x_\mu)dx_\mu = 0. \tag{2.264}$$

$x_\mu^{'}$ is a dummy variable, so let $x_\mu^{'} \to x_\mu$ to obtain

$$\int_{\Omega'} \tilde{L}\left(\eta_\rho'\left(x_\mu\right), \eta_{\rho,\nu}'\left(x_\mu\right), x_\mu\right) dx_\mu - \int_{\Omega} \tilde{L}\left(\eta_\rho\left(x_\mu\right), \eta_{\rho,\nu}\left(x_\mu\right), x_\mu\right) dx_\mu = 0 . \quad (2.265)$$

Under the transformations the first order difference between the integrals consists of two parts, one is an integral over Ω and the other is an integral over $\Omega' - \Omega$. For example, in one dimension,

$$\int_{a+\delta a}^{b+\delta b} [f(x) + \delta f(x)] dx - \int_a^b f(x) dx$$
$$= \int_a^b \delta f(x) dx + \int_b^{b+\delta b} [f(x) + \delta f(x)] dx - \int_a^{a+\delta a} [f(x) + \delta f(x)] dx . \quad (2.266)$$

To first order, the last two terms are

$$\int_b^{b+\delta b} f(x) dx - \int_a^{a+\delta a} f(x) dx = \delta b f(b) - \delta a f(a) . \quad (2.267)$$

The difference between integrals is

$$\int_b^b \delta f(x) dx + f(x) \delta x \Big|_a^b = \int_a^b \left[\delta f(x) + \frac{d}{dx}\left(\delta x f(x)\right) \right] dx . \quad (2.268)$$

Consequently, for the Lagrangians we have

$$\int_{\Omega'} \tilde{L}\left(\eta', x_\mu\right) dx_\mu - \int_{\Omega} \tilde{L}\left(\eta, x_\mu\right) dx_\mu$$
$$= \int_{\Omega'} \left[\tilde{L}\left(\eta', x_\mu\right) - \tilde{L}\left(\eta, x_\mu\right) \right] d\, x_\mu + \int_S \tilde{L}(\eta) \delta x_\mu dS_\mu = 0, \quad (2.269)$$

or

$$\int_{\Omega'} d\, x_\mu \left\{ \left[\tilde{L}\left(\eta', x_\mu\right) - \tilde{L}\left(\eta, x_\mu\right) \right] + \frac{d}{dx_\mu}\left(\tilde{L}\left(\eta, x_\mu\right) \delta x_\mu \right) \right\} = 0. \quad (2.270)$$

To first order,

$$\tilde{L}(\eta', x_\mu) - \tilde{L}(\eta, x_\mu) = \frac{\partial \tilde{L}}{\partial \eta_\rho} \bar{\delta}\eta_\rho + \frac{\partial \tilde{L}}{\partial \eta_{\rho,\nu}} \bar{\delta}\eta_{\rho,\nu}, \qquad (2.271)$$

or

$$\tilde{L}(\eta', x_\mu) - \tilde{L}(\eta, x_\mu) = \frac{d}{dx_\nu}\left(\frac{\partial \tilde{L}}{\partial \eta_{\rho,\nu}} \bar{\delta}\eta_\rho \right). \qquad (2.272)$$

The invariance condition is

$$\int dx_\mu \frac{d}{dx_\nu}\left\{ \frac{\partial \tilde{L}}{\partial \eta_{\rho,\nu}} \bar{\delta}\eta_\rho + \tilde{L}\delta x_\nu \right\} = 0. \qquad (2.273)$$

We define

$$\delta x_\nu = \varepsilon_r X_{r\nu},\ \delta\eta_\rho = \varepsilon_r \Psi_{r\rho}, \qquad (2.274)$$

because

$$\eta'_\rho(x'_\mu) = \eta_\rho(x_\mu) + \bar{\delta}\eta_\rho(x_\mu), \qquad (2.275)$$

and

$$\delta\eta_\rho = \bar{\delta}\eta_\rho + \frac{\partial \eta_\rho}{\partial x_\sigma}\delta x_\sigma, \qquad (2.276)$$

we have

$$\bar{\delta}\eta_\rho = \varepsilon_r \left(\Psi_{r\rho} - \eta_{\rho,\sigma} X_{r\sigma} \right). \qquad (2.277)$$

Equation 2.305 reduces to

$$\int \varepsilon_r \frac{d}{dx_\nu}\left\{ \left(\frac{\partial \tilde{L}}{\partial \eta_{\rho,\nu}} \eta_{\rho,\sigma} - \tilde{L}\delta_{\nu\sigma} \right) X_{r\sigma} - \frac{\partial \tilde{L}}{\partial \eta_{\rho,\nu}} \Psi_{r\rho} \right\} (dx_\mu) = 0. \qquad (2.278)$$

The result is Noether's theorem, which states that

$$\frac{d}{dx_\nu}\left\{\left(\frac{\partial\tilde{L}}{\partial\eta_{\rho,\nu}}\eta_{\rho,\sigma}-\tilde{L}\delta_{\nu\sigma}\right)X_{r\sigma}-\frac{\partial\tilde{L}}{\partial\eta_{\rho,\nu}}\Psi_{r\rho}\right\}=0\,. \qquad (2.279)$$

2.7 Bibliography

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