Chapter One

Background Material

We recall in this chapter basic facts concerning Riemannian geometry and nonlinear analysis on manifolds. For reasons of length, we are obliged to be succinct and partial. Possible references are Chavel [20], do Carmo [22], Gallot-Hulin-Lafontaine [36], Hebey [43], Jost [50], Kobayashi-Nomizu [53], Sakai [65], and Spivak [72]. As a general remark, we mention that Einstein's summation convention is adopted: an index occurring twice in a product is to be summed. This also holds for the rest of this book.

1.1 RIEMANNIAN GEOMETRY

We start with a few notions in differential geometry. Let M be a Hausdorff topological space. We say that M is a topological manifold of dimension n if each point of M possesses an open neighborhood that is homeomorphic to some open subset of the Euclidean space $\mathbb{R}^n$. A chart of M is then a couple (Ω, φ) where Ω is an open subset of M, and φ is a homeomorphism of Ω onto some open subset of $\mathbb{R}^n$. For $y \in \Omega$, the coordinates of $\varphi(y)$ in $\mathbb{R}^n$ are said to be the coordinates of y in (Ω, φ). An atlas of M is a collection of charts (Ω_i, φ_i), $i \in I$, such that $M = \bigcup_{i \in I} \Omega_i$. Given an atlas $(\Omega_i, \varphi_i)_{i \in I}$, the transition functions are

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(\Omega_i \cap \Omega_j) \to \varphi_j(\Omega_i \cap \Omega_j)$$

with the obvious convention that we consider $\varphi_j \circ \varphi_i^{-1}$ if and only if $\Omega_i \cap \Omega_j \neq \emptyset$. The atlas is then said to be of class C^k if the transition functions are of class C^k, and it is said to be C^k-complete if it is not contained in a (strictly) larger atlas of class C^k. As one can easily check, every atlas of class C^k is contained in a unique C^k-complete atlas. For our purpose, we will always assume in what follows that $k = +\infty$ and that M is connected. One then gets the following definition of a smooth manifold: A smooth manifold M of dimension n is a connected topological manifold M of dimension n together with a C^∞-complete atlas. Classical examples of smooth manifolds are the Euclidean space $\mathbb{R}^n$ itself, the torus T^n, the unit sphere S^n of $\mathbb{R}^{n+1}$, and the real projective space $\mathbb{P}^n(\mathbb{R})$.

Given two smooth manifolds, M and N, and a smooth map $f : M \to N$ from M to N, we say that f is differentiable (or of class C^k) if for any charts (Ω, φ) and $(\tilde{\Omega}, \tilde{\varphi})$ of M and N such that $f(\Omega) \subset \tilde{\Omega}$, the map

$$\tilde{\varphi} \circ f \circ \varphi^{-1} : \varphi(\Omega) \to \tilde{\varphi}(\tilde{\Omega})$$

is differentiable (or of class C^k). In particular, this allows us to define the notion of diffeomorphism and the notion of diffeomorphic manifolds.

We refer to the above definition of a manifold as the abstract definition of a smooth manifold. As a surface gives the idea of a two-dimensional manifold, a more concrete approach would have been to define manifolds as submanifolds of Euclidean space. According to a well-known result of Whitney, any paracompact (abstract) manifold of dimension n can be seen as a submanifold of some Euclidean space.

Let us now say some words about the tangent space of a manifold. Given M a smooth manifold and $x \in M$, let $\mathcal{F}_x$ be the vector space of functions $f : M \to \mathbb{R}$ which are differentiable at x. For $f \in \mathcal{F}_x$, we say that f is flat at x if for some chart (Ω, φ) of M at x, $D\big(f \circ \varphi^{-1}\big)_{\varphi(x)} = 0$. Let $\mathcal{N}_x$ be the vector space of such functions. A linear form X on $\mathcal{F}_x$ is then said to be a tangent vector of M at x if $\mathcal{N}_x \subset \mathrm{Ker}X$. We let $T_x(M)$ be the vector space of such tangent vectors. Given (Ω, φ) some chart at x, of associated coordinates x^i, we define $\big(\frac{\partial}{\partial x_i}\big)_x \in T_x(M)$ by, for any $f \in \mathcal{F}_x$,

$$\left(\frac{\partial}{\partial x_i}\right)_x \cdot (f) = D_i\big(f \circ \varphi^{-1}\big)_{\varphi(x)} \,.$$

As a simple remark, one gets that the $\big(\frac{\partial}{\partial x_i}\big)_x$'s form a basis of $T_x(M)$. Now, one defines the tangent bundle of M as the disjoint union of the $T_x(M)$'s, $x \in M$. If M is n-dimensional, one can show that $T(M)$ possesses a natural structure of a $2n$-dimensional smooth manifold. Given a chart (Ω, φ) of M,

$$\left(\bigcup_{x \in \Omega} T_x(M), \Phi\right)$$

is a chart of $T(M)$, where for $X \in T_x(M)$, $x \in \Omega$,

$$\Phi(X) = \big(\varphi^1(x), \dots, \varphi^n(x), X(\varphi^1), \dots, X(\varphi^n)\big)$$

[the coordinates of x in (Ω, φ) and the components of X in (Ω, φ), that is, the coordinates of X in the basis of $T_x(M)$ associated with (Ω, φ) by the process described above]. By definition, a vector field on M is a map $X : M \to T(M)$ such that for any $x \in M$, $X(x) \in T_x(M)$. Since M and $T(M)$ are smooth manifolds, the notion of a vector field of class C^k makes sense. A manifold M of dimension n is said to be parallelizable if there exist n smooth vector fields X_i, $i = 1, \dots, n$, such that for any $x \in M$, the $X_i(x)$'s, $i = 1, \dots, n$, define a basis of $T_x(M)$.

Given two smooth manifolds, M and N, a point x in M, and a differentiable map $f : M \to N$ at x, the tangent linear map of f at x (or the differential map of f at x), denoted by $f_\star(x)$, is the linear map from $T_x(M)$ to $T_{f(x)}(N)$ defined, for $X \in T_x(M)$ and $g : N \to \mathbb{R}$ differentiable at $f(x)$, by

$$\big(f_\star(x) \cdot (X)\big) \cdot (g) = X(g \circ f) \,.$$

By extension, if f is differentiable on M, one gets the tangent linear map of f, denoted by $f_\star$. That is the map $f_\star : T(M) \to T(N)$ defined, for $X \in T_x(M)$, by $f_\star(X) = f_\star(x).(X)$. As one can easily check, $f_\star$ is C^{k-1} if f is C^k. Similar to the construction of the tangent bundle, one can define the cotangent bundle of a smooth manifold M as the disjoint union of the $T_x(M)^\star$'s, $x \in M$. In a more general way, one can define $T_p^q(M)$ as the disjoint union of the $T_p^q\big(T_x(M)\big)$'s, where $T_p^q(T_x(M))$ is the space of (p,q)-tensors on $T_x(M)$. Then $T_p^q(M)$ possesses a natural structure of a smooth manifold of dimension $n\big(1+n^{p+q-1}\big)$. A map $T : M \to T_p^q(M)$ is then said to be a (p,q)-tensor field on M if for any $x \in M$, $T(x) \in T_p^q\big(T_x(M)\big)$. It is said to be of class C^k if it is of class C^k from the manifold M to the manifold $T_p^q(M)$. Given two manifolds M and N, a map $f : M \to N$ of class C^{k+1}, and a $(p,0)$-tensor field T of class C^k on N, one can define the pullback $f^\star T$ of T by f, that is, the $(p,0)$-tensor field of class C^k on M defined for $x \in M$ and $X_1, \dots, X_p \in T_x(M)$, by

$$\big(f^\star T\big)(x) \cdot \big(X_1, \dots, X_p\big) = T\big(f(x)\big) \cdot \big(f_\star(x) \cdot X_1, \dots, f_\star(x) \cdot X_p\big).$$

We now define the notion of a linear connection. Denote by $\Gamma(M)$ the space of differentiable vector fields on M. A linear connection D on M is a map $D : T(M) \times \Gamma(M) \to T(M)$ which satisfies a certain number of propositions. In local coordinates, given a chart (Ω, φ), this is equivalent to having n^3 smooth functions $\Gamma_{ij}^k : \Omega \to \mathbb{R}$, that we refer to as the Christoffel symbols of the connection in (Ω, φ). They characterize the connection in the sense that for $X \in T_x(M)$, $x \in \Omega$, and $Y \in \Gamma(M)$,

$$D_X(Y) = X^i (\nabla_i Y)(x) = X^i \left(\left(\frac{\partial Y^j}{\partial x_i} \right)_x + \Gamma_{i\alpha}^j(x) Y^\alpha(x) \right) \left(\frac{\partial}{\partial x_j} \right)_x$$

where the X^i's and Y^i's denote the components of X and Y in the chart (Ω, φ), and for $f : M \to \mathbb{R}$ differentiable at x,

$$\left(\frac{\partial f}{\partial x_i} \right)_x = D_i\big(f \circ \varphi^{-1}\big)_{\varphi(x)}.$$

As one can easily check, the Γ_{ij}^k's are not the components of a $(2,1)$-tensor field. An important remark is that linear connections have natural extensions to differentiable tensor fields. Given a differentiable (p,q)-tensor field, T, a point x in M, $X \in T_x(M)$, and a chart (Ω, φ) of M at x, $D_X(T)$ is the (p,q)-tensor on $T_x(M)$ defined by $D_X(T) = X^i(\nabla_i T)(x)$, where

$$\begin{aligned}\big(\nabla_i T\big)(x)_{i_1 \dots i_p}^{j_1 \dots j_q} = {} & \left(\frac{\partial T_{i_1 \dots i_p}^{j_1 \dots j_q}}{\partial x_i} \right)_x - \sum_{k=1}^p \Gamma_{i i_k}^\alpha(x) T(x)_{i_1 \dots i_{k-1} \alpha i_{k+1} \dots i_p}^{j_1 \dots j_q} \\ & + \sum_{k=1}^q \Gamma_{i\alpha}^{j_k}(x) T(x)_{i_1 \dots i_p}^{j_1 \dots j_{k-1} \alpha j_{k+1} \dots j_q}.\end{aligned}$$

The covariant derivative commutes with the contraction in the sense that

$$D_X\big(C_{k_1}^{k_2} T\big) = C_{k_1}^{k_2} D_X(T)$$

where $C_{k_1}^{k_2} T$ stands for the contraction of T of order (k_1, k_2). Given a (p,q)-tensor field of class C^{k+1}, T, we let ∇T be the $(p+1,q)$-tensor field of class C^k whose components in a chart are given by

$$\big(\nabla T\big)_{i_1 \dots i_{p+1}}^{j_1 \dots j_q} = \big(\nabla_{i_1} T\big)_{i_2 \dots i_{p+1}}^{j_1 \dots j_q}.$$

By extension, one can then define $\nabla^2 T$, $\nabla^3 T$, and so on. For $f : M \to \mathbb{R}$ a smooth function, one has that $\nabla f = df$ and, in any chart (Ω, φ) of M,

$$(\nabla^2 f)(x)_{ij} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_x - \Gamma_{ij}^k(x)\left(\frac{\partial f}{\partial x_k}\right)_x$$

where

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_x = D_{ij}^2\big(f \circ \varphi^{-1}\big)_{\varphi(x)} .$$

In the Riemannian context, $\nabla^2 f$ is called the Hessian of f and is sometimes denoted by $\mathrm{Hess}(f)$.

The torsion T of a linear connection D can be seen as the smooth $(2,1)$-tensor field on M whose components in any chart are given by the relation $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$. One says that the connection is torsion-free if $T \equiv 0$. The curvature R of D can be seen as the smooth $(3,1)$-tensor field on M whose components in any chart are given by the relation

$$R_{ijk}^l = \frac{\partial \Gamma_{ki}^l}{\partial x_j} - \frac{\partial \Gamma_{ji}^l}{\partial x_k} + \Gamma_{j\alpha}^l \Gamma_{ki}^\alpha - \Gamma_{k\alpha}^l \Gamma_{ji}^\alpha .$$

As one can easily check, $R_{ijk}^l = -R_{ikj}^l$. Moreover, when the connection is torsion-free, one has that

$$\begin{aligned} &R_{ijk}^l + R_{kij}^l + R_{jki}^l = 0 \text{ and} \\ &(\nabla_i R)_{mjk}^l + (\nabla_k R)_{mij}^l + (\nabla_j R)_{mki}^l = 0 . \end{aligned}$$

Such relations are referred to as the first Bianchi and second Bianchi identities.

We now discuss Riemannian geometry. Let M be a smooth manifold. A Riemannian metric g on M is a smooth $(2,0)$-tensor field on M such that for any $x \in M$, $g(x)$ is a scalar product on $T_x(M)$. A smooth Riemannian manifold is a pair (M,g) where M is a smooth manifold and g a Riemannian metric on M. According to Whitney, for any paracompact smooth n-manifold there exists a smooth embedding $f : M \to \mathbb{R}^{2n+1}$. One then gets that any smooth paracompact manifold possesses a Riemannian metric. Just think of $g = f^\star \xi$, where ξ is the Euclidean metric. Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to be isometric if there exists a diffeomorphism $f : M_1 \to M_2$ such that $f^\star g_2 = g_1$.

Given a smooth Riemannian manifold (M,g), and $\gamma : [a,b] \to M$ a curve of class C^1, the length of γ is

$$L(\gamma) = \int_a^b \sqrt{g(\gamma(t)) \cdot \left(\left(\frac{d\gamma}{dt}\right)_t, \left(\frac{d\gamma}{dt}\right)_t\right)}\, dt$$

where $(\frac{d\gamma}{dt})_t \in T_{\gamma(t)}(M)$ is such that $(\frac{d\gamma}{dt})_t \cdot f = \big(f \circ \gamma\big)'(t)$ for any $f : M \to \mathbb{R}$ differentiable at $\gamma(t)$. If γ is piecewise C^1, the length of γ may be defined as the sum of the lengths of its C^1 pieces. For x and y in M, let $\mathcal{C}_{xy}$ be the space of piecewise C^1 curves $\gamma : [a,b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then

$$d_g(x,y) = \inf_{\gamma \in \mathcal{C}_{xy}} L(\gamma)$$

defines a distance on M whose topology coincides with the original one of M. In particular, by Stone's theorem, a smooth Riemannian manifold is paracompact. By definition, d_g is the distance associated with g.

Let (M, g) be a smooth Riemannian manifold. There exists a unique torsion-free connection on M having the property that $\nabla g = 0$. Such a connection is the Levi-Civita connection of g. In any chart (Ω, φ) of M, of associated coordinates x^i, and for any $x \in \Omega$, its Christoffel symbols are given by the relations

$$\Gamma_{ij}^k(x) = \frac{1}{2}\left(\left(\frac{\partial g_{mj}}{\partial x_i}\right)_x + \left(\frac{\partial g_{mi}}{\partial x_j}\right)_x - \left(\frac{\partial g_{ij}}{\partial x_m}\right)_x\right) g(x)^{mk}$$

where the g^{ij}'s are such that $g_{im}g^{mj} = \delta_i^j$. Let R be the curvature of the Levi-Civita connection as introduced above. One defines

1. the *Riemann curvature* Rm_g of g as the smooth $(4, 0)$-tensor field on M whose components in a chart are $R_{ijkl} = g_{i\alpha}R_{jkl}^\alpha$,
2. the *Ricci curvature* Rc_g of g as the smooth $(2, 0)$-tensor field on M whose components in a chart are $R_{ij} = R_{\alpha i\beta j}g^{\alpha\beta}$, and
3. the *scalar curvature* S_g of g as the smooth real-valued function on M whose expression in a chart is $S_g = R_{ij}g^{ij}$.

As one can check, in any chart,

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij} \,,$$

and the two Bianchi identities are

$$\begin{aligned} &R_{ijkl} + R_{iljk} + R_{iklj} = 0 \,, \\ &\left(\nabla_i Rm_g\right)_{jklm} + \left(\nabla_m Rm_g\right)_{jkil} + \left(\nabla_l Rm_g\right)_{jkmi} = 0 \,. \end{aligned}$$

In particular, the Ricci curvature is symmetric, so that in any chart $R_{ij} = R_{ji}$.

Given a smooth Riemannian manifold (M, g), and its Levi-Civita connection D, a smooth curve $\gamma : [a, b] \to M$ is said to be a geodesic, if for all t,

$$D_{\left(\frac{d\gamma}{dt}\right)_t}\left(\frac{d\gamma}{dt}\right) = 0 \,.$$

This means again that in any chart, and for all k,

$$\left(\gamma^k\right)''(t) + \Gamma_{ij}^k\left(\gamma(t)\right)\left(\gamma^i\right)'(t)\left(\gamma^j\right)'(t) = 0 \,.$$

For any $x \in M$, and any $X \in T_x(M)$, there exists a unique geodesic $\gamma : [0, \epsilon] \to M$ such that $\gamma(0) = x$ and $(\frac{d\gamma}{dt})_0 = X$. Let $\gamma_{x,X}$ be this geodesic. For $\lambda > 0$ real, $\gamma_{x,\lambda X}(t) = \gamma_{x,X}(\lambda t)$. Hence, for $\|X\|$ small, where $\|\cdot\|$ stands for the norm in $T_x(M)$ associated with $g(x)$, one has that $\gamma_{x,X}$ is defined on $[0, 1]$. The exponential map at x is the map from a neighborhood of 0 in $T_x(M)$, with values in M, defined by $\exp_x(X) = \gamma_{x,X}(1)$. If M is n-dimensional and up to the assimilation of $T_x(M)$ to $\mathbb{R}^n$ via the choice of an orthonormal basis, one gets a chart $\left(\Omega, \exp_x^{-1}\right)$ of M at x. This chart is normal at x in the sense that the components g_{ij} of g in this chart are such that $g_{ij}(x) = \delta_{ij}$, with the additional property that the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection in this chart are such that $\Gamma_{ij}^k(x) = 0$. The coordinates associated with this chart are referred to as geodesic normal coordinates.

Given a smooth Riemannian n-manifold (M, g), one can define a natural positive Radon measure on M. In particular, the theory of the Lebesgue integral can be applied. For some atlas of M, $\left(\Omega_i, \varphi_i\right)_{i\in I}$, we shall say that a family $\left(\Omega_j, \varphi_j, \alpha_j\right)_{j\in J}$ is a partition of unity subordinate to $\left(\Omega_i, \varphi_i\right)_{i\in I}$ if the following holds:
(i) $(\alpha_j)_j$ is a smooth partition of unity subordinate to the covering $(\Omega_i)_i$,
(ii) $\left(\Omega_j, \varphi_j\right)_j$ is an atlas of M, and
(iii) for any j, supp$\alpha_j \subset \Omega_j$.
As one can easily check, for any atlas $\left(\Omega_i, \varphi_i\right)_{i\in I}$ of M, there exists a partition of unity $\left(\Omega_j, \varphi_j, \alpha_j\right)_{j\in J}$ subordinate to $\left(\Omega_i, \varphi_i\right)_{i\in I}$. One can then define the Riemannian measure as follows: Given a continuous map $f : M \to \mathbb{R}$ with compact support, and an atlas $\left(\Omega_i, \varphi_i\right)_{i\in I}$ of M,

$$\int_M f dv(g) = \sum_{j\in J} \int_{\varphi_j(\Omega_j)} \left(\alpha_j \sqrt{|g|} f\right) \circ \varphi_j^{-1} dx$$

where $\left(\Omega_j, \varphi_j, \alpha_j\right)_{j\in J}$ is a partition of unity subordinate to $\left(\Omega_i, \varphi_i\right)_{i\in I}$, $|g|$ stands for the determinant of the matrix whose elements are the components of g in $\left(\Omega_j, \varphi_j\right)$, and dx stands for the Lebesgue volume element of $\mathbb{R}^n$. One can prove that such a construction does not depend on the choice of the atlas $\left(\Omega_i, \varphi_i\right)_{i\in I}$ and the partition of unity $\left(\Omega_j, \varphi_j, \alpha_j\right)_{j\in J}$.

The Laplacian acting on functions of a smooth Riemannian manifold (M, g) is the operator Δ_g whose expression in a local chart of associated coordinates x^i is

$$\Delta_g u = -g^{ij}\left(\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial u}{\partial x_k}\right).$$

For u and v of class C^2 on M, one then has the following formula for integration by parts:

$$\int_M (\Delta_g u) v dv(g) = \int_M (\nabla u \nabla v)\, dv(g) = \int_M u\big(\Delta_g v\big) dv(g)$$

where $(\cdot, \cdot)$ is the scalar product associated with g for 1-forms.

Coming back to geodesics, one can define the injectivity radius of (M, g) at some point x, denoted by $i_g(x)$, as the largest positive real number r for which any geodesic starting from x and of length less than r is minimizing. One can then define the (global) injectivity radius by

$$i_g = \inf_{x\in M} i_g(x).$$

One has that $i_g > 0$ for a compact manifold, but it may be zero for a complete noncompact manifold. In a similar way, one can define the cut locus C_x of x, where C_x is a subset of M, and prove that C_x has measure zero, that $i_g(x) = d_g\big(x, C_x\big)$, and that $\exp_x$ is a diffeomorphism from some star-shaped domain of $T_x(M)$ at 0 onto $M\backslash C_x$. In particular, one gets that the distance function r to a given point is differentiable almost everywhere, with the additional property that $|\nabla r| = 1$ almost everywhere.

As is well known, curvature assumptions may give topological and diffeomorphic information on the manifold. A striking example of the relationship that exists between curvature and topology is given by the Gauss-Bonnet theorem, whose present form is actually due to the works of Allendoerfer [2], Allendoerfer-Weil [3], Chern [21], and Fenchel [35]. One has here that the Euler-Poincaré characteristic $\chi(M)$ of a compact manifold can be expressed as the integral of a universal polynomial in the curvature. For instance, when the dimension of M is 2,

$$\chi(M) = \frac{1}{4\pi}\int_M S_g dv(g)\,,$$

and when the dimension of M is 4, as shown by Avez [8],

$$\chi(M) = \frac{1}{16\pi^2}\int_M \left(\frac{1}{2}|W_g|^2 + \frac{1}{12}S_g^2 - |E_g|^2\right)dv(g)\,,$$

where $|\cdot|$ stands for the norm associated with g for tensors, and where W_g and E_g are, respectively, the Weyl tensor of g and the traceless Ricci tensor of g. In a local chart, the components of W_g are

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{n-2}\left(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\right) \\ &+ \frac{S_g}{(n-1)(n-2)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) \end{aligned}$$

where n stands for the dimension of the manifold. As another striking example of the relationship that exists between curvature and topology, one can refer to Hamilton's theorem [39]: any three-dimensional, compact, simply connected Riemannian manifold of positive Ricci curvature must be diffeomorphic to the unit sphere S^3. Conversely, by recent results of Lohkamp [59], negative sign assumptions on the Ricci curvature have no effect on the topology, since any compact manifold possesses a Riemannian metric of negative Ricci curvature. As another example of the relationship that exists between curvature and topology, we refer to the well-known sphere theorem of Berger [10], Klingenberg [51, 52], Rauch [61], and Tsukamoto [75].

1.2 BASICS IN NONLINEAR ANALYSIS

Given a smooth compact n-dimensional Riemannian manifold (M, g), one easily defines the Sobolev spaces $H_k^p(M)$, following what is done in the more traditional Euclidean context. For instance, when $k = 1$ and $p > 1$, one may define the Sobolev space $H_1^p(M)$ as follows: for $u \in C^\infty(M)$, we let

$$\|u\|_{H_1^p} = \|u\|_p + \|\nabla u\|_p$$

where $\|.\|_p$ is the L^p-norm with respect to the Riemannian measure dv_g. We then define $H_1^p(M)$ as the completion of $C^\infty(M)$ with respect to $\|.\|_{H_1^p}$. A similar definition holds for $H_k^p(M)$, with

$$\|u\|_{H_k^p} = \sum_{i=0}^{k} \|\nabla^i u\|_p \,.$$

Very useful properties of H_1^p are that Lipschitz functions on M do belong to the Sobolev spaces $H_1^p(M)$ for all p, and that if $u \in H_1^p(M)$ for some p, then we have that $|u| \in H_1^p(M)$ and $\big|\nabla|u|\big| = \big|\nabla u\big|$ almost everywhere.

As for bounded open subsets of the Euclidean space, the Sobolev embedding theorem (continuous embeddings), and the Rellich-Kondrakov theorem (compact embeddings), do hold. In order to fix ideas, we let $k = 1$ and $p = 2$. Let

$$2^\star = \frac{2n}{n-2}$$

be the critical Sobolev exponent. Then, for any $q \in [1, 2^\star]$, $H_1^2(M) \subset L^q(M)$ and this embedding is continuous, with the property that the embedding is also compact if $q < 2^\star$. The Sobolev inequality corresponding to the continuous embedding $H_1^2(M) \subset L^{2^\star}(M)$ is as follows: For any $u \in H_1^2(M)$,

$$\|u\|_{2^\star} \le A\|\nabla u\|_2 + B\|u\|_2$$

where A and B are positive constants independent of u, but that may depend on the manifold. Another very useful inequality is the so-called Poincaré inequality. When dealing with H_1^2, it reads as the existence of a positive constant A such that, for any $u \in H_1^2(M)$,

$$\|u - \overline{u}\|_2 \le A\|\nabla u\|_2$$

where

$$\overline{u} = \frac{1}{V_g}\int_M u dv_g$$

is the average of u, and V_g the volume of M with respect to g. In this particular case, thanks to the Rayleigh characterization of the first nonzero eigenvalue of the Laplacian, A may be taken to be the inverse of the square root of this eigenvalue. Combining the Sobolev inequality and the Poincaré inequality, one gets the so-called Sobolev-Poincaré inequality: for any $u \in H_1^2(M)$,

$$\|u - \overline{u}\|_{2^\star} \le A\|\nabla u\|_2 \,,$$

where A is a positive constant, independent of u as usual.

A very useful notion concerning Sobolev embeddings, which appeared to be crucial in many problems like the Yamabe problem, is that of best constants. In the particular case $k = 1$ and $p = 2$, the Sobolev inequality in the Euclidean space reads as

$$\left(\int_{\mathbb{R}^n} |u|^{2^\star} dx\right)^{\frac{1}{2^\star}} \le A\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{\frac{1}{2}} .$$

The best constant A in this inequality, which we denote by K_n, is

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}$$

where ω_n is the volume of the unit n-sphere. Taking $A = K_n$ in the above inequality, we get what we refer to as the sharp Euclidean Sobolev inequality. Its extremal functions are known. They are expressed as

$$u_{\lambda,a,\mu}(x) = \mu\left(\lambda + |x-a|^2\right)^{1-\frac{n}{2}}$$

where $\lambda > 0$, $\mu \in \mathbb{R}$, and $a \in \mathbb{R}^n$ are arbitrary. For compact manifolds, we consider the Sobolev inequality

$$\|u\|_{2^\star}^2 \leq A\|\nabla u\|_2^2 + B\|u\|_2^2 \,.$$

As is easily checked, any such constant A must satisfy $A \geq K_n^2$. Conversely, we refer to Hebey-Vaugon [47, 48], it can be proved that there exists a positive constant B such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq K_n^2\|\nabla u\|_2^2 + B\|u\|_2^2 \,.$$

More developments on Sobolev spaces, Sobolev inequalities, and the notion of best constants are in Druet-Hebey [29] and Hebey [44].

Let (M, g) be a smooth compact Riemannian manifold. The equations we will be interested in are basically of the form

$$\Delta_g u + au = f$$

where a, f are given functions on M. A function $u \in H_1^2(M)$ is said to be a weak solution of this equation if, for all $\varphi \in H_1^2(M)$,

$$\int_M \langle\nabla u, \nabla\varphi\rangle_g dv_g + \int_M a(x)u\varphi dv_g = \int_M f(x)\varphi dv_g \,.$$

Regularity results for this equation do hold. They are similar to the more traditional ones expressed in the Euclidean context (regularity is a local notion, so this is not very surprising). The regularity result we will mostly use is the following: if a is smooth, and $f \in H_k^p(M)$ for some $k \in \mathbb{N}$ and $p > 1$, then a weak solution u to the above equation is in $H_{k+2}^p(M)$. In particular, it follows from this result and the Sobolev embedding theorem, that when f is smooth u is also smooth. Needless to say, the "bible" for such topics is the exhaustive Gilbarg-Trudinger [37]. A simpler, but very nice reference is the lecture notes [41] by Han and Lin.

In parallel with regularity, the very useful maximum principles hold for the Laplacian on Riemannian manifolds. A currently used form is as follows: if a nonnegative $u \in C^2(M)$ is such that, for any $x \in M$,

$$\Delta_g u(x) \geq u(x)f(x, u(x))$$

for some continuous functions $f : M \times \mathbb{R} \to \mathbb{R}$, then either u is everywhere positive, or u is the zero function. This easily follows from the Hopf maximum principle, as usually stated.

In order to end this section, we give a basic example of a possible use of the above results. We discuss here the existence (and uniqueness) of a solution u to the Laplace equation

$$\Delta_g u = f$$

on a compact Riemannian manifold (M, g). Although not necessary, we assume here for convenience that $f : M \to \mathbb{R}$ is smooth. Integrating the Laplace equation, one sees that a necessary condition for the existence of a solution is that

$$\int_M f dv_g = 0 \, .$$

The elementary result we wish to briefly discuss here is that the Laplace equation possesses a smooth solution if and only if $\int_M f dv_g = 0$. Moreover, the solution is unique, up to the addition of a constant. In order to prove this claim, we proceed as follows. As already mentioned, the condition that f is of null average is a necessary condition. We prove now that it is also a sufficient condition. Let

$$\mathcal{H} = \left\{ u \in H_1^2(M) \text{ s.t. } \int_M u dv_g = 0 \text{ and } \int_M f u dv_g = 1 \right\}$$

and

$$\mu = \inf_{u \in \mathcal{H}} \int_M |\nabla u|^2 dv_g \, .$$

Clearly, $\mathcal{H} \neq \emptyset$. Consider a minimizing sequence $(u_i) \in \mathcal{H}$ for μ so that $u_i \in \mathcal{H}$ for all i, and

$$\lim_{i \to +\infty} \int_M |\nabla u_i|^2 dv_g = \mu \, .$$

By the Poincaré inequality we discussed above, there exists $A > 0$ such that, if $u \in H_1^2(M)$ is of null average, then

$$\int_M u^2 dv_g \le A \int_M |\nabla u|^2 dv_g \, .$$

It easily follows that the u_i's are bounded in $H_1^2(M)$. Since $H_1^2(M)$ is a reflexive space, and the embedding $H_1^2(M) \subset L^2(M)$ is compact by the Rellich-Kondrakov theorem (even when $n = 2$, noting that $H_1^2 \subset H_1^q$ for $q < 2$), there exists a function $u \in H_1^2(M)$ and a subsequence (u_i) of (u_i), such that

(1) (u_i) converges weakly to u in $H_1^2(M)$ and

(2) (u_i) converges to u in $L^2(M)$.

By (2), $u \in \mathcal{H}$. By (1), and a basic property of the weak limit (the norm of a weak limit is less than or equal to the infimum limit of the norms of the sequence), we get that

$$\int_M |\nabla u|^2 dv_g \le \mu \, .$$

Hence,

$$\int_M |\nabla u|^2 dv_g = \mu$$

and μ is attained. By a well-known theorem of Lagrange, this gives the existence of two constants α and β, the Lagrange multipliers, such that, for all $\varphi \in H_1^2(M)$,

$$\int_M \langle \nabla u, \nabla \varphi \rangle_g dv_g = \alpha \int_M \varphi dv_g + \beta \int_M f \varphi dv_g \,.$$

Taking $\varphi = 1$, one gets that $\alpha = 0$. Taking $\varphi = u$, one gets that $\beta = \mu$. Hence, u is a weak solution of the Laplace equation. By standard regularity results, u is smooth. The function $\mu^{-1} u$ is then the solution we were looking for. The proof of uniqueness is then very simple. If u and v are two solutions of the Laplace equation, then $\Delta_g(v - u) = 0$. Multiplying this relation by $v - u$ and integrating over M gives that

$$\int_M |\nabla(v - u)|^2 dv_g = 0 \,.$$

Hence, $v - u$ is constant, and this ends the proof of the above claim. As is easily checked, everything in this proof comes from the compactness of the embedding $H_1^2 \subset L^2$. The equations that we discuss below involve critical (noncompact) Sobolev embeddings.