Chapter One

Chamber Systems

In the first six chapters of this book we introduce Coxeter groups and study them via properties of their Cayley graphs. These Cayley graphs, it will turn out, are a special (but very important!) class of buildings. Arbitrary buildings will be described in terms of certain edge-colored graphs called chamber systems. (The edges of the Cayley graph of a Coxeter group are canonically 'colored' by the generators.) We could postpone introducing chamber systems until Chapter 7 when we really need them, but there is some idiosyncratic usage which comes along with chamber systems ('chambers' instead of 'vertices,' 'galleries' instead of 'paths,' etc.), and it makes sense to introduce this vocabulary right away.

Let $\Delta = (V, E)$ be a graph. By this we mean simply that V is a set and E is a subset of the set of two-element subsets of V. The elements of V are called *vertices* and the elements of E are called *edges*. (If E is the set of *all* two-element subsets of V, then Δ is called a *complete graph*.) A *subgraph* of Δ is a graph (V', E') such that $V' \subset V$ and $E' \subset E$. The subgraph *spanned* by a subset X of V is the subgraph (X, E_X), where E_X consists of *all* the two-element subsets of X which lie in E.

An *edge coloring* of a graph $\Delta = (V, E)$ is a map from the edge set E to a set I whose elements we think of as colors. We will always assume that this map is surjective, so that the set I is unambiguous. An *edge-colored graph* is a graph Δ endowed with an edge coloring. The image I of the edge coloring will be called the *index set* of the edge-colored graph Δ.

Suppose that $\Delta = (V, E)$ is an edge-colored graph with index set I. Rather than give a name to the edge coloring, we will write

$$x \sim_i y$$

(for $x, y \in V$ and $i \in I$) as abbreviation for the statement

'$\{x, y\}$ is an edge of Δ whose color is i.'

Two vertices x and y will be called *i-adjacent* (for some $i \in I$) if $x \sim_i y$, and two vertices will be called *adjacent* if they are i-adjacent for some $i \in I$. (Since E consists of two-element subsets of V, a vertex is never adjacent to itself.)

If a graph $\Delta = (V, E)$ has an edge coloring, we will refer to the elements of V as *chambers* rather than vertices and we will write Δ in place of V, so Δ will refer both to the set of chambers and to the edge-colored graph itself. A subgraph (V', E') of an edge-colored graph Δ will always be assumed to have the edge coloring obtained by restricting the edge coloring of Δ to E'; its index set is a subset of the index set of Δ.

For each set I, we will denote by M_I the *free monoid* on I, i.e. the set of all finite sequences of elements of I including the empty sequence, or equivalently, the set of all words in the alphabet I including the empty word, with multiplication given by concatenation.

Definition 1.1. Let Δ be an edge-colored graph with index set I, let $x, y \in \Delta$ and let $J \subset I$. A *gallery* of length k (for some $k \geqslant 0$) from x to y is a sequence $\gamma = (u_0, u_1, \ldots, u_k)$ of $k + 1$ chambers $u_0, u_1, \ldots, u_k$ such that $u_0 = x$, $u_k = y$ and

$$u_{j-1} \sim_{i_j} u_j$$

for some $i_j \in I$ for all $j \in [1, k]$, and the *type* of the gallery γ is the word $i_1 \cdots i_k$ (an element of the free monoid M_I). A J-gallery is a gallery whose type is in M_J. The *distance* from x to y is the length of a shortest gallery from x to y if there are galleries from x to y, and ∞ otherwise. We will denote the distance from x to y by $\text{dist}(x, y)$. A gallery from x to y is called *minimal* if its length is $\text{dist}(x, y)$. The *diameter* of Δ is the supremum of the set

$$\{\text{dist}(u, v) \mid u, v \in \Delta\};$$

it will be denoted by $\text{diam}\,\Delta$. Two chambers x and y of Δ are said to be *opposite* if $\text{dist}(x, y) = \text{diam}\,\Delta < \infty$ (so if $\text{diam}\,\Delta = \infty$, there are no opposite chambers in Δ).

Definition 1.2. Let Δ be an edge-colored graph with index set I and let $J \subset I$. Then Δ is *connected* (respectively, *J-connected*) if for any two chambers x and y there exists a gallery (respectively, a J-gallery) from x to y. A *connected component* of Δ is the subgraph spanned by an equivalence class with respect to the equivalence relation 'there exists a gallery from x to y' on Δ. A *J-residue* of Δ is a connected component of the subgraph of Δ obtained from Δ by discarding all the edges whose color is *not* in J. (Thus, in particular, for each chamber u of Δ there is a unique J-residue R such that u is a chamber of R.) A *residue* of Δ is a J-residue for some $J \subset I$.

1.3. Note that the notions 'distance,' 'diameter,' 'connected' and 'gallery' (but not the type of a gallery) as defined in 1.1 and 1.2 are (for a given graph Δ)

independent of the edge coloring and are really aspects of the graph Δ itself (i.e. without an edge coloring).

We come now to the main definition of this chapter.

Definition 1.4. A *chamber system* is an edge-colored graph Δ with index set I such that for each $i \in I$, all $\{i\}$-residues of Δ are complete graphs with at least two chambers.

An edge of a chamber system is usually permitted to have more than one color and $\{i\}$-residues of a chamber system are usually permitted to consist of a single chamber. Since neither of these possibilities occur in any of the examples which interest us, we have modified the usual definition of a chamber system to rule them out.

Definition 1.5. A *sub-chamber system* of a chamber system Δ is a subgraph (V', E') of Δ (with the edge coloring obtained by restricting the edge coloring of Δ to E'), which is also a chamber system. It would be more accurate to write 'sub-(chamber system)' rather than 'sub-chamber system,' but we will not, of course, do this.

Note that residues of a chamber system are sub-chamber systems.

Definition 1.6. Let Δ be a chamber system with index set I as defined in 1.4. The cardinality of I is called the *rank* of Δ. If R is a J-residue of Δ, then each chamber of R is contained in at least one edge of every color in J and hence J is the index set of R and the rank of R is $|J|$. We call the set J the *type* of R. A *panel* is a residue of rank one. More precisely, an *i-panel* or *panel of type i* (for some $i \in I$) is a residue of type $\{i\}$. We say that two chambers are *i-equivalent* for some $i \in I$ if they lie on the same i-panel. Thus two chambers are i-equivalent for some $i \in I$ if and only if they are either i-adjacent or equal.

By 1.4 and 1.6, the panels of a chamber system Δ are maximal complete monochromatic subgraphs, panels contain at least two chambers and i-equivalence is, in fact, an equivalence relation for each $i \in I$.

Definition 1.7. A chamber system is *thin* if every panel contains exactly two chambers, *thick* if every panel contains at least three chambers.

Note that a thin chamber system is just an edge-colored graph with the property that each chamber is contained in exactly one edge of each color.

A chamber system of rank zero is just a collection of chambers with no edges (and no colors).

A chamber system of rank one is a graph with edges all of the same color, each of whose connected components is a complete graph with at least two chambers.

A chamber system of rank two can be viewed as a bipartite graph. In fact, chamber systems of rank two and bipartite graphs are essentially the same thing, as we explain in the next two paragraphs.

1.8. Let $\Gamma = (V, E)$ be a bipartite graph (with no edge coloring). This means that there exists a partition of V into subsets V_1 and V_2 such that each edge contains one vertex of V_1 and one of V_2 (equivalently, such that each edge 'joins' a vertex of V_1 to a vertex of V_2). (If Γ is connected, then two vertices are in the same subset if and only if the distance between them is even. It follows that the partition of V is unique in this case.) Suppose, too, that every vertex of Γ has at least two neighbors. Let $\Delta_\Gamma = E$ and let $I = \{1, 2\}$. We set $x \sim_i y$ for $x, y \in \Delta_\Gamma$ and $i \in I$ whenever the edges x and y are distinct but have a vertex in V_i in common. This makes Δ_Γ into a chamber system with index set I which is connected if and only if Γ is connected—thin if and only if every vertex of Γ has exactly two neighbors and thick if and only if every vertex of Γ has at least three neighbors. The chamber system Δ_Γ depends on the choice of V_1 and V_2, but if Γ is connected, it is unique up to a relabeling of the index set.

1.9. Let Δ be a chamber system of rank two and let Γ_Δ denote the graph whose vertices are the panels of Δ, where two panels are joined by an edge if and only if they have a non-empty intersection. Then Γ_Δ is a bipartite graph (since two panels can have a non-empty intersection only if they have different types) all of whose vertices have at least two neighbors. If we restrict our attention to connected bipartite graphs and connected chamber systems of rank two, this construction is the inverse of the construction described in 1.8.

We will need the following definition in 7.32 and 7.33.

Definition 1.10. Let $\Delta_1, \ldots, \Delta_k$ be (a finite number of) edge-colored graphs with pairwise disjoint index sets $I_1, \ldots, I_k$. The *direct product* $\Delta_1 \times \cdots \times \Delta_k$ is the edge-colored graph with index set $I_1 \cup \cdots \cup I_k$ whose chambers are all k-tuples $(x_1, \ldots, x_k) \in \Delta_1 \times \cdots \times \Delta_k$ such that for each $t \in [1, k]$ and each $i \in I_t$,

$$(x_1, \ldots, x_k) \sim_i (y_1, \ldots, y_k)$$

whenever $x_t \sim_i y_t$ in Δ_t and $x_s = y_s$ for all $s \in [1, k]$ distinct from t. If $\Delta_1, \ldots, \Delta_k$ are chamber systems, then so is their direct product.

We now give a number of additional definitions. We have formulated them all for chamber systems rather than for arbitrary edge-colored graphs only because we will have no need for the more general case.

Definition 1.11. Let Δ be a chamber system and let

$$\gamma = (x_0, x_1, \ldots, x_k)$$

be a gallery in Δ. Then $[\gamma]$ denotes the (edge-colored) subgraph of Δ whose vertex set is

$$\{u \mid u = x_i \text{ for some } i \in [0, k]\}$$

and whose edge set is

$$\big\{\{u, v\} \mid \{u, v\} = \{x_{i-1}, x_i\} \text{ for some } i \in [1, k]\big\}.$$

Definition 1.12. Let Δ be a chamber system and let $\Delta' = (V', E')$ be a subgraph of Δ. Then Δ' will be called *convex* if for every two chambers u and v of Δ' and every minimal gallery $(x_0, x_1, \ldots, x_k)$ in Δ from u to v,

$$[\gamma] \subset \Delta',$$

where $[\gamma]$ is as defined in 1.11 (equivalently, $x_i \in V'$ for all $i \in [0, k]$ and $\{x_{i-1}, x_i\} \in E'$ for all $i \in [1, k]$).

Definition 1.13. Let

$$\gamma = (x_0, \ldots, x_{k-1}, x_k) \quad \text{and} \quad \gamma' = (x'_0, x'_1, \ldots, x'_m)$$

be galleries of a chamber system such that either $x_k = x'_0$ or x_k is adjacent to x'_0. The *concatenation* of γ and γ' is the gallery

$$(x_0, \ldots, x_{k-1}, x_k, x'_1, \ldots, x'_m)$$

in the first case and

$$(x_0, \ldots, x_{k-1}, x_k, x'_0, x'_1, \ldots, x'_m)$$

in the second; in both cases, it will be denoted by (γ, γ'). If u is a chamber adjacent to x_0 and v a chamber adjacent to x_k, then we will write simply (u, γ) and (γ, v) to denote the concatenations $((u), \gamma)$ and $(\gamma, (v))$.

Definition 1.14. Two chamber systems Δ and Δ' with index sets I and I' will be called *isomorphic* if there exist bijections σ from I to I' and ϕ from Δ to Δ' (i.e. from chambers to chambers) such that

$$x \sim_i y \text{ if and only if } \phi(x) \sim_{\sigma(i)} \phi(y)$$

for all $x, y \in \Delta$ and all $i \in I$. If (ϕ, σ) is such a pair of bijections, we will say that ϕ is a *σ-isomorphism* from Δ to Δ'. By *isomorphism* we mean a σ-isomorphism for some σ. An isomorphism is *special* (or *type preserving*) if $I = I'$ and the corresponding map σ from I to I is the identity map.

Since isomorphisms map galleries to galleries of the same length, they preserve the distance between chambers (i.e. they are *isometries* with respect to the distance defined in 1.1).

Notation 1.15. Let Δ be a chamber system with index set I. A *σ-automorphism* is a σ-isomorphism from Δ to itself for some permutation σ of I. An *automorphism* of Δ is a σ-automorphism for some σ. The set of all automorphisms of Δ forms a group (under composition of functions) which we denote by $\mathrm{Aut}(\Delta)$. We denote by $\mathrm{Aut}^\circ(\Delta)$ the subgroup of $\mathrm{Aut}(\Delta)$ consisting of just the special automorphisms, i.e. the σ-automorphisms such that $\sigma = 1$. The group $\mathrm{Aut}^\circ(\Delta)$ is a normal subgroup of $\mathrm{Aut}(\Delta)$.

Isomorphisms are a special case of homomorphisms.

Definition 1.16. Let Δ and Δ' be chamber systems with index sets I and I' and let σ be a map from I to I'. A *σ-homomorphism* from Δ to Δ' is a map ρ from chambers to chambers which sends each i-panel of Δ into some $\sigma(i)$-panel of Δ' for all $i \in I$, i.e. such that $\rho(x)$ and $\rho(y)$ are $\sigma(i)$-equivalent chambers of Δ' whenever x and y are i-adjacent chambers of Δ. A *σ-endomorphism* of Δ is a σ-homomorphism from Δ to itself. A *homomorphism* (or *endomorphism*) is a σ-homomorphism (or σ-endomorphism) for some σ. A homomorphism (or endomorphism) is *special* if $I = I'$ and the corresponding map σ from I to I is the identity map.

Definition 1.17. Let Δ be a chamber system. A *pre-gallery* in Δ is a sequence $(u_0, u_1, \ldots, u_k)$ of chambers $u_0, u_1, \ldots, u_k$ such that for each $i \in [1, k]$, the chamber u_i is either adjacent or equal to the chamber u_{i-1}. The *gallery underlying* the pre-gallery γ is the gallery obtained from γ by deleting every chamber which is equal to its successor.

Before going on to the next chapter, we make a few elementary observations about homomorphisms of chamber systems to which we will later refer a number of times, specifically, in the proofs of 3.18, 3.22, 8.5, 8.21 and 9.4.

1.18. Let Δ and Δ' be chamber systems with index sets I and I' and let ρ be a σ-homomorphism from Δ to Δ' for some map σ from I to I'. If γ is a gallery of type $i_1 \cdots i_k$ in Δ, then $\rho(\gamma)$ is a pre-gallery in Δ' and the type of its underlying gallery can be obtained from the word $\sigma(i_1)\cdots\sigma(i_k)$ by deleting letters. In particular,

$$\mathrm{dist}(\rho(x), \rho(y)) \leqslant \mathrm{dist}(x, y)$$

for all $x, y \in \Delta$ (where 'dist' on the left refers to distance in Δ' and 'dist' on the right refers to distance in Δ), and if x, y are two chambers both in the same J-residue of Δ for some $J \subset I$, then $\rho(x)$ and $\rho(y)$ lie in the same $\sigma(J)$-residue of Δ'.

Proposition 1.19. *Let ρ be a special endomorphism of a chamber system Δ and let R be a residue of Δ such that $R \cap \rho(R) \neq \emptyset$. Then $\rho(R) \subset R$.*

Proof. Let J denote the type of R, choose a chamber $x \in R$ such that $\rho(x) \in R$ and let $y \in R$ be arbitrary. Since $x, y \in R$, there is a J-gallery from x to y. By 1.18, therefore, there is a J-gallery from $\rho(x)$ to $\rho(y)$. Since $\rho(x) \in R$, also $\rho(y) \in R$. □