1
Introduction to Bond Markets

1.1 Bonds

A *bond* is a securitized form of loan.

The buyer of a bond lends the issuer an initial price P in return for a predetermined sequence of payments. These payments may be fixed in nominal terms (a fixed-interest bond) or the payments may be linked to some index (an index-linked bond): for example, the consumer (or retail) prices index.

In the UK, government bonds are called *gilt-edged securities* or *gilts* for short. In other countries they have other names such as *treasury bills* or *treasury notes* (both USA). Since government bonds are securitized, they can be traded freely in the stock market. Additionally, the main government bond markets are very liquid because of the large amount of stock in issue and the relatively small number of stocks in issue. For example, in the UK the number of gilts is less than 100 and the total value of the gilts market is about £300 billion.

Bonds are also issued by institutions other than national governments, such as regional governments, banks and companies (the latter giving rise to the name *corporate bonds*). Bonds that have identical characteristics but are sold by different issuers may not have the same price. For example, consider two bonds that have a term of 20 years and pay a coupon of 6% per annum payable twice yearly in arrears. One is issued by the government and the other by a company. The bond issued by the company will probably trade at a lower price than the government bond because the market makers will take into account the possibility of default on the coupon payments or on the redemption proceeds. In countries such as the USA and the UK it is generally assumed that government bonds are default free, whereas corporate bonds are subject to varying degrees of default risk depending upon the financial health of the issuing company.

We can see something of this in Figure 1.1. Germany, France and Italy all operate under the umbrella of the euro currency.[1] With standardized bond terms and in

[1] The group of European countries that use the euro currency are officially referred to as the *Eurozone*. However, market practitioners and others often refer to *Euroland*.

the absence of credit risk the three yield curves should coincide. Some differences (especially between Germany and France) may be down to differences in taxation and the terms of the contracts. Italy lies significantly above Germany and France, however, and this suggests that the international bond market has reduced prices to take account either of a perceived risk of default by the Italian government or that Italy will withdraw from the euro.

1.2 Fixed-Interest Bonds

1.2.1 Introduction

We will concentrate in this book on fixed-interest government bonds that have no probability of default.

The structure of a default-free, fixed-interest bond market can generally be characterized as follows. We pay a price P for a bond in return for a stream of payments $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$ from now respectively. The amounts of the payments are fixed at the time of issue.

For this bond, with a nominal value of 100, normally:

$$
\begin{aligned}
g &= \text{coupon rate per 100 nominal;}\\
n &= \text{number of coupon payments;}\\
\Delta t &= \text{fixed time between payments}\\
&\qquad\qquad \text{(equal to 0.5 years for UK and USA government bonds);}\\
t_1 &= \text{time of first payment } (t_1 \leqslant \Delta t);\\
t_j &= t_{j-1} + \Delta t \text{ for } j = 2, \ldots, n;\\
t_n &= \text{time to redemption;}\\
c_1 &= \begin{cases} g\Delta t & \text{normally the first coupon or interest payment,}\\ 0 & \text{if the security has gone ex-dividend;}\end{cases}\\
c_j &= g\Delta t \text{ for } j = 2, \ldots, n-1 \text{ (i.e. subsequent interest payments);}\\
c_n &= 100 + g\Delta t \text{ (final interest payment plus return of nominal capital).}
\end{aligned}
$$

Some markets also have irredeemable bonds (that is, $n = \infty$), but these bonds tend to trade relatively infrequently making quoted prices less reliable. Furthermore, they often have option characteristics which permit early redemption at the option of the government.

Details of government bond characteristics (Δt and ex-dividend rules) in many countries are given in Brown (1998).

1.2.2 Clean and Dirty Prices

Bond prices are often quoted in two different forms.

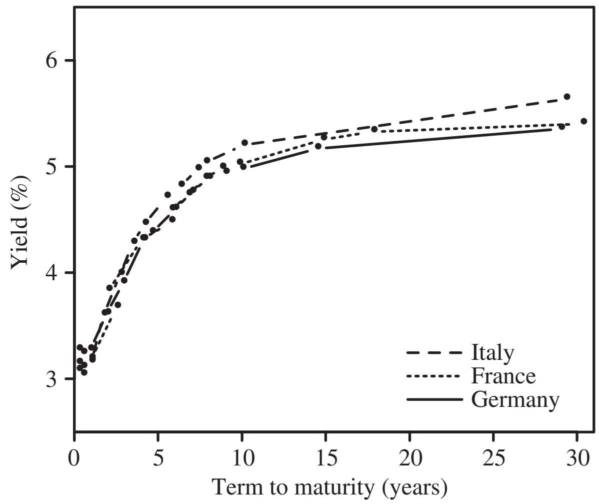

Figure 1.1. Benchmark yield curves for Germany, France and Italy on 3 January 2002.

The *dirty* price is the actual amount paid in return for the right to the full amount of each future coupon payment and the redemption proceeds. If the bond has gone ex-dividend, then the dirty price will give the buyer the right to the full coupon payable in just over six months (assuming twice-yearly coupon payments) but not the coupon due in a few days. As a consequence, the dirty price of a bond will drop by an amount approximately equal to a coupon payment at the time it goes ex-dividend. In addition, the dirty price of a bond will (everything else in the market being stable) rise steadily in between ex-dividend dates.

The *clean* price is an artificial price which is, however, the most-often-quoted price in the marketplace. It is equal to the dirty price minus the *accrued interest*. The accrued interest is equal to the amount of the next coupon payment multiplied by the proportion of the current inter-coupon period so far elapsed (according to certain conventions regarding the number of days in an inter-coupon period).[2] The popularity of the clean price relies on the fact that it does not jump at the time a bond goes ex-dividend, nor does it vary significantly (everything else in the market being stable) in between ex-dividend dates.

The evolution of clean and dirty prices relative to one another can be seen in Figure 1.2. In (a) market interest rates are left constant, demonstrating the sawtooth effect on dirty prices and the relative stability of clean prices. Randomness in interest rates creates volatility in both clean and dirty prices, although the same relationship between the two sets of prices is still quite clear.

[2] Further details for individual countries can be found in Brown (1998).

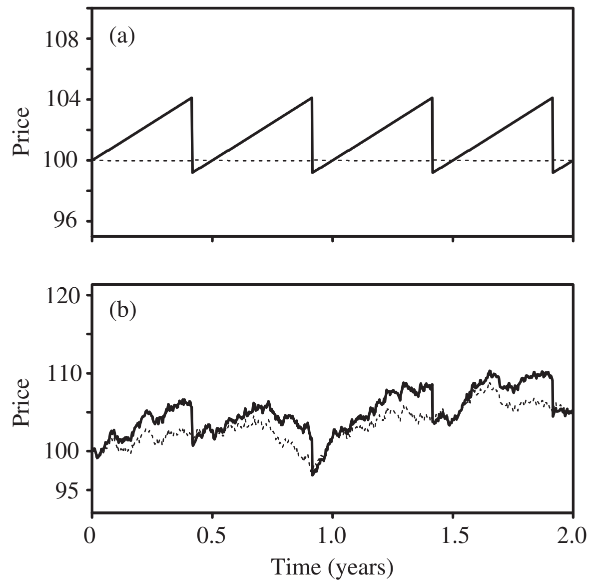

Figure 1.2. Evolution of clean (dotted line) and dirty (solid line) prices for a coupon bond over time. Coupon rate 10%. Maturity date $T = 10$. Coupons payable half-yearly. Ex-dividend period 30 days. (a) Constant market interest rates (10%). (b) Stochastic market interest rates.

1.2.3 Zero-Coupon Bonds

This type of bond has a coupon rate of zero and a nominal value of 1.

We will denote the price at time t of a zero-coupon bond that matures at time T by $P(t, T)$. In places we will call it the T-zero-coupon bond or T-bond for short.

Note that the value of 1 due immediately is, of course, 1, i.e. $P(t, t) = 1$ for all t.

Arbitrage considerations also indicate that $P(t, T) \leqslant 1$ for all T.

1.2.4 Spot Rates

The spot rate at time t for maturity at time T is defined as the yield to maturity of the T-bond:

$$R(t, T) = -\frac{\log P(t, T)}{T - t};$$

that is,

$$P(t, T) = \exp[-(T - t)R(t, T)].$$

The spot rate, $R(t, T)$, is interpreted in the following way. If we invest £1 at time t in the T-bond for $T - t$ years, then this will accumulate at an average rate of $R(t, T)$ over the whole period.

1.2.5 Forward Rates

The forward rate at time t (continuously compounding) which applies between times T and S ($t \leqslant T < S$) is defined as

$$F(t, T, S) = \frac{1}{S - T} \log \frac{P(t, T)}{P(t, S)}. \tag{1.1}$$

The forward rate arises within the terms of a forward contract. Under such a contract we agree at time t that we will invest £1 at time T in return for $e^{(S-T)F(t,T,S)}$ at time S. In other words we are fixing the rate of interest between times T and S in advance at time t.

The following simple no-arbitrage argument shows that the value of this contract must be zero. The forward contract imposes cashflows of -1 at time T and $+e^{(S-T)F(t,T,S)}$ at time S. By definition, this contract must have a value of zero at the time the contract is struck, time t, provided $F(t, T, S)$ is the fair forward rate.

We will now argue that the fair forward rate must be as defined in equation (1.1). Suppose that this is not true and that

$$F(t, T, S) > (S - T)^{-1} \log[P(t, T)/P(t, S)].$$

Then we could set up the following portfolio at time t: one forward contract (value zero at t by definition); +1 units of the T-bond; $-P(t, T)/P(t, S)$ units of the S-bond. The total cost of this portfolio at t is zero. This portfolio—if held to maturity of the respective contracts—will produce a net cashflow of zero at time T and $e^{(S-T)F(t,T,S)} - P(t, T)/P(t, S) > 0$ at time S. This is an example of an arbitrage: we have started with a portfolio worth zero at t and have a sure profit at S. Throughout this book we will assume that arbitrage opportunities like this do not exist. It follows that we cannot have $F(t, T, S) > (S-T)^{-1} \log[P(t, T)/P(t, S)]$. Equally, we cannot have $F(t, T, S) < (S - T)^{-1} \log[P(t, T)/P(t, S)]$ by constructing the reverse portfolio.

In summary, the forward rate $F(t, T, S)$ must satisfy equation (1.1) if we assume no arbitrage. This is an important example of a case where a price or relationship can be determined independently of the interest rate model being employed.

If the exercise date T for the forward contract is in fact equal to t, then the forward and spot rates must be equal, that is, $F(t, t, S) = R(t, S)$.

The *instantaneous* forward-rate curve (or just forward-rate curve) at time t is, for $T > t$,

$$\begin{aligned} f(t, T) &= \lim_{S \to T} F(t, T, S) = -\frac{\partial}{\partial T} \log P(t, T) = -\frac{\partial P(t, T)/\partial T}{P(t, T)} \\ \Rightarrow \quad P(t, T) &= \exp\left[-\int_t^T f(t, u)\, du\right]. \end{aligned}$$

In other words, we can make a contract at time t to earn a rate of interest of $f(t, T)$ per time unit between times T and $T + \mathrm{d}t$ (where $\mathrm{d}t$ is very small). This is, of course, a rather artificial concept. However, it is introduced for convenience as bond-price modelling is carried out much more easily with the instantaneous forward-rate curve $f(t, T)$ than the more cumbersome $F(t, S, T)$.

Arbitrage considerations indicate that $f(t, T)$ must be positive for all $T \geqslant t$. Hence $P(t, T)$ must be a decreasing function of T.

1.2.6 Risk-Free Rates of Interest and the Short Rate

$R(t, T)$ can be regarded as a risk-free rate of interest over the fixed period from t to T. When we talk about *the* risk-free rate of interest we mean the *instantaneous* risk-free rate:

$$r(t) = \lim_{T \to t} R(t, T) = R(t, t) = f(t, t).$$

The easiest way to think of $r(t)$ is to regard it as the rate of interest on a bank account: this can be changed on a daily basis by the bank with no control on the part of the investor or bank account holder. $r(t)$ is sometimes referred to as the *short rate*.

1.2.7 Par Yields

The par-yield curve $\rho(t, T)$ specifies the coupon rates, $100\rho(t, T)$, at which new bonds (issued at time t and maturing at time T) should be priced if they are to be issued at par. That is, they will have a price of 100 per 100 nominal.

The par yield for maturity at time T (with coupons payable annually, $\Delta t = 1$) can be calculated as follows (for $T = t + 1, t + 2, \dots$):

$$100 = 100\rho(t, T) \sum_{s=t+1}^{T} P(t, s) + 100P(t, T).$$

That is, with a coupon rate of $\rho(t, T)$ and a maturity date of T, the price at t for the coupon bond would be exactly 100. This implies that

$$\rho(t, T) = \frac{1 - P(t, T)}{\sum_{s=t+1}^{T} P(t, s)}.$$

1.2.8 Yield-to-Maturity or Gross Redemption Yield for a Coupon Bond

This term normally applies to coupon bonds. Consider the coupon bond described in Section 1.2.1 with coupon rate g, maturity date t_n and current price P. Let δ be a solution to the equation

$$P = \sum_{j=1}^{n} c_j \mathrm{e}^{-\delta t_j}.$$

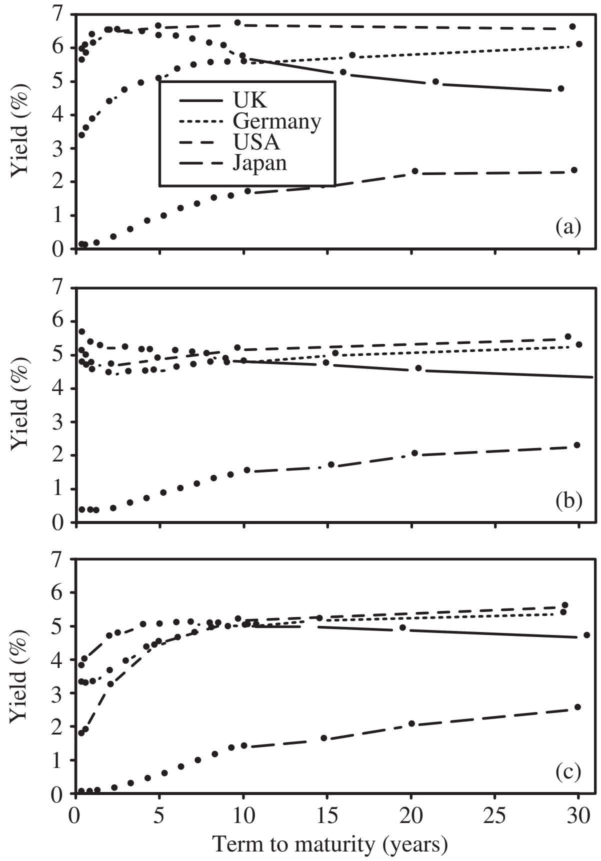

Figure 1.3. Benchmark yield curves (yields to maturity on benchmark coupon bonds) for the UK, Germany, the USA and Japan on (a) 26 January 2000, (b) 6 February 2001 and (c) 3 January 2002.

Since $P > 0$ and each of the c_j are positive there is exactly one solution to this equation in δ. This solution is the (continuously compounding) yield-to-maturity or gross redemption yield.

Typically, yields are quoted on an annual or semi-annual basis. The *Financial Times* (FT), for example, quotes half-yearly gross redemption yields. This reflects the fact that coupons on gilts are payable half yearly (that is, $\Delta t = 0.5$). Thus, the quoted rate is

$$y = 2[e^{\delta/2} - 1].$$

Thus, if $t_1 = 0.5$ and $P = 100$, y will be equal to the par yield.

Sometimes the expression *yield curve* is used, but this means different things to different people and should be avoided or described explicitly.

Gross redemption yields on benchmark bonds on 26 January 2000, 6 February 2001 and 3 January 2002 and are given for four different countries in Figure 1.3. These graphs illustrate the fact that yield curves in different currency zones can be quite different, reflecting the different states of each economy. They also show how the term structure of interest rates can vary over time. Finally, correlations between different countries can be seen, an example being a worldwide reduction in short-term interest rates following the terrorist attacks in the United States of America on 11 September 2001.

1.2.9 Relationships

For a given t, each of the curves $P(t,T)$, $f(t,T)$, $R(t,T)$, $\rho(t,T)$ (with coupons payable continuously) uniquely determines the other three. For example,

$$P(t,T) = \exp[-R(t,T)(T-t)] = \exp\left[-\int_t^T f(t,s)\,\mathrm{d}s\right].$$

In Figure 1.4 we give examples of the forward-rate, spot-rate and par-yield curves for the UK gilts market on three dates (1 September 1992, 1 September 1993 and 1 March 1996). These curves have been derived from the prices of coupon bonds (see Cairns (1998) for further details). The par-yield curve most closely matches the information available in the coupon-bond market (that is, gross redemption yields), whereas spot rates and forward rates are implied by the way in which the par-yield curve varies with term to maturity. The relationship between the shapes of the three types of curve is determined by the fact that the spot rate is the arithmetic average of the forward rate and the par yield is a weighted average of the spot rates.

1.2.10 Example

Suppose that the (continuously compounding) forward rates for the next five one-year periods are as follows:

T	1	2	3	4	5
$F(0, T-1, T)$	0.0420	0.0500	0.0550	0.0560	0.0530

Now the prices of zero-coupon bonds and spot rates at time 0 are given by

$$P(0,T) = \exp\left[-\sum_{t=1}^{T} F(0, t-1, t)\right] \quad \text{and} \quad R(0,T) = -\frac{\log P(0,T)}{T}.$$

Hence,

T	1	2	3	4	5
$P(0,T)$	0.958 87	0.912 11	0.863 29	0.816 28	0.774 14
$R(0,T)$	0.042 0	0.046 0	0.049 0	0.050 75	0.051 2

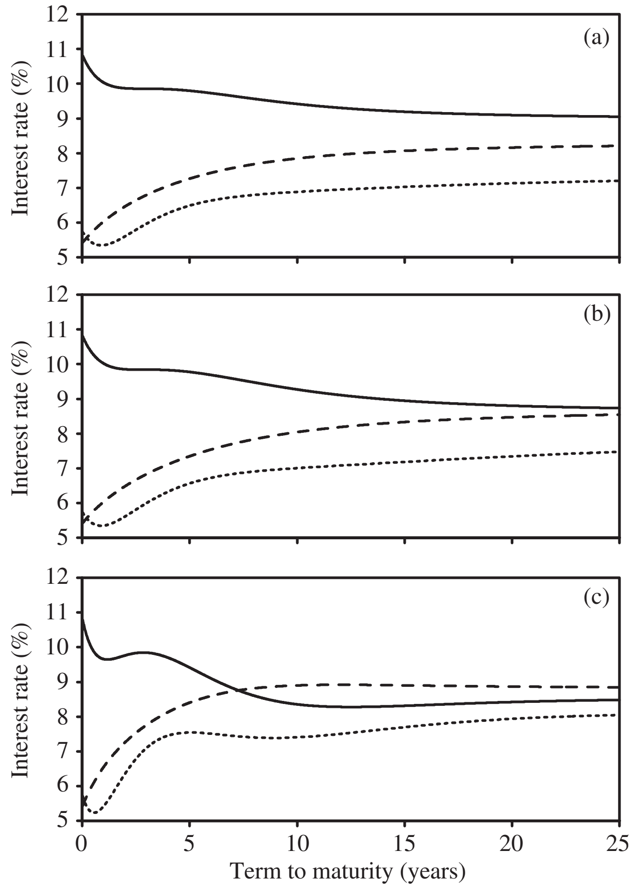

Figure 1.4. (a) Par-yield, (b) spot-rate and (c) forward-rate curves for the UK gilts market on 1 September 1992 (solid curves), 1 September 1993 (dotted curves) and 1 March 1996 (dashed curves). All curves are estimated from coupon-bond prices.

Finally, par yields (with coupons payable annually, $\Delta t = 1$) can be calculated according to the formula

$$\rho(0, T) = \frac{1 - P(0, T)}{\sum_{s=1}^{T} P(0, s)}.$$

Thus we have

T	1	2	3	4	5
$\rho(0, T)$	0.0429	0.0470	0.0500	0.0517	0.0522

(or 4.29 per 100 nominal, etc.).

1.3 STRIPS

STRIPS (Separate Trading of Registered Interest and Principal of Securities) are zero-coupon bonds that have been created out of coupon bonds by market makers rather than by the government. For example, in the UK, gilts maturing on 7 June or 7 December have been 'strippable' since November 1997. This means that a market exists in zero-coupon bonds which mature on the 7 June and the 7 December each year up to June 2032. More recently, strippable bonds have been added that have allowed the creation of zero-coupon bonds maturing on the 7 March and the 7 September each year up to March 2025. Where the date coincides with the maturity date of a strippable coupon bond there will be two types of zero-coupon bond available, depending upon whether or not it is made up of a coupon payment or a redemption payment, but they should have the same price because they are taxed on the same basis. In fact, stripped coupon interest and stripped principal do have slightly different prices, but their buying and selling spreads overlap, making arbitrage impossible.

1.4 Bonds with Built-in Options

In many countries the government bond market is complicated by the inclusion of a number of bonds which have option characteristics. Two examples common in the UK are as follows.

- Double-dated (or callable) bonds: the government has the right to redeem the bond *at par* at any time between two specified dates with three months notice. Thus, they will redeem if the price goes above 100 between the two redemption dates. This is similar to an American option. (An example is the UK gilt Treasury $7\frac{3}{4}\%$ 2012–15.)
- Convertible bonds: at the conversion date the holder has the right but not the obligation to convert the bond into a specified quantity of another bond.

Such bonds must be priced using the more sophisticated derivative-pricing techniques described later in this book.

1.5 Index-Linked Bonds

A number of countries including the UK and USA issue index-linked bonds. Let $\mathrm{CPI}(t)$ be the value of the consumer prices index (CPI) at time t. (In the UK this is called the Retail Prices Index or RPI.)

Suppose that a bond issued at time 0 has a nominal redemption value of 100 payable at time T and a nominal coupon rate of $g\%$ per annum, payable twice yearly. The payment on this bond at time t will be

$$\begin{array}{ll} \dfrac{\mathrm{CPI}(t-L)}{\mathrm{CPI}(-L)} \times g\Delta t & \text{for } t = \Delta t, 2\Delta t, \ldots, T-\Delta t, \\ \dfrac{\mathrm{CPI}(T-L)}{\mathrm{CPI}(-L)} \times (100 + g\Delta t) & \text{for } t = T. \end{array}$$

L is called the *time lag* and is typically about two months in most countries (including the USA), but sometimes eight months (in the UK for example).[3] The time lag of two months ensures that the relevant index value is known by the time a payment is due. The time lag of eight months ensures that the absolute amount of the next coupon payment is known immediately after the time of payment of the immediately preceding coupon. This makes the calculation of accrued interest precise (that is, the difference between the clean and dirty prices) but reduces the effectiveness of the security as a hedge against inflation.

1.6 General Theories of Interest Rates

In this section we will introduce four theories which attempt to explain the term structure of interest rates. The first three are based upon general economic reasoning, each containing useful ideas. The fourth theory, arbitrage-free pricing, introduces us to the approach that we will take in the rest of this book.

1.6.1 Expectations Theory

There are a number of variations on how this theory can be defined but the most popular form seems to be that

$$\mathrm{e}^{F(0,S,S+1)} = E[\mathrm{e}^{R(S,S+1)} \mid \mathcal{F}_0], \tag{1.2}$$

where $\mathcal{F}_t$ represents the information available at time t. Thus, the annualized one-year forward rate of interest for delivery over the period S to $S+1$ is conjectured to be equal to the expected value of the actual one-year rate of interest at time S.

Assume the conjecture to be true.

- Since e^x is a convex function, Jensen's inequality implies that $F(0, S, S+1) > E[R(S, S+1) \mid \mathcal{F}_0]$.
- Since $2F(0, S, S+2) = F(0, S, S+1) + F(0, S+1, S+2)$, it also follows from equation (1.2) that

$$\mathrm{e}^{2F(0,S,S+2)} = E[\mathrm{e}^{R(S,S+1)}]E[\mathrm{e}^{R(S+1,S+2)}].$$

[3] Note that the Debt-Management Office in the UK is currently considering reducing the time lag of eight months to two months.

The theory also suggests that $e^{F(0,S,S+2)} = E[e^{R(S,S+2)}]$, which then implies that $e^{R(S,S+1)}$ and $e^{R(S+1,S+2)}$ must be uncorrelated. This is very unlikely to be true.

An alternative version of the theory is based upon continuously compounding rates of interest, that is, for any $T < S$,

$$F(0, T, S) = E[R(T, S)].$$

This version of the theory does allow for correlation between $R(T, U)$ and $R(U, S)$, for any $T < U < S$.

The problem with this theory, on its own, is that the forward-rate curve is, more often than not, upward sloping. If the theory was true, then the curve would spend just as much time sloping downwards. However, we might conjecture that, for some reason, a forward rate is a biased expectation of future rates of interest. This is encapsulated by the next theory.

1.6.2 Liquidity Preference Theory

The background to this theory is that investors usually prefer short-term investments to long-term investments—they do not like to tie their capital up for too long. In particular, a small investor may incur a penalty on early redemption of a longer-term investment. In practice, bigger investors drive market prices. Furthermore, there is a very liquid market in bonds of all terms to maturity.

The theory has a better explanation, although this is not related to its name. The prices of longer-term bonds tend to be more volatile than short-term bonds. Investors will only invest in more volatile securities if they have a higher expected return, often referred to as the *risk premium*, to offset the higher risk. This leads to generally rising spot-rate and forward-rate curves.

We can see, therefore, that a combination of the expectations theory and the liquidity preference theory might explain what we see in the market.

1.6.3 Market Segmentation Theory

Each investor has in mind an appropriate set of bonds and maturity dates that are suitable for their purpose. For example, life insurance companies require long-term bonds to match their long-term liabilities. In contrast, banks are likely to prefer short-term bonds to reflect the needs of their customers.

Different groups of investors can act in different ways. The basic form of market segmentation theory says that there is no reason why there should be any interaction between different groups. This means that prices in different maturity bands will change in unrelated ways. More realistically, investors who prefer certain maturities may shift their investments if they think that bonds in a different maturity band are particularly cheap. This possibility therefore draws upon the risk–return aspect of liquidity-preference theory.

1.6.4 Arbitrage-Free Pricing Theory

The remainder of this book considers the pricing of bonds in a market which is free of arbitrage. The theory (which is very extensive) pulls together the expectation, liquidity-preference and market-segmentation theories in a mathematically precise way. Under this approach we can usually decompose forward rates into three components:

- the expected future risk-free rate of interest, $r(t)$;
- an adjustment for the market price of risk;[4]
- a convexity adjustment to reflect the fact that $E(\mathrm{e}^X) \geqslant \mathrm{e}^{E(X)}$ for any random variable X.

For example, consider the Vasicek model (see Section 4.5): given $r(0)$ we have

$$E[r(t)] = \mu + (r(0) - \mu)\mathrm{e}^{-\alpha t},$$

whereas the forward-rate curve at time 0 can be written as the sum of three components corresponding to those noted above. That is,

$$f(0, T) = \mu + (r(0) - \mu)\mathrm{e}^{-\alpha T} - \lambda\sigma(1 - \mathrm{e}^{-\alpha T})/\alpha - \tfrac{1}{2}\sigma^2[(1 - \mathrm{e}^{-\alpha T})/\alpha]^2,$$

where μ, α and σ are parameters in the model and λ is the market price of risk. For reasons which will be explained later, λ is normally negative.

The form of the two adjustments is not obvious. This is why we need arbitrage-free pricing theory to derive prices.

For a single-factor model—one with a single source of randomness, such as the Vasicek model—there is no place for market-segmentation theory. However, many models for the term structure of interest rates have more than one random factor (so-called multifactor models). These allow us to incorporate market-segmentation theory to some extent.

1.7 Exercises

Exercise 1.1. Prove that the gross redemption yield is uniquely defined for a fixed-interest coupon bond.

Exercise 1.2. One consequence of an arbitrage-free bond market is that the instantaneous risk-free rate, $r(T)$, must be non-negative for all T.

[4] We use the name *market price of risk* in the following sense. When we simulate prices under the real-world probability measure P, we will see in a later chapter that the excess expected return on a risky asset over the risk-free rate of interest is equal to the market price of risk multiplied by the volatility of the risky asset. Thus, the market price of risk is the excess expected return per unit of volatility.

(a) Why must the forward-rate curve, $f(t, t+s)$, also be non-negative?

(b) What are the consequences for $r(T)$, with $T > t$, if $f(t, t+s) = 0$ for some $s > 0$?

(c) What are the consequences for the form of $P(t, T)$?

Exercise 1.3. Show that the term structure is not necessarily arbitrage-free even if the spot-rate curve $R(t, t+s) \geqslant 0$ for all $s > 0$.

Exercise 1.4. Suppose that the UK government issues two bonds, Treasury 8% 2010–14 and Treasury 8% 2010, with earliest redemption date of the former coinciding with the fixed redemption date of the latter. Explain which bond will have the higher price?

Exercise 1.5. Suppose the UK government issues two bonds, Treasury 8% 2010 and Convertible 8% 2010. The bonds are redeemable on the same date. On 1 January 2006 (not a coupon-payment date) holders of Convertible 8% 2010 will be able to convert their stock into Treasury $8\frac{3}{4}$% 2017 on a one-for-one basis. Show that Convertible 8% 2010 will have a higher price than Treasury 8% 2010.

Exercise 1.6. Suppose that the spot rates (continuously compounding) for terms 1, 2 and 3 to maturity are

T	1	2	3
$R(0, T)$	6%	6.5%	7%

(a) Find the values of $F(0, 1, 2)$, $F(0, 1, 3)$ and $F(0, 2, 3)$.

(b) Assuming that coupons are payable annually in arrears, find the par yields for terms 1, 2 and 3 years.

Exercise 1.7. In a certain bond market coupons are payable annually. At time t, par yields $\rho(t, T)$ are given for maturities $T = t+1, t+2, \dots$. Derive recursive formulae for calculating the prices of zero-coupon bonds maturing at times $T = t+1, t+2, \dots$.

Exercise 1.8.

(a) In a certain bond market coupons are payable continuously. At time t, par yields $\rho(t, T)$ are given for all maturities, $T \in R$ with $t < T < t+s$. Show that the zero-coupon bond prices can be found by solving the ordinary differential equation

$$\frac{\partial P}{\partial T}(t, T) + P(t, T)\left(\rho(t, T) - \frac{1}{\rho(t, T)}\frac{\partial \rho}{\partial T}(t, T)\right) = -\frac{1}{\rho(t, T)}\frac{\partial \rho}{\partial T}(t, T).$$

(b) Hence find $P(0, T)$ given $\rho(0, T) = (1 - \frac{1}{2}T)^{-1}$ for $0 < T < 1$.

(c) Explain why, in part (b), T is limited above by 1.