CHAPTER ONE

The Confluence of Nature and Mathematical Modeling

> Great are the works of the Lord; they are pondered by all who delight in them.
>
> —Psalm 111:2

CONFLUENCE . . .

In recent years, as I have walked daily to and from work, I have started to train myself to observe the sky, the birds, butterflies, trees, and flowers, something I had not done previously in a conscious way (although I did watch out for fast-moving cars and unfriendly dogs). Despite living in suburbia, I find that there are many wonderful things to see: clouds exhibiting wave-like patterns, splotches of colored light some 22 degrees away from the sun (sundogs, or parhelia), wave after wave of Canada geese in "vee" formation, the way waves (and a following region of calm water) spread out on the surface of a puddle as a raindrop spoils its smooth surface, the occasional rainbow arc, even the iridescence on the neck of those rather annoying birds, pigeons, and many, many more nature-given delights. And so far I have not been late for my first class of the morning!

The idea for this book was driven by a fascination on my part for the way in which so many of the beautiful phenomena observable in the natural realm around us can be described in mathematical terms (at least in principle). What are some of these phenomena? Some have been already mentioned in the preface, but for a more complete list we might consider rainbows, "glories," halos (all atmospheric occurrences), waves in air, earth, oceans, rivers, lakes, and puddles (made by wind, ship, or duck), cloud formations (billows, lee waves), tree and leaf branching patterns (including phyllotaxis), the proportions of trees, the wind in the trees, mud-crack patterns, butterfly markings, leopard spots, and tiger stripes. In short, if you can see it outside, and a human didn't make it, it's probably described here! That, of course, is an exaggeration, but this book does attempt to answer on varied levels the fundamental question: what kind of scientific and mathematical principles undergird these patterns or regularities that I claim are so ubiquitous in nature?

Two of the most fundamental and widespread phenomena that occur in the realm of nature are the scattering of light and wave motion. Both may occur almost anywhere given the right circumstances, and both may be described in mathematical terms at varying levels of complexity. It is, for example, the scattering of light both by air molecules and by the much larger dust particles (or more generally, aerosols) that gives the amazing range of color, hues, and tints at sunrise or sunset that give us so much pleasure. The deep blue sky above and the red glow near the sun at the end of the day are due to molecular scattering of light, though dust or volcanic ash can render the latter quite spectacular at times.

The rainbow is formed by sunlight scattered in preferential directions by near-spherical raindrops: scattering in this context means refraction and reflection (although there many other fascinating features of light scattering that will not be discussed in great detail here). Using a simple mathematical description of this phenomenon, René Descartes in 1637 was able to "hang the rainbow in the sky" (i.e., deduce its location relative to the sun and observer), but to "paint" the rainbow required the genius of Isaac Newton some thirty years later. The bright primary and fainter secondary bows are well described by elementary mathematics, but the more subtle observable features require some of the most sophisticated techniques of mathematical physics to explain them. A related phenomenon is that of the "glory," the set of colored, concentric rainbow-like rings surrounding, for example, the shadow of an airplane on a cloud below. This, like the rainbow, is also a "backscatter" effect, and, intriguingly, both the rainbow and the glory have their counterparts in atomic and nuclear physics; mathematics is a unifying feature between these two widely differing contexts. The beautiful (and commonly circular) arcs known as halos, no doubt seen best in arctic climes, are formed by the refraction of sunlight through ice crystals of various shapes in the upper atmosphere. Sundogs, those colored splashes of light often seen on both sides of the sun when high cirrus clouds are present, are similarly formed.

Like the scattering of light, wave motion is ubiquitous, though we cannot always see it directly. It is manifested in the atmosphere, for example, by billow clouds and lee-wave clouds downwind from a hill or mountain. Waves on the surface of puddles, ponds, lakes, or oceans are governed by mathematical relationships between their speed, their wavelength, and the depth of the water. The wakes produced by ships or ducks generate strikingly similar patterns relative to their size; again, this correspondence is described by mathematical expressions of the physical laws that govern the motion. The situation is even more complex in the atmosphere: the "compressible" nature of a gas renders other types of wave motion possible. Sand dunes are another complex and beautiful example of waves. They can occur on a scale of centimeters to kilometers, and, like surface waves on bodies of water, it is only the waveform that actually moves; the body of sand is stationary (except at the surface).

In the plant world, the arrangement of leaves around a stem or seeds in a sunflower or daisy face shows, in the words of one mathematician (H.S.M. Coxeter), "a fascinatingly prevalent tendency" to form recurring numerical patterns, studied since medieval times. Indeed, these patterns are intimately linked with the "golden number" ($(1 + \sqrt{5})/2 \approx 1.618$) so beloved of Greek mathematicians long ago. The spiral arrangement of seeds in the daisy head is found to be present in the sweeping curve of the chambered nautilus shell and on its helical counterpart, the *Cerithium fasciatum* (a thin, pointy shell). The curl of a drying fern and the rolled-up tail of a chameleon all exhibit types of spiral arc.

In the animal and insect kingdoms, coat patterns (e.g., on leopards, cheetahs, tigers, and giraffes) and wing markings (e.g., on butterflies and moths) can be studied using mathematics, specifically by means of the properties and solutions of so-called reaction-diffusion equations (and other types of mathematical models). Reaction-diffusion equations describe the interactions between chemicals ("activators" and "inhibitors") that, depending on conditions, may produce spots, stripes, or more "splodgy" patterns. There are fascinating mathematical problems involved in this subject area, and also links with topics such as patterns on fish (e.g., angel fish) and seashells. In view of earlier comments, seashells combine both the effects of geometry and pattern formation mechanisms, and mathematical models can reproduce the essential features observed in many seashells.

Cracks also, whether formed in drying mud, tree bark, or rapidly cooling rock, have their own distinctive mathematical patterns; frequently they are hexagonal in nature. River meanders, far from being "accidents" of nature, define a form in which the river does the least work in turning (according to one class of models), which then defines the most probable form a river can take—no river, regardless of size, runs straight for more than ten times its average width.

Many other authors have written about these patterns in nature. Ian Stewart has noted in his popular book *Nature's Numbers* that "We live in a universe of patterns. . . . No two snowflakes appear to be the same, but all possess six-fold symmetry." Furthermore, he states that

> there is a formal system of thought for recognizing, classifying and exploiting patterns. . . . It is called mathematics. Mathematics helps us to organize and systemize our ideas about patterns; in so doing, not only can we admire and enjoy these patterns, but also we can use them to infer some of the underlying principles that govern the world of nature. . . . There is much beauty in nature's clues, and we can all recognize it without any mathematical training. There is beauty too in the

> mathematical stories that . . . deduce the underlying rules and regularities, but it is a different kind of beauty, applying to ideas rather than things. Mathematics is to nature as Sherlock Holmes is to evidence.

We may go further by asking questions like those posed by Peter S. Stevens in his lovely book *Patterns in Nature*. He asks,

> Why does nature appear to use only a few fundamental forms in so many different contexts? Why does the branching of trees resemble that of arteries and rivers? Why do crystal grains look like soap bubbles and the plates of a tortoise shell? Why do some fronds and fern tips look like spiral galaxies and hurricanes? Why do meandering rivers and meandering snakes look like the loop patterns in cables? Why do cracks in mud and markings on a giraffe arrange themselves like films in a froth of bubbles?

He concludes in part that "among nature's darlings are spirals, meanders, branchings, hexagons, and 137.5 degree angles. . . . Nature's productions are shoestring operations, encumbered by the constraints of three-dimensional space, the necessary relations among the size of things, and an eccentric sense of frugality.

In the book *By Nature's Design*, Pat Murphy expresses similar sentiments, writing,

> Nature, in its elegance and economy, often repeats certain forms and patterns . . . like the similarity between the spiral pattern in the heart of a daisy and the spiral of a seashell, or the resemblance between the branching pattern of a river and the branching pattern of a tree . . . ripples that flowing water leaves in the mud . . . the tracings of veins in an autumn leaf . . . the intricate cracking of tree bark . . . the colorful splashings of lichen on a boulder. . . . The first step to understanding—and one of the most difficult—is to see clearly. Nature modifies and adapts these basic patterns as needed, shaping them to the demands of a dynamic environment. But underlying all the modifications and adaptations is a hidden unity. Nature invariably seeks to accomplish the most with the least—the tightest fit, the shortest path, the least energy expended. Once you begin to see these basic patterns, don't be surprised if your view of the natural world undergoes a subtle shift.

Another fundamental (and philosophical) question has been asked by many—How can it be that mathematics, a product of human thought independent of experience, is so admirably adapted to the objects of reality? This fascinating question I do not address here; let it suffice to note that, hundreds of years ago, Galileo Galilei stated that the Universe "cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language." Mathematics is certainly the language of science, but it is far, far more than a mere tool, however valuable, for it is of course both a subject and a language in its own right. But lest any of us should balk at the apparent need for speaking a modicum of that language in order more fully to appreciate this book, the following reassuring statement from Albert Einstein, when writing to a young admirer at junior high school, should be an encouragement. He wrote "Do not worry about your difficulties in mathematics. I can assure you that mine are still greater." Obviously anyone, even scientists of great genius, can have difficulties in mathematics (one might add that it's all a matter of relativity in this regard).

Obviously a significant component of this book is the application of elementary mathematics to the natural world around us. As I have tried to show already, there are many mathematical patterns in the natural world that are accessible to us if we keep our eyes and ears open; indeed, the act of "asking questions of nature" can lead to many fascinating "thought trails," even if we do not always come up with the correct answers. First, though, let me remind you (unnecessarily, I am sure) that no one has all the answers to such questions. This is true for me at all times, of course (not just as a parent and a professor), but especially so in a subject as all-encompassing as "mathematics in nature." There will always be "displays" or phenomena in nature that any given individual will be unable to explain to the satisfaction of everyone, for the simple reason that none of us is ever in possession of all the relevant facts, physical intuition, mathematical techniques, or other requirements to do justice to the observed event. However, this does not mean that we cannot appreciate the broad principles that are exemplified in a rainbow, lenticular cloud, river meander, mud crack, or animal pattern. Most certainly we can.

It is these broad principles—undergirded by mathematics, much of it quite elementary—that I want us to perceive in a book of this admittedly rather free-ranging nature. My desire is that by asking mathematical questions of the phenomena we will gain both some understanding of the symbiosis that exists between the basic scientific principles involved and their mathematical description, and a deeper appreciation for the phenomenon itself, its beauty (obviously rather subjective), and its relationship to other events in the natural world around us. I have always found, for example, that my appreciation for a rainbow is greatly enhanced by my understanding of the mathematics and physics that undergird it (some of the mathematics can be extremely advanced; some references to this literature are provided in the bibliography). It is important to remember that this is a book on aspects of *applied mathematics*, and there will be at times some more advanced and even occasionally rigorous mathematics (in the form of theorems and sometimes proofs); for the most part, however, the writing style is intended to be informal. And so now, on to

. . . MODELING

An important question to be asked at the outset is *What is a mathematical model?* One basic answer is that it is the formulation in mathematical terms of the assumptions and their consequences believed to underlie a particular "real world" problem. The aim of mathematical modeling is the practical application of mathematics to help unravel the underlying mechanisms involved in, for example, economic, physical, biological, or other systems and processes. Common pitfalls include the indiscriminate, näive, or uninformed use of models, but, when developed and interpreted thoughtfully, mathematical models can provide insight into the nature of the problem, be useful in interpreting data, and stimulate experiments. There is not necessarily a "right" model, and obtaining results that are consistent with observations is only a first step; it does not imply that the model is the only one that applies, or even that it is "correct." Furthermore, mathematical descriptions are not explanations, and never on their own can they provide a complete solution to the biological (or other) problem—often there may be complementary levels of description possible within the particular scientific domain. Collaboration with scientists or engineers is needed for realism and help in modifying the model mechanisms to reflect the science more accurately. On the other hand, workers in nonmathematical subjects need to appreciate what mathematics (and its practitioners) can and cannot do. Inevitably, as always, good communication between the interested parties is a necessary (but not sufficient) recipe for success.

In the preface mention was made of fundamental steps necessary in developing a mathematical model (see figure 1.1): formulating a "real world" problem in mathematical terms using whatever appropriate simplifying assumptions may be necessary; solving the problem thus posed, or at least extracting sufficient information from it; and finally interpreting the solution in the context of the original problem (which as noted above may include validation of the model by testing both its consistency with known data and its predictive capability). Thus the art of good modeling relies on (i) a sound understanding and appreciation of the scientific or other problem; (ii) a realistic mathematical representation of the important phenomena; (iii) finding useful solutions, preferably quantitative ones; and (iv) interpretation of the mathematical results—insights, predictions, and so on. Sometimes the mathematics used can be very simple. The usefulness of a mathematical model should not be judged by the sophistication of the mathematics, but by different (and no less demanding) criteria.

Although techniques of statistical analysis may frequently be used in portraying and interpreting data, it is important to note two fundamentally distinct approaches to mathematical modeling, which differ somewhat in

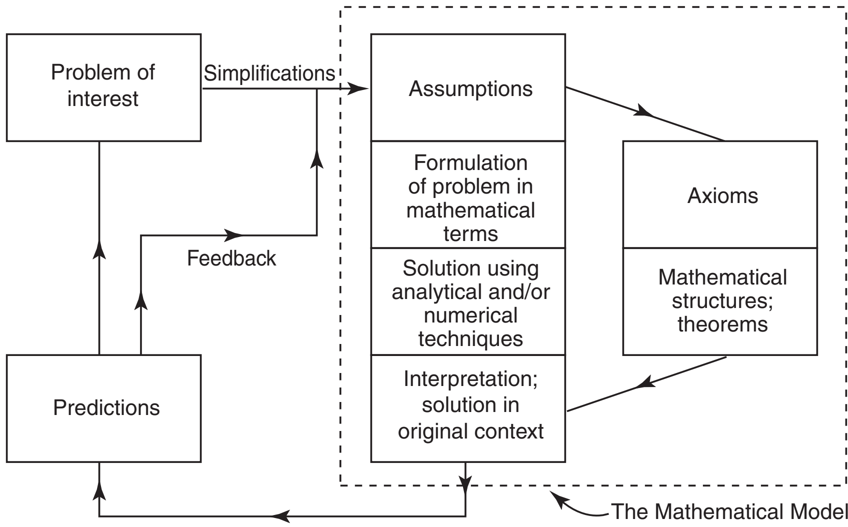

Figure 1.1. A generic flow chart illustrating stages and levels of mathematical modeling. Reprinted with permission of Birkhäuser, copyright 1997, from the book "A Survey of Models for Tumor-Immune System Dynamics," edited by J. A. Adam and N. Bellomo.

both mathematical and philosophical characteristics. These are deterministic models and probabilistic or statistical (sometimes referred to as stochastic) models. One's preference often depends upon the way one has been trained, which usually determines the way one looks at the world mathematically. Most of this book is written from the deterministic perspective. A general summary of the philosophy and methodology of this approach (with many references to applications in cancer biology) may be found in chapter 2 of the book edited by Adam and Bellomo (1997).

Deterministic models frequently possess the property that a system of interest (e.g., a population of cells, a mixture of chemicals or enzymes, a density or pressure imbalance in a gas) can be "observed" mathematically to evolve away from some initial configuration.

Frequently in deterministic models the assumption is made that the dependent variables are at least differentiable functions of their argument(s), and hence continuous. This assumption is quite reasonable when the magnitude of (say) the population of cells is very much larger than a typical change (increase or decrease) in the population. For a small tumor composed of about a billion cells, the individual cell cycles have become asynchronous, and so on the basis of a simple model the evolution of the tumor can be reasonably described using the techniques of differential and integral calculus. The situation is inevitably much more complex than this, and many other factors need to be included to derive a realistic model of tumor growth. Regardless of the complexity, deterministic models may be used to predict the size (or cell number) of the tumor as a function of time, in particular, predicting that the tumor will have $N(t)$ cells at time t. In practice, however, there is an element of uncertainty in every event, from the growth of a tumor to catching a bus. External factors, usually beyond our control, play a role in determining the outcome of the event. Thus diet, fitness, treatment regimen, and mental attitude may contribute to the eventual outcome for the cancer patient, while the faulty alarm clock used by the bus driver may influence whether the bus arrives on time. Similarly, in studying the relation between stature and heredity, differences in environment and nutrition may be sources of uncertainty in the results. Stochastic models enable researchers to identify and study many of these uncertainties.

Note also that in an experiment errors that arise may be classified as random or systematic. The word *error* as used here does not mean *mistake*; it is introduced into the results by external influences beyond the control of the experimenter. The static heard when listening to a radio with poor reception is generally random; the discovery of the 4°K microwave background radiation by Arno Penzias and Robert Wilson in 1964 resulted from *systematic* error (i.e., no matter what they tried, they could not eradicate it); and this in turn resulted in confirmation of a prediction by the scientist George Gamow (and others) in 1948. Systematic errors can lead to new discoveries! Probabilistic or stochastic models incorporate a measure of uncertainty; for example, they predict the *probability* that the tumor will have $N(t)$ cells at time t. Furthermore, if the cell population of interest is rather small, a typical change in cell number may well be a significant fraction of the total population, and so the "state" of the population must be represented, say, by an integer-valued random variable.

It has already been noted that mathematical models are not necessarily "right" (though they may be wrong as a result of ignoring fundamental processes). One model may be better than another in that it has better explanatory features: more specific predictions can be made that are subsequently confirmed, at least to some degree. Some of the models presented in this book are still controversial; in particular, the reaction-diffusion models of pattern formation presented in chapter 14 are not universally accepted by biomathematicians. There are other models, for example in the study of wound healing, that utilize more of the mechanical properties of the medium (skin in this case) and are therefore designated *mechanochemical* models (they are not discussed in this book). Some of the models cited are somewhat elderly or incomplete (or both), examples being those of sand dune formation and river meanders presented in chapters 6 and 12, respectively. Indeed, I venture to suggest that all mathematical models are flawed to some extent: many by virtue of inappropriate assumptions made in formulating the model, or (which may amount to the same thing) by the omission of certain terms in the governing equations, or even by misinterpretation of the mathematical conclusions in the original context of the problem. Occasionally models may be incorrect because of errors in the mathematical analysis, even if the underlying assumptions are valid. And, paradoxically, it can happen that even a less accurate model is preferable to a more mathematically sophisticated one; it was the mathematical statistician John Tukey who stated that "it is better to have an approximate answer to the right question than an exact answer to the wrong one."

This is well illustrated by Lee and Fraser in their comparison of the less accurate Airy theory of the rainbow with the more general and powerful Mie theory. They write, "Our point here is not that the exact Mie theory describes the natural rainbow inadequately, but rather that the approximate Airy theory can describe it quite well. Thus the supposedly outmoded Airy theory generates a more natural-looking map of real rainbow colors than Mie theory does, even though Airy theory makes substantial errors in describing the scattering of monochromatic light by isolated small drops. *As in many hierarchies of scientific models, the virtues of a simpler theory can, under the right circumstances, outweigh its vices*" (italics added).

With such provisos in mind, the aim of this book is *not* to present thorough mathematical descriptions of many naturally occurring phenomena—an impossible task—but instead to try to present a compilation and synthesis of several mathematical models that have been developed within these contexts. The extensive set of references in the bibliography is provided to encourage the inquisitive reader to pursue the original articles and books from which these models were first presented. There are undoubtedly many published papers on these topics of which I am unaware, and which certainly would have enhanced this book, and for that I must point out, regrettably, that one has to stop somewhere (the publisher requires it).

The organization of the book is as follows. Chapters 2 and 3 constitute a rather gentle introduction to the importance and usefulness of estimation (chapter 2) and the problem of shape, size, and scale, plus an introduction to the methods of dimensional analysis (chapter 3). It is important to note that most of the material in these two chapters is mathematically "fuzzy"; the conclusions drawn are not exact, and cannot be. They are intended to provide the reader with some guidelines, domains of validity, and basic principles to be borne in mind when constructing the next level of model, by which is meant a rather more sophisticated mathematical approach to the problem of interest, assuming this is appropriate. However, there may on occasion be specific counterexamples to the conclusions drawn, of the genre "My uncle smoked two packs of cigarettes every day for seventy years, and he never developed cancer"; "That tree must have had at least 10^5 leaves:

I just raked them all up"; or, perhaps more relevant to chapter 3, "We owned a horse that could beat any dog running uphill." Obviously such anecdotal comments, while perhaps true, do not vitiate broad conclusions based on solid statistical evidence (in the case of cigarette smoking) or general principles of bioengineering, provided that in each case we operate within the domains of validity of the underlying assumptions and procedures. The reader is reminded of the Tukey quote earlier in this section.

The next three chapters (4–6) are of a meteorological character and include straightforward mathematical (and physical) descriptions of shadow-related phenomena, rainbows, halos, mirages, and some aspects of clouds, hurricanes, and sand dunes. Chapters 7–9 are fluid dynamical in nature, introducing aspects of linear and nonlinear wave motion, respectively (chapters 7 and 9), separated by a chapter dealing with the "other side of the linear coin," so to speak, namely instability in some of its various forms. Chapter 10 examines some of the many properties of the golden ratio—an irrational number with fascinating connections to the plant world in particular. The topics of chapter 11 include honeycombs, soap bubbles (and foam), and mud cracks; they are loosely connected by the continual search for optimal solutions and principles of minimization that has been a major theme in mathematical research for millennia.

A not-unrelated theme meanders through part of chapter 12: river meanders and branching patterns, followed by some arboreal mathematical models—the application of engineering principles to establish the height/width relationship for a generic tree, and the "murmur" of the forest. Basic principles of bird flight are discussed in chapter 13, as is the underlying fluid dynamical theorem of Bernoulli. The final chapter contains selected models of pattern formation based on an examination of the diffusion equation (and related equations). Applications of these models to animal, insect, and seashell patterns and plankton blooms are briefly considered, followed by some applications of the diffusion equation that are primarily of historical interest. A short appendix on fractals is designed to whet the reader's appetite for more.

APPENDIX: A MATHEMATICAL MODEL OF SNOWBALL MELTING

The purpose of this appendix is to illustrate some features of mathematical modeling by means of a simple (some would say silly) example. Before doing so, however, readers may find helpful the following quotation from the above-mentioned book, *Wind Waves* by Blair Kinsman. It identifies both the importance and the ubiquity of assumptions made in the process of mathematical modeling.

> In the derivation to follow, it will be *assumed* that the Earth is flat, the water is of constant density, that the Coriolis force is negligible, that the density of the air can be neglected compared with the density of water, that the body of water is of infinite extent and completely covered by waves, and that viscosity and surface tension can be neglected. Moreover, *to simplify the problem*, it will be *assumed* that there is no variation in the wave properties in the y-direction. A few more *assumptions* will turn up along the way in deriving the equations to be solved, but the equations will still be unsolvable in closed form as they will be nonlinear. (italics added)

The following simplistic problem is expressed in the form of a question and an unhelpful answer to illustrate the fact that many problems we might wish to model may be, at best, ill-defined. It is a case study that may be helpful to the reader in revealing some of the more intuitive aspects of mathematical modeling.

Q: Half of a snowball melts in an hour. How long will it take for the remainder to melt?
A: I don't know.

Why may such a response be justified, at least in part? Because the question stated is not a precise one; it is ambiguous. Half of what? The mass of the snowball or its volume? Under what kinds of assumptions can we formulate a mathematical model and will it be realistic? This type of problem is often posed in "Calculus I" textbooks, and as such requires only a little basic mathematical material, for example, the chain rule and elementary integration. However, it is what we do with all this that makes it an interesting and informative exercise in mathematical modeling. There are several reasonable assumptions that can be made in order to formulate a model of snowball melting; however, unjustifiable assumptions are also a possibility! The reader may consider some or all of these to be in the latter category, but ultimately the test of a model is how well it fits known data and predicts new phenomena. The model here is less ambitious (and not a particularly good one either), for we merely wish to illustrate how one might approach the problem. It can lead to a good discussion in the classroom setting, especially during the winter. Some plausible assumptions (and the questions they generate) might be as follows:

i. Assume that the snowball is a sphere of radius $r(t)$ at all times. This is almost certainly never the case, but the question becomes one of simplicity. Is the snowball roughly spherical initially? Subsequently? Is there likely to be preferential warming and melting on one side even if it starts life as a sphere? The answer to this last question is yes: preferential melting will probably occur in the direction of direct sunlight unless the snowball is in the shade or the sky is uniformly overcast. If we can make this assumption, then the resulting surface area and volume considerations involve only the one spatial variable r.

ii. Assume that the density of the snow/ice mixture is constant throughout the snowball, so there are no differences in "snow-packing." This may be reasonable for small snowballs (hand-sized ones), but large ones formed by rolling will probably become more densely packed as their weight increases. A major advantage of the constant density assumption is that the mass (and weight) of the snowball is then directly proportional to its volume.

iii. Assume that the mass of the snowball decreases at a rate proportional to its surface area, and only this. This appears to make sense since it is the outside surface of the snowball that is in contact with the warmer air that induces melting. In other words, the transfer of heat occurs at the surface. This assumption in particular will be examined in the light of the model's prediction. But even if it is a good assumption to make, is the "constant" of proportionality really constant? Might it not depend on the humidity of the air, the angle of incidence and intensity of sunlight, the external temperature, and so on?

iv. Assume that no external factors change during the "lifetime" of the snowball. This is related to assumption (iii), and is probably the weakest of them all; unless the melting time is very much less than a day it is safe to say that external factors will vary! Obviously the angle and intensity of sunlight will change over time, and possibly other factors as noted above. Let us proceed, nevertheless, on the basis of these four assumptions, and formulate a model by examining some of the mathematical consequences of these assumptions. We may do so by asking further questions, for example:

i. What are expressions for the mass, volume, and surface area of the snowball?
ii. How do we formulate the governing equations? What are the appropriate initial and/or boundary conditions? How do we incorporate the information provided?
iii. Can we obtain a solution (analytic, approximate, or numerical) of the equations?
iv. What is the physical interpretation of the solution and does it make sense? That is, is it consistent with the information provided and are the predictions from the model reasonable?
v. Does a unique solution exist?

We will answer questions (i)–(iv) first, and briefly comment on (v) at the end. It is an important question that is of a more theoretical nature than the rest, and its consequences are far reaching for models in general. Let the initial radius of the snowball be $r(0) = R$. If we denote the uniform density of the snowball by ρ, its mass by $M(t)$, and its volume by $V(t)$, and measure time t in hours, then the mass of the snowball at any time t is

$$M(t) = \frac{4}{3}\pi\rho r^3(t). \tag{1}$$

It follows that the instantaneous rate of change of mass or time derivative is

$$\frac{dM}{dt} = 4\pi\rho r^2 \frac{dr}{dt}. \tag{2}$$

By assumption (iii)

$$\frac{dM}{dt} = -4\pi r^2 k, \tag{3}$$

where k is a positive constant of proportionality, the negative sign implying that the mass is *decreasing* with time! By equating the last two expressions it follows that

$$\frac{dr}{dt} = -\frac{k}{\rho} = -\alpha, \text{ say.} \tag{4}$$

Thus, according to this model, the radius of the snowball decreases uniformly with time. Upon integrating this differential equation and invoking the initial condition we obtain

$$r(t) = R - \alpha t = R\left(1 - \frac{t}{t_m}\right) = 0 \quad \text{when } t = \frac{R}{\alpha} = t_m, \tag{5}$$

where t_m is the time for the original snowball to melt, which occurs when its radius is zero! We do not know the value of α, since that information was not provided, but we *are* informed that after one hour half the snowball has melted, so we have from equation (5) that $r(1) = R - \alpha$. A sketch of the linear equation in (5) and use of similar triangles in figure 1.2a shows that

$$t_m = \frac{R}{R - r(1)}$$

and furthermore

$$\frac{V(1)}{V(0)} = \frac{1}{2} = \frac{r^3(1)}{R^3},$$

so that

$$r(1) = 2^{-1/3}R \approx 0.79R.$$

Hence $t_m \approx 4.8$ hours, so that according to this model the snowball will take a little less than 4 more hours to melt away completely. This is a rather long time, and certainly the sun's position will have changed during that

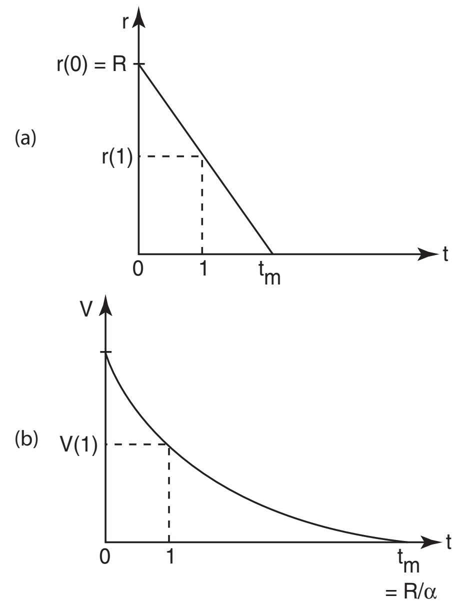

Figure 1.2. The time dependence of the radius $r(t)$ and volume $V(t)$ for the model of snowball melting.

time (through an arc of roughly 60°), so in retrospect assumption (iv) is not really justified. A further implication of equation (5) is that both the volume and mass of the snowball (by assumption (ii)) decrease like a cubic polynomial in t, i.e.,

$$V(t) = V(0)\left(1 - \frac{t}{t_m}\right)^3.$$

(See figure 1.2b). Note that $V'(t) < 0$ as required, and $V'(t_m) = 0$. Since $V''(t) > 0$ it is clear that the snowball melts more quickly at first, when $|V'|$ is larger, than at later times, as the value for t_m attests. I recall being told as a child by my mother that "snow waits around for more," but this model is hardly a "proof" of that, despite further revelations below! It may be adequate under some circumstances, but there are obvious deficiencies given the initial "data" (which to be honest, I invented). What other factors have been ignored here? Here are some:

We are all familiar with the fact that the consistency of snow varies depending on whether it is "wet" or "dry"; snowballs are more easily made with the former (at least, I have found it to be so). Wet snow can be packed more easily and a layer of ice may be formed on the outside. This can in turn cool a thin layer of air around the surface, which will insulate (somewhat) the snowball from the warmer air beyond that. A nice clean snowball, as opposed to one made with dirty snow, may be highy reflective of sunlight (it has a high *albedo*), and this will reduce the rate of melting further. There are no doubt several other factors missing.

Some other aspects of the model are more readily appreciated if we generalize the original problem by suggesting instead that "a fraction β of a snowball melts in h hours . . . ". The melting time is then found to be

$$t_m = \frac{h}{1 - \sqrt[3]{1-\beta}}, \tag{6}$$

which depends linearly on h and in a monotonically decreasing manner on β. The dependence on h is not surprising; if a given fraction β melts in half the time, the total melting time is also halved. For a given value of h, the dependence on β is also plausible: the larger the fraction that melts in time h, the shorter the melting time.

A final point concerns question (v) on the existence and uniqueness of the solution to this mathematical model. While the existence of a solution is clear in this rather trivial example, it is certainly of fundamental importance in general terms, as is the uniqueness (or not) of a solution and the *stability* of such solution(s) to small variations in the initial and/or boundary conditions. Such considerations are outside the scope and theme of this book, but the interested reader can find information on these topics by consulting many undergraduate and most graduate texts on ordinary and partial differential equations. For completeness, we state and apply the relevant theorem for this particular example, namely:

> **Theorem:** there exists a unique solution to the ordinary differential equation
>
> $$\frac{dy}{dx} = f(x,y)$$
>
> satisfying the initial condition
>
> $$y(x_0) = y_0,$$
>
> provided that both the function $f(x,y)$ and its partial derivative $\partial f/\partial y$ are continuous functions of x and y in a neighborhood of the initial point (x_0, y_0).

In many cases of practical interest, these continuity conditions are satisfied for all values of (x_0, y_0).

So what has this to do with our snowball? Everything! Our problem has reduced to

$$\frac{dr}{dt} = f(r,t) \equiv -\alpha, \quad r(0) = r_0 \equiv R,$$

where α is a positive constant. Since constants and their derivatives are continuous everywhere (!), the theorem applies, and hence a unique solution exists for the problem as posed. Of course, like the familiar saying about snowflakes, the snowball is probably unique as well.

As a final comment, it should be pointed out that in a majority of the mathematical models that follow in this book, the assumptions and their consequences will not be formally laid out as they have been in the above pedagogic example. Indeed, that organizational style is not generally used (in the author's experience) even in the modeling literature.