# Problem 1.1

## Stability and composition of transfer functions

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### 1 INTRODUCTION

As far as the frequency-described continuous linear time-invariant systems are concerned, the study of control-oriented properties (like stability) resulting from the substitution of the complex Laplace variable $s$ by rational transfer functions have been little studied by the Automatic Control community. However, some interesting results have recently been published:

Concerning the study of the so-called uniform systems, i.e., **LTI** systems consisting of identical components and amplifiers, it was established in [8] a general criterion for robust stability for rational functions of the form $D(f(s))$, where $D(s)$ is a polynomial and $f(s)$ is a rational transfer function. By applying such a criterium, it gave a generalization of the celebrated Kharitonov's theorem [7], as well as some robust stability criteria under $H_\infty$-uncertainty. The results given in [8] are based on the so-called *H-domains*.[1]

As far as robust stability of polynomial families is concerned, some Kharito-

[1] The *H-domain* of a function $f(s)$ is defined to be the set of points $h$ on the complex plane for which the function $f(s) - h$ has no zeros on the open right-half complex plane.

nov's like results [7] are given in [9] (for a particular class of polynomials), when interpreting substitutions as nonlinearly correlated perturbations on the coefficients.

More recently, in [1], some results for proper and stable real rational SISO functions and coprime factorizations were proved, by making substitutions with $\alpha(s) = (as+b)/(cs+d)$, where $a$, $b$, $c$, and $d$ are strictly positive real numbers, and with $ad - bc \neq 0$. But these results are limited to the bilinear transforms, which are very restricted.

In [4] is studied the preservation of properties linked to control problems (like weighted nominal performance and robust stability) for Single-Input Single-Output systems, when performing the substitution of the Laplace variable (in transfer functions associated to the control problems) by strictly positive real functions of zero relative degree. Some results concerning the preservation of control-oriented properties in Multi-Input Multi-Output systems are given in [5], while [6] deals with the preservation of solvability conditions in algebraic Riccati equations linked to robust control problems.

Following our interest in substitutions we propose in section 22.2 three interesting problems. The motivations concerning the proposed problems are presented in section 22.3.

## 2 DESCRIPTION OF THE PROBLEMS

In this section we propose three closely related problems. The first one concerns the characterization of a transfer function as a composition of transfer functions. The second problem is a modified version of the first problem: the characterization of a transfer function as the result of substituting the Laplace variable in a transfer function by a strictly positive real transfer function of zero relative degree. The third problem is in fact a conjecture concerning the preservation of stability property in a given polynomial resulting from the substitution of the coefficients in the given polynomial by a polynomial with non-negative coefficients evaluated in the substituted coefficients.

**Problem 1:** *Let a Single Input Single Output* (SISO) *transfer function* $G(s)$ *be given. Find transfer functions* $G_0(s)$ *and* $H(s)$ *such that:*

1. $G(s) = G_0(H(s))$;

2. $H(s)$ *preserves proper stable transfer functions under substitution of the variable* $s$ *by* $H(s)$*, and:*

3. *The degree of the denominator of* $H(s)$ *is the maximum with the properties 1 and 2.*

**Problem 2:** *Let a* SISO *transfer function $G(s)$ be given. Find a transfer function $G_0(s)$ and a Strictly Positive Real transfer function of zero relative degree (SPR0), say $H(s)$, such that:*

1. $G(s) = G_0(H(s))$ *and:*
2. *The degree of the denominator of $H(s)$ is the maximum with the property 1.*

**Problem 3:** *(Conjecture) Given any stable polynomial:*

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

*and given any polynomial $q(s)$ with non-negative coefficients, then the polynomial:*

$$q(a_n)s^n + q(a_{n-1})s^{n-1} + \cdots + q(a_1)s + q(a_0)$$

*is stable (see* [3]*).*

## 3 MOTIVATIONS

Consider the closed-loop control scheme:

$$y(s) = G(s)\,u(s) + d(s), \quad u(s) = K(s)\,(r(s) - y(s)),$$

where: $P(s)$ denotes the SISO plant; $K(s)$ denotes a *stabilizing controller*; $u(s)$ denotes the control input; $y(s)$ denotes the control input; $d(s)$ denotes the disturbance and $r(s)$ denotes the reference input. We shall denote the closed-loop transfer function from $r(s)$ to $y(s)$ as $\mathcal{F}_r(G(s), K(s))$ and the closed-loop transfer function from $d(s)$ to $y(s)$ as $\mathcal{F}_d(G(s), K(s))$.

- Consider the closed-loop system $\mathcal{F}_r(G(s), K(s))$, and suppose that the plant $G(s)$ results from a particular substitution of the $s$ Laplace variable in a transfer function $G_0(s)$ by a transfer function $H(s)$, i.e., $G(s) = G_0(H(s))$. It has been proved that a controller $K_0(s)$ which stabilizes the closed-loop system $\mathcal{F}_r(G_0(s), K_0(s))$ is such that $K_0(H(s))$ stabilizes $\mathcal{F}_r(G(s), K_0(H(s)))$ (see [2] and [8]). Thus, the simplification of procedures for the synthesis of stabilizing controllers (profiting from transfer function compositions) justifies problem 1.
- As far as problem 2 is concerned, consider the synthesis of a controller $K(s)$ stabilizing the closed-loop transfer function $\mathcal{F}_d(G(s), K(s))$, and such that $\|\mathcal{F}_d(G(s), K(s))\|_\infty < \gamma$, for a fixed given $\gamma > 0$. If we known that $G(s) = G_0(H(s))$, being $H(s)$ a SPR0 transfer function, the solution of problem 2 would arise to the following procedure:
    1. Find a controller $K_0(s)$ which stabilizes the closed-loop transfer function $\mathcal{F}_d(G_0(s), K_0(s))$ and such that:

    $$\|\mathcal{F}_d(G_0(s), K_0(s))\|_\infty < \gamma.$$

2. The composed controller $K(s) = K_0(H(s))$ stabilizes the closed-loop system $\mathcal{F}_d(G(s), K(s))$ and:

$$\|\mathcal{F}_d(G(s), K(s))\|_\infty < \gamma$$

(see [2], [4], and [5]).

It is clear that condition 3 in the first problem, or condition 2 in the second problem, can be relaxed to the following condition: *the degree of the denominator of $H(s)$ is as high as be possible with the appropriate conditions*. With this new condition, the open problems are a bit less difficult.

- Finally, problem 3 can be interpreted in terms of robustness under positive polynomial perturbations in the coefficients of a stable transfer function.

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