1 The Genesis of Fourier Analysis

> Regarding the researches of d'Alembert and Euler could one not add that if they knew this expansion, they made but a very imperfect use of it. They were both persuaded that an arbitrary and discontinuous function could never be resolved in series of this kind, and it does not even seem that anyone had developed a constant in cosines of multiple arcs, the first problem which I had to solve in the theory of heat.
>
> *J. Fourier,* 1808-9

In the beginning, it was the problem of the vibrating string, and the later investigation of heat flow, that led to the development of Fourier analysis. The laws governing these distinct physical phenomena were expressed by two different partial differential equations, the wave and heat equations, and these were solved in terms of Fourier series.

Here we want to start by describing in some detail the development of these ideas. We will do this initially in the context of the problem of the vibrating string, and we will proceed in three steps. First, we describe several physical (empirical) concepts which motivate corresponding mathematical ideas of importance for our study. These are: the role of the functions $\cos t$, $\sin t$, and e^{it} suggested by simple harmonic motion; the use of separation of variables, derived from the phenomenon of standing waves; and the related concept of linearity, connected to the superposition of tones. Next, we derive the partial differential equation which governs the motion of the vibrating string. Finally, we will use what we learned about the physical nature of the problem (expressed mathematically) to solve the equation. In the last section, we use the same approach to study the problem of heat diffusion.

Given the introductory nature of this chapter and the subject matter covered, our presentation cannot be based on purely mathematical reasoning. Rather, it proceeds by plausibility arguments and aims to provide the motivation for the further rigorous analysis in the succeeding chapters. The impatient reader who wishes to begin immediately with the theorems of the subject may prefer to pass directly to the next chapter.

1 The vibrating string

The problem consists of the study of the motion of a string fixed at its end points and allowed to vibrate freely. We have in mind physical systems such as the strings of a musical instrument. As we mentioned above, we begin with a brief description of several observable physical phenomena on which our study is based. These are:

- simple harmonic motion,
- standing and traveling waves,
- harmonics and superposition of tones.

Understanding the empirical facts behind these phenomena will motivate our mathematical approach to vibrating strings.

Simple harmonic motion

Simple harmonic motion describes the behavior of the most basic oscillatory system (called the simple harmonic oscillator), and is therefore a natural place to start the study of vibrations. Consider a mass $\{m\}$ attached to a horizontal spring, which itself is attached to a fixed wall, and assume that the system lies on a frictionless surface.

Choose an axis whose origin coincides with the center of the mass when it is at rest (that is, the spring is neither stretched nor compressed), as shown in Figure 1. When the mass is displaced from its initial equilibrium

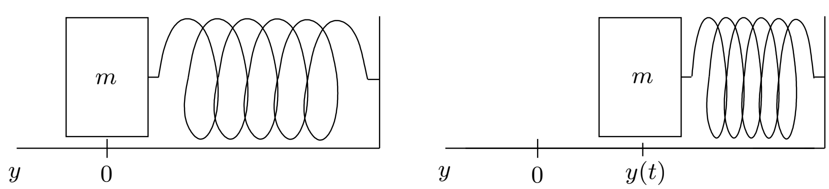

Figure 1. Simple harmonic oscillator

position and then released, it will undergo **simple harmonic motion**. This motion can be described mathematically once we have found the differential equation that governs the movement of the mass.

Let $y(t)$ denote the displacement of the mass at time t. We assume that the spring is ideal, in the sense that it satisfies Hooke's law: the restoring force F exerted by the spring on the mass is given by $F = -ky(t)$. Here

$k > 0$ is a given physical quantity called the spring constant. Applying Newton's law (force = mass × acceleration), we obtain

$$-ky(t) = my''(t),$$

where we use the notation y'' to denote the second derivative of y with respect to t. With $c = \sqrt{k/m}$, this second order ordinary differential equation becomes

$$y''(t) + c^2 y(t) = 0. \tag{1}$$

The general solution of equation (1) is given by

$$y(t) = a \cos ct + b \sin ct\,,$$

where a and b are constants. Clearly, all functions of this form solve equation (1), and Exercise 6 outlines a proof that these are the only (twice differentiable) solutions of that differential equation.

In the above expression for $y(t)$, the quantity c is given, but a and b can be any real numbers. In order to determine the particular solution of the equation, we must impose two initial conditions in view of the two unknown constants a and b. For example, if we are given $y(0)$ and $y'(0)$, the initial position and velocity of the mass, then the solution of the physical problem is unique and given by

$$y(t) = y(0) \cos ct + \frac{y'(0)}{c} \sin ct\,.$$

One can easily verify that there exist constants $A > 0$ and $\varphi \in \mathbb{R}$ such that

$$a \cos ct + b \sin ct = A \cos(ct - \varphi).$$

Because of the physical interpretation given above, one calls $A = \sqrt{a^2 + b^2}$ the "amplitude" of the motion, c its "natural frequency," φ its "phase" (uniquely determined up to an integer multiple of 2π), and $2\pi/c$ the "period" of the motion.

The typical graph of the function $A\cos(ct - \varphi)$, illustrated in Figure 2, exhibits a wavelike pattern that is obtained from translating and stretching (or shrinking) the usual graph of $\cos t$.

We make two observations regarding our examination of simple harmonic motion. The first is that the mathematical description of the most elementary oscillatory system, namely simple harmonic motion, involves

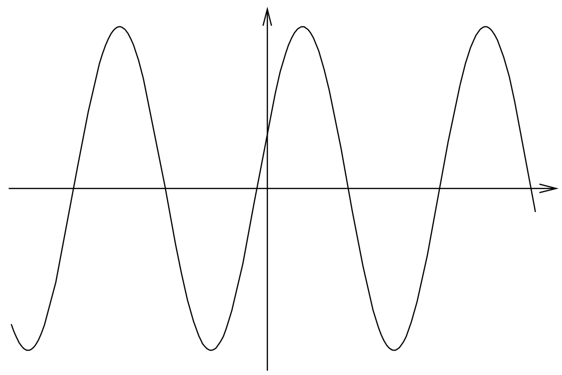

Figure 2. The graph of $A\cos(ct-\varphi)$

the most basic trigonometric functions $\cos t$ and $\sin t$. It will be important in what follows to recall the connection between these functions and complex numbers, as given in Euler's identity $e^{it} = \cos t + i \sin t$. The second observation is that simple harmonic motion is determined as a function of time by two initial conditions, one determining the position, and the other the velocity (specified, for example, at time $t = 0$). This property is shared by more general oscillatory systems, as we shall see below.

Standing and traveling waves

As it turns out, the vibrating string can be viewed in terms of one-dimensional wave motions. Here we want to describe two kinds of motions that lend themselves to simple graphic representations.

- First, we consider **standing waves**. These are wavelike motions described by the graphs $y = u(x,t)$ developing in time t as shown in Figure 3.

  In other words, there is an initial profile $y = \varphi(x)$ representing the wave at time $t = 0$, and an amplifying factor $\psi(t)$, depending on t, so that $y = u(x,t)$ with

  $$u(x,t) = \varphi(x)\psi(t).$$

  The nature of standing waves suggests the mathematical idea of "separation of variables," to which we will return later.

- A second type of wave motion that is often observed in nature is that of a **traveling wave**. Its description is particularly simple:

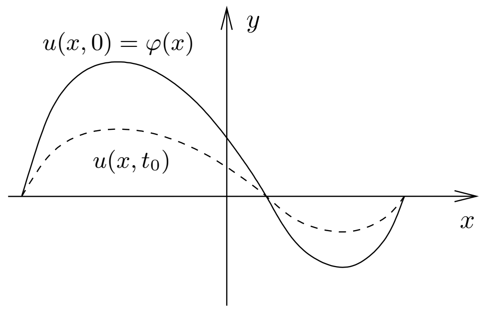

Figure 3. A standing wave at different moments in time: $t = 0$ and $t = t_0$

there is an initial profile $F(x)$ so that $u(x,t)$ equals $F(x)$ when $t = 0$. As t evolves, this profile is displaced to the right by ct units, where c is a positive constant, namely

$$u(x,t) = F(x - ct).$$

Graphically, the situation is depicted in Figure 4.

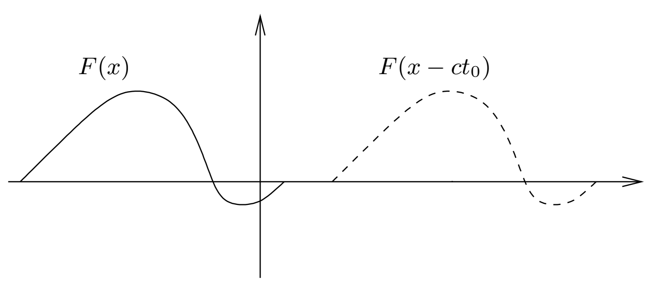

Figure 4. A traveling wave at two different moments in time: $t = 0$ and $t = t_0$

Since the movement in t is at the rate c, that constant represents the **velocity** of the wave. The function $F(x - ct)$ is a one-dimensional traveling wave moving to the right. Similarly, $u(x,t) = F(x + ct)$ is a one-dimensional traveling wave moving to the left.

Harmonics and superposition of tones

The final physical observation we want to mention (without going into any details now) is one that musicians have been aware of since time immemorial. It is the existence of harmonics, or overtones. The **pure tones** are accompanied by combinations of **overtones** which are primarily responsible for the timbre (or tone color) of the instrument. The idea of combination or superposition of tones is implemented mathematically by the basic concept of linearity, as we shall see below.

We now turn our attention to our main problem, that of describing the motion of a vibrating string. First, we derive the wave equation, that is, the partial differential equation that governs the motion of the string.

1.1 Derivation of the wave equation

Imagine a homogeneous string placed in the (x, y)-plane, and stretched along the x-axis between $x = 0$ and $x = L$. If it is set to vibrate, its displacement $y = u(x, t)$ is then a function of x and t, and the goal is to derive the differential equation which governs this function.

For this purpose, we consider the string as being subdivided into a large number N of masses (which we think of as individual particles) distributed uniformly along the x-axis, so that the n^{th} particle has its x-coordinate at $x_n = nL/N$. We shall therefore conceive of the vibrating string as a complex system of N particles, each oscillating in the *vertical direction only*; however, unlike the simple harmonic oscillator we considered previously, each particle will have its oscillation linked to its immediate neighbor by the tension of the string.

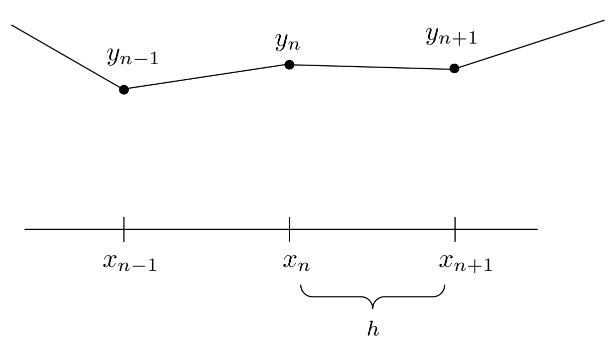

Figure 5. A vibrating string as a discrete system of masses

We then set $y_n(t) = u(x_n, t)$, and note that $x_{n+1} - x_n = h$, with $h = L/N$. If we assume that the string has constant density $\rho > 0$, it is reasonable to assign mass equal to ρh to each particle. By Newton's law, $\rho h y_n''(t)$ equals the force acting on the n-th particle. We now make the simple assumption that this force is due to the effect of the two nearby particles, the ones with x-coordinates at x_{n-1} and x_{n+1} (see Figure 5). We further assume that the force (or tension) coming from the right of the n^{th} particle is proportional to $(y_{n+1} - y_n)/h$, where h is the distance between x_{n+1} and x_n; hence we can write the tension as

$$\left(\frac{\tau}{h}\right)(y_{n+1} - y_n),$$

where $\tau > 0$ is a constant equal to the coefficient of tension of the string. There is a similar force coming from the left, and it is

$$\left(\frac{\tau}{h}\right)(y_{n-1} - y_n).$$

Altogether, these forces which act in opposite directions give us the desired relation between the oscillators $y_n(t)$, namely

$$\rho h y_n''(t) = \frac{\tau}{h}\left\{y_{n+1}(t) + y_{n-1}(t) - 2y_n(t)\right\}. \tag{2}$$

On the one hand, with the notation chosen above, we see that

$$y_{n+1}(t) + y_{n-1}(t) - 2y_n(t) = u(x_n + h, t) + u(x_n - h, t) - 2u(x_n, t).$$

On the other hand, for any reasonable function $F(x)$ (that is, one that has continuous second derivatives) we have

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} \to F''(x) \quad \text{as } h \to 0.$$

Thus we may conclude, after dividing by h in (2) and letting h tend to zero (that is, N goes to infinity), that

$$\rho \frac{\partial^2 u}{\partial t^2} = \tau \frac{\partial^2 u}{\partial x^2},$$

or

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad \text{with } c = \sqrt{\tau/\rho}.$$

This relation is known as the **one-dimensional wave equation**, or more simply as the **wave equation**. For reasons that will be apparent later, the coefficient $c > 0$ is called the **velocity** of the motion.

In connection with this partial differential equation, we make an important simplifying mathematical remark. This has to do with **scaling**, or in the language of physics, a "change of units." That is, we can think of the coordinate x as $x = aX$ where a is an appropriate positive constant. Now, in terms of the new coordinate X, the interval $0 \le x \le L$ becomes $0 \le X \le L/a$. Similarly, we can replace the time coordinate t by $t = bT$, where b is another positive constant. If we set $U(X,T) = u(x,t)$, then

$$\frac{\partial U}{\partial X} = a\frac{\partial u}{\partial x}, \qquad \frac{\partial^2 U}{\partial X^2} = a^2\frac{\partial^2 u}{\partial x^2},$$

and similarly for the derivatives in t. So if we choose a and b appropriately, we can transform the one-dimensional wave equation into

$$\frac{\partial^2 U}{\partial T^2} = \frac{\partial^2 U}{\partial X^2},$$

which has the effect of setting the velocity c equal to 1. Moreover, we have the freedom to transform the interval $0 \le x \le L$ to $0 \le X \le \pi$. (We shall see that the choice of π is convenient in many circumstances.) All this is accomplished by taking $a = L/\pi$ and $b = L/(c\pi)$. Once we solve the new equation, we can of course return to the original equation by making the inverse change of variables. Hence, we do not sacrifice generality by thinking of the wave equation as given on the interval $[0,\pi]$ with velocity $c = 1$.

1.2 Solution to the wave equation

Having derived the equation for the vibrating string, we now explain two methods to solve it:

- using traveling waves,

- using the superposition of standing waves.

While the first approach is very simple and elegant, it does not directly give full insight into the problem; the second method accomplishes that, and moreover is of wide applicability. It was first believed that the second method applied only in the simple cases where the initial position and velocity of the string were themselves given as a superposition of standing waves. However, as a consequence of Fourier's ideas, it became clear that the problem could be worked either way for all initial conditions.

Traveling waves

To simplify matters as before, we assume that $c = 1$ and $L = \pi$, so that the equation we wish to solve becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{on } 0 \leq x \leq \pi, \text{ with } t \geq 0.$$

The crucial observation is the following: if F is any twice differentiable function, then $u(x,t) = F(x+t)$ and $u(x,t) = F(x-t)$ solve the wave equation. The verification of this is a simple exercise in differentiation. Note that the graph of $u(x,t) = F(x-t)$ at time $t = 0$ is simply the graph of F, and that at time $t = 1$ it becomes the graph of F translated to the right by 1. Therefore, we recognize that $F(x-t)$ is a traveling wave which travels to the right with speed 1. Similarly, $u(x,t) = F(x+t)$ is a wave traveling to the left with speed 1. These motions are depicted in Figure 6.

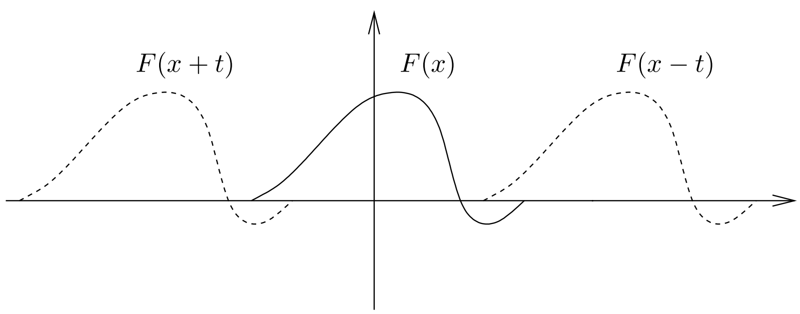

Figure 6. Waves traveling in both directions

Our discussion of tones and their combinations leads us to observe that the wave equation is **linear**. This means that if $u(x,t)$ and $v(x,t)$ are particular solutions, then so is $\alpha u(x,t) + \beta v(x,t)$, where α and β are any constants. Therefore, we may superpose two waves traveling in opposite directions to find that whenever F and G are twice differentiable functions, then

$$u(x,t) = F(x+t) + G(x-t)$$

is a solution of the wave equation. In fact, we now show that all solutions take this form.

We drop for the moment the assumptions that $0 \leq x \leq \pi$ and $t \geq 0$, and suppose that u is a twice differentiable function which solves the wave equation for all real x and t. Consider the following new set of variables $\xi = x + t$, $\eta = x - t$, and define $v(\xi, \eta) = u(x, t)$. The change of variables formula shows that v satisfies

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = 0.$$

Integrating this relation twice gives $v(\xi, \eta) = F(\xi) + G(\eta)$, which then implies

$$u(x, t) = F(x + t) + G(x - t),$$

for some functions F and G.

We must now connect this result with our original problem, that is, the physical motion of a string. There, we imposed the restrictions $0 \leq x \leq \pi$, the initial shape of the string $u(x, 0) = f(x)$, and also the fact that the string has fixed end points, namely $u(0, t) = u(\pi, t) = 0$ for all $t \geq 0$. To use the simple observation above, we first extend f to all of $\mathbb{R}$ by making it odd[1] on $[-\pi, \pi]$, and then periodic[2] in x of period 2π, and similarly for $u(x, t)$, the solution of our problem. Finally, set $u(x, t) = u(x, -t)$ whenever $t < 0$. Then the extension u solves the wave equation on all of $\mathbb{R}$, and $u(x, 0) = f(x)$ for all $x \in \mathbb{R}$. Therefore, $u(x, t) = F(x + t) + G(x - t)$, and setting $t = 0$ we find that

$$F(x) + G(x) = f(x).$$

Since many choices of F and G will satisfy this identity, this suggests imposing another initial condition on u (similar to the two initial conditions in the case of simple harmonic motion), namely the initial velocity of the string which we denote by $g(x)$:

$$\frac{\partial u}{\partial t}(x, 0) = g(x),$$

where of course $g(0) = g(\pi) = 0$. Again, we extend g to $\mathbb{R}$ first by making it odd over $[-\pi, \pi]$, and then periodic of period 2π. The two initial conditions of position and velocity now translate into the following system:

$$\begin{cases} F(x) + G(x) = f(x)\,, \\ F'(x) - G'(x) = g(x)\,. \end{cases}$$

[1] A function f defined on a set U is **odd** if $-x \in U$ whenever $x \in U$ and $f(-x) = -f(x)$, and **even** if $f(-x) = f(x)$.

[2] A function f on $\mathbb{R}$ is **periodic** of period ω if $f(x + \omega) = f(x)$ for all x.

Differentiating the first equation and adding it to the second, we obtain

$$2F'(x) = f'(x) + g(x).$$

Similarly

$$2G'(x) = f'(x) - g(x),$$

and hence there are constants C_1 and C_2 so that

$$F(x) = \frac{1}{2}\left[f(x) + \int_0^x g(y)\,dy\right] + C_1$$

and

$$G(x) = \frac{1}{2}\left[f(x) - \int_0^x g(y)\,dy\right] + C_2.$$

Since $F(x) + G(x) = f(x)$ we conclude that $C_1 + C_2 = 0$, and therefore, our final solution of the wave equation with the given initial conditions takes the form

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy.$$

The form of this solution is known as **d'Alembert's formula**. Observe that the extensions we chose for f and g guarantee that the string always has fixed ends, that is, $u(0,t) = u(\pi,t) = 0$ for all t.

A final remark is in order. The passage from $t \geq 0$ to $t \in \mathbb{R}$, and then back to $t \geq 0$, which was made above, exhibits the time reversal property of the wave equation. In other words, a solution u to the wave equation for $t \geq 0$, leads to a solution u^- defined for negative time $t < 0$ simply by setting $u^-(x,t) = u(x,-t)$, a fact which follows from the invariance of the wave equation under the transformation $t \mapsto -t$. The situation is quite different in the case of the heat equation.

Superposition of standing waves

We turn to the second method of solving the wave equation, which is based on two fundamental conclusions from our previous physical observations. By our considerations of standing waves, we are led to look for special solutions to the wave equation which are of the form $\varphi(x)\psi(t)$. This procedure, which works equally well in other contexts (in the case of the heat equation, for instance), is called **separation of variables** and constructs solutions that are called pure tones. Then by the linearity of the wave equation, we can expect to combine these pure tones into a more complex combination of sound. Pushing this idea further, we can hope ultimately to express the general solution of the wave equation in terms of sums of these particular solutions.

Note that one side of the wave equation involves only differentiation in x, while the other, only differentiation in t. This observation provides another reason to look for solutions of the equation in the form $u(x,t) = \varphi(x)\psi(t)$ (that is, to "separate variables"), the hope being to reduce a difficult partial differential equation into a system of simpler ordinary differential equations. In the case of the wave equation, with u of the above form, we get

$$\varphi(x)\psi''(t) = \varphi''(x)\psi(t),$$

and therefore

$$\frac{\psi''(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}.$$

The key observation here is that the left-hand side depends only on t, and the right-hand side only on x. This can happen only if both sides are equal to a constant, say λ. Therefore, the wave equation reduces to the following

$$(3) \qquad \begin{cases} \psi''(t) - \lambda\psi(t) \;= 0 \\ \varphi''(x) - \lambda\varphi(x) = 0. \end{cases}$$

We focus our attention on the first equation in the above system. At this point, the reader will recognize the equation we obtained in the study of simple harmonic motion. Note that we need to consider only the case when $\lambda < 0$, since when $\lambda \geq 0$ the solution ψ will not oscillate as time varies. Therefore, we may write $\lambda = -m^2$, and the solution of the equation is then given by

$$\psi(t) = A\cos mt + B\sin mt.$$

Similarly, we find that the solution of the second equation in (3) is

$$\varphi(x) = \tilde{A}\cos mx + \tilde{B}\sin mx.$$

Now we take into account that the string is attached at $x = 0$ and $x = \pi$. This translates into $\varphi(0) = \varphi(\pi) = 0$, which in turn gives $\tilde{A} = 0$, and if $\tilde{B} \neq 0$, then m must be an integer. If $m = 0$, the solution vanishes identically, and if $m \leq -1$, we may rename the constants and reduce to the case $m \geq 1$ since the function $\sin y$ is odd and $\cos y$ is even. Finally, we arrive at the guess that for each $m \geq 1$, the function

$$u_m(x,t) = (A_m \cos mt + B_m \sin mt)\sin mx,$$

which we recognize as a **standing wave**, is a solution to the wave equation. Note that in the above argument we divided by φ and ψ, which sometimes vanish, so one must actually check by hand that the standing wave u_m solves the equation. This straightforward calculation is left as an exercise to the reader.

Before proceeding further with the analysis of the wave equation, we pause to discuss standing waves in more detail. The terminology comes from looking at the graph of $u_m(x,t)$ for each fixed t. Suppose first that $m = 1$, and take $u(x,t) = \cos t \sin x$. Then, Figure 7 (a) gives the graph of u for different values of t.

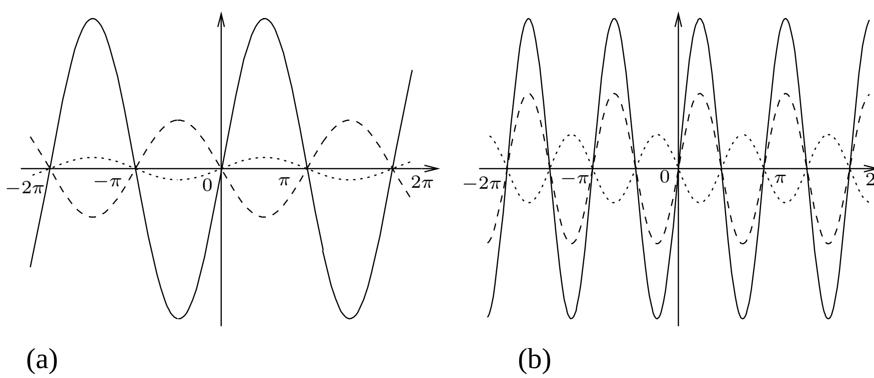

Figure 7. Fundamental tone (a) and overtones (b) at different moments in time

The case $m = 1$ corresponds to the **fundamental tone** or **first harmonic** of the vibrating string.

We now take $m = 2$ and look at $u(x,t) = \cos 2t \sin 2x$. This corresponds to the **first overtone** or **second harmonic**, and this motion is described in Figure 7 (b). Note that $u(\pi/2, t) = 0$ for all t. Such points, which remain motionless in time, are called nodes, while points whose motion has maximum amplitude are named anti-nodes.

For higher values of m we get more overtones or higher harmonics. Note that as m increases, the frequency increases, and the period $2\pi/m$

decreases. Therefore, the fundamental tone has a lower frequency than the overtones.

We now return to the original problem. Recall that the wave equation is linear in the sense that if u and v solve the equation, so does $\alpha u + \beta v$ for any constants α and β. This allows us to construct more solutions by taking linear combinations of the standing waves u_m. This technique, called **superposition**, leads to our final guess for a solution of the wave equation

$$u(x,t) = \sum_{m=1}^{\infty} (A_m \cos mt + B_m \sin mt) \sin mx. \tag{4}$$

Note that the above sum is infinite, so that questions of convergence arise, but since most of our arguments so far are formal, we will not worry about this point now.

Suppose the above expression gave *all* the solutions to the wave equation. If we then require that the initial position of the string at time $t = 0$ is given by the shape of the graph of the function f on $[0, \pi]$, with of course $f(0) = f(\pi) = 0$, we would have $u(x, 0) = f(x)$, hence

$$\sum_{m=1}^{\infty} A_m \sin mx = f(x).$$

Since the initial shape of the string can be any reasonable function f, we must ask the following basic question:

> Given a function f on $[0, \pi]$ (with $f(0) = f(\pi) = 0$), can we find coefficients A_m so that

$$f(x) = \sum_{m=1}^{\infty} A_m \sin mx\ ? \tag{5}$$

This question is stated loosely, but a lot of our effort in the next two chapters of this book will be to formulate the question precisely and attempt to answer it. This was the basic problem that initiated the study of Fourier analysis.

A simple observation allows us to guess a formula giving A_m if the expansion (5) were to hold. Indeed, we multiply both sides by $\sin nx$

and integrate between $[0, \pi]$; working formally, we obtain

$$\begin{aligned}\int_0^\pi f(x)\sin nx\,dx &= \int_0^\pi \left(\sum_{m=1}^\infty A_m \sin mx\right)\sin nx\,dx \\ &= \sum_{m=1}^\infty A_m \int_0^\pi \sin mx\,\sin nx\,dx = A_n \cdot \frac{\pi}{2},\end{aligned}$$

where we have used the fact that

$$\int_0^\pi \sin mx\,\sin nx\,dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi/2 & \text{if } m = n. \end{cases}$$

Therefore, the guess for A_n, called the n^{th} Fourier sine coefficient of f, is

$$A_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx. \tag{6}$$

We shall return to this formula, and other similar ones, later.

One can transform the question about Fourier sine series on $[0, \pi]$ to a more general question on the interval $[-\pi, \pi]$. If we could express f on $[0, \pi]$ in terms of a sine series, then this expansion would also hold on $[-\pi, \pi]$ if we extend f to this interval by making it odd. Similarly, one can ask if an even function $g(x)$ on $[-\pi, \pi]$ can be expressed as a cosine series, namely

$$g(x) = \sum_{m=0}^\infty A'_m \cos mx.$$

More generally, since an arbitrary function F on $[-\pi, \pi]$ can be expressed as $f + g$, where f is odd and g is even,[3] we may ask if F can be written as

$$F(x) = \sum_{m=1}^\infty A_m \sin mx + \sum_{m=0}^\infty A'_m \cos mx,$$

or by applying Euler's identity $e^{ix} = \cos x + i \sin x$, we could hope that F takes the form

$$F(x) = \sum_{m=-\infty}^\infty a_m e^{imx}.$$

[3] Take, for example, $f(x) = [F(x) - F(-x)]/2$ and $g(x) = [F(x) + F(-x)]/2$.

By analogy with (6), we can use the fact that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}e^{-inx}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m, \end{cases}$$

to see that one expects that

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)e^{-inx}\,dx.$$

The quantity a_n is called the n^{th} **Fourier coefficient** of F.

We can now reformulate the problem raised above:

> **Question:** Given any reasonable function F on $[-\pi, \pi]$, with Fourier coefficients defined above, is it true that

$$F(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx}\ ? \tag{7}$$

This formulation of the problem, in terms of complex exponentials, is the form we shall use the most in what follows.

Joseph Fourier (1768-1830) was the first to believe that an "arbitrary" function F could be given as a series (7). In other words, his idea was that any function is the linear combination (possibly infinite) of the most basic trigonometric functions $\sin mx$ and $\cos mx$, where m ranges over the integers.[4] Although this idea was implicit in earlier work, Fourier had the conviction that his predecessors lacked, and he used it in his study of heat diffusion; this began the subject of "Fourier analysis." This discipline, which was first developed to solve certain physical problems, has proved to have many applications in mathematics and other fields as well, as we shall see later.

We return to the wave equation. To formulate the problem correctly, we must impose two initial conditions, as our experience with simple harmonic motion and traveling waves indicated. The conditions assign the initial position and velocity of the string. That is, we require that u satisfy the differential equation and the two conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x),$$

[4] The first proof that a general class of functions can be represented by Fourier series was given later by Dirichlet; see Problem 6, Chapter 4.

where f and g are pre-assigned functions. Note that this is consistent with (4) in that this requires that f and g be expressible as

$$f(x) = \sum_{m=1}^{\infty} A_m \sin mx \quad \text{and} \quad g(x) = \sum_{m=1}^{\infty} mB_m \sin mx.$$

1.3 Example: the plucked string

We now apply our reasoning to the particular problem of the plucked string. For simplicity we choose units so that the string is taken on the interval $[0, \pi]$, and it satisfies the wave equation with $c = 1$. The string is assumed to be plucked to height h at the point p with $0 < p < \pi$; this is the initial position. That is, we take as our initial position the triangular shape given by

$$f(x) = \begin{cases} \dfrac{xh}{p} & \text{for } 0 \leq x \leq p \\[2ex] \dfrac{h(\pi - x)}{\pi - p} & \text{for } p \leq x \leq \pi, \end{cases}$$

which is depicted in Figure 8.

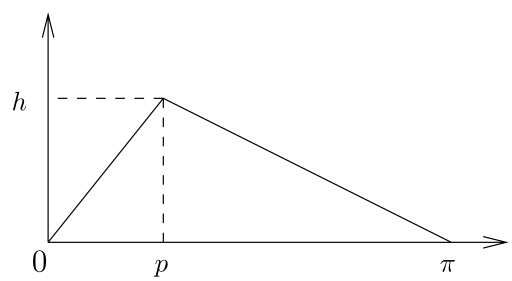

Figure 8. Initial position of a plucked string

We also choose an initial velocity $g(x)$ identically equal to 0. Then, we can compute the Fourier coefficients of f (Exercise 9), and assuming that the answer to the question raised before (5) is positive, we obtain

$$f(x) = \sum_{m=1}^{\infty} A_m \sin mx \quad \text{with} \quad A_m = \frac{2h}{m^2} \frac{\sin mp}{p(\pi - p)}.$$

Thus

$$u(x,t) = \sum_{m=1}^{\infty} A_m \cos mt \, \sin mx, \tag{8}$$

and note that this series converges absolutely. The solution can also be expressed in terms of traveling waves. In fact

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2}. \tag{9}$$

Here $f(x)$ is defined for all x as follows: first, f is extended to $[-\pi, \pi]$ by making it odd, and then f is extended to the whole real line by making it periodic of period 2π, that is, $f(x + 2\pi k) = f(x)$ for all integers k.

Observe that (8) implies (9) in view of the trigonometric identity

$$\cos v \sin u = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right].$$

As a final remark, we should note an unsatisfactory aspect of the solution to this problem, which however is in the nature of things. Since the initial data $f(x)$ for the plucked string is not twice continuously differentiable, neither is the function u (given by (9)). Hence u is not truly a solution of the wave equation: while $u(x,t)$ does represent the position of the plucked string, it does not satisfy the partial differential equation we set out to solve! This state of affairs may be understood properly only if we realize that u does solve the equation, but in an appropriate generalized sense. A better understanding of this phenomenon requires ideas relevant to the study of "weak solutions" and the theory of "distributions." These topics we consider only later, in Books III and IV.

2 The heat equation

We now discuss the problem of heat diffusion by following the same framework as for the wave equation. First, we derive the time-dependent heat equation, and then study the steady-state heat equation in the disc, which leads us back to the basic question (7).

2.1 Derivation of the heat equation

Consider an infinite metal plate which we model as the plane $\mathbb{R}^2$, and suppose we are given an initial heat distribution at time $t = 0$. Let the temperature at the point (x, y) at time t be denoted by $u(x, y, t)$.

Consider a small square centered at (x_0, y_0) with sides parallel to the axis and of side length h, as shown in Figure 9. The amount of heat energy in S at time t is given by

$$H(t) = \sigma \iint_S u(x, y, t)\, dx\, dy\,,$$

where $\sigma > 0$ is a constant called the specific heat of the material. Therefore, the heat flow into S is

$$\frac{\partial H}{\partial t} = \sigma \iint_S \frac{\partial u}{\partial t}\, dx\, dy\,,$$

which is approximately equal to

$$\sigma h^2 \frac{\partial u}{\partial t}(x_0, y_0, t),$$

since the area of S is h^2. Now we apply Newton's law of cooling, which states that heat flows from the higher to lower temperature at a rate proportional to the difference, that is, the gradient.

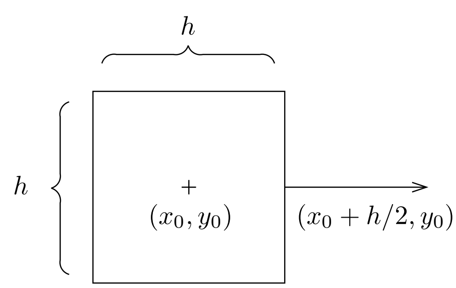

Figure 9. Heat flow through a small square

The heat flow through the vertical side on the right is therefore

$$-\kappa h \frac{\partial u}{\partial x}(x_0 + h/2, y_0, t)\,,$$

where $\kappa > 0$ is the conductivity of the material. A similar argument for the other sides shows that the total heat flow through the square S is given by

$$\kappa h \left[\frac{\partial u}{\partial x}(x_0 + h/2, y_0, t) - \frac{\partial u}{\partial x}(x_0 - h/2, y_0, t) \right.$$
$$\left. + \frac{\partial u}{\partial y}(x_0, y_0 + h/2, t) - \frac{\partial u}{\partial y}(x_0, y_0 - h/2, t) \right].$$

Applying the mean value theorem and letting h tend to zero, we find that

$$\frac{\sigma}{\kappa} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2};$$

this is called the **time-dependent heat equation**, often abbreviated to the heat equation.

2.2 Steady-state heat equation in the disc

After a long period of time, there is no more heat exchange, so that the system reaches thermal equilibrium and $\partial u/\partial t = 0$. In this case, the time-dependent heat equation reduces to the **steady-state heat equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{10}$$

The operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ is of such importance in mathematics and physics that it is often abbreviated as $\triangle$ and given a name: the Laplace operator or **Laplacian**. So the steady-state heat equation is written as

$$\triangle u = 0,$$

and solutions to this equation are called **harmonic functions**.

Consider the unit disc in the plane

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

whose boundary is the unit circle C. In polar coordinates (r, θ), with $0 \le r$ and $0 \le \theta < 2\pi$, we have

$$D = \{(r, \theta) : 0 \le r < 1\} \quad \text{and} \quad C = \{(r, \theta) : r = 1\}.$$

The problem, often called the **Dirichlet problem** (for the Laplacian on the unit disc), is to solve the steady-state heat equation in the unit disc subject to the boundary condition $u = f$ on C. This corresponds to fixing a predetermined temperature distribution on the circle, waiting a long time, and then looking at the temperature distribution inside the disc.

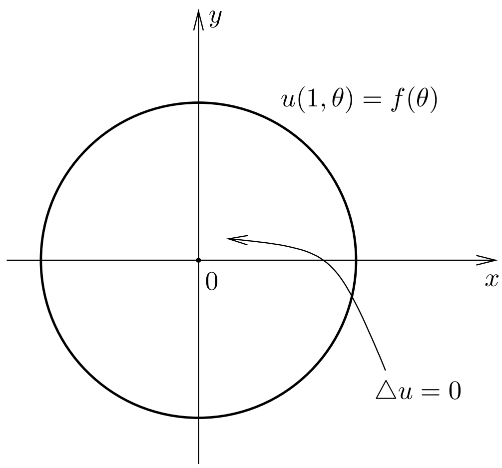

Figure 10. The Dirichlet problem for the disc

While the method of separation of variables will turn out to be useful for equation (10), a difficulty comes from the fact that the boundary condition is not easily expressed in terms of rectangular coordinates. Since this boundary condition is best described by the coordinates (r, θ), namely $u(1, \theta) = f(\theta)$, we rewrite the Laplacian in polar coordinates. An application of the chain rule gives (Exercise 10):

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

We now multiply both sides by r^2, and since $\triangle u = 0$, we get

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} = -\frac{\partial^2 u}{\partial \theta^2}.$$

Separating these variables, and looking for a solution of the form $u(r, \theta) = F(r)G(\theta)$, we find

$$\frac{r^2 F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}.$$

Since the two sides depend on different variables, they must both be constant, say equal to λ. We therefore get the following equations:

$$\begin{cases} G''(\theta) + \lambda G(\theta) = 0\,, \\ r^2 F''(r) + rF'(r) - \lambda F(r) = 0. \end{cases}$$

Since G must be periodic of period 2π, this implies that $\lambda \geq 0$ and (as we have seen before) that $\lambda = m^2$ where m is an integer; hence

$$G(\theta) = \tilde{A}\cos m\theta + \tilde{B}\sin m\theta.$$

An application of Euler's identity, $e^{ix} = \cos x + i\sin x$, allows one to rewrite G in terms of complex exponentials,

$$G(\theta) = Ae^{im\theta} + Be^{-im\theta}.$$

With $\lambda = m^2$ and $m \neq 0$, two simple solutions of the equation in F are $F(r) = r^m$ and $F(r) = r^{-m}$ (Exercise 11 gives further information about these solutions). If $m = 0$, then $F(r) = 1$ and $F(r) = \log r$ are two solutions. If $m > 0$, we note that r^{-m} grows unboundedly large as r tends to zero, so $F(r)G(\theta)$ is unbounded at the origin; the same occurs when $m = 0$ and $F(r) = \log r$. We reject these solutions as contrary to our intuition. Therefore, we are left with the following special functions:

$$u_m(r, \theta) = r^{|m|}e^{im\theta}, \qquad m \in \mathbb{Z}.$$

We now make the important observation that (10) is *linear*, and so as in the case of the vibrating string, we may superpose the above special solutions to obtain the presumed general solution:

$$u(r, \theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}.$$

If this expression gave all the solutions to the steady-state heat equation, then for a reasonable f we should have

$$u(1, \theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} = f(\theta).$$

We therefore ask again in this context: given any reasonable function f on $[0, 2\pi]$ with $f(0) = f(2\pi)$, can we find coefficients a_m so that

$$f(\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} \ ?$$

Historical Note: D'Alembert (in 1747) first solved the equation of the vibrating string using the method of traveling waves. This solution was elaborated by Euler a year later. In 1753, D. Bernoulli proposed the solution which for all intents and purposes is the Fourier series given by (4), but Euler was not entirely convinced of its full generality, since this could hold only if an "arbitrary" function could be expanded in Fourier series. D'Alembert and other mathematicians also had doubts. This viewpoint was changed by Fourier (in 1807) in his study of the heat equation, where his conviction and work eventually led others to a complete proof that a general function could be represented as a Fourier series.

3 Exercises

1. If $z = x + iy$ is a complex number with $x, y \in \mathbb{R}$, we define

$$|z| = (x^2 + y^2)^{1/2}$$

and call this quantity the **modulus** or **absolute value** of z.

(a) What is the geometric interpretation of $|z|$?

(b) Show that if $|z| = 0$, then $z = 0$.

(c) Show that if $\lambda \in \mathbb{R}$, then $|\lambda z| = |\lambda||z|$, where $|\lambda|$ denotes the standard absolute value of a real number.

(d) If z_1 and z_2 are two complex numbers, prove that

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad |z_1 + z_2| \leq |z_1| + |z_2|.$$

(e) Show that if $z \neq 0$, then $|1/z| = 1/|z|$.

2. If $z = x + iy$ is a complex number with $x, y \in \mathbb{R}$, we define the **complex conjugate** of z by

$$\overline{z} = x - iy.$$

(a) What is the geometric interpretation of $\overline{z}$?

(b) Show that $|z|^2 = z\overline{z}$.

(c) Prove that if z belongs to the unit circle, then $1/z = \overline{z}$.

3. A sequence of complex numbers $\{w_n\}_{n=1}^{\infty}$ is said to converge if there exists $w \in \mathbb{C}$ such that

$$\lim_{n\to\infty} |w_n - w| = 0,$$

and we say that w is a limit of the sequence.

(a) Show that a converging sequence of complex numbers has a unique limit.

The sequence $\{w_n\}_{n=1}^{\infty}$ is said to be a **Cauchy sequence** if for every $\epsilon > 0$ there exists a positive integer N such that

$$|w_n - w_m| < \epsilon \qquad \text{whenever } n, m > N.$$

(b) Prove that a sequence of complex numbers converges if and only if it is a Cauchy sequence. [Hint: A similar theorem exists for the convergence of a sequence of real numbers. Why does it carry over to sequences of complex numbers?]

A series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge if the sequence formed by the partial sums

$$S_N = \sum_{n=1}^{N} z_n$$

converges. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers such that the series $\sum_n a_n$ converges.

(c) Show that if $\{z_n\}_{n=1}^{\infty}$ is a sequence of complex numbers satisfying $|z_n| \le a_n$ for all n, then the series $\sum_n z_n$ converges. [Hint: Use the Cauchy criterion.]

4. For $z \in \mathbb{C}$, we define the **complex exponential** by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(a) Prove that the above definition makes sense, by showing that the series converges for every complex number z. Moreover, show that the convergence is uniform[5] on every bounded subset of $\mathbb{C}$.

(b) If z_1, z_2 are two complex numbers, prove that $e^{z_1}e^{z_2} = e^{z_1+z_2}$. [Hint: Use the binomial theorem to expand $(z_1 + z_2)^n$, as well as the formula for the binomial coefficients.]

[5]A sequence of functions $\{f_n(z)\}_{n=1}^{\infty}$ is said to be uniformly convergent on a set S if there exists a function f on S so that for every $\epsilon > 0$ there is an integer N such that $|f_n(z) - f(z)| < \epsilon$ whenever $n > N$ and $z \in S$.

(c) Show that if z is purely imaginary, that is, $z = iy$ with $y \in \mathbb{R}$, then

$$e^{iy} = \cos y + i \sin y.$$

This is Euler's identity. [Hint: Use power series.]

(d) More generally,

$$e^{x+iy} = e^x(\cos y + i \sin y)$$

whenever $x, y \in \mathbb{R}$, and show that

$$|e^{x+iy}| = e^x.$$

(e) Prove that $e^z = 1$ if and only if $z = 2\pi k i$ for some integer k.

(f) Show that every complex number $z = x + iy$ can be written in the form

$$z = re^{i\theta},$$

where r is unique and in the range $0 \le r < \infty$, and $\theta \in \mathbb{R}$ is unique up to an integer multiple of 2π. Check that

$$r = |z| \quad \text{and} \quad \theta = \arctan(y/x)$$

whenever these formulas make sense.

(g) In particular, $i = e^{i\pi/2}$. What is the geometric meaning of multiplying a complex number by i? Or by $e^{i\theta}$ for any $\theta \in \mathbb{R}$?

(h) Given $\theta \in \mathbb{R}$, show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

These are also called Euler's identities.

(i) Use the complex exponential to derive trigonometric identities such as

$$\cos(\theta + \vartheta) = \cos\theta\cos\vartheta - \sin\theta\sin\vartheta,$$

and then show that

$$\begin{aligned} 2\sin\theta\sin\varphi &= \cos(\theta - \varphi) - \cos(\theta + \varphi), \\ 2\sin\theta\cos\varphi &= \sin(\theta + \varphi) + \sin(\theta - \varphi). \end{aligned}$$

This calculation connects the solution given by d'Alembert in terms of traveling waves and the solution in terms of superposition of standing waves.

5. Verify that $f(x) = e^{inx}$ is periodic with period 2π and that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Use this fact to prove that if $n, m \geq 1$ we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \, \cos mx \, dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & n = m, \end{cases}$$

and similarly

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \, \sin mx \, dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & n = m. \end{cases}$$

Finally, show that

$$\int_{-\pi}^{\pi} \sin nx \, \cos mx \, dx = 0 \qquad \text{for any } n, m.$$

[Hint: Calculate $e^{inx}e^{-imx} + e^{inx}e^{imx}$ and $e^{inx}e^{-imx} - e^{inx}e^{imx}$.]

6. Prove that if f is a twice continuously differentiable function on $\mathbb{R}$ which is a solution of the equation

$$f''(t) + c^2 f(t) = 0,$$

then there exist constants a and b such that

$$f(t) = a\cos ct + b\sin ct.$$

This can be done by differentiating the two functions $g(t) = f(t)\cos ct - c^{-1}f'(t)\sin ct$ and $h(t) = f(t)\sin ct + c^{-1}f'(t)\cos ct$.

7. Show that if a and b are real, then one can write

$$a\cos ct + b\sin ct = A\cos(ct - \varphi),$$

where $A = \sqrt{a^2 + b^2}$, and φ is chosen so that

$$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin\varphi = \frac{b}{\sqrt{a^2+b^2}}.$$

8. Suppose F is a function on (a, b) with two continuous derivatives. Show that whenever x and $x + h$ belong to (a, b), one may write

$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi(h)\,,$$

where $\varphi(h) \to 0$ as $h \to 0$.

Deduce that

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} \to F''(x) \qquad \text{as } h \to 0.$$

[Hint: This is simply a Taylor expansion. It may be obtained by noting that

$$F(x+h) - F(x) = \int_x^{x+h} F'(y)\, dy,$$

and then writing $F'(y) = F'(x) + (y-x)F''(x) + (y-x)\psi(y-x)$, where $\psi(h) \to 0$ as $h \to 0$.]

9. In the case of the plucked string, use the formula for the Fourier sine coefficients to show that

$$A_m = \frac{2h}{m^2} \frac{\sin mp}{p(\pi - p)}.$$

For what position of p are the second, fourth, ... harmonics missing? For what position of p are the third, fifth, ... harmonics missing?

10. Show that the expression of the Laplacian

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is given in polar coordinates by the formula

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Also, prove that

$$\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2.$$

11. Show that if $n \in \mathbb{Z}$ the only solutions of the differential equation

$$r^2 F''(r) + rF'(r) - n^2 F(r) = 0,$$

which are twice differentiable when $r > 0$, are given by linear combinations of r^n and r^{-n} when $n \neq 0$, and 1 and $\log r$ when $n = 0$.

[Hint: If F solves the equation, write $F(r) = g(r)r^n$, find the equation satisfied by g, and conclude that $rg'(r) + 2ng(r) = c$ where c is a constant.]

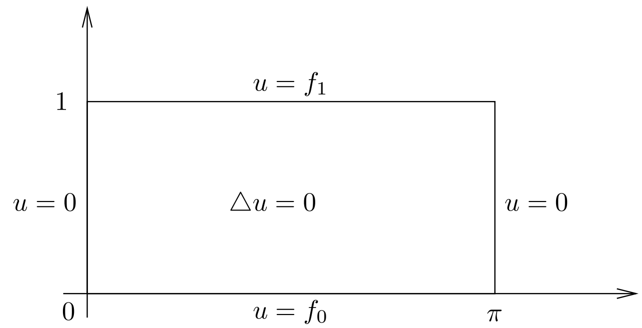

Figure 11. Dirichlet problem in a rectangle

4 Problem

1. Consider the Dirichlet problem illustrated in Figure 11.

More precisely, we look for a solution of the steady-state heat equation $\triangle u = 0$ in the rectangle $R = \{(x, y) : \ 0 \le x \le \pi, \quad 0 \le y \le 1\}$ that vanishes on the vertical sides of R, and so that

$$u(x, 0) = f_0(x) \quad \text{and} \quad u(x, 1) = f_1(x)\,,$$

where f_0 and f_1 are initial data which fix the temperature distribution on the horizontal sides of the rectangle.

Use separation of variables to show that if f_0 and f_1 have Fourier expansions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \quad \text{and} \quad f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx,$$

then

$$u(x, y) = \sum_{k=1}^{\infty} \left(\frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx.$$

We recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Compare this result with the solution of the Dirichlet problem in the strip obtained in Problem 3, Chapter 5.