

## Chapter One

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### What Is the Theory of Games?

The most casual observer of the social behavior of mathematicians will remark that the solitary mathematician is a puzzle-solver while mathematicians together consume an extraordinary amount of time playing games. Coupling this observation with the obvious economic interest of many games of skill and chance, it does not seem strange that the basis for a mathematical theory of games of strategy was created by the mathematician John von Neumann [1]\* and elaborated with the economist Oskar Morgenstern [2]. In this series of lectures we shall attempt to present the salient results of this theory.

Although our primary concern will be with the mathematical aspects, a complete understanding of the theory can only come from the recognition that it is an interpreted system. This means that, like any satisfactory physical theory, it consists of (1) an axiom system and (2) a set of semantical rules for its interpretation, that is, a set of prescriptions that connect the mathematical symbols of the axioms with the objects of our experience. In the immediate interpretation, these objects are ordinary parlor games such as chess, poker, or bridge, and the success of the axiom system that we choose is measured by the number of interesting facts that we can derive about these games. There are, of course, other interpretations of the axioms, and one of the more ambitious hopes of the theory is that eventually these will include a substantial portion of the science of economics [3]. In these lectures we will restrict ourselves, with few exceptions, to the principal interpretation in terms of games.

Ideally, such an interpreted system could be presented by giving its axioms and the rules for their interpretation and then deriving all the important theorems of the theory by the application of mathematical techniques. But practically the situation is such that most fields of science seem at the present time to be not yet developed to a degree which would suggest this strict form of presentation; only certain branches of physics and geometry qualify by their apparent completeness. Fortunately for the researcher, the theory of games is by no means complete. Indeed, in some portions, such as games played by more than two players, it seems likely that the final formulation will be far removed from the present theory. Accordingly, the ideal order of presentation will be reversed in these lectures, and it will only be after a number of specific games have been examined in detail that the axioms of the theory will appear.

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\*Numbers in square brackets refer to explanatory material appended to each chapter. In some cases this material leads to sources and advanced topics in the theory.

The central problem of game theory was posed by von Neumann as early as 1926 in Göttingen. It is the following:

If  $n$  players,  $P_1, \dots, P_n$ , play a given game  $\Gamma$ , how must the  $i^{\text{th}}$  player,  $P_i$ , play to achieve the most favorable result for himself?

Stated in this form, of course, the question is very vague and our first problem will be to sharpen the statement until we can agree upon a meaning. First of all, we mean by a game any of the ordinary objects designated by this name, for instance, chess, checkers, tic-tac-toe, gin rummy. However, unfortunately, the word “game” is used with two meanings in everyday English. We speak of “the game of checkers,” referring to the game defined by its rules, and we say that we played “five games of checkers,” meaning that we completed five particular contests of checkers. We shall separate these meanings, reserving *game* for the first meaning only and using the word *play* for an individual contest. The usage in French is more fortuitous than in English; *un jeu* is used in the sense that we have just defined for “game” while *une partie* is a play. A similar distinction must be made between the occasion of a selection of one of several alternatives, which we shall call a *move*, and the particular selection, which we will call a *choice*. Thus, when the record of a chess game begins:

1.  $P-K4$
2.  $P-K4$
3.  $Kt-KB3$

the moves are 1, 2, and 3 while the choices are  $P-K4$ ,  $P-K4$ , and  $Kt-KB3$ .

In order to give meaning to the phrase, “the most favorable result,” we shall assume that, at the end of each play of  $\Gamma$ , each of the players,  $P_i$ , will receive an amount of money,  $h_i$ , called the *payoff to player  $P_i$* , and that the only object of a player will be to maximize the amount of money he receives [4]. In most parlor games, the payoffs satisfy the condition

$$(1) \quad h_1 + h_2 + \dots + h_n = 0 \text{ for all plays.}$$

This important category of games will be called the *zero-sum* games.

Thus we find that, by merely introducing this terminology, we can classify games according to the number of players who participate, the number of moves in the game, and whether the game is zero-sum or not. There is one other immediate property that can be used for classification. All parlor games have the characteristic property that, at each move, only a finite number of alternatives are presented to a player. Also, the “length” of a play or the number of possible choices occurring in a play is bounded by a “stop rule.” This rule is simple in some cases (in tic-tac-toe, the maximum number of choices is nine, the number of blank squares in the game diagram), while in other games it is quite complicated (in chess, the length of a play is made finite by rules preventing repeated cycles of choices). Such games

will be called *finite* games. However, there certainly exist activities that seem to be games in which people are presented with an infinite number of alternatives, and one can also conceive of games in which a play would consist of an infinite sequence of choices. An example of the first type is provided by a duel in which the duelist has the alternatives of firing at any instant in a fixed interval of the infinite time continuum, while it is a simple matter to construct conceptual games of the second type. Such games will be called *infinite* games.

The preceding classification suggests that we should attack the central problem posed above for the simplest type of game in this hierarchy. Leaving the case of one player aside for the moment, this simplest game must have two players, two moves, each with a finite number of alternatives, and be zero-sum. It is this class of games that we shall analyze in the next chapter.

### Notes

1. The birth of the theory of games (or perhaps the birth announcement) was the paper of von Neumann, "Zur Theorie der Gesellschaftsspiele," *Math. Annalen*, **100** (1928), 295–300. This paper contains the min-max theorem for zero-sum two-person games and the core of the characteristic function treatment of games with three or more players.

2. The infant of 1929 appears full grown in the impressive volume of von Neumann and Morgenstern, *Theory of Games and Economic Behavior*, Princeton, 1944, 2nd ed. 1947. Here we find the economic interpretation given full justice.

3. The immediate hope inspired by the appearance of the book is reflected in the reviews in economic and mathematical journals. The following might be mentioned as being representative:

Hurwicz, L., "The theory of economic behavior," *American Economic Review*, **35** (1945), 909–925.

Marschak, J., "Neumann's and Morgenstern's new approach to static economics," *Journal of Political Economy*, **54** (1946), 97–115.

Kaysen, C., "A revolution in economic theory?," *Review of Economic Studies*, **14/1** (1946–47), 1–15.

Stone, R., "The theory of games," *Economic Journal*, **58** (1948), 185–201.

Copeland, A. H., Review of "The Theory of Games and Economic Behavior," *Bulletin of the American Mathematical Society*, **51** (1945), 498–504.

4. While this assumption seems natural enough for the primary interpretation in terms of parlor games, it is quite a subtle step in an economic model from

preferences to measurable utility or money. See, for example, the Appendix and Section 3 of the book of von Neumann and Morgenstern cited above.

The reader may find it profitable at this point to gain a comprehensive non-technical view of the theory of games from the following more or less popular articles:

McDonald, J., "Poker: an American game," *Fortune*, **37** (1948), 128–131 and 181–187.

McDonald, J., "The theory of strategy," *Fortune*, **38** (1949), 100–110.

McDonald, J., *Strategy in Poker, Business, and War*, W.W. Norton, New York, 1950.

Morgenstern, O., "The theory of games," *Scientific American* (May, 1949), 22–25.

Bellman, R., Blackwell, D., "Red dog, blackjack, and poker," *Scientific American* (January, 1951), 44–47.

Dresher, M., "Games of strategy," *Math. Magazine*, **25** (1951), 93–99.