5

Foundations of Neoclassical Growth

The Solow growth model is predicated on a constant saving rate. It would be more informative to specify the preference orderings of households (individuals), as in standard general equilibrium theory, and derive their decisions from these preferences. This specification would enable us both to have a better understanding of the factors that affect savings decisions and also to discuss the optimality of equilibria—in other words, to pose and answer questions related to whether the (competitive) equilibria of growth models can be improved. The notion of improvement here is based on the standard concept of Pareto optimality, which asks whether some households can be made better off without others being made worse off. Naturally, we can only talk of households being "better off" if we have some information about well-defined preference orderings.

5.1 Preliminaries

To prepare for this analysis, let us consider an economy consisting of a unit measure of infinitely-lived households. By a "unit measure of households" I mean an uncountable number of households with total measure normalized to 1; for example, the set of households $\mathcal{H}$ could be represented by the unit interval [0, 1]. This abstraction is adopted for simplicity, to emphasize that each household is infinitesimal and has no effect on aggregates. Nothing in this book hinges on this assumption. If the reader instead finds it more convenient to think of the set of households, $\mathcal{H}$, as a countable set, for example, $\mathcal{H} = \mathbb{N}$, this can be done without any loss of generality. The advantage of having a unit measure of households is that averages and aggregates are the same, enabling us to economize on notation. It would be even simpler to have $\mathcal{H}$ as a finite set of the form $\{1, 2, \ldots, M\}$. While this form would be sufficient in many contexts, overlapping generations models in Chapter 9 require the set of households to be infinite.

Households in this economy may be truly "infinitely lived," or alternatively they may consist of overlapping generations with full (or partial) altruism linking generations within the household. Throughout I equate households with individuals and thus ignore all possible sources of conflict or different preferences within the household. In other words, I assume that households have well-defined preference orderings.

As in basic general equilibrium theory, let us suppose that preference orderings can be represented by utility functions. In particular, suppose that there is a unique consumption good, and each household h has an *instantaneous utility function* given by

$$u^h(c^h(t)),$$

where $c^h(t)$ is the consumption of household h, and $u^h : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave. I take the domain of the utility function to be $\mathbb{R}_+$ rather than $\mathbb{R}$, so that negative levels of consumption are not allowed. Even though some well-known economic models allow negative consumption, this is not easy to interpret in general equilibrium or in growth theory. Thus this restriction is sensible in most models.

The instantaneous utility function captures the utility that an individual (or household) derives from consumption at time t. It is therefore not the same as a utility function specifying a complete preference ordering over all commodities—here all commodities corresponding to consumption levels at all dates. For this reason, the instantaneous utility function is sometimes also referred to as the "felicity function."

There are two major assumptions in writing an instantaneous utility function. First, it supposes that the household does not derive any utility from the consumption of other households, so consumption externalities are ruled out. Second, in writing the instantaneous utility function, I have already imposed the condition that overall utility is *time-separable and stationary;* that is, instantaneous utility at time t is independent of the consumption levels at past or future dates and is represented by the same utility function u^h at all dates. This second feature is important in enabling us to develop tractable models of dynamic optimization.

Finally, let us introduce a third assumption and suppose that households discount the future exponentially—or proportionally. In discrete time and ignoring uncertainty, this assumption implies that household preferences or utility (starting at time $t = 0$) can be represented as

$$U^h\left(c^h(1), c^h(2), \ldots, c^h(T)\right) \equiv \sum_{t=0}^{T} (\beta^h)^t u^h(c^h(t)), \tag{5.1}$$

where $\beta^h \in (0, 1)$ is the discount factor of household h, and the horizon T could be finite or equal to infinity (so that $T = \infty$ is allowed). Here U^h denotes the utility function of household h defined over the entire stream of consumption levels, while u^h is still the instantaneous utility function. The distinction between these two concepts is important to bear in mind. The functional form of the utility function U^h incorporates exponential discounting and time separability. It implies that the weight given to tomorrow's utility u^h is a fraction β^h of today's utility, and the weight given to the utility the day after tomorrow is a fraction $(\beta^h)^2$ of today's utility, and so on. Exponential discounting and time separability are convenient because they naturally ensure time-consistent behavior.

A solution $\{x(t)\}_{t=0}^{T}$ (possibly with $T = \infty$) to a dynamic optimization problem is *time-consistent* if the following is true: when $\{x(t)\}_{t=0}^{T}$ is a solution starting at time $t = 0$, $\{x(t)\}_{t=t'}^{T}$ is a solution to the continuation dynamic optimization problem starting from time $t = t' > 0$. If a problem is not time-consistent, it is *time-inconsistent*. Time-consistent problems are much more straightforward to work with and satisfy all the standard axioms of rational decision making. Although time-inconsistent preferences may be useful in the modeling of certain behaviors, such as problems of addiction or self-control, time-consistent preferences are ideal for the focus in this book, since they are tractable, relatively flexible, and provide a good approximation to reality in the context of aggregative models. It is also worth noting that many classes of preferences that do not feature exponential and time-separable discounting nonetheless lead to time-consistent behavior. Exercise 5.1 discusses issues of time-consistency further and shows how nonexponential discounting may lead to time-inconsistent behavior, while Exercise 5.2 introduces some common non-time-separable preferences that lead to time-consistent behavior.

There is a natural analogue to (5.1) in continuous time, again incorporating exponential discounting, which is introduced and discussed below and further in Chapter 7.

Equation (5.1) ignores uncertainty in the sense that it assumes the sequence of consumption levels for household h, $\{c^h(t)\}_{t=0}^T$, is known with certainty. If this sequence were uncertain, we would need to look at expected utility maximization. Most growth models do not necessitate an analysis of behavior under uncertainty, but a stochastic version of the neoclassical growth model is the workhorse of much of the rest of modern macroeconomics and will be presented in Chapter 17. For now, it suffices to say that in the presence of uncertainty, $u^h(\cdot)$ should be interpreted as a *Bernoulli utility function*, defined over risky acts, so that the preferences of household h at time $t = 0$ can be represented by the following *(von Neumann-Morgenstern) expected utility function* U^h:

$$U^h \equiv \mathbb{E}_0^h \sum_{t=0}^{T} (\beta^h)^t u^h(c^h(t)),$$

where $\mathbb{E}_0^h$ is the expectation operator with respect to the information set available to household h at time $t = 0$. In this expression, I did not explicitly write the argument of the expected utility function U^h, since this argument is now more involved; it no longer corresponds to a given sequence of consumption levels but to a probability distribution over consumption sequences. At this point, there is no need to introduce the notation for these distributions (see Chapter 16).

The formulation so far indexes the individual utility function $u^h(\cdot)$ and the discount factor β^h by "h" to emphasize that these preference parameters are potentially different across households. Households could also differ according to their income paths. For example, each household could have effective labor endowments of $\{e^h(t)\}_{t=0}^T$ and thus a sequence of labor income of $\{e^h(t)w(t)\}_{t=0}^T$, where $w(t)$ is the equilibrium wage rate per unit of effective labor.

Unfortunately, at this level of generality, this problem is not tractable. Even though we can establish the existence of equilibrium under some regularity conditions, it would be impossible to go beyond that. Proving the existence of equilibrium in this class of models is of some interest, but our focus is on developing workable models of economic growth that generate insights into the process of growth over time and cross-country income differences. I therefore follow the standard approach in macroeconomics and assume the existence of a representative household.

5.2 The Representative Household

An economy *admits a representative household* when the preference (demand) side of the economy can be represented as if there were a single household making the aggregate consumption and saving decisions (and also the labor supply decisions when these are endogenized) subject to an aggregate budget constraint. The major convenience of the representative household assumption is that rather than modeling the preference side of the economy as resulting from equilibrium interactions of many heterogeneous households, it allows us to model it as a solution to a single maximization problem. Note that this description is purely *positive*—it asks the question of whether the aggregate behavior can be represented as if it were generated by a single household. A stronger notion, the *normative* representative household, would also allow us to use the representative household's utility function for welfare comparisons and is introduced later in this section.

Let us start with the simplest case that leads to the existence of a representative household. For concreteness, suppose that all households are infinitely-lived and identical; that is, each household has the same discount factor β, the same sequence of effective labor endowments $\{e(t)\}_{t=0}^{\infty}$, and the same instantaneous utility function

$$u(c^h(t)),$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave, and $c^h(t)$ is the consumption of household h. Therefore there really is a representative household in this case. Consequently, again ignoring uncertainty, the demand side of the economy can be represented as the solution to the following maximization problem starting at time $t = 0$:

$$\max \sum_{t=0}^{\infty} \beta^t u(c(t)), \tag{5.2}$$

where $\beta \in (0, 1)$ is the common discount factor of all the households, and $c(t)$ is the consumption level of the representative household.

The economy described so far admits a representative household rather trivially; all households are identical. In this case, the representative household's preferences, (5.2), can be used not only for positive analysis (e.g., to determine the level of savings) but also for normative analysis, such as evaluating the optimality of equilibria.

The assumption that the economy is inhabited by a set of identical households is not very appealing. Instead, we would like to know when an economy with heterogeneity can be modeled as if aggregate consumption levels were generated by the optimization decision of a representative household. To illustrate the potential difficulties that the "as if" perspective might encounter, let us consider a simple exchange economy with a finite number of commodities and state an important theorem from general equilibrium theory. Recall that in an exchange economy, the equilibrium can be characterized in terms of excess demand functions for different commodities (or more generally, in terms of excess demand correspondences; see Appendix A). Let the equilibrium of the economy be represented by the aggregate excess demand function $\mathbf{x}(p)$ when the vector of prices is p. The demand side of an economy admits a representative household if $\mathbf{x}(p)$ can be obtained as a solution to the maximization problem of a single household. The next theorem shows that this is not possible in general.

Theorem 5.1 (Debreu-Mantel-Sonnenschein Theorem) *Let $\varepsilon > 0$ and $N \in \mathbb{N}$. Consider a set of prices $\mathbf{P}_\varepsilon = \{p \in \mathbb{R}_+^N : p_j / p_{j'} \geq \varepsilon \text{ for all } j \text{ and } j'\}$ and any continuous function $\mathbf{x} : \mathbf{P}_\varepsilon \to \mathbb{R}_+^N$ that satisfies Walras's Law and is homogeneous of degree 0. Then there exists an exchange economy with N commodities and $H < \infty$ households, where the aggregate excess demand is given by $\mathbf{x}(p)$ over the set $\mathbf{P}_\varepsilon$.*

Proof. See Debreu (1974) or Mas-Colell, Whinston, and Green (1995, Proposition 17.E.3). ■

Therefore the fact that excess demands result from aggregating the optimizing behavior of households places few restrictions on the form of these demands. In particular, recall from basic microeconomics that individual (excess) demands satisfy the weak axiom of revealed preference and have Slutsky matrices that are symmetric and negative semidefinite. These properties do not necessarily hold for the aggregate excess demand function $\mathbf{x}(p)$. Thus without imposing further structure, it is impossible to derive $\mathbf{x}(p)$ from the maximization behavior of a single household. Theorem 5.1 therefore raises a severe warning against the use of the representative household assumption.

Nevertheless, this result is an outcome of strong income effects, which can create unintuitive results even in basic consumer theory (recall, e.g., Giffen goods). Special but approximately realistic preference functions, as well as restrictions on the distribution of income across households, enable us to rule out arbitrary aggregate excess demand functions. To show that the representative household assumption is not as hopeless as Theorem 5.1 suggests, I now present a special but relevant case in which aggregation of individual preferences is possible and enables the modeling of the economy as if the demand side were generated by a representative household.

To prepare for this theorem, consider an economy with a finite number N of commodities and recall that an indirect utility function for household h, $v^h(p, w^h)$, specifies the household's (ordinal) utility as a function of the price vector $p = (p_1, \ldots, p_N)$ and the household's income w^h. Naturally, any indirect utility function $v^h(p, w^h)$ has to be homogeneous of degree 0 in p and w.

Theorem 5.2 (Gorman's Aggregation Theorem) *Consider an economy with $N < \infty$ commodities and a set $\mathcal{H}$ of households. Suppose that the preferences of each household $h \in \mathcal{H}$ can be represented by an indirect utility function of the form*

$$v^h(p, w^h) = a^h(p) + b(p)w^h \tag{5.3}$$

and that each household $h \in \mathcal{H}$ has a positive demand for each commodity. Then these preferences can be aggregated and represented by those of a representative household, with indirect utility

$$v(p, w) = a(p) + b(p)w,$$

where $a(p) \equiv \int_{h\in\mathcal{H}} a^h(p)dh$, and $w \equiv \int_{h\in\mathcal{H}} w^h dh$ is aggregate income.

Proof. See Exercise 5.3. ■

This theorem implies that when preferences can be represented by the special linear indirect utility functions (5.3), aggregate behavior can indeed be represented as if it resulted from the maximization of a single household. This class of preferences are referred to as "Gorman preferences" after W. M. (Terence) Gorman, who was among the first economists studying issues of aggregation and proposed the special class of preferences used in Theorem 5.2. These preferences are convenient because they lead to linear Engel curves. Recall that *Engel curves* represent the relationship between expenditure on a particular commodity and income (for given prices). Gorman preferences imply that the Engel curve of each household (for each commodity) is linear and has the same slope as the Engel curve of the other households for the same commodity. In particular, assuming that $a^h(p)$ and $b(p)$ are differentiable, *Roy's Identity* implies that household h's demand for commodity j is given by

$$x_j^h(p, w^h) = -\frac{1}{b(p)}\frac{\partial a^h(p)}{\partial p_j} - \frac{1}{b(p)}\frac{\partial b(p)}{\partial p_j} w^h.$$

Therefore a linear relationship exists between demand and income (or between expenditure and income) for each household, and the slope of this relationship is independent of the household's identity. This property is in fact indispensable for the existence of a representative household and for Theorem 5.2, unless we wish to impose restrictions on the distribution of income. In particular, let us say that an economy admits a *strong representative household* if redistribution of income or endowments across households does not affect the demand side. The strong representative household applies when preferences take the Gorman form as shown by Theorem 5.2. Moreover, it is straightforward to see that, since without the Gorman form the Engel curves of some households have different slopes, there exists a specific scheme of income redistribution across households that would affect the aggregate demand for different goods. This reasoning establishes the following converse to Theorem 5.2: Gorman preferences (with the same $b(p)$ for all households) are necessary for the economy to admit a strong representative household.[1]

Notice that instead of the summation, Theorem 5.2 is stated with the integral over the set $\mathcal{H}$ to allow for the possibility that the set of households may be a continuum. The integral should be thought of as the Lebesgue integral, so that when $\mathcal{H}$ is a finite or countable set, $\int_{h\in\mathcal{H}} w^h dh$ is indeed equivalent to the summation $\sum_{h\in\mathcal{H}} w^h$.[2] Although Theorem 5.2 is stated for an economy with a finite number of commodities, this limitation is only for simplicity, and the results in this theorem hold in economies with an infinite number or a continuum of commodities.

Finally, note that Theorem 5.2 does not require that the indirect utility must take the form of (5.3). Instead, they must have a representation of the Gorman form. Recall that in the absence of uncertainty, monotone transformations of the utility or the indirect function have no effect on behavior, so all that is required in models without uncertainty is that there exist a monotone transformation of the indirect utility function that takes the form given in (5.3).

Many commonly used preferences in macroeconomics are special cases of Gorman preferences, as illustrated in the next example.

Example 5.1 (CES Preferences) *A very common class of preferences used in industrial organization and macroeconomics are the CES preferences, also referred to as "Dixit-Stiglitz preferences" after the two economists who first used these preferences. Suppose that each household $h \in \mathcal{H}$ has total income w^h and preferences defined over $j = 1, \ldots, N$ goods given by*

$$U^h\left(x_1^h, \ldots, x_N^h\right) = \left[\sum_{j=1}^{N} \left(x_j^h - \xi_j^h\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{5.4}$$

where $\sigma \in (0, \infty)$ and $\xi_j^h \in [-\bar{\xi}, \bar{\xi}]$ is a household-specific term, which parameterizes whether the particular good is a necessity for the household. For example, $\xi_j^h > 0$ may mean that household h needs to consume at least a certain amount of good j to survive. The utility function (5.4) is referred to as a CES function for the following reason: if we define the level

1. Naturally, we can obtain a broader class of preferences for which aggregate behavior can be represented as if it resulted from the maximization problem of a single representative household once attention is restricted to specific distributions of income (wealth) across households. The most extreme version of this restriction would be to impose the existence of a representative household by assuming that all households have identical utility functions and endowments.

2. Throughout the book I avoid the use of measure theory when I can, but I refer to the Lebesgue integral a number of times. It is a generalization of the standard Riemann integral. The few references to Lebesgue integrals simply signal that we should think of integration in a slightly more general context, so that the integrals could represent expectations or averages even with discrete random variables and discrete distributions (or mixtures of continuous and discrete distributions). References to some introductory treatments of measure theory and the Lebesgue integral are provided at the end of Chapter 16.

of consumption of each good as $\hat{x}_j^h = x_j^h - \xi_j^h$*, then the elasticity of substitution between any two* $\hat{x}_j^h$ *and* $\hat{x}_{j'}^h$ *(with* $j \neq j'$*) is equal to* σ*.*

Each consumer faces a vector of prices $p=(p_1, \ldots, p_N)$*, and we assume that for all* h*,*

$$\sum_{j=1}^{N} p_j \bar{\bar{\xi}} < w^h,$$

so that each household $h \in \mathcal{H}$ *can afford a bundle such that* $\hat{x}_j^h \geq 0$ *for all* j*. In Exercise 5.6 you are asked to derive the optimal consumption levels for each household and show that their indirect utility function is given by*

$$v^h(p, w^h) = \frac{\left[-\sum_{j=1}^{N} p_j \xi_j^h + w^h\right]}{\left[\sum_{j=1}^{N} p_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}, \tag{5.5}$$

which satisfies the Gorman form (and is also homogeneous of degree 0 in p *and* w*). Therefore this economy admits a representative household with an indirect utility function given by*

$$v(p, w) = \frac{\left[-\sum_{j=1}^{N} p_j \xi_j + w\right]}{\left[\sum_{j=1}^{N} p_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}},$$

where $w \equiv \int_{h \in \mathcal{H}} w^h dh$ *is the aggregate income level in the economy, and* $\xi_j \equiv \int_{h \in \mathcal{H}} \xi_j^h dh$*. It can be verified that the utility function leading to this indirect utility function is*

$$U(x_1, \ldots, x_N) = \left[\sum_{j=1}^{N} (x_j - \xi_j)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}. \tag{5.6}$$

We will see in Chapter 8 that preferences closely related to the CES preferences presented here play a special role not only in aggregation, but also in ensuring balanced growth.

Most—but importantly, not all—macro models assume more than the existence of a representative household. First, many models implicitly assume the existence of a strong representative household, thus abstracting from the distribution of income and wealth among households and its implications for aggregate behavior. Second, most approaches also impose the existence of a *normative representative household:* not only does there exist a representative household whose maximization problem generates the relevant aggregate demands but also the utility function of this household can be used for welfare analysis.[3] More specifically, recall that an allocation is *Pareto optimal* (Pareto efficient) if no household can be made strictly better off without some other household being made worse off (see Definition 5.2 below). Equivalently, a Pareto optimal allocation is a solution to the maximization of a weighted average of the utilities of the households in the economy subject to the resource and technology constraints (and typically, different weights give different Pareto optimal allocations).

3. To emphasize the pitfalls of imposing this restriction without ensuring that the economy admits a normative representative household, I can do no better than directly quote Deaton and Muellbauer (1980, p. 163): "It is extremely dangerous to deduce microeconomic behavior on the basis of macroeconomic observations, particularly if such deductions are then used to make judgments about economic welfare."

When the economy admits a normative representative household, then we can model the demand side in a simple manner and use this modeling to make statements about whether a particular allocation is Pareto optimal and how it can be improved. The existence of a normative representative household is significantly stronger than the existence of a (positive) representative household. Nevertheless, the Gorman preferences in Theorem 5.2 not only imply the existence of a strong representative household (and thus aggregation of individual demands regardless of the exact distribution of income), but they also generally imply the existence of a normative representative household. The next theorem states a simple form of this result.

Theorem 5.3 (Existence of a Normative Representative Household) *Consider an economy with a finite number $N < \infty$ of commodities, a set $\mathcal{H}$ of households, and a convex aggregate production possibilities set Y. Suppose that the preferences of each household $h \in \mathcal{H}$ can be represented by Gorman form $v^h(p, w^h) = a^h(p) + b(p)w^h$, where $p = (p_1, \ldots, p_N)$ is the price vector, and that each household $h \in \mathcal{H}$ has a positive demand for each commodity.*

1. *Then any feasible allocation that maximizes the utility of the representative household, $v(p, w) = \sum_{h\in\mathcal{H}} a^h(p) + b(p)w$, with $w \equiv \sum_{h\in\mathcal{H}} w^h$, is Pareto optimal.*
2. *Moreover, if $a^h(p) = a^h$ for all p and all $h \in \mathcal{H}$, then any Pareto optimal allocation maximizes the utility of the representative household.*

Proof. Let Y represent the aggregate production possibilities set inclusive of endowments, and let $Y_j(p)$ denote the set of profit-maximizing levels of net supply of commodity j when the price vector is p. Since Y is convex, the planner can equivalently choose p and an element in $Y_j(p)$ for each j rather than directly choosing $y \in Y$ (see Theorems 5.4 and 5.7 below). Then a Pareto optimal allocation can be represented as

$$\max_{\{y_j\}_{j=1}^N, p, \{w^h\}_{h\in\mathcal{H}}} \sum_{h\in\mathcal{H}} \alpha^h v^h\big(p, w^h\big) = \max_{\{y_j\}_{j=1}^N, p, \{w^h\}_{h\in\mathcal{H}}} \sum_{h\in\mathcal{H}} \alpha^h\big(a^h(p) + b(p)w^h\big) \tag{5.7}$$

subject to

$$-\frac{1}{b(p)}\left(\sum_{h\in\mathcal{H}} \frac{\partial a^h(p)}{\partial p_j} + \frac{\partial b(p)}{\partial p_j} w\right) = y_j \in Y_j(p) \text{ for } j = 1, \ldots, N,$$

$$\sum_{h\in\mathcal{H}} w^h = w \equiv \sum_{j=1}^N p_j y_j,$$

$$\sum_{j=1}^N p_j \omega_j = w,$$

$$p_j \geq 0 \quad \text{for all } j,$$

where $\{\alpha^h\}_{h\in\mathcal{H}}$ are nonnegative Pareto weights with $\sum_{h\in\mathcal{H}} \alpha^h = 1$. The first set of constraints uses Roy's Identity to express the total demand for good j and set it equal to the supply of good j, which is given by some y_j in $Y_j(p)$. The second equation defines total income as the value of net supplies. The third equation makes sure that total income in the economy is equal to the value of the endowments. The fourth set of constraints requires that all prices be nonnegative.

Now compare the maximization problem (5.7) to the following problem:

$$\max_{\{y_j\}_{j=1}^N, p, \{w^h\}_{h\in\mathcal{H}}} \sum_{h\in\mathcal{H}} a^h(p) + b(p)w \tag{5.8}$$

subject to the same set of constraints. The only difference between the two problems is that in the latter, each household has been assigned the same weight. Let $\mathbf{w} \in \mathbb{R}^{|\mathcal{H}|}$ (note that here, w is a number, whereas $\mathbf{w} = (w^1, \ldots, w^{|\mathcal{H}|})$ is a vector).

Let $(p^*, \mathbf{w}^*)$ be a solution to (5.8) with $w^h = w^*/|\mathcal{H}|$ for all $h \in \mathcal{H}$ so that all households have the same income (together with an associated vector of net supplies $\{y_j\}_{j=1}^N$). By definition it is feasible and also a solution to (5.7) with $\alpha^h = \alpha$, and therefore it is Pareto optimal, which establishes the first part of the theorem.

To establish the second part, suppose that $a^h(p) = a^h$ for all p and all $h \in \mathcal{H}$. To obtain a contradiction, suppose that some feasible $(p_\alpha^{**}, \mathbf{w}_\alpha^{**})$, with associated net supplies $\{y_j\}_{j=1}^N$ is a solution to (5.7) for some weights $\{\alpha^h\}_{h\in\mathcal{H}}$, and suppose that it is not a solution to (5.8). Let

$$\alpha^M = \max_{h\in\mathcal{H}} \alpha^h$$

and

$$\mathcal{H}^M = \left\{h \in \mathcal{H} \mid \alpha^h = \alpha^M\right\}$$

be the set of households given the maximum Pareto weight. Let $(p^*, \mathbf{w}^*)$ be a solution to (5.8) such that

$$w^{h*} = 0 \quad \text{for all } h \notin \mathcal{H}^M, \tag{5.9}$$

and $w^* = \sum_{h\in\mathcal{H}^M} w^{h*}$. Note that such a solution exists, since the objective function and the constraint set in the second problem depend only on the vector $(w^1, \ldots, w^{|\mathcal{H}|})$ through $w = \sum_{h\in\mathcal{H}} w^h$.

Since by definition $(p_\alpha^{**}, \mathbf{w}_\alpha^{**})$ is in the constraint set of (5.8) and is not a solution,

$$\begin{aligned} \sum_{h\in\mathcal{H}} a^h + b(p^*)w^* &> \sum_{h\in\mathcal{H}} a^h + b(p_\alpha^{**})w_\alpha^{**} \\ b(p^*)w^* &> b(p_\alpha^{**})w_\alpha^{**}. \end{aligned} \tag{5.10}$$

The hypothesis that $(p_\alpha^{**}, \mathbf{w}_\alpha^{**})$ is a solution to (5.7) implies that

$$\begin{aligned} \sum_{h\in\mathcal{H}} \alpha^h a^h + \sum_{h\in\mathcal{H}} \alpha^h b(p_\alpha^{**})w_\alpha^{h**} &\geq \sum_{h\in\mathcal{H}} \alpha^h a^h + \sum_{h\in\mathcal{H}} \alpha^h b(p^*)w^{h*} \\ \sum_{h\in\mathcal{H}} \alpha^h b(p_\alpha^{**})w_\alpha^{h**} &\geq \sum_{h\in\mathcal{H}} \alpha^h b(p^*)w^{h*}, \end{aligned} \tag{5.11}$$

where w^{h*} and w_α^{h**} denote the hth components of the vectors $\mathbf{w}^*$ and $\mathbf{w}_\alpha^{**}$.

Note also that any solution $(p^{**}, \mathbf{w}^{**})$ to (5.7) satisfies $w^{h**} = 0$ for any $h \notin \mathcal{H}^M$. In view of this and the choice of $(p^*, \mathbf{w}^*)$ in (5.9), equation (5.11) implies

$$\begin{aligned} \alpha^M b(p_\alpha^{**}) \sum_{h\in\mathcal{H}} w_\alpha^{h**} &\geq \alpha^M b(p^*) \sum_{h\in\mathcal{H}} w^{h*} \\ b(p_\alpha^{**})w_\alpha^{**} &\geq b(p^*)w^*, \end{aligned}$$

which contradicts (5.10) and establishes that, under the stated assumptions, any Pareto optimal allocation maximizes the utility of the representative household. ■

5.3 Infinite Planning Horizon

Another important aspect of the standard preferences used in growth theory and macroeconomics concerns the planning horizon of individuals. Although some growth models are formulated with finitely-lived households (see, e.g., Chapter 9), most growth and macro models assume that households have an infinite planning horizon as in (5.2) or (5.16) below. A natural question is whether this is a good approximation to reality. After all, most individuals we know are not infinitely lived.

There are two reasonable microfoundations for this assumption. The first comes from the "Poisson death model" or the *perpetual youth model*, which is discussed in greater detail in Chapter 9. The general idea is that, while individuals are finitely lived, they are not aware of when they will die. Even somebody who is 100 years old cannot consume all his assets, since there is a fair chance that he will live for another 5 or 10 years. At the simplest level, we can consider a discrete-time model and assume that each individual faces a constant probability of death equal to $\nu > 0$. This is a strong simplifying assumption, since the likelihood of survival to the next age in reality is not a constant, but a function of the age of the individual (a feature best captured by actuarial life tables, which are of great importance to the insurance industry). Nevertheless it is a good starting point, since it is relatively tractable and also implies that individuals have an expected lifespan of $1/\nu < \infty$ periods, which can be used to get a sense of what the value of ν should be.

Suppose also that each individual has a standard instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}$, and a true or pure discount factor $\hat{\beta}$, meaning that $\hat{\beta}$ is the discount factor that the individual would apply between consumption today and tomorrow if he or she were sure to live between the two dates. Moreover, let us normalize $u(0) = 0$ to be the utility after death. Now consider an individual who plans to have a consumption sequence $\{c(t)\}_{t=0}^{\infty}$ (conditional on living). Clearly, after the individual dies, future consumption plans do not matter. Standard arguments imply that this individual would have expected utility at time $t = 0$ given by

$$\begin{aligned} U_0(c(0), c(1), \ldots) &= u(c(0)) + \hat{\beta}(1-\nu)u(c(1)) + \hat{\beta}\nu u(0) \\ &\quad + \hat{\beta}^2(1-\nu)^2 u(c(2)) + \hat{\beta}^2(1-\nu)\nu u(0) + \ldots \\ &= \sum_{t=0}^{\infty} (\hat{\beta}(1-\nu))^t u(c(t)) \\ &\equiv \sum_{t=0}^{\infty} \beta^t u(c(t)), \end{aligned} \tag{5.12}$$

where the second line collects terms and uses $u(0) = 0$, and the final line defines $\beta \equiv \hat{\beta}(1-\nu)$ as the effective discount factor of the individual. With this formulation, the model with finite lives and random deaths leads to an individual maximization problem identical to that in the model with infinitely-lived households (though the reasonable values of β in this case would differ; see also Exercise 5.7 for a similar derivation in continuous time). While until now agents faced no uncertainty, the possibility of death implies that there is a nontrivial (in fact quite important!) uncertainty in individuals' lives. As a result, instead of the standard ordinal utility theory, we have to use the expected utility theory as developed by von Neumann and Morgenstern. In particular, (5.12) is already the expected utility of the individual, since probabilities have been substituted in and there is no need to include explicit expectations.[4]

A second justification for the infinite planning horizon comes from intergenerational altruism or the "bequest" motive. At the simplest level, imagine an individual who lives for one period and has a single offspring (who will also live for a single period and will beget a single offspring, and so on). Suppose that this individual not only derives utility from his consumption but also from the bequest he leaves to his offspring. For example, we could posit

$$U(c(t), b(t)) = u(c(t)) + U^b(b(t)),$$

where $c(t)$ is his consumption and $b(t)$ denotes the bequest left to his offspring. Suppose also that the individual has total income $y(t)$, so that his budget constraint is

$$c(t) + b(t) \leq y(t).$$

The function $U^b(\cdot)$ contains information about how much the individual values the bequest left to his offspring. In general, there may be various reasons why individuals leave bequests (including accidental bequests that will be left by an individual facing random death probability, as in the example discussed in the previous paragraph). Nevertheless, a natural benchmark might be one in which the individual is purely altruistic, so that he cares about the utility of his offspring (with some discount factor).[5] Let the discount factor between generations be β. Also assume that the offspring will have an income of w without the bequest. Then the utility of the individual can be written as

$$u(c(t)) + \beta V(b(t) + w),$$

where $V(\cdot)$ can now be interpreted as the continuation value, equal to the utility that the offspring will obtain from receiving a bequest of $b(t)$ (together with his own income of w). Naturally, the value of the individual at time t can in turn be written as

$$V(y(t)) = \max_{c(t)+b(t)\leq y(t)} \{u(c(t)) + \beta V(b(t) + w)\},$$

which defines the current value of the individual starting with income $y(t)$ and takes into account what the continuation value will be. The next chapter shows that this equation is the canonical form of a dynamic programming representation of an infinite-horizon maximization problem. In particular, under some mild technical assumptions, this dynamic programming representation is equivalent to maximizing

$$\sum_{s=0}^{\infty} \beta^s u(c_{t+s})$$

at time t. Intuitively, even though an individual lives for one period, he cares about the utility of his offspring, and realizes that in turn his offspring cares about the utility of his own offspring, and so on. Thus each individual internalizes the utility of all future members of the "dynasty."

4. Throughout, except in the stochastic growth analysis in Chapters 16 and 17, I directly specify the expected utility function of households rather than explicitly introducing expectations.

5. The alternative to purely altruistic preferences are those in which a parent receives utility from specific types of bequests or from a subcomponent of the utility of his offspring. Models with such impure altruism are sometimes quite convenient and are discussed in Chapters 9 and 21.

This argument establishes that fully altruistic behavior within a dynasty (so-called dynastic preferences) also leads to a situation in which decision makers act as if they have an infinite planning horizon.

5.4 The Representative Firm

The previous section discussed how an economy admits a representative household only under special circumstances. The other assumption commonly used in growth models, and already introduced in Chapter 2, is the representative firm assumption. In particular, recall from Chapter 2 that the entire production side of the economy was represented by an aggregate production possibilities set, which can be thought of as the production possibilities set or the production function of a representative firm. One may think that this representation also requires quite stringent assumptions on the production structure of the economy. This is not the case, however. While not all economies admit a representative household, the standard assumptions adopted in general equilibrium theory or dynamic general equilibrium analysis (in particular, the assumptions of no production externalities and competitive markets) are sufficient to ensure that the formulation with a representative firm is without loss of generality.

This result is stated in the next theorem. To prepare for this result, let us introduce the following notation: for two vectors of the same dimension, p and y, $p \cdot y$ denotes their *inner product* (i.e., if $p = (p_1, \ldots, p_N)$ and $y = (y_1, \ldots, y_N)$, then $p \cdot y = \sum_{j=1}^{N} p_j y_j$). In addition, let $\mathcal{F}$ be the set of firms in the economy and

$$Y \equiv \left\{ \sum_{f \in \mathcal{F}} y^f : y^f \in Y^f \text{ for each } f \in \mathcal{F} \right\}. \tag{5.13}$$

denote the *aggregate production possibilities set* of the economy.

Theorem 5.4 (Representative Firm Theorem) *Consider a competitive production economy with $N \in \mathbb{N} \cup \{+\infty\}$ commodities and a countable set $\mathcal{F}$ of firms, each with a production possibilities set $Y^f \subset \mathbb{R}^N$. Let $p \in \mathbb{R}_+^N$ be the price vector in this economy and denote the set of profit-maximizing net supplies of firm $f \in \mathcal{F}$ by $\hat{Y}^f(p) \subset Y^f$ (so that for any $\hat{y}^f \in \hat{Y}^f(p)$, we have $p \cdot \hat{y}^f \geq p \cdot y^f$ for all $y^f \in Y^f$). Then there exists a representative firm with production possibilities set $Y \subset \mathbb{R}^N$ and a set of profit-maximizing net supplies $\hat{Y}(p)$ such that for any $p \in \mathbb{R}_+^N$, $\hat{y} \in \hat{Y}(p)$ if and only if $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$.*

Proof. Let Y be the aggregate production possibilities set defined in (5.13). To prove the "if" part of the theorem, fix $p \in \mathbb{R}_+^N$ and construct $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$. Suppose, to obtain a contradiction, that $\hat{y} \notin \hat{Y}(p)$, so that there exists y' such that $p \cdot y' > p \cdot \hat{y}$. By definition of the set Y, this implies that there exists $\{y^f\}_{f \in \mathcal{F}}$ with $y^f \in Y^f$ such that

$$p \cdot \left(\sum_{f \in \mathcal{F}} y^f \right) > p \cdot \left(\sum_{f \in \mathcal{F}} \hat{y}^f \right)$$

$$\sum_{f \in \mathcal{F}} p \cdot y^f > \sum_{f \in \mathcal{F}} p \cdot \hat{y}^f,$$

so that there exists at least one $f' \in \mathcal{F}$ such that

$$p \cdot y^{f'} > p \cdot \hat{y}^{f'},$$

which contradicts the hypothesis that $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$ and completes this part of the proof.

To prove the "only if" part of the theorem, let $\hat{y} \in \hat{Y}(p)$ be a profit-maximizing choice for the representative firm. Then, since $\hat{Y}(p) \subset Y$, we have

$$\hat{y} = \sum_{f \in \mathcal{F}} y^f$$

for some $y^f \in Y^f$ for each $f \in \mathcal{F}$. Let $\hat{y}^f \in \hat{Y}^f(p)$. Then

$$\sum_{f \in \mathcal{F}} p \cdot y^f \le \sum_{f \in \mathcal{F}} p \cdot \hat{y}^f,$$

which implies that

$$p \cdot \hat{y} \le p \cdot \sum_{f \in \mathcal{F}} \hat{y}^f. \tag{5.14}$$

Since by hypothesis, $\sum_{f \in \mathcal{F}} \hat{y}^f \in Y$ and $\hat{y} \in \hat{Y}(p)$, we have

$$p \cdot \hat{y} \ge p \cdot \sum_{f \in \mathcal{F}} \hat{y}^f.$$

Therefore inequality (5.14) must hold with equality, which implies $y^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$. This completes the proof of the theorem. ■

Theorem 5.4 implies that when there are no externalities and all factors are priced competitively, focusing on the aggregate production possibilities set of the economy—or equivalently on the representative firm—is without any loss of generality (naturally, assuming that the representative firm acts taking prices as given). Why is there such a difference between the representative household and representative firm assumptions? The answer is related to income effects. The reason why the representative household assumption is restrictive is that changes in prices create income effects, which affect different households differently. A representative household exists only when these income effects can be ignored, which is what the Gorman preferences guarantee. Since there are no income effects in producer theory, the representative firm assumption can be made without loss of generality.

Naturally, the fact that the production side of an economy can be modeled via a representative firm does not mean that heterogeneity among firms is uninteresting or unimportant. On the contrary, productivity differences across firms and firms' attempts to increase their productivity relative to others are central phenomena in this study of economic growth. Theorem 5.4 simply says that, when there is price-taking behavior, the production side of the economy can be equivalently represented by a single representative firm or an aggregate production possibilities set. I return to the issue of firm heterogeneity in the context of monopolistic competition in Part IV.

5.5 Problem Formulation

Let us now consider a discrete-time infinite-horizon economy and suppose that the economy admits a representative household. In particular, once again ignoring uncertainty, the utility of the representative household (starting at time $t = 0$) is given by

$$\sum_{t=0}^{\infty} \beta^t u(c(t)), \tag{5.15}$$

where $\beta \in (0, 1)$ is again the discount factor.

In continuous time, the utility function (5.15) of the representative household becomes

$$\int_0^{\infty} \exp(-\rho t) u(c(t)) dt, \tag{5.16}$$

where $\rho > 0$ is now the discount rate of the household.

Where does the exponential form of the discounting in (5.16) come from? Discounting in the discrete-time case was already referred to as "exponential," so the link should be apparent. More explicitly, let us calculate the value of \$1 after time T. Divide the interval $[0, T]$ into $T/\Delta t$ equally sized subintervals. Let the interest rate in each subinterval be equal to $r\Delta t$. It is important that the quantity r is multiplied by Δt, otherwise as we vary Δt, we would be changing the interest rate per unit of time. Using the standard compound interest rate formula, the value of \$1 in T periods at this interest rate is given by

$$v(T \mid \Delta t) = (1 + r\Delta t)^{T/\Delta t}.$$

Next, let us approach the continuous time limit by letting $\Delta t \to 0$ to obtain

$$v(T) = \lim_{\Delta t \to 0} v(T \mid \Delta t) = \lim_{\Delta t \to 0} (1 + r\Delta t)^{T/\Delta t} = \lim_{\Delta t \to 0} \left[\exp\left(\log\,(1 + r\Delta t)^{T/\Delta t} \right) \right],$$

where the last equality uses the fact that $\exp(\log x) = x$ for any $x > 0$. Thus

$$\begin{aligned} v(T) &= \exp\left[\lim_{\Delta t \to 0} \log(1 + r\Delta t)^{T/\Delta t} \right] \\ &= \exp\left[\lim_{\Delta t \to 0} \frac{T}{\Delta t} \log(1 + r\Delta t) \right]. \end{aligned}$$

The last term in square brackets has a limit of the form $0/0$ (or of the form $\infty \times 0$). To evaluate this limit, write it as

$$\lim_{\Delta t \to 0} \frac{\log(1 + r\Delta t)}{\Delta t / T} = \lim_{\Delta t \to 0} \frac{r/(1 + r\Delta t)}{1/T} = rT,$$

where the first equality follows from l'Hôpital's Rule (Theorem A.21 in Appendix A). Therefore

$$v(T) = \exp(rT).$$

Conversely, \$1 in T periods from now is worth $\exp(-rT)$ today. The same reasoning applies to discounting utility, so the utility of consuming $c(t)$ in period t evaluated at time $t = 0$ is $\exp(-\rho t)u(c(t))$, where ρ denotes the (subjective) discount rate. Equivalently, one could also go from exponential discounting in continuous time to discrete-time discounting. In particular, given a discount rate $\rho > 0$, the discount factor that applies during a time interval of length Δt is $\beta_{\Delta t} = \exp(-\rho \Delta t)$.

5.6 Welfare Theorems

We are ultimately interested in equilibrium growth. But there is a close connection between Pareto optima and competitive equilibria. These connections could not be exploited in the preceding chapters, because household (individual) preferences were not specified. I now introduce the First and Second Welfare Theorems and develop the relevant connections between the theory of economic growth and dynamic general equilibrium models.

Let us start with models that have a finite number of households, so that in terms of the notation in Sections 5.1 and 5.2 the set $\mathcal{H}$ is finite. Throughout I allow an infinite number of commodities, since growth models almost always feature an infinite number of time periods and thus an infinite number of commodities. The results stated in this section have analogues for economies with a continuum of commodities (corresponding to dynamic economies in continuous time), but for the sake of brevity and to reduce technical details, I focus on economies with a countable number of commodities.

Let the commodities be indexed by $j \in \mathbb{N}$, $x^h \equiv \{x_j^h\}_{j=0}^{\infty}$ be the consumption bundle of household h, and $\omega^h \equiv \{\omega_j^h\}_{j=0}^{\infty}$ be its endowment bundle. In addition, let us assume that feasible x^h values must belong to some *consumption set* $X^h \subset \mathbb{R}_+^{\infty}$. The most relevant interpretation for us is that there is an infinite number of dates, say indexed by t, and at each date $t = 0, 1, \ldots,$ each household consumes a finite-dimensional vector of products, so that $\tilde{x}_t^h = \{\tilde{x}_{1,t}^h, \ldots, \tilde{x}_{N,t}^h\} \in \tilde{X}_t^h \subset \mathbb{R}_+^N$ for some $N \in \mathbb{N}$, and $x^h = \{\tilde{x}_t^h\}_{t=0}^{\infty}$. The consumption sets are introduced to ensure that households do not have negative consumption levels and are thus subsets of $\mathbb{R}_+^{\infty}$ (this restriction can be relaxed by allowing some components of the vector, e.g., those corresponding to different types of labor supply, to be negative; this extension is straightforward, and I do not do it here to conserve notation).

Let $\mathbf{X} \equiv \prod_{h \in \mathcal{H}} X^h$ be the Cartesian product of these consumption sets, which can be thought of as the aggregate consumption set of the economy. I also use the notation $\mathbf{x} \equiv \{x^h\}_{h \in \mathcal{H}}$ and $\boldsymbol{\omega} \equiv \{\omega^h\}_{h \in \mathcal{H}}$ to describe the entire set of consumption allocations and endowments in the economy. Feasibility of a consumption allocation requires that $\mathbf{x} \in \mathbf{X}$.

Each household in $\mathcal{H}$ has a well-defined preference ordering over consumption bundles. Suppose again that for each household $h \in \mathcal{H}$, preferences can be represented by a real-valued utility function $U^h : X^h \to \mathbb{R}$. The domain of this function is $X^h \subset \mathbb{R}_+^{\infty}$. I also assume that U^h is nondecreasing in each of its arguments for each $h \in \mathcal{H}$. Let $\mathbf{U} \equiv \{U^h\}_{h \in \mathcal{H}}$ be the set of utility functions.

Let us next describe the production side. Suppose that there is a finite number of firms represented by the set $\mathcal{F}$ and that each firm $f \in \mathcal{F}$ is characterized by a production set Y^f, which specifies levels of output firm f can produce from specified levels of inputs. In other words, $y^f \equiv \{y_j^f\}_{j=0}^{\infty}$ is a feasible production plan for firm f if $y^f \in Y^f$. For example, if there were only two commodities, labor and a final good, Y^f would include pairs $(-l, z)$ such that with labor input l (hence the negative sign), the firm can produce at most z. As is usual in general equilibrium theory, let us take each Y^f to be a cone, so that if $y^f \in Y^f$, then $\lambda y^f \in Y^f$ for any $\lambda \in \mathbb{R}_+$. This implies two important features: first, $\underline{0} \in Y^f$ for each $f \in \mathcal{F}$ (where $\underline{0}$ denotes the infinite sequence whose elements consist of 0); and second, each Y^f exhibits constant returns to scale. If there are diminishing returns to scale because of the presence of some scarce factors, such as entrepreneurial talent, this is added as an additional factor of production, and Y^f is still interpreted as a cone. Let $\mathbf{Y} \equiv \prod_{f \in \mathcal{F}} Y^f$ represent the aggregate production set in this economy, and let $\mathbf{y} \equiv \{y^f\}_{f \in \mathcal{F}}$ be such that $y^f \in Y^f$ for all f, or equivalently, $\mathbf{y} \in \mathbf{Y}$.[6]

The final object that needs to be described is the ownership structure of firms. In particular, if firms make profits, these profits should be distributed to some agents in the economy. We capture this distribution by assuming that there exists a sequence of profit shares represented by $\boldsymbol{\theta} \equiv \{\theta_f^h\}_{f \in \mathcal{F}, h \in \mathcal{H}}$ such that $\theta_f^h \geq 0$ for all f and h, and $\sum_{h \in \mathcal{H}} \theta_f^h = 1$ for all $f \in \mathcal{F}$. The number θ_f^h is the share of profits of firm f that will accrue to household h.

An economy $\mathcal{E}$ is described by preferences, endowments, production sets, consumption sets, and allocation of shares, that is, $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$. An allocation in this economy is $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x}$ and $\mathbf{y}$ are feasible: $\mathbf{x} \in \mathbf{X}$, $\mathbf{y} \in \mathbf{Y}$, and

$$\sum_{h \in \mathcal{H}} x_j^h \leq \sum_{h \in \mathcal{H}} \omega_j^h + \sum_{f \in \mathcal{F}} y_j^f$$

for all $j \in \mathbb{N}$. The last requirement implies that the total consumption of each commodity cannot be more than the sum of its total endowment and net production. Once an economy $\mathcal{E}$ is specified, we can discuss how resources are (or should be) allocated in it. For example, we can think of *dictatorial allocations*, which would be chosen by a single individual (according to his or her preferences). Alternatively, we can think of the choices of a social planner wishing to maximize the weighted sum of the utilities of the households in the economy (which, as noted above, are closely connected to Pareto optimal allocations). Our main interest, however, is with *competitive equilibria,* which correspond to allocations resulting from a specific set of institutions combined with household maximizing behavior. These institutions are those of competitive markets, where, because of the existence of a large number of participants, households and firms take prices as given, and these prices are determined to clear markets. The additional, implicit, assumption is that of *complete markets*, which means that there exists a separate market for each commodity. In particular, this assumption implicitly rules out externalities, since externalities result from the nonmarket impact of one agent's actions on the utility or productivity of another agent. Though clearly an abstraction from reality, competitive equilibria are a good approximation to behavior in a range of circumstances, motivating the practice of using such equilibria as the benchmark in much of economic analysis.

The key component of a competitive equilibrium is the price system. A *price system* is a sequence $p \equiv \{p_j\}_{j=0}^{\infty}$ such that $p_j \geq 0$ for all j, with one of the commodities chosen as the numeraire and its price normalized to 1. Recall that $p \cdot z$ is again the inner product of sequences p and z (where, e.g., $z = x^h$ or y^f), so that $p \cdot z \equiv \sum_{j=0}^{\infty} p_j z_j$.[7] Then a competitive equilibrium—where externalities are absent, all commodities are traded competitively, all firms maximize profits, all households maximize their utility given their budget sets, and all markets clear—can be defined as follows.

Definition 5.1 *A* competitive equilibrium *for economy* $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ *is given by an allocation* $(\mathbf{x}^* = \{x^{h*}\}_{h \in \mathcal{H}}, \mathbf{y}^* = \{y^{f*}\}_{f \in \mathcal{F}})$ *and a price system* p^* *such that*

6. In some dynamic models, it is also useful to explicitly insist that the aggregate production possibilities set $\mathbf{Y}$ should allow for the transformation of date t goods into date $t+1$ capital. However, this stipulation is not typically necessary in the models studied in this book.

7. You may note that such an inner product may not always exist in infinite-dimensional spaces. This issue will be explicitly dealt with in the proofs of Theorems 5.6 and 5.7 below.

1. *The allocation* $(\mathbf{x}^*, \mathbf{y}^*)$ *is feasible, that is,* $x^{h*} \in X^h$ *for all* $h \in \mathcal{H}$, $y^{f*} \in Y^f$ *for all* $f \in \mathcal{F}$, *and*

$$\sum_{h\in\mathcal{H}} x_j^{h*} \le \sum_{h\in\mathcal{H}} \omega_j^h + \sum_{f\in\mathcal{F}} y_j^{f*} \quad \text{for all } j \in \mathbb{N}.$$

2. *For every firm* $f \in \mathcal{F}$, y^{f*} *maximizes profits:*

$$p^* \cdot y^{f*} \ge p^* \cdot y^f \quad \text{for all } y^f \in Y^f.$$

3. *For every household* $h \in \mathcal{H}$, x^{h*} *maximizes utility:*

$$U^h(x^{h*}) \ge U^h(x^h) \quad \text{for all } x \text{ such that } x^h \in X^h \text{ and } p^* \cdot x^h \le p^* \cdot \left(\omega^h + \sum_{f\in\mathcal{F}} \theta_f^h y^f\right).$$

A major focus of general equilibrium theory is to establish the existence of a competitive equilibrium under reasonable assumptions. When there is a finite number of commodities and standard convexity assumptions are made on preferences and production sets, this is straightforward (in particular, the proof of existence involves simple applications of Theorems A.16, A.17, and A.18 in Appendix A). When an infinite number of commodities exists, as in infinite-horizon growth models, proving the existence of a competitive equilibrium is somewhat more difficult and requires more sophisticated arguments. Here I present the First and Second Welfare Theorems, which concern the efficiency properties of competitive equilibria, when they exist, and the decentralization of efficient (Pareto optimal) allocations as competitive equilibria. These results are more important than existence theorems for the focus in this book, both because in most growth models we will be able to characterize competitive equilibrium explicitly and also because the Second Welfare Theorem indirectly establishes the existence of a competitive equilibrium. Let us first recall the standard definition of Pareto optimality.

Definition 5.2 *A feasible allocation* $(\mathbf{x}, \mathbf{y})$ *for economy* $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ *is* Pareto optimal *if there exists no other feasible allocation* $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ *such that* $\hat{x}^h \in X^h$ *for all* $h \in \mathcal{H}$, $\hat{y}^f \in Y^f$ *for all* $f \in \mathcal{F}$,

$$\sum_{h\in\mathcal{H}} \hat{x}_j^h \le \sum_{h\in\mathcal{H}} \omega_j^h + \sum_{f\in\mathcal{F}} \hat{y}_j^f \quad \text{for all } j \in \mathbb{N},$$

and

$$U^h\left(\hat{x}^h\right) \ge U^h\left(x^h\right) \quad \text{for all } h \in \mathcal{H}$$

with at least one strict inequality in the preceding relationship.

Our next result is the celebrated First Welfare Theorem for competitive economies. Before presenting this result, we need the following definition.

Definition 5.3 *Household* $h \in \mathcal{H}$ *is* locally nonsatiated *if, at each* $x^h \in X^h$, $U^h(x^h)$ *is strictly increasing in at least one of its arguments and* $U^h(x^h) < \infty$.

The latter requirement in this definition is already implied by the fact that $U^h : X^h \to \mathbb{R}$, but it is included for additional emphasis, since it is important for the proof and also because if in fact $U^h(x^h) = \infty$, we could not meaningfully talk of $U^h(x^h)$ being strictly increasing. Also note that local nonsatiation at a price vector p implies that $p \cdot x^h < \infty$ (see Exercise 5.5).

Theorem 5.5 (First Welfare Theorem I) *Suppose that* $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ *is a competitive equilibrium of economy* $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ *with* $\mathcal{H}$ *finite. Assume that all households are locally nonsatiated. Then* $(\mathbf{x}^*, \mathbf{y}^*)$ *is Pareto optimal.*

Proof. Suppose that $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium. To obtain a contradiction, suppose that there exists a feasible $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that $U^h(\hat{x}^h) \geq U^h(x^{h*})$ for all $h \in \mathcal{H}$ and $U^h(\hat{x}^h) > U^h(x^{h*})$ for all $h \in \mathcal{H}'$, where $\mathcal{H}'$ is a nonempty subset of $\mathcal{H}$.

Since $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium, it must be the case that for all $h \in \mathcal{H}$,

$$\begin{aligned} p^* \cdot \hat{x}^h &\geq p^* \cdot x^{h*} \\ &= p^* \cdot \left(\omega^h + \sum_{f \in \mathcal{F}} \theta_f^h y^{f*} \right) \end{aligned} \tag{5.17}$$

and for all $h \in \mathcal{H}'$,

$$p^* \cdot \hat{x}^h > p^* \cdot \left(\omega^h + \sum_{f \in \mathcal{F}} \theta_f^h y^{f*} \right). \tag{5.18}$$

Inequality (5.18) follows immediately because x^{h*} is the utility-maximizing choice for household h; thus if $\hat{x}^h$ is strictly preferred, then it cannot be in the budget set. Inequality (5.17) follows with a similar reasoning. Suppose that it did not hold. Then, by the hypothesis of local nonsatiation, U^h must be strictly increasing in at least one of its arguments, let us say the j'th component of x. Then construct $\hat{x}^h(\varepsilon)$ such that $\hat{x}_j^h(\varepsilon) = \hat{x}_j^h$ and $\hat{x}_{j'}^h(\varepsilon) = \hat{x}_{j'}^h + \varepsilon$ for $\varepsilon > 0$. For ε sufficiently small, $\hat{x}^h(\varepsilon)$ is in household h's budget set and yields strictly greater utility than the original consumption bundle $\hat{x}^h$, contradicting the hypothesis that household h is maximizing utility. Also note that local nonsatiation implies that $U^h(x^h) < \infty$, and thus the right-hand sides of (5.17) and (5.18) are finite (and in particular, $p^* \cdot x^{h*} < \infty$ for all $h \in \mathcal{H}$).

Now summing (5.17) over $\mathcal{H} \backslash \mathcal{H}'$ and (5.18) over $\mathcal{H}'$ and combining the two, we have

$$\begin{aligned} p^* \cdot \sum_{h \in \mathcal{H}} \hat{x}^h &> p^* \cdot \sum_{h \in \mathcal{H}} \left(\omega^h + \sum_{f \in \mathcal{F}} \theta_f^h y^{f*} \right) \\ &= p^* \cdot \left(\sum_{h \in \mathcal{H}} \omega^h + \sum_{f \in \mathcal{F}} y^{f*} \right), \end{aligned} \tag{5.19}$$

where the second line uses the fact that the sums are finite, so that the order of summation can be exchanged, and that by the definition of the shares $\sum_{h \in \mathcal{H}} \theta_f^h = 1$ for all $f \in \mathcal{F}$. Finally, since $\mathbf{y}^*$ is profit maximizing at prices p^*, we have

$$p^* \cdot \sum_{f \in \mathcal{F}} y^{f*} \geq p^* \cdot \sum_{f \in \mathcal{F}} y^f \quad \text{for any } \left\{ y^f \right\}_{f \in \mathcal{F}} \text{ with } y^f \in Y^f \text{ for all } f \in \mathcal{F}. \tag{5.20}$$

However, by feasibility of $\hat{x}^h$ (Condition 1 of Definition 5.1),

$$\sum_{h \in \mathcal{H}} \hat{x}_j^h \leq \sum_{h \in \mathcal{H}} \omega_j^h + \sum_{f \in \mathcal{F}} \hat{y}_j^f \quad \text{for all } j,$$

and therefore, by taking the inner products of both sides with p^*, and exploiting (5.20) and the fact that $p^* \geq \underline{0}$, we conclude

$$p^* \cdot \sum_{h \in \mathcal{H}} \hat{x}_j^h \leq p^* \cdot \left(\sum_{h \in \mathcal{H}} \omega_j^h + \sum_{f \in \mathcal{F}} \hat{y}_j^f \right)$$

$$\leq p^* \cdot \left(\sum_{h \in \mathcal{H}} \omega_j^h + \sum_{f \in \mathcal{F}} y_j^{f*} \right),$$

which contradicts (5.19), establishing that any competitive equilibrium allocation $(\mathbf{x}^*, \mathbf{y}^*)$ is Pareto optimal. ■

The proof of the First Welfare Theorem is both intuitive and simple. The proof is based on two simple ideas. First, if another allocation Pareto dominates the competitive equilibrium prices, then it must be nonaffordable in the competitive equilibrium for at least one household. Second, profit maximization implies that any competitive equilibrium already maximizes the set of affordable allocations. The proof is also simple, since it only uses the summation of the values of commodities at a given price vector. In particular, it makes no convexity assumption. However, the proof also highlights the importance of the feature that the relevant sums exist and are finite. Otherwise, the last step would lead to the conclusion that "$\infty < \infty$," which may or may not be a contradiction. The fact that these sums exist, in turn, follows from two assumptions: finiteness of the number of individuals and nonsatiation. However, as noted before, working with economies that have only a finite number of households (even if there is an infinite number of commodities) is not always sufficient for our purposes. For this reason, the next theorem provides a version of the First Welfare Theorem with an infinite number of households. For simplicity, here I take $\mathcal{H}$ to be a countably infinite set (e.g., $\mathcal{H} = \mathbb{N}$). The next theorem generalizes the First Welfare Theorem to this case. It makes use of an additional assumption to take care of infinite sums.

Theorem 5.6 (First Welfare Theorem II) *Suppose that* $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ *is a competitive equilibrium of the economy* $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ *with* $\mathcal{H}$ *countably infinite. Assume that all households are locally nonsatiated and*

$$p^* \cdot \omega^* \equiv \sum_{h \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^h < \infty.$$

Then $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ *is Pareto optimal.*

Proof. The proof is the same as that of Theorem 5.5 with a major difference. Local nonsatiation does not guarantee that the summations are finite in (5.19), since the summations are over an infinite number of households. In particular, (5.17) and (5.18) from the proof of Theorem 5.5 still apply, and we have $p^* \cdot x^{h*} < \infty$ for each $h \in \mathcal{H}$. Moreover, by profit maximization, $p^* \cdot \sum_{f \in \mathcal{F}} y^{f*} < \infty$. Now summing (5.17) over $\mathcal{H} \backslash \mathcal{H}'$ and (5.18) over $\mathcal{H}'$ yields (5.19), provided that

$$p^* \cdot \omega^* \equiv \sum_{h \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^h < \infty.$$

Then the remaining relations in the proof of Theorem 5.5 apply and yield a contradiction, establishing the desired result. ■

Theorem 5.6 is particularly useful in the analysis of overlapping generations models in Chapter 9. The assumption that $\sum_{h\in\mathcal{H}}\sum_{j=0}^{\infty} p_j^*\omega_j^h < \infty$ is not very restrictive; for example, in dynamic models, discounting ensures that this condition is generally satisfied. The reader may also note that when we apply Theorem 5.5 to a infinite-horizon economy with infinitely-lived agents, this condition is satisfied automatically (since, otherwise, local nonsatiation would be violated). However there also exist reasonable and important economic models, such as the overlapping generations models, that can lead to equilibria where $\sum_{h\in\mathcal{H}}\sum_{j=0}^{\infty} p_j^*\omega_j^h = \infty$.

Let us next turn to the Second Welfare Theorem, which is the converse of the First Welfare Theorem. It answers the question of whether a Pareto optimal allocation can be decentralized as a competitive equilibrium. The Second Welfare Theorem requires a number of additional assumptions on preferences and technology, such as the convexity of consumption and production sets and of preferences. When the set of commodities is infinite, this theorem also requires several technical assumptions (the equivalents of which are trivially satisfied when the number of commodities is finite). Convexity assumptions are necessary because the Second Welfare Theorem implicitly contains an argument for the existence of equilibrium, which runs into problems in the presence of nonconvexities. Before stating this theorem, recall that the consumption set of each household $h \in \mathcal{H}$ is $X^h \subset \mathbb{R}_+^{\infty}$, so a typical element of X^h is $x^h = (x_0^h, x_1^h, x_2^h, \ldots)$, where x_t^h can be interpreted as the (finite-dimensional) vector of consumption of individual h at time t, that is, $x_t^h = (x_{1,t}^h, x_{2,t}^h, \ldots, x_{N,t}^n)$. Similarly, a typical element of the production set of firm $f \in \mathcal{F}$, Y^f, is of the form $y^f = (y_0^f, y_1^f, y_2^f, \ldots)$.

Let us also define $x^h[T] = (x_0^h, x_1^h, x_2^h, \ldots, x_T^h, 0, 0, \ldots)$ and $y^f[T] = (y_0^f, y_1^f, y_2^f, \ldots, y_T^f, 0, 0, \ldots)$. In other words, these are truncated sequences that involve zero consumption or zero production after some date T. It can be verified that $\lim_{T\to\infty} x^h[T] = x^h$ and $\lim_{T\to\infty} y^f[T] = y^f$ in the product topology (see Section A.4 in Appendix A). Finally, since in this case each x^h (or y^f) is an N-dimensional vector, with a slight abuse of notation, I use $p \cdot x^h$ for an appropriately defined inner product, for example,

$$p \cdot x^h = \sum_{t=0}^{\infty}\sum_{j=1}^{N} p_{j,t}x_{j,t}^h.$$

Theorem 5.7 (Second Welfare Theorem) *Consider a Pareto optimal allocation* $(\mathbf{x}^*, \mathbf{y}^*)$ *in an economy with endowment vector* ω, *production sets* $\{Y^f\}_{f\in\mathcal{F}}$, *consumption sets* $\{X^h\}_{h\in\mathcal{H}}$, *and utility functions* $\{U^h(\cdot)\}_{h\in\mathcal{H}}$. *Suppose that all production and consumption sets are convex, all production sets are cones, and all utility functions are continuous and quasi-concave and satisfy local nonsatiation. Moreover, suppose also that (i) there exists* $\chi < \infty$ *such that* $\sum_{h\in\mathcal{H}} x_{j,t}^h < \chi$ *for all* j *and* t; *(ii)* $\underline{0} \in X^h$ *for each* h; *(iii) for any* h *and* $x^h, \bar{x}^h \in X^h$ *such that* $U^h(x^h) > U^h(\bar{x}^h)$, *there exists* $\bar{T}$ *(possibly as a function of* h, x^h, *and* $\bar{x}^h$*) such that* $U^h(x^h[T]) > U^h(\bar{x}^h)$ *for all* $T \ge \bar{T}$; *and (iv) for any* f *and* $y^f \in Y^f$, *there exists* $\tilde{T}$ *such that* $y^f[T] \in Y^f$ *for all* $T \ge \tilde{T}$. *Then there exist a price vector* p^* *and endowment and share allocations* $(\boldsymbol{\omega}^*, \boldsymbol{\theta}^*)$ *such that in economy* $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \boldsymbol{\omega}^*, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta}^*)$:

1. *the endowment allocation* $\boldsymbol{\omega}^*$ *satisfies* $\omega = \sum_{h\in\mathcal{H}} \omega^{h*}$;
2. *for all* $f \in \mathcal{F}$,

$$p^* \cdot y^{f*} \ge p^* \cdot y^f \quad \text{for any } y^f \in Y^f; \quad \textit{and}$$

3. *for all* $h \in \mathcal{H}$,

$$\text{if } U^h(x^h) > U^h(x^{h*}) \quad \text{for some } x^h \in X^h \text{ then } p^* \cdot x^h \ge p^* \cdot w^{h*},$$

where

$$w^{h*} \equiv \omega^{h*} + \sum_{f \in \mathcal{F}} \theta_f^{h*} y^{f*}.$$

Moreover, if $p^* \cdot w^{h*} > 0$ *for each* $h \in \mathcal{H}$*, then economy* $\mathcal{E}$ *has a competitive equilibrium* $(\mathbf{x}^*, \mathbf{y}^*, p^*)$.

The proof of this theorem involves the application of the Geometric Hahn-Banach Theorem (Theorem A.27). It is somewhat long and involved. For this reason, its proof is provided in the next (starred) section. Here notice that if, instead of an infinite-dimensional economy, we were dealing with an economy with a finite commodity space, say with N commodities, then parts ii–iv of the hypothesis in the theorem, would be satisfied automatically by taking $\bar{T} = \tilde{T} = N$. In fact, this condition is not imposed in the statement of the Second Welfare Theorem in economies with a finite number of commodities. Its role in dynamic economies is that changes in allocations that are very far in the future should not have a large effect on utility. This condition is naturally satisfied in infinite-horizon economies with discounted utility and separable production structure. Intuitively, if a sequence of consumption levels x^h is strictly preferred to the sequence $\bar{x}^h$, then setting the elements of x^h and $\bar{x}^h$ to 0 in the very far (and thus heavily discounted) future should not change this conclusion (since discounting implies that x^h could not be strictly preferred to $\bar{x}^h$ because of higher consumption under x^h in the arbitrarily far future). Similarly, if some production vector y^f is feasible, the separable production structure implies that $y^f[T]$, which involves zero production after some date T, must also be feasible. Exercise 5.13 demonstrates these claims more formally. One difficulty in applying this theorem is that U^h may not be defined when the vector x^h involves zeros (e.g., when instantaneous utility of consumption is given by $\log c$). Exercise 5.14 shows that the theorem can be generalized to the case in which there exists a sufficiently small positive scalar $\varepsilon > 0$ and a sequence $\underline{\varepsilon}$ with each element equal to ε such that X^h (for all $h \in \mathcal{H}$) is restricted to $x^h \geq \underline{\varepsilon}$.

The conditions for the Second Welfare Theorem are more difficult to satisfy than those for the First Welfare Theorem because of the convexity requirements. In many ways, it is also the more important of the two theorems. While the First Welfare Theorem is celebrated as a formalization of Adam Smith's invisible hand, the Second Welfare Theorem establishes the stronger result that any Pareto optimal allocation can be decentralized as a competitive equilibrium. An immediate corollary of this property is an existence result; since the Pareto optimal allocation can be decentralized as a competitive equilibrium, a competitive equilibrium must exist (at least for the endowments leading to Pareto optimal allocations).

The Second Welfare Theorem motivates many macroeconomists to look for the set of Pareto optimal allocations instead of explicitly characterizing competitive equilibria. This approach is especially useful in dynamic models, in which competitive equilibria can sometimes be quite difficult to characterize or even to specify, while the characterization of Pareto optimal allocations is typically more straightforward.

The real power of the Second Welfare Theorem in dynamic macro models comes when it is combined with a normative representative household. Recall that Theorem 5.3 shows an equivalence between Pareto optimal allocations and optimal allocations for the representative household. In certain models, including many—but not all—growth models studied in this book, the combination of a normative representative household and the Second Welfare Theorem enables us to characterize *the optimal growth path* that maximizes the utility of the representative household and assert that this path corresponds to a competitive equilibrium.

5.7 Proof of the Second Welfare Theorem (Theorem 5.7) *

In this section, I provide a proof of the Second Welfare Theorem. The most important part of the theorem is proved using the Geometric Hahn-Banach Theorem (Theorem A.27).

Proof of Theorem 5.7. First, I establish that there exist a price vector p^* and an endowment and share allocation $(\boldsymbol{\omega}^*, \boldsymbol{\theta}^*)$ that satisfy conditions 1–3 of Theorem 5.7. This proof has two parts.

(Part 1) This part follows from the Geometric Hahn-Banach Theorem (Theorem A.27). Define the "more preferred" sets for each $h \in \mathcal{H}$ as

$$P^h = \left\{x^h \in X^h : U^h\left(x^h\right) > U^h\left(x^{h*}\right)\right\}.$$

Clearly, each P^h is convex. Let $P = \sum_{h\in\mathcal{H}} P^h$ and $Y' = \sum_{f\in\mathcal{F}} Y^f + \{\omega\}$, where recall that $\omega = \sum_{h\in\mathcal{H}} \omega^{h*}$, so that Y' is the sum of the production sets shifted by the endowment vector. Both P and Y' are convex (since each P^h and each Y^f is convex). Let the sequences of production plans for each firm be elements of the vector space ℓ_∞^N, which includes infinite sequences of vectors of the form $y^f = (y_0^f, y_1^f, \ldots)$, with each $y_j^f \in \mathbb{R}_+^N$. Since each production set is a cone, $Y' = \sum_{f\in\mathcal{F}} Y^f + \{\omega\}$ has an interior point (the argument is identical to that of Exercise A.31 in Appendix A). Moreover, let $x^* = \sum_{h\in\mathcal{H}} x^{h*}$ (and similarly $x = \sum_{h\in\mathcal{H}} x^h$). By feasibility and local nonsatiation, $x^* = \sum_{f\in\mathcal{F}} y^{f*} + \omega$. Then $x^* \in Y'$ and also $x^* \in \overline{P}$ (where recall that $\overline{P}$ is the closure of P).

Next observe that $P \cap Y' = \emptyset$. Otherwise, there would exist $\tilde{y} \in Y'$, which is also in P. The existence of such a $\tilde{y}$ would imply that, if distributed appropriately across the households, $\tilde{y}$ would make all households equally well off and at least one of them would be strictly better off (e.g., by the definition of the set P, there would exist $\{\tilde{x}^h\}_{h\in\mathcal{H}}$ such that $\sum_{h\in\mathcal{H}} \tilde{x}^h = \tilde{y}$, $\tilde{x}^h \in X^h$, and $U^h(\tilde{x}^h) \geq U^h(x^{h*})$ for all $h \in \mathcal{H}$ with at least one strict inequality). This would contradict the hypothesis that $(\mathbf{x}^*, \mathbf{y}^*)$ is a Pareto optimum.

Since Y' has an interior point, P and Y' are convex, and $P \cap Y' = \emptyset$, Theorem A.27 implies that there exists a nonzero continuous linear functional ϕ such that

$$\phi(y) \leq \phi(x^*) \leq \phi(x) \quad \text{for all } y \in Y' \quad \text{and all } x \in P. \tag{5.21}$$

(Part 2) I next show that the linear functional ϕ can be interpreted as a price vector, in particular, that it has an inner product representation. Consider the functional

$$\bar{\phi}(x) = \lim_{T\to\infty} \phi(x[T]), \tag{5.22}$$

where recall that for $x^h = (x_0^h, x_1^h, x_2^h, \ldots)$, $x^h[T] = (x_0^h, x_1^h, x_2^h, \ldots, x_T^h, 0, 0, \ldots)$. The main step of this part of the theorem involves showing that $\bar{\phi}$ is a well-defined continuous linear functional, also separating Y' and P as in (5.21). In what follows, let $\|x\|$ be the sup norm of x (short for $\|\cdot\|_\infty$) and let $\|\phi\|$ be the norm of the linear operator ϕ (see Appendix A).

First, let us define $\underline{x}_t^h \equiv (0, 0, \ldots, x_t^h, 0, \ldots,)$. That is, $\underline{x}_t^h$ is the same as the sequence x^h with zeros everywhere except its tth element. Next, note that by the linearity of ϕ,

$$\phi(x[T]) = \sum_{t=0}^{T} \phi(\underline{x}_t).$$

Clearly, if $\lim_{T\to\infty} \sum_{t=0}^{T} \left|\phi(\underline{x}_t)\right|$ exists and is well defined, then $\sum_{t=0}^{T} \phi(\underline{x}_t)$ would be absolutely convergent and thus $\lim_{T\to\infty} \phi(x[T])$ would exist (see Fact A.7). To show this, let us also define $\underline{z}^{\phi} \equiv (\underline{z}_0^{\phi}, \underline{z}_1^{\phi}, \ldots)$, where

$$\underline{z}_t^{\phi} \equiv \begin{cases} \underline{x}_t & \text{if } \phi(\underline{x}_t) \geq 0, \\ -\underline{x}_t & \text{if } \phi(\underline{x}_t) < 0. \end{cases}$$

Then by definition,

$$\begin{aligned} \sum_{t=0}^{T} \left|\phi(\underline{x}_t)\right| &= \phi(\underline{z}^{\phi}[T]) \\ &\leq \|\phi\| \left\|\underline{z}^{\phi}[T]\right\| \\ &= \|\phi\| \, \|x[T]\| \\ &\leq \|\phi\| \, \|x\|, \end{aligned}$$

where the first line uses the definition of $\underline{z}^{\phi}$, the second line uses the fact that ϕ is a linear functional, the third line exploits the fact that the norm $\|\cdot\|$ does not depend on whether the elements are negative or positive, and the final line uses $\|x\| \geq \|x[T]\|$. This string of relationships implies that the sequence $\left\{\sum_{t=0}^{T}\left|\phi(\underline{x}_t)\right|\right\}_{T=1}^{\infty}$, which naturally dominates the sequence $\{\phi\ (x[T])\}_{T=1}^{\infty}$, is bounded (by $\|\phi\|\ \|x\| < \|\phi\|\ \chi < \infty$, since $\|x\| < \chi$ by hypothesis). This establishes that $\{\phi\ (x[T])\}_{T=1}^{\infty}$ converges and thus $\bar{\phi}(x)$ in (5.22) is well defined. The last inequality above also implies that $\bar{\phi}(x) \leq \|\phi\|\ \|x\|$, so $\bar{\phi}$ is a bounded, and thus continuous, linear functional (see Theorem A.26).

Next, for $t \in \mathbb{N}$, define $\bar{\phi}_t : X_t \to \mathbb{R}$ as $\bar{\phi}_t : x_t \mapsto \phi(\underline{x}_t)$ (where recall that $x = (x_0, x_1, \ldots, x_t, \ldots)$, $\underline{x}_t = (0, 0, \ldots, x_t, 0, \ldots)$, and $X_t \subset \mathbb{R}_+^N$, with $x_t \in X_t$). Clearly, $\bar{\phi}_t$ is a linear functional (since ϕ is linear) and moreover, since the domain of $\bar{\phi}_t$ is a subset of a Euclidean space, it has an inner product representation, and in particular, there exists $p_t^* \in \mathbb{R}^N$ such that

$$\bar{\phi}_t(x_t) = p_t^* \cdot x_t \quad \text{for all } x_t \in \mathbb{R}^N.$$

This representation also implies that

$$\bar{\phi}(x) = \lim_{T\to\infty} \phi(x[T]) = \lim_{T\to\infty} \sum_{t=0}^{T} \bar{\phi}_t(x_t) = \lim_{T\to\infty} \sum_{t=0}^{T} p_t^* \cdot x_t,$$

so that $\bar{\phi}$ is a continuous linear functional with an inner product representation.

To complete this part of the proof, we need to show that $\bar{\phi}(x) = \sum_{j=0}^{\infty} \bar{\phi}(x_j)$ can be used instead of ϕ as the continuous linear functional in (5.21). We will first establish the following four steps:

$$\begin{aligned} &\text{(a)} \quad \phi(x^*) \leq \bar{\phi}(x) \quad \text{for all } x \in P, \\ &\text{(b)} \quad \phi(x^*) \geq \bar{\phi}(y') \quad \text{for all } y' \in Y', \\ &\text{(c)} \quad \bar{\phi}(x^*) \leq \phi(x^*), \\ &\text{(d)} \quad \phi(x^*) \leq \bar{\phi}(x^*). \end{aligned}$$

The desired result follows by combining these four steps. To prove each of these steps, we will make use of the hypotheses that $\underline{0} \in X^h$ for each $h \in \mathcal{H}$, and moreover that (i) for any $h \in \mathcal{H}$ and $x^h, \bar{x}^h \in X^h$ with $U^h(x^h) > U^h(\bar{x}^h)$, there exists $\bar{T}^h$ such that $U^h(x^h[T]) > U^h(\bar{x}^h)$ for all $T \geq \bar{T}^h$, and (ii) for any $f \in \mathcal{F}$ and $y^f \in Y^f$, there exists $\tilde{T}$ such that $y^f[T] \in Y^f$ for all $T \geq \tilde{T}$.

In particular, take $x \in P$ and recall that $x = \sum_{h\in\mathcal{H}} x^h$, with $x^h \in P^h$. Let $T^h(T) \equiv \max\{\bar{T}^h, \tilde{T}, T\}$, and drop the dependence on T to simplify notation. Since each x^h has the property that $U^h(x^h) > U^h(x^{h*})$ for each $h \in \mathcal{H}$, we have that $U^h(x^h[T^h]) > U^h(x^{h*})$ for each $h \in \mathcal{H}$. Moreover, since $\sum_{h\in\mathcal{H}} x^h[T^h]$ is in P, we also have

$$\phi(x^*) \leq \phi\left(\sum_{h\in\mathcal{H}} x^h\left[T^h\right]\right),$$

(where again recall that $x^* = \sum_{h\in\mathcal{H}} x^{h*}$). Since ϕ is linear, we also have

$$\phi\left(\sum_{h\in\mathcal{H}} x^h\left[T^h\right]\right) = \sum_{h\in\mathcal{H}} \phi\left(x^h\left[T^h\right]\right).$$

By definition $\lim_{T\to\infty} \phi(x^h[T^h]) = \bar{\phi}(x^h)$ (where recall that $T^h = T^h(T)$). Since each $\bar{\phi}(x^h)$ is well defined and ϕ is linear, this implies that as $T \to \infty$, $\phi\left(\sum_{h\in\mathcal{H}} x^h[T^h]\right) \to \bar{\phi}(x)$ and establishes step (a).

Next, take $y' \in Y'$. By hypothesis, $y'[T] \in Y'$ for T sufficiently large. Then

$$\phi(x^*) \geq \phi(y'[T]) = \bar{\phi}(y'[T]) \text{ for } T \text{ sufficiently large.}$$

Taking the limit as $T \to \infty$ establishes step (b).

Now, take $y' \in \operatorname{Int} Y'$ and construct the sequence $\{y'_n\}$ with $y'_n = (1 - 1/n)x^* + y'/n$. Clearly, $y'_n \in Y'$ for each n and again by hypothesis, $y'_n[T] \in Y'$ for T sufficiently large. Thus with the same argument as in the previous paragraph, $\phi(x^*) \geq \bar{\phi}(y'_n[T])$ for each n (and for T sufficiently large). Taking the limit as $T \to \infty$ and then $n \to \infty$, we obtain step (c).

Finally, let $x^*_n = (1 + 1/n)x^*$. By local nonsatiation $x^*_n \in P$ for each n. This implies that $x^*_n[T] \in P$ for T sufficiently large and therefore

$$\phi(x^*_n[T]) = \bar{\phi}(x^*_n[T]) \geq \phi(x^*)$$

for each n and for T sufficiently large. Taking the limit as $T \to \infty$ and then $n \to \infty$, we obtain step (d).

Combining steps (a)–(d) establishes that $\bar{\phi}(x)$ can be used as the continuous linear functional separating P and Y'. As shown above, $\bar{\phi}(x)$ has an inner product representation of the form $\bar{\phi}(x) = \sum_{t=0}^{\infty} \phi_t(x_t) = p^* \cdot x$. Moreover, since each U^h is nondecreasing in its arguments, we have that $p^* \geq \underline{0}$. Thus, p^* can be interpreted as a price vector (functional).

Parts 1 and 2 have therefore established that there exists a price vector p^* such that conditions 1–3 in the theorem hold. Then, condition 1 of Definition 5.1 is satisfied by feasibility. Condition 2 in the theorem is sufficient to establish that all firms maximize profits at the price vector p^* (condition 2 of Definition 5.1). To show that all households maximize utility at the price vector p^* (condition 3 of Definition 5.1), use the hypothesis that $p^* \cdot w^{h*} > 0$ for each $h \in \mathcal{H}$. We know from condition 3 of the theorem that if $x^h \in X^h$ involves $U^h(x^h) > U^h(x^{h*})$, then $p^* \cdot x^h \geq p^* \cdot w^{h*}$. It is then straightforward to show that there cannot exist such an x^h that is strictly preferred to x^{h*} and satisfies $p^* \cdot x^h \leq p^* \cdot w^{h*}$. In particular, let

$\underline{\varepsilon} = (0, 0, \ldots, , \varepsilon, 0, \ldots)$, with ε corresponding to some $x^h_{j,t} > 0$ (such a strictly positive $x^h_{j,t}$ exists, since $p^* \cdot x^h \geq p^* \cdot w^{h*} > 0$). Then for $\varepsilon > 0$ and small enough,

$$x^h - \underline{\varepsilon} \in X^h, \quad U^h\left(x^h - \underline{\varepsilon}\right) > U^h\left(x^{h*}\right), \quad \text{and} \quad p^* \cdot \left(x^h - \underline{\varepsilon}\right) < p^* \cdot w^{h*},$$

thus violating condition 3 of the theorem. This establishes that for all $x^h \in X^h$ with $p^* \cdot x^h \leq p^* \cdot w^{h*}$, we have $U^h(x^h) \leq U^h(x^{h*})$, and thus condition 3 of Definition 5.1 holds and all households maximize utility at the price vector p^*. Thus $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium. ■

5.8 Sequential Trading

A final issue that is useful to discuss at this point relates to *sequential trading*. Standard general equilibrium models, in particular, the *Arrow-Debreu equilibrium* notion, assume that all commodities are traded at a given point in time—and once and for all. In the context of a dynamic model, where some of the different commodities correspond to the same product at different times, this assumption implies that trading takes place at the initial date, and there are no further trades in the future. This is not a good approximation to reality, and as we have already seen in the context of the Solow growth model in Chapter 2, growth models typically assume that firms hire capital and labor at each t, and households make their consumption decisions for time t at time t. Does the presence of this type of sequential trading make any difference to the insights of general equilibrium analysis? If so, then the applicability of the lessons from general equilibrium theory to dynamic macroeconomic models would be limited. Fortunately, in the presence of complete markets, sequential trading gives the same result as trading at a single point in time.

More explicitly, in the Arrow-Debreu equilibrium of a dynamic general equilibrium model at time $t = 0$, households agree on all future trades (including trades of goods that are not yet produced). Sequential trading, on the other hand, corresponds to separate markets opening at each t and households trading labor, capital, and consumption goods in each such market at each period. Clearly, both for mathematical convenience and descriptive realism, we would like to think of macroeconomic models as involving sequential trading, with separate markets at each date.

The key result concerning the comparison of models with trading at a single point in time and those with sequential trading is due to Kenneth Arrow (1964). Arrow's focus was on trading across states of nature. However, his results also imply that with complete markets trading at a single point in time and sequential trading are equivalent. The easiest way of seeing this equivalence is to consider the Arrow securities already discussed in Chapter 2. (Basic) Arrow securities provide an economical means of transferring resources across different dates and different states of nature. Instead of completing all trades at a single point in time, say at time $t = 0$, households can trade Arrow securities and then use these securities to purchase goods at different dates or after different states of nature have been revealed. While Arrow securities are most useful when there is uncertainty as well as a temporal dimension, for our purposes it is sufficient to focus on the transfer of resources across different dates.

The reason sequential trading with Arrow securities achieves the same result as trading at a single point in time is simple: by the definition of a competitive equilibrium, households correctly anticipate all prices that they will be facing at different dates (and in different states of nature) and purchase sufficient Arrow securities to cover the expenses that they will incur once the time to trade comes. In other words, instead of buying claims at time $t = 0$ for $x^h_{j,t'}$

units of commodity $j = 1, \ldots, N$ at date $t' > 0$ at prices $(p_{1,t'}, \ldots, p_{N,t'})$, it is sufficient for household h to have an income of $\sum_{j=1}^{N} p_{j,t'} x_{j,t'}^h$ and know that it can purchase as many units of each commodity as it wishes at time t' at the price vector $(p_{1,t'}, \ldots, p_{N,t'})$.

This result can be stated in a slightly more formal manner. Let us consider a dynamic exchange economy running across periods $t = 0, 1, \ldots, T$, possibly with $T = \infty$ (with the convention that when $T = \infty$, all summations are assumed to take finite values). Nothing here depends on the assumption that we are focusing on an exchange economy, but suppressing production simplifies notation. Imagine that there are N goods at each date, denoted by $(x_{1,t}, \ldots, x_{N,t})$, and let the consumption of good j by household h at time t be denoted by $x_{j,t}^h$. Suppose that these goods are perishable, so that they are indeed consumed at time t. Denote the set of households by $\mathcal{H}$ and suppose that each household $h \in \mathcal{H}$ has a vector of endowments $(\omega_{1,t}^h, \ldots, \omega_{N,t}^h)$ at time t and preferences given by the time-separable function of the form

$$\sum_{t=0}^{T} (\beta^h)^t u^h (x_{1,t}^h, \ldots, x_{N,t}^h),$$

for some $\beta^h \in (0, 1)$. These preferences imply that there are no externalities and preferences are time-consistent. I also assume that all markets are open and competitive.

Let an Arrow-Debreu equilibrium be given by $(\mathbf{p}^*, \mathbf{x}^*)$, where $\mathbf{x}^*$ is the complete list of consumption vectors of each household $h \in \mathcal{H}$, that is,

$$\mathbf{x}^* = (x_{1,0}, \ldots x_{N,0}, \ldots, x_{1,T}, \ldots x_{N,T}),$$

with $x_{j,t} = \{x_{j,t}^h\}_{h \in \mathcal{H}}$ for each j and t, and $\mathbf{p}^*$ is the vector of complete prices

$$\mathbf{p}^* = (p_{1,0}^*, \ldots, p_{N,0}^*, \ldots, p_{1,T}^*, \ldots, p_{N,T}^*),$$

with one of the prices, say $p_{1,0}^*$, chosen as the numeraire, so that $p_{1,0}^* = 1$. In the Arrow-Debreu equilibrium, each household $h \in \mathcal{H}$ purchases and sells claims on each of the commodities only at $t = 0$ and thus simply chooses an allocation that satisfies the budget constraint

$$\sum_{t=0}^{T} \sum_{j=1}^{N} p_{j,t}^* x_{j,t}^h \le \sum_{t=0}^{T} \sum_{j=1}^{N} p_{j,t}^* \omega_{j,t}^h.$$

Market clearing then requires

$$\sum_{h \in \mathcal{H}} x_{j,t}^h \le \sum_{h \in \mathcal{H}} \omega_{j,t}^h$$

for each $j = 1, \ldots, N$ and $t = 0, 1, \ldots, T$.

In the equilibrium with sequential trading, markets for goods dated t open at time t. In addition, there are T bonds—Arrow securities—that are in zero net supply and can be traded among the households at time $t = 0$.[8] The bond indexed by t pays 1 unit of one of the goods, say good $j = 1$ at time t. Let the prices of bonds be denoted by $(q_1, \ldots, q_T)$, again expressed in units of good $j = 1$ (at time $t = 0$). Thus a household can purchase a unit of bond t at time

8. Note that the Arrow securities do not correspond to technologies for transforming goods dated t into goods dated $t' > t$. Instead they are simply units of account specifying what the income levels of different households are at different dates.

0 by paying q_t units of good 1, and in return, it will receive 1 unit of good 1 at time t (or conversely can sell short 1 unit of such a bond) The purchase of bond t by household h is denoted by $b_t^h \in \mathbb{R}$, and since each bond is in zero net supply, market clearing requires that

$$\sum_{h \in \mathcal{H}} b_t^h = 0 \quad \text{for each } t = 0, 1, \ldots, T.$$

Notice that this specification assumes that there are only T bonds (Arrow securities). More generally, we could have introduced additional bonds, for example, bonds traded at time $t > 0$ for delivery of good 1 at time $t' > t$. This restriction to only T bonds is without loss of any generality (see Exercise 5.10).

Sequential trading corresponds to each individual using their endowment plus (or minus) the proceeds from the corresponding bonds at each date t. Since there is a market for goods at each t, it turns out to be convenient (and possible) to choose a separate numeraire for each date t. Let us again suppose that this numeraire is good 1, so that $p_{1,t}^{**} = 1$ for all t. Therefore the budget constraint of household $h \in \mathcal{H}$ at time t, given the equilibrium price vector for goods and bonds, $(\mathbf{p}^{**}, \mathbf{q}^{**})$, can be written as

$$\sum_{j=1}^{N} p_{j,t}^{**} x_{j,t}^h \leq \sum_{j=1}^{N} p_{j,t}^{**} \omega_{j,t}^h + b_t^h \tag{5.23}$$

for $t = 0, 1, \ldots, T$, together with $\sum_{t=0}^{T} q_t^{**} b_t^h \leq 0$ and the normalization $q_0^{**} = 1$. Let an equilibrium of the sequential trading economy be denoted by $(\mathbf{p}^{**}, \mathbf{q}^{**}, \mathbf{x}^{**}, \mathbf{b}^{**})$, where once again $\mathbf{p}^{**}$ and $\mathbf{x}^{**}$ denote the entire lists of prices and quantities of consumption by each household, and $\mathbf{q}^{**}$ and $\mathbf{b}^{**}$ denote the vectors of bond prices and bond purchases by each household. Given this specification, the following theorem can be established.

Theorem 5.8 (Sequential Trading) *For the above-described economy, if* $(\mathbf{p}^*, \mathbf{x}^*)$ *is an Arrow-Debreu equilibrium, then there exists a sequential trading equilibrium* $(\mathbf{p}^{**}, \mathbf{q}^{**}, \mathbf{x}^{**}, \mathbf{b}^{**})$*, such that* $\mathbf{x}^* = \mathbf{x}^{**}$, $p_{j,t}^{**} = p_{j,t}^* / p_{1,t}^*$ *for all* j *and* t*, and* $q_t^{**} = p_{1,t}^*$ *for all* $t > 0$*. Conversely, if* $(\mathbf{p}^{**}, \mathbf{q}^{**}, \mathbf{x}^{**}, \mathbf{b}^{**})$ *is a sequential trading equilibrium, then there exists an Arrow-Debreu equilibrium* $(\mathbf{p}^*, \mathbf{x}^*)$ *with* $\mathbf{x}^* = \mathbf{x}^{**}$, $p_{j,t}^* = p_{j,t}^{**} p_{1,t}^*$ *for all* j *and* t*, and* $p_{1,t}^* = q_t^{**}$ *for all* $t > 0$.

Proof. See Exercise 5.9. ■

This theorem implies that all the results concerning Arrow-Debreu equilibria apply to economies with sequential trading. In most of the models studied in this book the focus is on economies with sequential trading and (except in the context of models with explicit financial markets and with possible credit market imperfections) we assume that there exist Arrow securities to transfer resources across dates. These securities might be riskless bonds in zero net supply, or in models without uncertainty, this role is typically played by the capital stock. We also follow the approach leading to Theorem 5.8 and normalize the price of one good at each date to 1. Thus in economies with a single consumption good, like the Solow or the neoclassical growth models, the price of the consumption good in each date is normalized to 1, and the interest rates directly give the intertemporal relative prices. This is the justification for focusing on interest rates as the key relative prices in macroeconomic (economic growth) models. It should also be emphasized that the presence of Arrow securities to transfer resources across dates also implies that capital (financial) markets are perfect, and in particular, that there are no credit constraints. When such constraints exist, we need to be much more explicit about whether and how each household can transfer resources across different dates (see, e.g., Chapter 21).

One final point implicit in the argument leading to Theorem 5.8 should be highlighted. The equivalence of the Arrow-Debreu equilibria and sequential trading equilibria is predicated on the requirement that the budget constraints facing households are the same under both formulations. Though this stipulation seems obvious, it is not always trivial to ensure that households face exactly the same budget constraints in sequential trading equilibria as in the Arrow-Debreu equilibria. This issue is discussed further at the beginning of Chapter 8, when we introduce the neoclassical growth model.

5.9 Optimal Growth

Motivated by the discussion at the end of Section 5.6, let us start with an economy characterized by an aggregate production function, and a normative representative household (recall Theorem 5.3). The *optimal growth problem* in this context refers to characterizing the allocation of resources that maximizes the utility of the representative household. For example, if the economy consists of a number of identical households, then this problem corresponds to the Pareto optimal allocation giving the same (Pareto) weight to all households (recall Definition 5.2).[9] Therefore the optimal growth problem in discrete time with no uncertainty, no population growth, and no technological progress can be written as follows:

$$\max_{\{c(t),k(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t)), \tag{5.24}$$

subject to

$$k(t+1) = f(k(t)) + (1-\delta)k(t) - c(t), \tag{5.25}$$

with $k(t) \geq 0$ and given $k(0) > 0$. Here $u : \mathbb{R}_+ \to \mathbb{R}$ is the instantaneous utility function of the representative household. The objective function represents the discounted sum of instantaneous utilities. The constraint (5.25) is also straightforward to understand: total output per capita produced with capital-labor ratio $k(t)$, $f(k(t))$, together with a fraction $1-\delta$ of the capital that is undepreciated make up the total resources of the economy at date t. Out of these resources, $c(t)$ is spent as consumption per capita and the rest becomes next period's capital-labor ratio, $k(t+1)$.

The optimal growth problem requires that the social planner chooses an entire sequence of consumption levels and capital stocks, subject only to the resource constraint (5.25). There are no additional equilibrium constraints. The initial level of capital stock $k(0) > 0$ has been specified as one boundary condition. But in contrast to the basic Solow model, the solution to this problem involves two, not one, dynamic (difference or differential) equations and thus necessitates two boundary conditions. The additional boundary condition does take the form of an initial condition but comes from the optimality of a dynamic plan in the form of a transversality condition. The relevant transversality conditions for this class of problems will be discussed in the next two chapters.

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9. One can also imagine allocations in which ex ante identical households receive different weights and utility levels. Throughout, whenever the economy admits a normative representative household, I follow the standard practice of focusing on optima with equal Pareto weights.

This maximization problem can be solved in a number of ways, for example, by setting up an infinite-dimensional Lagrangian. But the most convenient and common way of approaching it is by using dynamic programming, which we will study in the next chapter.

An important question for us is whether the solution to the optimal growth problem can be decentralized as a competitive equilibrium; that is, whether the Second Welfare Theorem (Theorem 5.7) can be applied to this environment. The answer to this question is yes. In fact, one of the main motivations for developing Theorem 5.7 in this chapter has been its use in discounted growth problems, such as the baseline neoclassical growth model presented in this section. The details of how this theorem can be applied to the optimal growth problem are developed in Exercises 5.12–5.14.

It is also useful to note that even if we wished to bypass the Second Welfare Theorem and directly solve for competitive equilibria, we would have to solve a problem similar to the maximization of (5.24) subject to (5.25). In particular, to characterize the equilibrium, we would need to start with the maximizing behavior of households. Since the economy admits a representative household, we need only look at the maximization problem of this household. Assuming that the representative household has 1 unit of labor supplied inelastically and denoting its assets at time t by $a(t)$, this problem can be written as

$$\max_{\{c(t),a(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t))$$

subject to some given $a(0) > 0$ and

$$a(t+1) = (1 + r(t))a(t) - c(t) + w(t), \tag{5.26}$$

where $r(t)$ is the net rate of return on assets (so that $1 + r(t)$ is the gross rate of return) and $w(t)$ is the equilibrium wage rate (and thus the wage earnings of the representative household). Market clearing then requires that $a(t) = k(t)$. The constraint (5.26) is the flow budget constraint, meaning that it links tomorrow's assets to today's. Here we need an additional condition to ensure that the flow budget constraint eventually converges (so that $a(t)$ should not go to negative infinity). This can be ensured by imposing a lifetime budget constraint. Since a flow budget constraint in the form of (5.26) is both more intuitive and often more convenient to work with, we will not work with the lifetime budget constraint but instead augment the flow budget constraint with a limiting condition, which is introduced and discussed in the next three chapters.

The formulation of the optimal growth problem in continuous time is very similar and takes the form

$$\max_{[c(t),k(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp(-\rho t) u(c(t)) dt \tag{5.27}$$

subject to

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t), \tag{5.28}$$

with $k(t) \geq 0$ and given $k(0) > 0$. The objective function (5.27) is the direct continuous-time analogue of (5.24), and (5.28) gives the resource constraint of the economy, similar to (5.25) in discrete time. Once again this problem lacks one boundary condition, which is supplied by the transversality condition. The most convenient way of characterizing the solution to this problem is by optimal control theory, which is developed in Chapter 7.

5.10 Taking Stock

This chapter introduced the preliminaries necessary for an in-depth study of equilibrium and optimal growth theory. At some level it can be thought of as an "odds-and-ends" chapter, introducing the reader to the notions of representative household, dynamic optimization, welfare theorems, and optimal growth. However, the material here is more than odds and ends, since a good understanding of the general equilibrium foundations of economic growth and the welfare theorems is necessary for what is to come in Part III and later.

The most important messages from this chapter are as follows. First, the set of models in this book are examples of more general dynamic general equilibrium models. It is therefore important to understand which features of the growth models are general (in the sense that they do not depend on the specific simplifying assumptions) and which results depend on the further simplifying assumptions. In this respect, the First and the Second Welfare Theorems are essential. They show that, provided that all product and factor markets are competitive and that there are no externalities in production or consumption (and under some relatively mild technical assumptions), dynamic competitive equilibria are Pareto optimal and any Pareto optimal allocation can be decentralized as a dynamic competitive equilibrium. These results are especially relevant in Part III, where the focus is on competitive economies. Importantly, these results do not directly apply in our analysis of technological change, where product markets are monopolistic, or in our study of economic development, where various market imperfections play an important role.

Second, the most general class of dynamic general equilibrium models are not tractable enough for us to derive sharp results about the process of economic growth. For this reason, we often adopt a range of simplifying assumptions. The most important of these is the representative household assumption, which enables us to model the demand side of the economy as if it were generated by the optimizing behavior of a single household. We saw how this assumption is generally not satisfied but also how a certain class of preferences (the Gorman preferences) enable us to model economies as if they admitted a representative household, even with arbitrary distributions of wealth and income.

In addition, this chapter introduced the first formulation of the optimal growth problems in discrete and in continuous time. These are used as examples in the next two chapters.

5.11 References and Literature

This chapter covered a great deal of ground and often many details were omitted for brevity. Many readers will be familiar with some of the material in this chapter. Deaton and Muellbauer (1980), Hildenbrand and Kirman (1988), and Mas-Colell, Whinston, and Green (1995) provide excellent discussions of the issues related to aggregation and the representative household assumption. Some of the original contributions on this topic are contained in Gorman (1953, 1959, 1976, 1980) and Pollak (1971). These and many other relevant results on separability and aggregation appear in the works of W. M. (Terence) Gorman. Deaton and Muellbauer (1980) provide an excellent discussion of Gorman's work and the implications of Gorman preferences. Caselli and Ventura (2000) use Gorman preferences in the context of a model of capital accumulation with heterogeneous agents. Mas-Colell, Whinston, and Green also discuss the concepts of positive and normative representative households. The concept of normative representative household in Theorem 5.3 is motivated by the use of the representative household assumption in dynamic macroeconomic models (which focus on the maximization of the utility of a representative household to characterize all Pareto optimal allocations and competitive equilibria). This concept is stronger than the one in Mas-Colell, Whinston, and Green, who define a normative representative household for a given social welfare function.

The Debreu-Mantel-Sonnenschein Theorem (Theorem 5.1) was originally proved by Sonnenschein (1972) and then extended by Debreu (1974) and Mantel (1976). Both Mas-Colell, Whinston, and Green (1995) and Hildenbrand and Kirman (1988) present this theorem and sketch its proof. Both Deaton and Muellbauer (1980) and Hildenbrand and Kirman (1988) also show how such aggregation is possible under weaker assumptions on utility functions together with certain restrictions on the distribution of income (or endowments).

Some basic concepts from microeconomic theory were assumed in this chapter, and the reader can find a thorough exposition of these in Mas-Colell, Whinston, and Green (1995). These include Roy's Identity, used following Theorem 5.2 and then again in Theorem 5.3, and Walras's Law, the concept of a numeraire, and expected utility theory of von Neumann and Morgenstern, used throughout the analysis. The reader is also referred to Chapter 16 of Mas-Colell, Whinston, and Green and to Bewley (2007) for clear expositions of the different representation of Pareto optima (including the result that every Pareto optimal allocation is a solution to the maximization of the weighted average of utilities of households in the economy).

The Representative Firm Theorem (Theorem 5.4) presented here is quite straightforward, but I am not aware of any discussion of this theorem in the literature (or at least in the macroeconomics literature). It is important to distinguish the subject matter of this theorem from the Cambridge controversy in early growth theory, which revolved around the issue of whether different types of capital goods could be aggregated into a single capital index (see, e.g., Wan, 1971). The Representative Firm Theorem says nothing about this issue.

The best reference for the analysis of the existence of competitive equilibria and the welfare theorems with a finite number of households and a finite number of commodities is still Debreu's (1959) *Theory of Value*. This short book introduces all mathematical tools necessary for general equilibrium theory and gives a very clean exposition. Equally lucid and more modern are the treatments of the same topics in Mas-Colell, Whinston, and Green (1995) and Bewley (2007). The reader may also wish to consult Mas-Colell, Whinston, and Green (their Chapter 16) for a proof of the Second Welfare Theorem with a finite number of commodities (Theorem 5.7 in this chapter is more general, because it covers the case of an infinite number of commodities). Both of these books also have an excellent discussion of the necessary restrictions on preferences to allow preferences to be represented by utility functions. Mas-Colell, Whinston, and Green (their Chapter 19) also gives a very clear discussion of the role of Arrow securities and the relationship between trading at a single point in time and sequential trading. The classic reference on Arrow securities is Arrow (1964).

Neither Debreu (1959) nor Mas-Colell, Whinston, and Green (1995) discuss infinite-dimensional economies. The seminal reference for infinite-dimensional welfare theorems is Debreu (1954). Bewley (2007) contains a number of useful results on infinite-dimensional economies. Stokey, Lucas, and Prescott (1989, their Chapter 15) present existence and welfare theorems for economies with a finite number of households and countably infinite number of commodities. The mathematical prerequisites for their treatment are greater than what has been assumed here, but their treatment is both thorough and straightforward, once the reader makes the investment in the necessary mathematical techniques. The most accessible references for the Hahn-Banach Theorem, which is necessary for a proof of Theorem 5.7 in infinite-dimensional spaces, are Luenberger (1969), Kolmogorov and Fomin (1970), and Kreyszig (1978). Luenberger (1969) is also an excellent source for all the mathematical techniques used in Stokey, Lucas, and Prescott (1989) and also contains much material useful for appreciating continuous-time optimization.

On the distinction between the coefficient of relative risk aversion and the intertemporal elasticity of substitution discussed in Exercise 5.2, see Kreps (1988), Epstein and Zin (1989), and Becker and Boyd (1997).

5.12 Exercises

5.1 Recall that a solution $\{x(t)\}_{t=0}^{T}$ to a dynamic optimization problem is time-consistent if the following is true: if $\{x(t)\}_{t=0}^{T}$ is a solution starting at time $t = 0$, then $\{x(t)\}_{t=t'}^{T}$ is a solution to the continuation dynamic optimization problem starting from time $t = t' > 0$.

(a) Consider the following optimization problem

$$\max_{\{x(t)\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(x(t))$$

subject to

$$x(t) \in [0, \bar{x}], \quad \text{and}$$
$$G(x(0), \ldots, x(T)) \le 0.$$

Although you do not need to, you may assume that G is continuous and convex, and u is continuous and concave. Prove that any solution $\{x^*(t)\}_{t=0}^{T}$ to this problem is time-consistent.

(b) Consider the optimization problem

$$\max_{\{x(t)\}_{t=0}^{T}} u(x(0)) + \delta \sum_{t=1}^{T} \beta^t u(x(t))$$

subject to

$$x(t) \in [0, \bar{x}],$$
$$G(x(0), \ldots, x(T)) \le 0.$$

Suppose that the objective function at time $t = 1$ becomes $u(x(1)) + \delta \sum_{t=2}^{T} \beta^{t-1} u(x(t))$. Interpret this objective function (sometimes referred to as "hyperbolic discounting").

(c) Let $\{x^*(t)\}_{t=0}^{T}$ be a solution to the maximization problem in part b. Assume that the individual chooses $x^*(0)$ at $t = 0$ and then is allowed to reoptimize at $t = 1$, that is, she can now solve the problem

$$\max_{\{x(t)\}_{t=1}^{T}} u(x(1)) + \delta \sum_{t=2}^{T} \beta^{t-1} u(x(t))$$

subject to

$$x(t) \in [0, \bar{x}], \quad \text{and}$$
$$G(x^*(0), \ldots, x(T)) \le 0.$$

Prove that the solution from $t = 1$ onward, $\{x^{**}(t)\}_{t=1}^{T}$, is not necessarily the same as $\{x^*(t)\}_{t=1}^{T}$.

(d) Explain which standard axioms of preferences in basic general equilibrium theory are violated by the preferences in parts b and c of this exercise.

5.2 This exercise asks you to work through an example that illustrates the difference between the coefficient of relative risk aversion and the intertemporal elasticity of substitution. Consider a household with the following non-time-separable preferences over consumption levels at two dates:

$$V(c_1, c_2) = \mathbb{E}\left[\left(\frac{c_1^{1-\theta}-1}{1-\theta}\right)^{\frac{\alpha-1}{\alpha}} + \beta\left(\frac{c_2^{1-\theta}-1}{1-\theta}\right)^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}},$$

where $\mathbb{E}$ is the expectations operator. The budget constraint of the household is

$$c_1 + \frac{1}{1+r}c_2 \leq W,$$

where r is the interest rate and W is its total wealth, which may be stochastic.

(a) First suppose that W is nonstochastic and equal to $W_0 > 0$. Characterize the utility maximizing choice of c_1 and c_2.

(b) Now suppose that W is distributed over the support $[\underline{W}, \overline{W}]$ with some distribution function $G(W)$, where $0 < \underline{W} < \overline{W} < \infty$. Characterize the utility maximizing choice of c_1 (made before the realization of W). Define the coefficient of relative risk aversion and the intertemporal elasticity of substitution in this case. Explain why these two measures are not necessarily the same.

5.3 Prove Theorem 5.2.

* 5.4 Generalize Theorem 5.3 to an economy with a continuum of commodities.

5.5 Show that if a household chooses a consumption bundle x^h at price vector p and is locally nonsatiated, then $p \cdot x^h < \infty$.

5.6 (a) Derive the utility-maximizing demands for households in Example 5.1 and show that the resulting indirect utility function for each household is given by (5.5).

(b) Show that maximization of (5.6) leads to the indirect utility function corresponding to the representative household.

(c) Now suppose that $U^h(x_1^h, \ldots, x_N^h) = \sum_{j=1}^{N}(x_j^h - \xi_j^h)^{\frac{\sigma-1}{\sigma}}$. Repeat the same computations as in parts a and b and verify that the resulting indirect utility function is homogeneous of degree 0 in p and y but does not satisfy the Gorman form. Show, however, that a monotone transformation of the indirect utility function satisfies the Gorman form. Is this sufficient to ensure that the economy admits a representative household?

5.7 Construct a continuous-time version of the model with finite lives and random deaths (recall (5.12) in the text). In particular suppose that an individual faces a constant (Poisson) flow rate of death equal to $\nu > 0$ and has a true discount factor equal to ρ. Show that this individual behaves as if she were infinitely lived with an effective discount factor of $\rho + \nu$.

5.8 (a) Will dynastic preferences, such as those discussed in Section 5.2, lead to infinite-horizon maximization if the instantaneous utility functions of future generations are different ($u_t(\cdot)$ potentially different for each generation t)?

(b) How would the results be different if an individual cares about the continuation utility of his offspring with discount factor β, but also cares about the continuation utility of the offspring of his offspring with a smaller discount factor δ?

5.9 Prove Theorem 5.8.

5.10 Consider the sequential trading model discussed in Section 5.8 and suppose now that households can trade bonds at time t that deliver one unit of good t at time t'. Denote the price of such bonds by $q_{t,t'}$.

(a) Rewrite the budget constraint of household h at time t, (5.23), including these bonds.

(b) Prove an equivalent of Theorem 5.8 in this environment with the extended set of bonds.

5.11 Consider a two-period economy consisting of two types of households. N_A households have the utility function

$$u(c_1^h) + \beta_A u(c_2^h),$$

where c_1^h and c_2^h denote the consumption of household h in the two periods. The remaining N_B households have the utility function

$$u(c_1^h) + \beta_B u(c_2^h),$$

with $\beta_B < \beta_A$. The two groups have, respectively, incomes w_A and w_B at date 1 and can save this income to the second date at some exogenously given gross interest rate $R > 0$. Show that for general $u(\cdot)$, this economy does not admit a strong representative household, that is, a representative household without restricting the distribution of incomes. [Hint: show that different distributions of income will lead to different demands.]

5.12 Consider an economy consisting of H households each with a utility function at time $t = 0$ given by

$$\sum_{t=0}^{\infty} \beta^t u(c^h(t)),$$

with $\beta \in (0, 1)$, where $c^h(t)$ denotes the consumption of household h at time t. Suppose that $u(0) = 0$. The economy starts with an endowment of $y > 0$ units of the final good and has access to no production technology. This endowment can be saved without depreciating or gaining interest rate between periods.

(a) What are the Arrow-Debreu commodities in this economy?

(b) Characterize the set of Pareto optimal allocations of this economy.

(c) Prove that the Second Welfare Theorem (Theorem 5.7) can be applied to this economy.

(d) Consider an allocation of y units to the households, $\{y^h\}_{h=1}^H$, such that $\sum_{h=1}^H y^h = y$. Given this allocation, find the unique competitive equilibrium price vector and the corresponding consumption allocations.

(e) Are all competitive equilibria Pareto optimal?

(f) Derive a redistribution scheme for decentralizing the entire set of Pareto optimal allocations.

5.13 (a) Suppose that utility of household h given by

$$U(x^h(0), x^h(1), \ldots) = \sum_{t=0}^{\infty} \beta^t v^h(x^h(t))$$

where $x^h(t) \in X \subset \mathbb{R}_+^N$, $v^h : X \to \mathbb{R}$ is continuous, X is compact, and $\beta < 1$. Show that the hypothesis that for any $x^h, \bar{x}^h \in X^h$ with $U^h(x^h) > U^h(\bar{x}^h)$, there exists $\bar{T}$ (as a function of x^h and $\bar{x}^h$) such that $U^h(x^h[T]) > U^h(\bar{x}^h)$ for all $T \geq \bar{T}$ in Theorem 5.7 (hypothesis iii) is satisfied.

(b) Suppose that the production structure is given by a neoclassical production function, where the production vector at time t is only a function of inputs at time t and capital stock chosen at time $t-1$, that higher capital stock contributes to greater production, and there is free disposal. Show that the second hypothesis in Theorem 5.7, which states that for each $y^f \in Y^f$, there exists $\tilde{T}$ such that $y^f[T] \in Y^f$ for all $T \geq \tilde{T}$, is satisfied.

* 5.14 (a) Show that Theorem 5.7 does not cover the one-good neoclassical growth model with instantaneous preferences given by $u(c) = (c^{1-\theta} - 1)/(1-\theta)$ with $\theta \geq 1$.

(b) For $\varepsilon > 0$, construct the sequence $\underline{\varepsilon}$ with each element equal to ε. Reformulate and prove a version of Theorem 5.7 such that that X^h (for all $h \in \mathcal{H}$) is restricted to have elements $x^h \geq \underline{\varepsilon}$ for $\varepsilon > 0$ sufficiently small. [Hint: redefine $x^h[T]$ to have ε rather than 0 after the Tth element and reformulate the hypothesis of the theorem accordingly.]

(c) Show that this modified version of Theorem 5.7 covers the economy in part a of this exercise.