Chapter One

SESQUILINEAR FORMS, ASSOCIATED OPERATORS, AND SEMIGROUPS

1.1 BOUNDED SESQUILINEAR FORMS

Let H be a Hilbert space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We denote by $(.;.)$ the inner product of H and by $\|.\|$ the corresponding norm. Let $\mathfrak{a}$ be a sesquilinear form on H, i.e., $\mathfrak{a}$ is an application from $H \times H$ into $\mathbb{K}$ such that for every $\alpha \in \mathbb{K}$ and $u, v, h \in H$:

$$\mathfrak{a}(\alpha u + v, h) = \alpha \mathfrak{a}(u, h) + \mathfrak{a}(v, h) \text{ and } \mathfrak{a}(u, \alpha v + h) = \overline{\alpha}\mathfrak{a}(u, v) + \mathfrak{a}(u, h). \tag{1.1}$$

Here $\overline{\alpha}$ denotes the conjugate number of α. Of course, $\overline{\alpha} = \alpha$ if $\mathbb{K} = \mathbb{R}$ and in this case the form $\mathfrak{a}$ is then bilinear. For simplicity, we will not distinguish the two cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. We will use the sesquilinear term in both cases and also write conjugate, real part, imaginary part, and so forth of elements in $\mathbb{K}$ as if we had $\mathbb{K} = \mathbb{C}$. These quantities have their obvious meaning if $\mathbb{K} = \mathbb{R}$.

DEFINITION 1.1 *A sesquilinear form $\mathfrak{a} : H \times H \to \mathbb{K}$ is continuous if there exists a constant M such that*

$$|\mathfrak{a}(u, v)| \le M\|u\|\|v\| \text{ for all } u, v \in H.$$

Every continuous form can be represented by a unique bounded linear operator. More precisely,

PROPOSITION 1.2 *Assume that $\mathfrak{a} : H \times H \to \mathbb{K}$ is a continuous sesquilinear form. There exists a unique bounded linear operator T acting on H such that*

$$\mathfrak{a}(u, v) = (Tu; v) \ \text{ for all } u, v \in H.$$

Proof. Fix $u \in H$ and consider the linear continuous functional

$$\phi(v) := \overline{\mathfrak{a}(u, v)}, \ v \in H.$$

By the Riesz representation theorem, there exists a unique vector $Tu \in H$, such that

$$\phi(v) = (v; Tu) \text{ for all } v \in H.$$

The fact that T is a linear and continuous operator on H follows easily from the linearity and continuity of the form $\mathfrak{a}$. The uniqueness of T is obvious. □

The bounded operator T is the operator associated with the form $\mathfrak{a}$. One can study the invertibility of T (or its adjoint T^*) using the form. More precisely, the following basic result holds.

LEMMA 1.3 *(Lax-Milgram) Let $\mathfrak{a}$ be a continuous sesquilinear form on H. Assume that $\mathfrak{a}$ is coercive, that is, there exists a constant $\delta > 0$ such that*

$$\Re\mathfrak{a}(u, u) \geq \delta \|u\|^2 \ for \ all \ u \in H.$$

Let ϕ be a continuous linear functional on H. Then there exists a unique $v \in H$ such that

$$\phi(u) = \mathfrak{a}(u, v) \ \ for \ all \ u \in H.$$

Proof. It suffices to prove that the adjoint operator T^* is invertible on H. Indeed, by the Riesz representation theorem, there exists a unique $g \in H$ such that

$$\phi(u) = (u; g) \ \text{ for all } u \in H,$$

and hence by writing $g = T^*v$ for some $v \in H$, it follows that

$$\phi(u) = (u; T^*v) = (Tu; v) = \mathfrak{a}(u, v) \text{ for all } u \in H.$$

Now we prove that T^* is invertible. Let $v \in H$ be such that $T^*v = 0$. Thus,

$$0 = (v; T^*v) = (Tv; v) = \Re\mathfrak{a}(v, v) \geq \delta \|v\|^2.$$

Hence $v = 0$ and so T^* is injective.

It remains to show that T^* has range $R(T^*) = H$. We first prove that $R(T^*)$ is dense. If $u \in H$ is such that

$$(u; T^*v) = 0 \text{ for all } v \in H,$$

then by taking $v = u$ and using again the coercivity assumption, we obtain $u = 0$. Finally, we prove that $R(T^*)$ is closed. For this, let $v_k = T^*u_k$ be a sequence which converges to v in H. We have

$$\begin{aligned} \delta \|u_k - u_j\|^2 &\leq \Re\mathfrak{a}(u_k - u_j, u_k - u_j) \\ &\leq |(u_k - u_j; T^*u_k - T^*u_j)| \\ &\leq \|u_k - u_j\| \|v_k - v_j\|. \end{aligned}$$

From this, it follows that $(u_k)_k$ is a Cauchy sequence and hence it converges in H. If u denotes the limit, then $v = T^*u$ by continuity of T^*. This proves that $R(T^*)$ is closed. □

1.2 UNBOUNDED SESQUILINEAR FORMS AND THEIR ASSOCIATED OPERATORS

1.2.1 Closed and closable forms

In this section, we consider sesquilinear forms which do not act on the whole space H, but only on subspaces of H. These forms are unbounded sesquilinear forms. They play an important role in the study of elliptic or parabolic equations (cf. Chapters 4 and 5). We will say, for simplicity, sesquilinear forms rather than "unbounded sesquilinear forms."

Let H be as in the previous section and consider a sesquilinear form $\mathfrak{a}$ defined on a linear subspace $D(\mathfrak{a})$ of H, called the domain of $\mathfrak{a}$. That is,

$$\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{K}$$

is a map which satisfies (1.1) for $u, v, h \in D(\mathfrak{a})$.

DEFINITION 1.4 *Let $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{K}$ be a sesquilinear form. We say that:*
1) $\mathfrak{a}$ is densely defined if

$$D(\mathfrak{a}) \textit{ is dense in } H. \tag{1.2}$$

2) $\mathfrak{a}$ is accretive if

$$\Re\mathfrak{a}(u,u) \geq 0 \textit{ for all } u \in D(\mathfrak{a}). \tag{1.3}$$

3) $\mathfrak{a}$ is continuous if there exists a non-negative constant M such that

$$|\mathfrak{a}(u,v)| \leq M\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}} \textit{ for all } u, v \in D(\mathfrak{a}) \tag{1.4}$$

where $\|u\|_{\mathfrak{a}} := \sqrt{\Re\mathfrak{a}(u,u) + \|u\|^2}$.
4) $\mathfrak{a}$ is closed if

$$(D(\mathfrak{a}), \|.\|_{\mathfrak{a}}) \textit{ is a complete space.} \tag{1.5}$$

If $\mathfrak{a}$ satisfies (1.2)−(1.5), one checks easily that $\|.\|_{\mathfrak{a}}$ is a norm on $D(\mathfrak{a})$. It is called the norm associated with the form $\mathfrak{a}$.

DEFINITION 1.5 *Let $\mathfrak{a}$ be a sesquilinear form on H. The adjoint form of $\mathfrak{a}$ is the sesquilinear form $\mathfrak{a}^*$ defined by:*

$$\mathfrak{a}^*(u,v) := \overline{\mathfrak{a}(v,u)} \textit{ with domain } D(\mathfrak{a}^*) = D(\mathfrak{a}).$$

The symmetric part of $\mathfrak{a}$ *is defined by*

$$\mathfrak{b} := \frac{1}{2}(\mathfrak{a} + \mathfrak{a}^*),\ D(\mathfrak{b}) = D(\mathfrak{a}).$$

We say that $\mathfrak{a}$ *is a symmetric form if* $\mathfrak{a}^* = \mathfrak{a}$, *that is,*

$$\mathfrak{a}(u,v) = \overline{\mathfrak{a}(v,u)}\ for\ all\ u,v \in D(\mathfrak{a}).$$

Let $\mathfrak{a}$ be a sesquilinear form which satisfies (1.2)−(1.5). Then $D(\mathfrak{a})$ is a Hilbert space. The inner product is given by

$$(u;v)_{\mathfrak{a}} := \frac{1}{2}[\mathfrak{a}(u,v) + \mathfrak{a}^*(u,v)] + (u;v) \text{ for all } u,v \in D(\mathfrak{a}).$$

The norm $\|.\|_{\mathfrak{a}}$ is the same as $\|.\|_{\mathfrak{b}}$, where $\mathfrak{b}$ is the symmetric part of $\mathfrak{a}$.

On a complex Hilbert space H, every sesquilinear form $\mathfrak{a}$ can be written in terms of symmetric forms $\mathfrak{b}$ and $\mathfrak{c}$ as follows:

$$\mathfrak{a} = \mathfrak{b} + i\mathfrak{c},\ D(\mathfrak{a}) = D(\mathfrak{b}) = D(\mathfrak{c}). \tag{1.6}$$

It suffices indeed to take $\mathfrak{b} := \frac{1}{2}(\mathfrak{a}+\mathfrak{a}^*)$ and $\mathfrak{c} := \frac{1}{2i}(\mathfrak{a}-\mathfrak{a}^*)$. In this way, the symmetric part $\mathfrak{b}$ is seen as the real part of the form $\mathfrak{a}$ and $\mathfrak{c}$ as the imaginary part.

In the present chapter we will consider only accretive forms (i.e., forms that satisfy (1.3)). We could instead consider forms that are merely bounded from below, that is,

$$\Re\mathfrak{a}(u,u) \geq -\gamma(u;u) \text{ for all } u \in D(\mathfrak{a})$$

for some positive constant γ. The general theory of such forms does not differ much from that of accretive ones. A simple perturbation argument (which consists of considering the form $\mathfrak{a} + \gamma$, defined by $(\mathfrak{a} + \gamma)(u,v) := \mathfrak{a}(u,v) + \gamma(u;v)$ for $u,v \in D(\mathfrak{a})$) allows us to consider only accretive forms. According to Section 1.2.3 below, if B denotes the operator associated with the accretive form $\mathfrak{a} + \gamma$, then $A = B - \gamma I$ is the operator associated with $\mathfrak{a}$. Here I denotes the identity operator on H.

If $\mathfrak{a}$ is a symmetric form, the accretivity property (1.3) means that $\mathfrak{a}$ is non-negative, that is,

$$\mathfrak{a}(u,u) \geq 0 \text{ for all } u \in D(\mathfrak{a}).$$

Thus, for symmetric forms, we use both terms non-negative or accretive to refer to the property (1.3).

The condition (1.4) means that the sesquilinear form $\mathfrak{a}$ is continuous on the space $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$. The smallest possible constant M for which (1.4) holds is of some interest (see, e.g., Theorem 1.52).

PROPOSITION 1.6 *Let $\mathfrak{a} : H \times H \to \mathbb{K}$ be a closed accretive sesquilinear form. Then the norms $\|.\|$ and $\|.\|_{\mathfrak{a}}$ are equivalent on H.*

Proof. We have for every $u \in H$

$$\|u\| \le \|u\|_{\mathfrak{a}} = [\|u\|^2 + \Re\mathfrak{a}(u,u)]^{1/2}.$$

In other words, the identity operator $I : (H, \|.\|_{\mathfrak{a}}) \to H$ is continuous. Since I is bijective, its inverse $I^{-1} = I$ is continuous by the closed graph theorem. Hence, there exists a non-negative constant C such that

$$\|u\|_{\mathfrak{a}} \le C\|u\| \text{ for all } u \in H.$$

This shows that the two norms are equivalent. □

A stronger assumption than continuity is sectoriality, which we introduce in the following definition.

DEFINITION 1.7 *A sesquilinear form $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{C}$, acting on a complex Hilbert space H, is called sectorial if there exists a non-negative constant C, such that*

$$|\Im\mathfrak{a}(u,u)| \le C\Re\mathfrak{a}(u,u) \ for\ all\ u \in D(\mathfrak{a}). \tag{1.7}$$

The numerical range of $\mathfrak{a}$ is the set

$$\mathcal{N}(\mathfrak{a}) := \{\mathfrak{a}(u,u),\ u \in D(\mathfrak{a})\ with\ \|u\| = 1\}. \tag{1.8}$$

Clearly, $\mathfrak{a}$ satisfies (1.7) if and only if the numerical range $\mathcal{N}(\mathfrak{a})$ is contained in the closed sector $\{z \in \mathbb{C}, |\arg z| \le \arctan C\}$.

PROPOSITION 1.8 *Every sectorial form acting on a complex Hilbert space H is continuous. More precisely, if*

$$|\Im\mathfrak{a}(u,u)| \le C\Re\mathfrak{a}(u,u) \ for\ all\ u \in D(\mathfrak{a}),$$

where $C \ge 0$ is a constant, then

$$|\mathfrak{a}(u,v)| \le (1+C)(\Re\mathfrak{a}(u,u))^{1/2}(\Re\mathfrak{a}(v,v))^{1/2} \ for\ all\ u,v \in D(\mathfrak{a}).$$

Proof. By (1.6) we have $\mathfrak{a} = \mathfrak{b} + i\mathfrak{c}$, where $\mathfrak{b}$ and $\mathfrak{c}$ are symmetric forms and $\mathfrak{b}$ is non-negative. By the Cauchy-Schwarz inequality,

$$|\mathfrak{b}(u,v)| \le \mathfrak{b}(u,u)^{1/2}\mathfrak{b}(v,v)^{1/2}.$$

It remains to estimate $|\mathfrak{c}(u,v)|$. Changing v into $e^{i\psi}v$ for some ψ, we may assume without loss of generality that $\mathfrak{c}(u,v)$ is real. In this case, we have

$$\mathfrak{c}(u,v) = \frac{1}{4}[\mathfrak{c}(u+v,u+v) - \mathfrak{c}(u-v,u-v)].$$

The sectoriality assumption gives

$$|\mathfrak{c}(u,v)| \le \frac{C}{4}[\mathfrak{b}(u+v,u+v)+\mathfrak{b}(u-v,u-v)]$$
$$= \frac{C}{2}[\mathfrak{b}(u,u)+\mathfrak{b}(v,v)].$$

Replacing u by $\sqrt{\varepsilon}u$ in the last estimate gives

$$|\mathfrak{c}(u,v)| \le \frac{C}{2}\left[\sqrt{\varepsilon}\mathfrak{b}(u,u)+\frac{1}{\sqrt{\varepsilon}}\mathfrak{b}(v,v)\right].$$

If $\mathfrak{b}(u,u) \neq 0$, we choose $\varepsilon = \frac{\mathfrak{b}(v,v)}{\mathfrak{b}(u,u)}$ and obtain

$$|\mathfrak{c}(u,v)| \le C\mathfrak{b}(u,u)^{1/2}\mathfrak{b}(v,v)^{1/2} = C(\Re\mathfrak{a}(u,u))^{1/2}(\Re\mathfrak{a}(v,v))^{1/2}.$$

If $\mathfrak{b}(u,u) = 0$, then $\mathfrak{c}(u,v) \le \frac{C}{2}\mathfrak{b}(v,v)$. Replacing v by λv for $\lambda > 0$ and letting $\lambda \to 0$, one obtains $\mathfrak{c}(u,v) = 0$, which gives again the desired conclusion. □

A converse to Proposition 1.8 is given by the following simple lemma.

LEMMA 1.9 *If $\mathfrak{a}$ is an accretive and continuous sesquilinear form on a complex Hilbert space H, then $1+\mathfrak{a}$ is sectorial. More precisely, if $\mathfrak{a}$ satisfies (1.4) with some constant M, then*

$$|\Im[(u;u)+\mathfrak{a}(u,u)]| \le M\Re[(u;u)+\mathfrak{a}(u,u)] \; for\ all\ u \in D(\mathfrak{a})$$

Proof. The lemma follows immediately from

$$|\Im[(u;u)+\mathfrak{a}(u,u)]| = |\Im\mathfrak{a}(u,u)| \le |\mathfrak{a}(u,u)|$$

and the continuity assumption (1.4). □

Note that the continuity assumption of the form $\mathfrak{a}$ is sometimes written in the following way:

$$|\mathfrak{a}(u,v)| \le M'[\Re\mathfrak{a}(u,u)+w\|u\|^2]^{1/2}[\Re\mathfrak{a}(v,v)+w\|v\|^2]^{1/2}$$

for some constants w and M'. It is clear that the norms $[\Re\mathfrak{a}(u,u)+w\|u\|^2]^{1/2}$ and $[\Re\mathfrak{a}(u,u)+\|u\|^2]^{1/2}$ are equivalent. For this reason, we have chosen to write (1.4) and $\|.\|_{\mathfrak{a}}$ without the extra constant w.

It may happen in some problems that the starting form $\mathfrak{a}$ satisfies the properties (1.2)−(1.4) but not (1.5). In this case, one tries to find an extension of $\mathfrak{a}$ which is a closed form and acts on a subspace of H.

DEFINITION 1.10 *A densely defined accretive sesquilinear form $\mathfrak{a}$ is called closable if there exists a closed accretive form $\mathfrak{c}$, acting on a subspace $D(\mathfrak{c})$ of H, such that $D(\mathfrak{a}) \subseteq D(\mathfrak{c})$ and $\mathfrak{a}(u,v) = \mathfrak{c}(u,v)$ for all $u,v \in D(\mathfrak{a})$.*

A closed extension, when it exists, is not unique in general.[1] Nevertheless, in that case, one can define the smallest closed extension $\overline{\mathfrak{a}}$. It is tempting to define $\overline{\mathfrak{a}}$ as follows:

$$D(\overline{\mathfrak{a}}) := \{u \in H \text{ s.t. } \exists u_n \in D(\mathfrak{a}) : u_n \to u \text{ (in } H) \text{ and } \mathfrak{a}(u_n - u_m, u_n - u_m) \to 0 \text{ as } n, m \to \infty\}$$

and

$$\overline{\mathfrak{a}}(u, v) := \lim_{n\to\infty} \mathfrak{a}(u_n, v_n) \tag{1.9}$$

for $u, v \in D(\overline{\mathfrak{a}})$, where $(u_n)_n$ and $(v_n)_n$ are any sequences of elements of $D(\mathfrak{a})$ which converge respectively to u and v (with respect to the norm of H) and satisfy $\mathfrak{a}(u_n - u_m, u_n - u_m) \to 0$ and $\mathfrak{a}(v_n - v_m, v_n - v_m) \to 0$ as $n, m \to \infty$.

PROPOSITION 1.11 *Let $\mathfrak{a}$ be a densely defined, accretive, and continuous sesquilinear form. If $\mathfrak{a}$ is closable, then $\overline{\mathfrak{a}}$ is well defined and satisfies (1.2)–(1.5). In addition, every closed extension of $\mathfrak{a}$ is also an extension of $\overline{\mathfrak{a}}$.*

Proof. Fix two sequences $(u_n)_n$ and $(v_n)_n$ of elements of $D(\mathfrak{a})$ which converge for the norm of H and satisfy $\mathfrak{a}(u_n - u_m, u_n - u_m) \to 0$ and $\mathfrak{a}(v_n - v_m, v_n - v_m) \to 0$ as $n, m \to \infty$. In order to see that $\lim_{n\to\infty} \mathfrak{a}(u_n, v_n)$ exists, we write using the continuity assumption (1.4):

$$|\mathfrak{a}(u_n, v_n) - \mathfrak{a}(u_m, v_m)| = |\mathfrak{a}(u_n - u_m, v_n) + \mathfrak{a}(u_m, v_n - v_m)| \le M\|u_n - u_m\|_{\mathfrak{a}}\|v_n\|_{\mathfrak{a}} + M\|v_n - v_m\|_{\mathfrak{a}}\|u_m\|_{\mathfrak{a}}.$$

Since $\|u_n - u_m\|_{\mathfrak{a}}$ and $\|v_n - v_m\|_{\mathfrak{a}} \to 0$ as $n, m \to \infty$, the sequences $(\|u_n\|_{\mathfrak{a}})_n$ and $(\|v_n\|_{\mathfrak{a}})_n$ are bounded. It follows from the previous inequality that $\mathfrak{a}(u_n, v_n)$ is a Cauchy sequence, thus it is convergent.

The fact that $\lim_{n\to\infty} \mathfrak{a}(u_n, v_n)$ is independent of the chosen sequences $(u_n)_n$ and $(v_n)_n$ follows by a similar argument. Indeed, if $(u'_n)_n$ and $(v'_n)_n$ satisfy the same properties as $(u_n)_n$ and $(v_n)_n$, then

$$|\mathfrak{a}(u_n, v_n) - \mathfrak{a}(u'_n, v'_n)| = |\mathfrak{a}(u_n - u'_n, v_n) + \mathfrak{a}(u'_n, v_n - v'_n)| \le M\|u_n - u'_n\|_{\mathfrak{a}}\|v_n\|_{\mathfrak{a}} + M\|v_n - v'_n\|_{\mathfrak{a}}\|u'_n\|_{\mathfrak{a}}.$$

Now, if $\mathfrak{a}_1$ is a closed extension of $\mathfrak{a}$ then $\|u_n - u'_n\|_{\mathfrak{a}} = \|u_n - u'_n\|_{\mathfrak{a}_1} \to 0$ as $n \to \infty$, since $(u_n)_n$ and $(u'_n)_n$ converge to the same limit in the Hilbert

[1] A simple example is given by the form $\mathfrak{a}(u, v) = \int_{(0,1)} \frac{d}{dx}u \frac{d}{dx}v dx$, $D(\mathfrak{a}) = C_c^\infty(0, 1)$. The same expression with domains the Sobolev spaces $H_0^1(0, 1)$ and $H^1(0, 1)$ gives two different closed extensions. See Chapter 4 for more examples.

space $(D(\mathfrak{a}_1), \|.\|_{\mathfrak{a}_1})$. Applying the same argument to $\|v_n - v'_n\|_{\mathfrak{a}}$, we obtain $|\mathfrak{a}(u_n, v_n) - \mathfrak{a}(u'_n, v'_n)| \to 0$ as $n \to \infty$.

By construction, $D(\mathfrak{a})$ is dense in $(D(\overline{\mathfrak{a}}), \|.\|_{\overline{\mathfrak{a}}})$ and hence (1.2)–(1.4) hold for the form $\overline{\mathfrak{a}}$. This density shows also that every closed extension of $\mathfrak{a}$ is an extension of $\overline{\mathfrak{a}}$.

Finally, we show that $\overline{\mathfrak{a}}$ is closed. Let $(u_n)_n \in D(\mathfrak{a})$ be a Cauchy sequence for the norm of $\overline{\mathfrak{a}}$. It converges with respect to the norm of H to some $u \in H$. It follows from the definition of $\overline{\mathfrak{a}}$ that $u \in D(\overline{\mathfrak{a}})$. In addition,

$$\overline{\mathfrak{a}}(u_n - u, u_n - u) = \lim_m \mathfrak{a}(u_n - u_m, u_n - u_m).$$

Thus,

$$\lim_n \overline{\mathfrak{a}}(u_n - u, u_n - u) = 0,$$

which means that the sequence $(u_n)_n$ is convergent in $(D(\overline{\mathfrak{a}}), \|.\|_{\overline{\mathfrak{a}}})$. This together with the density of $D(\mathfrak{a})$ in $(D(\overline{\mathfrak{a}}), \|.\|_{\overline{\mathfrak{a}}})$ show that $\overline{\mathfrak{a}}$ is a closed form. □

DEFINITION 1.12 *If the form $\mathfrak{a}$ is closable, then $\overline{\mathfrak{a}}$ defined by (1.9) with domain $D(\overline{\mathfrak{a}})$ is called the closure of the form $\mathfrak{a}$.*

Remark. 1) The proof of Proposition 1.11 shows that if $\mathfrak{a}$ is any sesquilinear form satisfying (1.2)–(1.4) and $(u_n)_n, (v_n)_n$ are convergent sequences in H, such that $\mathfrak{a}(u_n - u_m, u_n - u_m)$ and $\mathfrak{a}(v_n - v_m, v_n - v_m) \to 0$ as $n, m \to \infty$, then the limit in the right-hand side of (1.9) exists. In addition, if $\mathfrak{a}$ is closed then this limit is $\mathfrak{a}(u, v)$, where u and v are the limits in H of $(u_n)_n$ and $(v_n)_n$, respectively.

2) It follows also from the same proof that if $\mathfrak{a}$ is a sesquilinear form satisfying (1.2)–(1.4), then the form $\overline{\mathfrak{a}}$ is closed whenever it is well defined (i.e., the limit in the right-hand side of (1.9) does not depend on the chosen sequences $(u_n)_n$ and $(v_n)_n$).

PROPOSITION 1.13 *Let $\mathfrak{a}$ be a densely defined, accretive, and continuous sesquilinear form. Then $\mathfrak{a}$ is closable if and only if it satisfies the following property:*
If $(u_n)_n \in D(\mathfrak{a}), u_n \to 0$ in H and $\mathfrak{a}(u_n - u_m, u_n - u_m) \to 0$ $(as\ n, m \to \infty)$, then $\mathfrak{a}(u_n, u_n) \to 0$ as $n \to \infty$.

Proof. Assume that $\mathfrak{a}$ is closable and let $\mathfrak{a}_1$ be a closed extension. If $u_n \to 0$ in H and $\mathfrak{a}(u_n - u_m, u_n - u_m) \to 0$, then $(u_n)_n$ converges to 0 in $(D(\mathfrak{a}_1), \|.\|_{\mathfrak{a}_1})$. The above proposition (or the above Remark 1)) implies that $\mathfrak{a}(u_n, u_n) = \mathfrak{a}_1(u_n, u_n) \to 0$.

Now we prove the converse. We construct a closed extension by taking the completion of $D(\mathfrak{a})$ with respect to the norm $\|.\|_{\mathfrak{a}}$. That is, we prove that the form $\overline{\mathfrak{a}}$ given by (1.9) with domain $D(\overline{\mathfrak{a}})$ is well defined (by Remark 2) above, $\overline{\mathfrak{a}}$ will be a closed extension of $\mathfrak{a}$). As mentioned in Remark 1) above, the limit in the right-hand side of (1.9) exists. It remains to prove that the limit is independent of the chosen sequences $(u_n)_n$ and $(v_n)_n$. Let $(u'_n)_n$ and $(v'_n)_n$ be two other sequences satisfying $u'_n \to u, v'_n \to v$ in H and $\|u'_n - u'_m\|_{\mathfrak{a}}, \|v'_n - v'_m\|_{\mathfrak{a}} \to 0$ as $n, m \to \infty$. We write again

$$|\mathfrak{a}(u_n, v_n) - \mathfrak{a}(u'_n, v'_n)| = |\mathfrak{a}(u_n - u'_n, v_n) + \mathfrak{a}(u'_n, v_n - v'_n)| \\ \leq M\|u_n - u'_n\|_{\mathfrak{a}}\|v_n\|_{\mathfrak{a}} + M\|v_n - v'_n\|_{\mathfrak{a}}\|u'_n\|_{\mathfrak{a}}.$$

The sequence $w_n := u_n - u'_n$ satisfies $w_n \to 0$ in H and

$$\|w_n - w_m\|_{\mathfrak{a}} \leq \|u_n - u_m\|_{\mathfrak{a}} + \|u'_n - u'_m\|_{\mathfrak{a}} \to 0 \text{ as } n, m \to \infty.$$

By assumption, this implies $\mathfrak{a}(w_n, w_n) \to 0$ as $n \to \infty$. The same argument applies to $v_n - v'_n$. Hence, $|\mathfrak{a}(u_n, v_n) - \mathfrak{a}(u'_n, v'_n)| \to 0$ as $n \to \infty$. □

Lemma 1.29 below guarantees the closability for a class of sesquilinear forms. There are several examples of sesquilinear forms which are not closable.

Example 1.2.1 *Consider on $L^2(\mathbb{R})$ (endowed with the Lebesgue measure dx) the symmetric form*

$$\mathfrak{a}(u, v) = u(0)\overline{v(0)}, \; D(\mathfrak{a}) = C_c(\mathbb{R}), \tag{1.10}$$

where $C_c(\mathbb{R})$ is the space of continuous and compactly supported functions on $\mathbb{R}$. Then $\mathfrak{a}$ is densely defined, symmetric, and non-negative but not closable. Indeed, choose a sequence $(u_n)_n \in C_c(\mathbb{R})$ such that $u_n(0) = 1$ for all n and such that $u_n \to 0$ in $L^2(\mathbb{R})$. Thus, $\mathfrak{a}(u_n - u_m, u_n - u_m) = 0$ and $u_n \to 0$ in $L^2(\mathbb{R})$ but $\mathfrak{a}(u_n, u_n) = 1$ for all n. Proposition 1.13 shows that $\mathfrak{a}$ is not closable.

Example 1.2.2 *Consider now on the real space $L^2(\mathbb{R})$ (endowed again with the Lebesgue measure dx) the form*

$$\mathfrak{a}(u, v) = \int_{\mathbb{R}} \frac{du}{dx}(x)v(x)dx, \; D(\mathfrak{a}) = H^1(\mathbb{R}). \tag{1.11}$$

The form $\mathfrak{a}$ is not closable. Otherwise, the fact that $\mathfrak{a}(u, u) = 0$ for all $u \in H^1(\mathbb{R})$ implies that $D(\overline{\mathfrak{a}}) = L^2(\mathbb{R})$ and one deduces from Proposition 1.2 that there exists a bounded linear operator T on $L^2(\mathbb{R})$ such that $\mathfrak{a}(u, v) = (Tu; v)$ for all $u, v \in H^1(\mathbb{R})$. This is not possible since $Tu = \frac{du}{dx}$ for $u \in H^1(\mathbb{R})$ and T cannot be extended to a bounded operator on $L^2(\mathbb{R})$.

Example 1.2.2 shows also that the conclusion of Proposition 1.13 cannot hold if the form is not continuous.

DEFINITION 1.14 *Let* $\mathfrak{a}$ *be a densely defined accretive sesquilinear form on* H*. A linear subspace* D *of* $D(\mathfrak{a})$ *is called a core of* $\mathfrak{a}$ *if* D *is dense in* $D(\mathfrak{a})$ *endowed with the norm* $\|.\|_{\mathfrak{a}}$.

Let D be a linear subspace of $D(\mathfrak{a})$. The restriction of $\mathfrak{a}$ to D is the form $\mathfrak{a}_{|D}$, defined by

$$\mathfrak{a}_{|D}(u,v) = \mathfrak{a}(u,v),\ D(\mathfrak{a}_{|D}) = D.$$

A relationship between closability and the notion of core is given by the following.

PROPOSITION 1.15 *Let* $\mathfrak{a}$ *be a densely defined, accretive, continuous, and closed sesquilinear form. Denote by* D *a linear subspace of* $D(\mathfrak{a})$*. Then* D *is a core of* $\mathfrak{a}$ *if and only if the closure of* $\mathfrak{a}_{|D}$ *is* $\mathfrak{a}$*, i.e.,* $\overline{\mathfrak{a}_{|D}} = \mathfrak{a}$.

Proof. The form $\mathfrak{a}$ is a closed extension of $\mathfrak{a}_{|D}$, hence it is an extension of $\overline{\mathfrak{a}_{|D}}$.

Assume that D is a core of $\mathfrak{a}$ and let $u \in D(\mathfrak{a})$. There exists a sequence $(u_n) \in D$ such that $\|u_n - u\|_{\mathfrak{a}} \to 0$ as $n \to \infty$. Hence, (u_n) converges to u in H and $\mathfrak{a}(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$. This shows that $u \in D(\overline{\mathfrak{a}_{|D}})$. Therefore $\overline{\mathfrak{a}_{|D}} = \mathfrak{a}$.

Conversely, assume that $\overline{\mathfrak{a}_{|D}} = \mathfrak{a}$ and let $u \in D(\mathfrak{a}) = D(\overline{\mathfrak{a}_{|D}})$. It follows from the definition of the closure that there exists a sequence (u_n) in D which converges in H to u and such that $\mathfrak{a}_{|D}(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$. This means that (u_n) converges to u with respect to the norm $\|.\|_{\mathfrak{a}}$, which shows that D is a core of $\mathfrak{a}$. □

1.2.2 Perturbation of sesquilinear forms

In this section, we study perturbations of forms. The main questions concern closability and continuity of the sum of two sesquilinear forms.

The sum $\mathfrak{a} + \mathfrak{b}$ of two sesquilinear forms $\mathfrak{a}$ and $\mathfrak{b}$ on H is defined by

$$[\mathfrak{a} + \mathfrak{b}](u,v) := \mathfrak{a}(u,v) + \mathfrak{b}(u,v), \quad D(\mathfrak{a} + \mathfrak{b}) = D(\mathfrak{a}) \cap D(\mathfrak{b}).$$

THEOREM 1.16 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two accretive sesquilinear forms on* H*. Then the sum* $\mathfrak{a} + \mathfrak{b}$ *is accretive. In addition,*
1) If $\mathfrak{a}$ *and* $\mathfrak{b}$ *are continuous, then so is* $\mathfrak{a} + \mathfrak{b}$.
2) If $\mathfrak{a}$ *and* $\mathfrak{b}$ *are closed, then so is* $\mathfrak{a} + \mathfrak{b}$.
3) If $\mathfrak{a}$ *and* $\mathfrak{b}$ *are closable, then so is* $\mathfrak{a} + \mathfrak{b}$.

Proof. The accretivity of the sum as well as assertion 1) are obvious. Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are closed. Let $(u_n)_n \in D(\mathfrak{a}) \cap D(\mathfrak{b})$ be a Cauchy sequence for the norm $\|.\|_{\mathfrak{a}+\mathfrak{b}}$. The inequalities $\|.\|_{\mathfrak{a}} \le \|.\|_{\mathfrak{a}+\mathfrak{b}}$ and $\|.\|_{\mathfrak{b}} \le \|.\|_{\mathfrak{a}+\mathfrak{b}}$ imply that $(u_n)_n$ is a Cauchy sequence for the norms $\|.\|_{\mathfrak{a}}$ and for $\|.\|_{\mathfrak{b}}$. It follows that $(u_n)_n$ converges both in $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$ and $(D(\mathfrak{b}), \|.\|_{\mathfrak{b}})$. The limit in both spaces is the same since the convergence in each space implies the convergence in H. The limit belongs then to $D(\mathfrak{a}) \cap D(\mathfrak{b})$. The inequality $\|.\|_{\mathfrak{a}+\mathfrak{b}} \le \|.\|_{\mathfrak{a}} + \|.\|_{\mathfrak{b}}$ implies that $(u_n)_n$ converges with respect to the norm $\|.\|_{\mathfrak{a}+\mathfrak{b}}$. Hence, $\mathfrak{a}+\mathfrak{b}$ is closed.

If both forms $\mathfrak{a}$ and $\mathfrak{b}$ are closable, then the sum of their closures $\overline{\mathfrak{a}} + \overline{\mathfrak{b}}$ is a closed form by assertion 2). Thus, $\overline{\mathfrak{a}} + \overline{\mathfrak{b}}$ is a closed extension of $\mathfrak{a}+\mathfrak{b}$. The latter is then a closable form, its closure $\overline{\mathfrak{a}+\mathfrak{b}}$ is a restriction of $\overline{\mathfrak{a}} + \overline{\mathfrak{b}}$. □

DEFINITION 1.17 *Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed sesquilinear form on H. A sesquilinear form $\mathfrak{a}'$ with domain $D(\mathfrak{a}')$ is $\mathfrak{a}$-form bounded if $D(\mathfrak{a}) \subseteq D(\mathfrak{a}')$ and there exist non-negative constants α and β such that*

$$|\mathfrak{a}'(u,u)| \le \alpha|\mathfrak{a}(u,u)| + \beta\|u\|^2 \text{ for all } u \in D(\mathfrak{a}). \tag{1.12}$$

The infimum of all possible constants α for which the inequality holds is called the $\mathfrak{a}$-bound of $\mathfrak{a}'$.

Under closability assumptions on the forms, $\mathfrak{a}'$ is $\mathfrak{a}$-bounded as soon as $D(\mathfrak{a}) \subseteq D(\mathfrak{a}')$. More precisely,

PROPOSITION 1.18 *Let $\mathfrak{a}$ and $\mathfrak{a}'$ be accretive and continuous forms. Assume that $\mathfrak{a}$ is closed, $\mathfrak{a}'$ is closable, and $D(\mathfrak{a}) \subseteq D(\mathfrak{a}')$. Then $\mathfrak{a}'$ is $\mathfrak{a}$-bounded.*

Proof. Since the form $\mathfrak{a}'$ is closable, its restriction to $D(\mathfrak{a})$, $\mathfrak{a}'_{|D(\mathfrak{a})} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{K}$ is also closable. Thus, $\mathfrak{a}'_{|D(\mathfrak{a})}$ is a closable form acting on the Hilbert space $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$. Its closure (as a form on $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$) is itself. By Proposition 1.6, there exists a non-negative constant M such that for all $u \in D(\mathfrak{a})$

$$|\mathfrak{a}'(u,u)| \le M\|u\|_{\mathfrak{a}}^2 = M[\|u\|^2 + \Re\mathfrak{a}(u,u)].$$

This proves that $\mathfrak{a}'$ is $\mathfrak{a}$-bounded. □

Assume that $\mathfrak{a}$ is an accretive and continuous sesquilinear form. Denote by $\mathfrak{b} := \frac{1}{2}[\mathfrak{a} + \mathfrak{a}^*]$ the symmetric part of $\mathfrak{a}$. Recall that $\mathfrak{b}(u,u) = \Re\mathfrak{a}(u,u)$ for all $u \in D(\mathfrak{a})$. Using the continuity of $\mathfrak{a}$, we have for all $u \in D(\mathfrak{a})$

$$\mathfrak{b}(u,u) \le |\mathfrak{a}(u,u)| \le M\|u\|_{\mathfrak{a}}^2 = M[\|u\|^2 + \mathfrak{b}(u,u)].$$

It follows from this that a form $\mathfrak{a}'$ is $\mathfrak{a}$-bounded if and only if it is $\mathfrak{b}$-bounded.

The next theorem shows that continuity and closability properties carry over from a form $\mathfrak{a}$ to $\mathfrak{a} + \mathfrak{a}'$ provided the form $\mathfrak{a}'$ is $\mathfrak{a}$-bounded with small bound.

THEOREM 1.19 *Let $\mathfrak{a}$ be an accretive and continuous sesquilinear form on a complex Hilbert space H. Assume that $\mathfrak{a}'$ is a sesquilinear form such that $D(\mathfrak{a}) \subseteq D(\mathfrak{a}')$ and*

$$|\mathfrak{a}'(u,u)| \le \alpha \Re \mathfrak{a}(u,u) + \beta \|u\|^2 \text{ for all } u \in D(\mathfrak{a}), \tag{1.13}$$

where α, β are non-negative constants with $\alpha < 1$. Then the form sum $\mathfrak{t} := \mathfrak{a} + \mathfrak{a}' + \beta$ with domain $D(\mathfrak{t}) = D(\mathfrak{a})$ is accretive and continuous. Moreover,
1) $\mathfrak{t}$ is closed if and only if $\mathfrak{a}$ is closed.
2) $\mathfrak{t}$ is closable if and only if $\mathfrak{a}$ is closable.

Proof. The domain of $\mathfrak{t}$ is $D(\mathfrak{a})$, since $D(\mathfrak{a}) \subseteq D(\mathfrak{a}')$. By (1.13) and the fact that $\alpha < 1$, we have for $u \in D(\mathfrak{a})$,

$$\Re \mathfrak{t}(u,u) = \Re \mathfrak{a}(u,u) + \Re \mathfrak{a}'(u,u) + \beta \|u\|^2 \ge (1-\alpha) \Re \mathfrak{a}(u,u) \ge 0.$$

Thus, $\mathfrak{t}$ is accretive.

Using the continuity of $\mathfrak{a}$, we obtain by Lemma 1.9 and (1.13)

$$\begin{aligned} |\Im \mathfrak{t}(u,u)| &\le |\Im \mathfrak{a}(u,u)| + |\Im \mathfrak{a}'(u,u)| \\ &\le \alpha \Re \mathfrak{a}(u,u) + \beta \|u\|^2 + M[\Re \mathfrak{a}(u,u) + \|u\|^2] \\ &\le \frac{M+\alpha}{1-\alpha} \Re \mathfrak{t}(u,u) + (M+\beta)\|u\|^2 \\ &\le C \Re[\mathfrak{t}(u,u) + \|u\|^2] \end{aligned}$$

for some non-negative constant C. Hence, the form $\mathfrak{t}+1$ is sectorial. Proposition 1.8 implies that $\mathfrak{t}$ is continuous.

The inequalities

$$\Re \mathfrak{t}(u,u) \ge (1-\alpha)\Re \mathfrak{a}(u,u) \text{ and } \Re \mathfrak{t}(u,u) \le (1+\alpha)\Re \mathfrak{a}(u,u) + 2\beta \|u\|^2$$

show that the norms $\|.\|_{\mathfrak{a}}$ and $\|.\|_{\mathfrak{t}}$ are equivalent on $D(\mathfrak{a})$. From this, assertion 1) follows immediately. To prove assertion 2), let us assume that $\mathfrak{a}$ is closable and let $(u_n) \in D(\mathfrak{a})$ such that $u_n \to 0$ in H and $\mathfrak{t}(u_n - u_m, u_n - u_m) \to 0$ as $m, n \to \infty$. Then, $\Re \mathfrak{a}(u_n - u_m, u_n - u_m) \to 0$ as $m, n \to \infty$, since the norms $\|.\|_{\mathfrak{a}}$ and $\|.\|_{\mathfrak{t}}$ are equivalent. By continuity of $\mathfrak{a}$, it follows that $\mathfrak{a}(u_n - u_m, u_n - u_m) \to 0$. Proposition 1.13 asserts that $\mathfrak{a}(u_n, u_n) \to 0$ as $n \to \infty$. As previously, we obtain from this and continuity of $\mathfrak{t}$, that $\mathfrak{t}(u_n, u_n) \to 0$ as $n \to \infty$ and we conclude by Proposition 1.13 that $\mathfrak{t}$ is closable. The converse holds for the same reasons. □

The following proposition is extracted from the previous proof.

PROPOSITION 1.20 *Let $\mathfrak{a}$ and $\mathfrak{a}'$ be two accretive and continuous sesquilinear forms on a Hilbert space H. Assume that $D(\mathfrak{a}) = D(\mathfrak{a}')$ and the norms $\|.\|_{\mathfrak{a}}$ and $\|.\|_{\mathfrak{a}'}$ are equivalent. Then*
1) $\mathfrak{a}$ is closed if and only if $\mathfrak{a}'$ is closed.
2) $\mathfrak{a}$ is closable if and only if $\mathfrak{a}'$ is closable.

1.2.3 Associated operator

Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed sesquilinear form on H. One can define in terms of $\mathfrak{a}$ an unbounded operator A, defined on a linear subspace $D(A)$ of H as follows:

$u \in D(\mathfrak{a})$ is in the domain $D(A)$ of A, if and only if there exists $v \in H$ such that the equality $\mathfrak{a}(u, \phi) = (v; \phi)$ holds for all $\phi \in D(\mathfrak{a})$. We then set $Au := v$.

We rewrite this as

$$D(A) = \{u \in H \text{ s.t. } \exists v \in H : \mathfrak{a}(u, \phi) = (v; \phi)\ \forall \phi \in D(\mathfrak{a})\},\ Au := v.$$

Observe also that $D(A)$ is the set of vectors $u \in D(\mathfrak{a})$ for which the mapping $\phi \mapsto \mathfrak{a}(u, \phi)$ is continuous on $D(\mathfrak{a})$ with respect to the norm of H.

DEFINITION 1.21 *The linear operator A, defined above, is called the operator associated with the form $\mathfrak{a}$.*

There are several important properties of operators which are associated with sesquilinear forms. We start with the following result.

PROPOSITION 1.22 *Denote by A the operator associated with a densely defined, accretive, continuous, and closed sesquilinear form $\mathfrak{a}$. Then A is densely defined and for every $\lambda > 0$, the operator $\lambda I + A$ is invertible (from $D(A)$ into H) and its inverse $(\lambda I + A)^{-1}$ is a bounded operator on H (here I is the identity operator). In addition,*

$$\|\lambda(\lambda I + A)^{-1} f\| \leq \|f\| \ for\ all\ \lambda > 0, f \in H.$$

Proof. Fix $\lambda > 0$ and put

$$\|u\|_\lambda := \sqrt{\Re\mathfrak{a}(u, u) + \lambda\|u\|^2},\ u \in D(\mathfrak{a}).$$

The norm $\|.\|_\lambda$ is equivalent to the norm $\|.\|_{\mathfrak{a}}$ and hence $V := (D(\mathfrak{a}), \|.\|_\lambda)$ is a Hilbert space. It follows from (1.4) that the form $\lambda + \mathfrak{a}^*$ (defined by $(\lambda + \mathfrak{a}^*)(u, v) = \lambda(u; v) + \mathfrak{a}^*(u, v)$) is bounded on V. It is in addition coercive on V.

Let $f \in H$ and define

$$\phi(v) := (v; f), \ v \in V.$$

Clearly, ϕ is a linear continuous functional on V. Thus by Lemma 1.3, there exists a unique $u \in V$ such that

$$\phi(v) = \mathfrak{a}^*(v, u) + \lambda(v; u) = \overline{\mathfrak{a}(u, v)} + \lambda(v; u) \text{ for all } v \in V.$$

It follows from this and the definition of A that $u \in D(A)$ and $(\lambda I + A)u = f$. Thus, $\lambda I + A$ has range $R(\lambda I + A) = H$. The accretivity assumption (1.3) implies easily that $\lambda I + A$ is injective and hence invertible.

Let now $f \in H$ and let $u \in D(A)$ be such that $(\lambda I + A)u = f$. Taking the inner product with u, and using

$$\Re(Au; u) = \Re\mathfrak{a}(u, u) \geq 0,$$

it follows that

$$\Re(f; u) \geq \lambda \|u\|^2.$$

This implies that $\lambda\|u\| \leq \|f\|$. That is,

$$\|\lambda(\lambda I + A)^{-1} f\| \leq \|f\|.$$

Finally, we show that $D(A)$ is dense in H. Let $v \in H$ be such that

$$(v; u) = 0 \ \text{ for all } u \in D(A).$$

Since $I + A$ is invertible, there exists $\psi \in D(A)$ such that $v = (I + A)\psi$. Applying the above equality with $u = \psi$, we obtain

$$0 = (v; \psi) = ((I + A)\psi; \psi) = \|\psi\|^2 + (A\psi; \psi).$$

This together with the fact that $\Re(A\psi; \psi) = \Re\mathfrak{a}(\psi, \psi) \geq 0$ implies that $\psi = 0$ and hence $v = 0$. □

Note that if the sesquilinear form $\mathfrak{a}$ satisfies (1.2)–(1.5), then the adjoint form $\mathfrak{a}^*$ satisfies the same conditions. One then associates an operator with $\mathfrak{a}^*$. It turns out that this operator is the adjoint A^* of A. Let us recall the definition of the adjoint for unbounded operators.

DEFINITION 1.23 *Let B be a densely defined operator acting in H. The adjoint of B is the operator B^* defined by*

$$D(B^*) = \{u \in H \ s.t. \ \exists v \in H : (B\phi; u) = (\phi; v) \ for \ all \ \phi \in D(B)\},$$
$$B^* u := v.$$

A symmetric operator B is an operator such that $D(B) \subseteq D(B^*)$ and $Bu = B^*u$ for all $u \in D(B)$.

The operator B is self-adjoint if $B^* = B$. This means that $D(B) = D(B^*)$ and $Bu = B^*u$ for all $u \in D(B)$.

PROPOSITION 1.24 *The operator associated with $\mathfrak{a}^*$ is A^*. In particular, if $\mathfrak{a}$ is symmetric then A is self-adjoint.*

Proof. Denote by B the operator associated with $\mathfrak{a}^*$ and let $u \in D(B)$. By definition,

$$\mathfrak{a}^*(u, \phi) = (Bu; \phi) \text{ for all } \phi \in D(\mathfrak{a}^*) = D(\mathfrak{a}).$$

Hence

$$(Bu; \phi) = \mathfrak{a}^*(u, \phi) = \overline{\mathfrak{a}(\phi, u)} = \overline{(A\phi; u)} \text{ for all } \phi \in D(A).$$

This shows that $u \in D(A^*)$ and $A^*u = Bu$. It remains to prove that $D(A^*) \subseteq D(B)$. For this, fix $v \in D(A^*)$. By Proposition 1.22, there exists $\psi \in D(B)$ such that $(I + A^*)v = (I + B)\psi$. Hence $(I + A^*)v = (I + A^*)\psi$. Thus,

$$(v - \psi; (I + A)u) = ((I + A^*)(v - \psi); u) = 0 \text{ for all } u \in D(A).$$

Since $I + A$ is invertible, this implies that $v = \psi \in D(B)$. □

We have seen in Proposition 1.22 that the operator A associated with $\mathfrak{a}$ is densely defined in H. It is also densely defined in $D(\mathfrak{a})$, endowed with the norm $\|.\|_{\mathfrak{a}}$. This is formulated in the following lemma whose proof is postponed to Section 1.4.2.

LEMMA 1.25 *Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed sesquilinear form and denote by A its associated operator. Then $D(A)$ is a core of $\mathfrak{a}$.*

Using this lemma and Proposition 1.15 one concludes that the form $\mathfrak{a}$ coincides with the closure of the restriction of $\mathfrak{a}$ to $D(A)$, that is, $\mathfrak{a} = \overline{\mathfrak{a}_{|D(A)}}$.

DEFINITION 1.26 *1) An operator $B : D(B) \subseteq H \to H$ is called sectorial if there exists a non-negative constant C, such that*

$$|\Im(Bu; u)| \leq C\Re(Bu; u) \; for \; all \; u \in D(B). \tag{1.14}$$

2) The numerical range of an operator B on H is the set

$$\mathcal{N}(B) := \{(Bu; u), u \in D(B) \; with \; \|u\| = 1\}.$$

Clearly, B satisfies (1.14) if and only if its numerical range $\mathcal{N}(B)$ is contained in the sector $\{z \in \mathbb{C}, |\arg z| \leq \arctan C\}$.

It is also clear that the operator associated with a sectorial form is a sectorial operator. The converse is also true. We formulate this in the following proposition.

PROPOSITION 1.27 *Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed sesquilinear form acting on a complex Hilbert space H. Denote by A the operator associated with $\mathfrak{a}$. The following assertions are equivalent:*
1) $\mathfrak{a}$ is a sectorial form.
2) A is a sectorial operator.

The proposition is an immediate consequence of the following lemma.

LEMMA 1.28 *Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed sesquilinear form acting on a complex Hilbert space H and denote by A its associated operator. Then, the numerical range $\mathcal{N}(A)$ of A is dense in the numerical range $\mathcal{N}(\mathfrak{a})$ of $\mathfrak{a}$.*

Proof. Apply Lemma 1.25. □

LEMMA 1.29 *Let A be a densely defined operator on a Hilbert space H such that $\Re(Au; u) \geq 0$ for all $u \in D(A)$. Assume that either:*
1) H is complex and there exists a constant $\alpha \geq 0$, such that the operator $\alpha I + A$ is sectorial,
or
2) A is a symmetric operator (here H may be real).
Then the form defined by

$$\mathfrak{a}(u, v) := (Au; v) \; with \; domain \; D(\mathfrak{a}) = D(A)$$

is closable.

Proof. Assume that 1) is satisfied. Write

$$\mathfrak{a}(u, v) = ((\alpha I + A)u; v) - \alpha(u; v).$$

By Proposition 1.8, the sectorial form $(u, v) \to ((\alpha I + A)u; v)$ is continuous and hence $\mathfrak{a}$ is continuous, too.

If 2) is satisfied, then $\mathfrak{a}$ is continuous. This follows from the Cauchy-Schwarz inequality.

In order to prove that $\mathfrak{a}$ is closable, we apply Proposition 1.13. Assume that $(u_n) \in D(A)$ is such that $u_n \to 0$ in H and $\mathfrak{a}(u_n - u_m, u_n - u_m) \to 0$ (as $n, m \to \infty$). By continuity of the form, we have

$$\begin{aligned} |\mathfrak{a}(u_n, u_n)| &\leq |\mathfrak{a}(u_n - u_m, u_n)| + |\mathfrak{a}(u_m, u_n)| \\ &\leq M\|u_n - u_m\|_{\mathfrak{a}}\|u_n\|_{\mathfrak{a}} + |(Au_m; u_n)|. \end{aligned}$$

By assumption, $\|u_n - u_m\|_{\mathfrak{a}} \to 0$ as $m, n \to \infty$ and thus $\|u_n\|_{\mathfrak{a}}$ is a bounded sequence. In addition, for each $m, |(Au_m; u_n)| \to 0$ as $n \to \infty$. These properties imply that $\mathfrak{a}(u_n, u_n) \to 0$ as $n \to \infty$. This proves that $\mathfrak{a}$ is closable. □

The role of assumptions 1) and 2) in Lemma 1.29 is to guarantee the continuity of the form

$$\mathfrak{a}(u, v) := (Au; v), \; D(\mathfrak{a}) = D(A). \tag{1.15}$$

The proof shows that this form is closable whenever it is continuous. The assumption of continuity cannot be removed. Also, on a real Hilbert space, the single assumption that $(Au, u) \geq 0$ for all $u \in D(A)$, is not enough to guarantee the closability of the form $\mathfrak{a}$. All this can be seen from Example 1.2.2.

When the form defined by (1.15) is closable, the operator associated with its closure is clearly an extension of A.

Definition 1.30 *The operator associated with the closure $\overline{\mathfrak{a}}$ of the form $\mathfrak{a}$ defined by (1.15) is called the Friedrichs extension of A.*

Proposition 1.31 *Let $B : D(B) \subseteq H \to H$ be a closed operator (see Definition 1.33 below) with dense domain $D(B)$. Then B^*B defined by*

$$D(B^*B) = \{u \in D(B), Bu \in D(B^*)\}, \; B^*Bu = B^*(Bu)$$

is a densely defined self-adjoint operator.

Proof. Define the symmetric form

$$\mathfrak{a}(u, v) = (Bu; Bv), \quad D(\mathfrak{a}) = D(B).$$

Since B is a closed operator, $\mathfrak{a}$ is a closed form. Thus, there exists a self-adjoint (and densely defined) operator A associated with $\mathfrak{a}$. By definition,

$$D(A) = \{u \in D(\mathfrak{a}), \exists v \in H : \mathfrak{a}(u, \phi) = (v; \phi) \; \forall \phi \in D(\mathfrak{a})\}, \; Au = v.$$

Thus,

$$\begin{aligned} D(A) &= \{u \in D(B), \exists v \in H : (Bu; B\phi) = (v; \phi) \; \forall \phi \in D(B)\} \\ &= \{u \in D(B), Bu \in D(B^*)\}, \end{aligned}$$

and $Au = B^*(Bu)$. This shows that $A = B^*B$ and proves the proposition. □

We finish this section with the following lemma, which is related to the results of the previous section. Its proof requires certain results of the present section.

LEMMA 1.32 *Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed form on a Hilbert space H. Assume that $(u_n)_n$ is a bounded sequence in $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$ which converges in H to u. Then $u \in D(\mathfrak{a})$ and we have $\Re\mathfrak{a}(u,u) \leq \liminf_n \Re\mathfrak{a}(u_n, u_n)$.*

Proof. The sequence $(u_n)_n$ is bounded in the Hilbert space $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$, thus it has a weakly convergent subsequence. Let (u_{n_k}) be this subsequence and $\phi \in D(\mathfrak{a})$ be its weak limit. For every $v \in D(\mathfrak{a})$,

$$(u_{n_k}; v) + \mathfrak{b}(u_{n_k}, v) \to (\phi; v) + \mathfrak{b}(\phi, v) \text{ as } n_k \to \infty, \tag{1.16}$$

where $\mathfrak{b}$ denotes the symmetric part of $\mathfrak{a}$. Denote by B the self-adjoint operator associated with $\mathfrak{b}$. The above convergence holds for $v \in D(B)$ and hence

$$(u_{n_k}; (I+B)v) \to (\phi; (I+B)v).$$

Now the fact that u_{n_k} converges to u in H implies that

$$(u; (I+B)v) = (\phi; (I+B)v) \text{ for all } v \in D(B).$$

By Proposition 1.22, $I + B$ is invertible and hence $u = \phi \in D(\mathfrak{a})$.

Taking $v = u$ in (1.16), yields $\mathfrak{b}(u,u) = \lim_k \mathfrak{b}(u_{n_k}, u)$. This and the Cauchy-Schwarz inequality imply $\mathfrak{b}(u,u) \leq \liminf \mathfrak{b}(u_{n_k}, u_{n_k})$. It follows that $\mathfrak{b}(u,u) \leq \liminf b(u_n, u_n)$ since we can replace $(u_n)_n$ in the above arguments by any subsequence. □

Remark. We have used in the proof only that some subsequence of (u_n) converges weakly to u. Therefore, the conclusion of the lemma holds under the weaker assumption that (u_n) is a bounded sequence in $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$ which converges weakly in H to u.

1.3 SEMIGROUPS AND UNBOUNDED OPERATORS

1.3.1 Closed and closable operators

Throughout this section, E denotes a Banach space (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with norm $\|.\|$. By $\mathcal{L}(E)$, we denote the space of all bounded linear operators on E.

DEFINITION 1.33 *An operator $B : D(B) \subseteq E \to E$ is called a closed operator if the graph*

$$G(B) := \{(u; Bu), u \in D(B)\}$$

is closed in $E \times E$.

This definition can be rephrased as follows:

If $(x_n)_n \in D(B)$ is such that $x_n \to x$ and $Bx_n \to y$ in E (as $n \to \infty$), then $x \in D(B)$ and $y = Bx$.

Note also that B is a closed operator if and only if $D(B)$ endowed with the graph norm $\|.\| + \|B.\|$ is a complete space.

DEFINITION 1.34 *Let $B : D(B) \subseteq E \to E$ be an operator on E. A scalar $\lambda \in \mathbb{K}$ is in the resolvent set of B if $\lambda I - B$ is invertible (from $D(B)$ into E) and its inverse $(\lambda I - B)^{-1}$ is a bounded operator on E. For such λ, the operator $(\lambda I - B)^{-1}$ is the resolvent of B at λ.*
The set

$$\rho(B) := \{\lambda \in \mathbb{K}, \lambda I - B \; is \; invertible \; and \; (\lambda I - B)^{-1} \in \mathcal{L}(E)\}$$

is called the resolvent set of B.

The complement of $\rho(B)$ in $\mathbb{K}$

$$\sigma(B) := \mathbb{K} \setminus \rho(B)$$

is called the spectrum of B.

PROPOSITION 1.35 *1) Assume that B is a closed operator on a Banach space E. Then a scalar λ is in $\rho(B)$ if and only if $\lambda I - B$ is invertible (from $D(B)$ into E).*
2) If the resolvent set $\rho(B)$ is not empty, then B is a closed operator.

Proof. In order to prove the first assertion we have to prove that $(\lambda I - B)^{-1}$ is a continuous operator for every λ such that $\lambda I - B$ is invertible. Let $(y_n)_n$ be a sequence in E which converges to y and such that $(\lambda I - B)^{-1} y_n$ converges to z. Set $x_n := (\lambda I - B)^{-1} y_n$. We have $x_n \in D(B)$ for each n and $(\lambda I - B)x_n$ converges to y. Since B is a closed operator, it follows that $z \in D(B)$ and $y = (\lambda I - B)z$, that is, $z = (\lambda I - B)^{-1} y$. We conclude now by the closed graph theorem that $(\lambda I - B)^{-1}$ is continuous on E.

Assume that $\lambda \in \rho(B)$ for some λ. Let (x_n) be a sequence in $D(B)$ such that $x_n \to x$ and $Bx_n \to y$ in E. Thus, $(\lambda I - B)x_n \to \lambda x - y$ and by continuity of $(\lambda I - B)^{-1}$, we have $x_n \to (\lambda I - B)^{-1}(\lambda x - y)$. Thus, $x = (\lambda I - B)^{-1}(\lambda x - y)$. This implies that $x \in D(B)$ and $Bx = y$. This shows that B is a closed operator. □

DEFINITION 1.36 *An operator B on a Banach space E is closable if there exists a closed operator $C : D(C) \subseteq E \to E$ such that $D(B) \subseteq D(C)$ and $Bu = Cu$ for all $u \in D(B)$. In other words, B has a closed extension C.*

Assume that B is a closable operator on a Banach space E. One can define the smallest closed extension $\overline{B}$ of B as follows:

$$D(\overline{B}) = \{u \in E \; s.t. \; \exists u_n \in D(B) : \lim_n u_n = u, \; \lim_{n,m}[Bu_n - Bu_m] = 0\}, \tag{1.17}$$

and if u and $(u_n)_n$ are as in (1.17) we set

$$\overline{B}u := \lim_n Bu_n, \tag{1.18}$$

where the limits are taken with respect to the norm of E.

One shows easily that $\overline{B}$ is a closed operator and every closed extension of B is also an extension of $\overline{B}$.

If B is an operator such that $\overline{B}$, defined by (1.17) and (1.18), is well defined (i.e., $\overline{B}u = \lim_n Bu_n$ does not depend on the choice of the sequence (u_n)), then $\overline{B}$ is a closed extension of B. Consequently, B is closable if and only if $\overline{B}$ is a well defined operator.

Let now $u \in D(\overline{B})$ and let $u_n \in D(B), v_n \in D(B)$ be two sequences which converge to u and such that $Bu_n - Bu_m \to 0$ and $Bv_n - Bv_m \to 0$ as $n, m \to \infty$. Thus, Bu_n and Bv_n converge to some w and w' in E. Now, $\overline{B}$ is well defined if and only if $w = w'$. Thus, we have proved the following characterization of closable operators.

PROPOSITION 1.37 *A linear operator B on E is closable if and only if it satisfies the following property:*
if $(u_n) \in D(B)$ is any sequence such that $u_n \to 0$ and $Bu_n \to v$ (in E), then $v = 0$.

DEFINITION 1.38 *Assume that B is a closable operator on a Banach space E. The operator $\overline{B}$ defined by (1.17) and (1.18) is called the closure of B.*

DEFINITION 1.39 *Let B be an operator with domain $D(B)$ on a Banach space E. A linear subspace of $D(B)$ is called a core of B if it is dense in $D(B)$, endowed with the graph norm $\|.\| + \|B.\|$.*

Let B act on a Banach space E and D a linear subspace of $D(B)$. The restriction of B to D is the operator

$$B_{|D}u := Bu \text{ for } u \in D = D(B_{|D}).$$

The next result follows easily from the previous definitions.

PROPOSITION 1.40 *Let B be a closed operator on a Banach space E and D a linear subspace of $D(B)$. Then, D is a core of B if and only if the closure of $B_{|D}$ is B, i.e., $\overline{B_{|D}} = B$.*

1.3.2 A rapid course on semigroup theory

In this subsection, we give some definitions and recall several important results and properties of semigroups. Semigroup theory is a well documented subject and we shall not give a detailed study. For more details and proofs of the classical results given below see, e.g., Arendt et al. [ABHN01], Davies [Dav80], Goldstein [Gol85], Engel and Nagel [EnNa99], Kato [Kat80], Nagel et al. [Nag86], Pazy [Paz83], Yosida [Yos65].

DEFINITION 1.41 *1) A semigroup on a Banach space E is a family of bounded linear operators $(T(t))_{t\geq 0}$ acting on E such that*

$$T(0) = I \text{ and } T(t+s) = T(t)T(s) \text{ for all } t, s \geq 0.$$

2) A semigroup $(T(t))_{t\geq 0}$ is called a contraction semigroup (or a contractive semigroup) if $T(t)$ is a contraction operator on E for each $t \geq 0$.
3) We say that a semigroup $(T(t))_{t\geq 0}$ is strongly continuous if for every $u \in E$, we have

$$\lim_{t\downarrow 0} T(t)u = u.$$

Note that the property in 3) is precisely the strong continuity at $t_0 = 0$. From this and the semigroup property it follows that $(T(t))_{t\geq 0}$ is strongly continuous at each $t_0 \in [0, \infty)$.

DEFINITION 1.42 *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on E. The generator of $(T(t))_{t\geq 0}$ is the operator B defined by*

$$D(B) := \{u \in E, \lim_{t\downarrow 0} \frac{1}{t}(T(t)u - u) \text{ exists}\},$$

$$Bu := \lim_{t\downarrow 0} \frac{1}{t}(T(t)u - u) \text{ for all } u \in D(B).$$

The theory of strongly continuous semigroups was developed in order to study existence and uniqueness of solutions to the evolution equations (or the Cauchy problem)

$$(CP) \begin{cases} \frac{d}{dt}u(t) = & Bu(t), \quad t \geq 0, \\ \quad u(0) = & f, \end{cases}$$

where $u : [0, \infty) \to E$ satisfies $u(t) \in D(B)$ for all $t > 0$ is the searched for solution.

If B is the generator of the strongly continuous semigroup $(T(t))_{t\geq 0}$, then for every $f \in D(B)$, (CP) has a unique solution. The latter is given by $u(t) = T(t)f$.

The following is a central theorem in semigroup theory.

THEOREM 1.43 *(Hille-Yosida) Let B be a densely defined operator on E. The following assertions are equivalent:*
i) B is the generator of a strongly continuous semigroup.
ii) There exists a constant w such that $(w, \infty) \subseteq \rho(B)$ and

$$\sup_{\lambda > w, n \in \mathbb{N}} \|(\lambda - w)^n (\lambda I - B)^{-n}\|_{\mathcal{L}(E)} < \infty.$$

If the operator B is bounded on E, then it generates a strongly continuous semigroup. In addition, this semigroup is given by

$$e^{tB} = \sum_{k \geq 0} \frac{t^k B^k}{k!}.$$

By analogy to this case, we will denote by $(e^{tB})_{t \geq 0}$ the strongly continuous semigroup generated by the operator B, even when B is not bounded.[2]

The resolvent of the generator B coincides with the Laplace transform of the semigroup, that is,

$$(\lambda I - B)^{-1} = \int_0^\infty e^{-\lambda t} e^{tB} dt \text{ for all } \lambda > w.$$

Conversely, the semigroup can be written in terms of the resolvent. This is given by the exponential formula

$$e^{tB} u = \lim_n (I - \frac{t}{n} B)^{-n} u \text{ for all } u \in E.$$

DEFINITION 1.44 *Let $\psi \in (0, \frac{\pi}{2}]$ and denote by $\Sigma(\psi)$ the open sector*

$$\Sigma(\psi) := \{z \in \mathbb{C}, z \neq 0 \; and |\arg z| < \psi\}.$$

A strongly continuous semigroup $(T(t))_{t \geq 0}$ acting on E is called a bounded holomorphic semigroup on the sector $\Sigma(\psi)$ if $(T(t))_{t \geq 0}$ admits a holomorphic extension $(T(z))_{z \in \Sigma(\psi)}$ such that for each $\theta \in (0, \psi)$, $(T(z))_{z \in \Sigma(\theta)}$ is uniformly bounded and strongly continuous at 0.

If the boundedness assumption on smaller sectors is not required, we say that $(T(t))_{t \geq 0}$ is a holomorphic semigroup on $\Sigma(\psi)$. Finally, by a holomorphic semigroup we mean a semigroup that is holomorphic on some sector of angle > 0.

Note that a holomorphic semigroup on the sector $\Sigma(\psi)$ satisfies

$$T(z + z') = T(z) T(z') \text{ for all } z, z' \in \Sigma(\psi).$$

[2] This notation makes sense for self-adjoint operators by the functional calculus.

This is a consequence of holomorphy and the corresponding property for z and $z' \geq 0$.

Holomorphic semigroups play an important role in the theory of evolution equations and functional calculi. In particular, if the semigroup generated by B is holomorphic, then

$$(CP) \begin{cases} \frac{d}{dt}u(t) = & Bu(t), \;\; t > 0, \\ \;\;\; u(0) = & f \end{cases}$$

have a unique solution for every initial data $f \in E$.

The following theorem characterizes generators of bounded holomorphic semigroups (for a proof see, e.g., [ABHN01], Theorem 3.7.11, [Nag86], Theorem 1.12 Chap. A II, or [EnNa99], Theorem 4.5 Chap. II).

THEOREM 1.45 *Let B be a densely defined operator on a complex Banach space E. Then B generates a semigroup which is bounded holomorphic on $\Sigma(\psi)$ if and only if $\Sigma(\psi + \frac{\pi}{2}) \subseteq \rho(B)$ and for every $\theta \in (0, \psi)$, one has*

$$\sup_{\lambda \in \Sigma(\theta + \frac{\pi}{2})} \|\lambda(\lambda I - B)^{-1}\|_{\mathcal{L}(E)} < \infty.$$

Assume that B generates a semigroup $(T(t))_{t \geq 0}$ which is bounded holomorphic on $\Sigma(\psi)$ for some $\psi > 0$. An application of the Cauchy formula shows that there exists a constant M such that

$$\|BT(t)u\| \leq \frac{M}{t}\|u\| \text{ for all } u \in E \text{ and } t > 0. \tag{1.19}$$

The holomorphy of the semigroup $(T(t))_{t \geq 0}$ on the sector $\Sigma(\psi)$ also implies that for every $\theta \in (-\psi, \psi)$, $(T(e^{i\theta}t))_{t \geq 0}$ is a strongly continuous semigroup on E whose generator is $e^{i\theta}B$. These results and more information on holomorphic semigroups can be found in the books mentioned at the beginning of this section.

1.3.3 Accretive operators on Hilbert spaces

Denote again by H a Hilbert space with inner product $(.;.)$ and norm $\|.\|$. Let A be an operator on H, with domain $D(A)$.

DEFINITION 1.46 *We say that A is an accretive operator if*

$$\Re(Au; u) \geq 0 \; for \; all \; u \in D(A).$$

An operator A is m-accretive (or maximal accretive) if it is accretive and $1 \in \rho(-A)$.

It is clear that the operator associated with an accretive form is an accretive operator. If the sesquilinear form is densely defined, accretive, continuous, and closed, then its associated operator is m-accretive (see Proposition 1.22).

Note also that if A is an accretive operator that satisfies the assumptions of Lemma 1.29, its Friedrichs extension is an m-accretive operator. In particular, every densely defined symmetric accretive operator has an m-accretive symmetric extension. Therefore, every densely defined symmetric accretive operator has a self-adjoint extension.

LEMMA 1.47 *Let A be a densely defined accretive operator on H. Then A is closable, its closure $\overline{A}$ is accretive, and for every $\lambda \in \mathbb{K}$, the range $R(\lambda I + A)$ is dense in $R(\lambda I + \overline{A})$.*

Proof. Let $(u_n)_n \in D(A)$ be a sequence such that u_n converges to 0 and Au_n converges to v in H. Take $w \in D(A)$ and apply the accretivity assumption to obtain

$$\begin{aligned} 0 &\leq \Re(A(u_n + w); u_n + w) \\ &= \Re(Au_n; u_n) + \Re(Au_n; w) + \Re(Aw; u_n) + \Re(Aw; w). \end{aligned}$$

Letting $n \to \infty$, we obtain $\Re(v; w) + \Re(Aw; w) \geq 0$. We apply this with λw in place of w for $\lambda > 0$ and let $\lambda \to 0$ to obtain $\Re(v; w) \geq 0$. The same inequality applied to $-w$ allows us to conclude that $\Re(v; w) = 0$ and hence $(v; w) = 0$. Since this holds for all $w \in D(A)$, which is dense in H, it follows that $v = 0$. We conclude by Proposition 1.37 that A is closable.

The accretivity of $\overline{A}$ as well as the density of $R(\lambda + A)$ in $R(\lambda + \overline{A})$ follows easily from the definition of $\overline{A}$ and simple approximation arguments.
□

LEMMA 1.48 *Let A be a densely defined operator on H.*
1) Assume that A is closed and accretive. Then $I + A$ is injective and has closed range. In particular, A is m-accretive if and only if $I + A$ has dense range.
2) If A is m-accretive, then $(0, \infty) \subseteq \rho(-A)$ and $\lambda(\lambda I + A)^{-1}$ is a contraction operator on H for every $\lambda > 0$.
3) Assume that A is accretive and denote by $\overline{A}$ its closure (cf. Lemma 1.47). Then $\overline{A}$ is m-accretive if and only if there exists $\lambda > 0$ such that $\lambda I + A$ has dense range.

Proof. 1) Let $u \in D(A)$ be such that $u + Au = 0$. The accretivity of A implies that

$$(u; u) \leq \Re(u + Au; u) = 0$$

and hence $u = 0$. This shows that $I + A$ is injective.

In order to show that $I + A$ has closed range, we let $(u_n)_n \in D(A)$ be such that the sequence $u_n + Au_n$ converges to $v \in H$. Since $(u_n; u_n) \leq \Re(u_n + Au_n; u_n)$ it follows that u_n is a bounded sequence. Writing

$$\begin{aligned}(u_n - u_m; u_n - u_m) &\leq \Re(u_n - u_m; u_n - u_m + Au_n - Au_m)\\ &\leq \|u_n - u_m\| \|u_n + Au_n - u_m - Au_m\|,\end{aligned}$$

we see that $(u_n)_n$ is a Cauchy sequence. If u denotes the limit of $(u_n)_n$, then Au_n converges to $v - u$. The fact that A is closed implies that $u \in D(A)$ and $v = u + Au \in R(I + A)$.

By Proposition 1.35, $1 \in \rho(-A)$ if and only if $I + A$ is invertible. Hence, A is m-accretive if and only if $I + A$ has dense range.

2) Assume that A is m-accretive and let $\lambda > 0$. By Proposition 1.35, A is a closed operator. Thus, by the same proposition, $\lambda \in \rho(-A)$ if and only if $\lambda I + A$ is invertible. Applying assertion 1) to the accretive operator $\lambda^{-1}A$, we see that it is enough to prove that $\lambda I + A$ has dense range. Let $f \in H$ be such that

$$(f; \lambda u + Au) = 0 \text{ for all } u \in D(A).$$

Since A is m-accretive we can find $v \in D(A)$ such that $f = v + Av$. Applying the previous equality with $u = v$, gives $v = 0$ and hence $f = 0$. Thus, $R(\lambda I + A)$ is dense.

Now fix $f \in H$ and let $u \in D(A)$ be such that $f = \lambda u + Au$. We write

$$\begin{aligned}\|f\|^2 &= \Re(\lambda u + Au; \lambda u + Au)\\ &\geq \lambda^2 \|u\|^2 + 2\lambda \Re(Au; u)\\ &\geq \lambda^2 \|u\|^2.\end{aligned}$$

This implies that $\lambda(\lambda I + A)^{-1}$ is a contraction operator on H.

3) By assertion 2), if $\overline{A}$ is m-accretive then $\lambda I + \overline{A}$ is invertible for $\lambda > 0$. By Lemma 1.47, the range $R(\lambda + A)$ is dense in $R(\lambda I + \overline{A}) = H$.

Conversely, assume that $\lambda I + A$ has dense range for some $\lambda > 0$. This implies that $I + \lambda^{-1}A$ has dense range. Thus, by 1), $\lambda^{-1}\overline{A}$ is m-accretive. Assertion 2) implies now that $\alpha I + \lambda^{-1}\overline{A}$ is invertible for all $\alpha > 0$. This implies in particular that $I + \overline{A}$ is invertible and hence $\overline{A}$ is m-accretive. ☐

The following result is a particular case of the well-known Lumer-Phillips theorem for generators of contraction semigroups.

THEOREM 1.49 *Let A be a densely defined operator on H. The following assertions are equivalent:*
1) The operator A is closable and $-\overline{A}$ is the generator of a strongly continuous contraction semigroup on H.

2) $\overline{A}$ *is m-accretive.*
3) A *is accretive and there exists* $\lambda > 0$ *such that* $\lambda I + A$ *has dense range.*

Proof. The fact that 2) and 3) are equivalent is already stated in Lemma 1.48.

Assume now that 2) holds. By Lemma 1.48, $(0, \infty) \subseteq \rho(-\overline{A})$ and $\lambda(\lambda I + \overline{A})^{-1}$ is a contraction operator on H. The Hille-Yosida theorem implies that $-A$ generates a strongly continuous semigroup $(e^{-t\overline{A}})_{t\geq 0}$ on H. Moreover, for every $u \in D(\overline{A})$, we have

$$\frac{d}{dt}\|e^{-t\overline{A}}u\|^2 = -2\Re(\overline{A}e^{-t\overline{A}}u; e^{-t\overline{A}}u) \leq 0.$$

Hence, $\|e^{-t\overline{A}}u\|^2 \leq \|u\|^2$ for all $t \geq 0$. From this and the density of $D(\overline{A})$, it follows that $e^{-t\overline{A}}$ is a contraction operator on H for every $t \geq 0$. This shows assertion 1).

Conversely, assume that A is closable and $-\overline{A}$ generates a strongly continuous contraction semigroup $(e^{-t\overline{A}})_{t\geq 0}$. Hence, for $t \geq 0$

$$\Re(u - e^{-t\overline{A}}u; u) \geq 0 \text{ for all } u \in H.$$

Applying this to $u \in D(\overline{A})$ yields

$$\Re(\overline{A}u; u) = \lim_{t\downarrow 0} \frac{1}{t}\Re(u - e^{-t\overline{A}}u; u) \geq 0,$$

which shows that $\overline{A}$ is accretive. Since $-\overline{A}$ is the generator of a strongly continuous contraction semigroup, $(0, \infty) \subseteq \rho(-A)$. Thus $\overline{A}$ is m-accretive. □

Let A be a densely defined accretive operator on H and denote by $\overline{A}$ its closure. The operator $\overline{A}$ is accretive, too. The next theorem gives a sufficient condition under which the operator $\overline{A}$ is m-accretive.

THEOREM 1.50 *Let* A *be an accretive operator on* H*. Assume that* S *is an m-accretive operator satisfying the following two conditions:*
1) $D(S) \subseteq D(A)$.
2) There exists a constant $a \in \mathbb{R}$ *such that*

$$\Re(Au; Su) \geq -a(u; Su) \; for \; all \; u \in D(S).$$

Then the closure $\overline{A}$ *of* A *is m-accretive and* $D(S)$ *is a core of* $\overline{A}$.

Proof. Of course, we can assume $a \geq 0$. Considering now $A + aI$ instead of A and applying assertion 3) of Lemma 1.48, we see that we can assume $a = 0$.

Let $B_n := A(I + \frac{1}{n}S)^{-1}$ for $n \geq 1$. For each n, the operator B_n is bounded on H (apply assumption 1) and the closed graph theorem). In addition,

$$\Re\left(Au; \left(I + \frac{1}{n}S\right)u\right) = \Re(Au; u) + \frac{1}{n}\Re(Au; Su) \geq 0 \text{ for all } u \in D(S).$$

This implies that B_n is accretive. On the other hand, the resolvent set of the bounded operator B_n is not empty; thus by assertion 3) of Lemma 1.48 we can conclude that B_n is m-accretive. From the formula

$$I + A + \frac{1}{n}S = (I + B_n)\left(I + \frac{1}{n}S\right)$$

it follows that the operator $I + A + \frac{1}{n}S$ with domain $D(S)$ is invertible. Hence for every $f \in H$ and every n, there exists $u_n \in D(S)$ such that

$$u_n + Au_n + \frac{1}{n}Su_n = f. \tag{1.20}$$

We claim that

$$\|u_n\| \leq \|f\| \text{ and } \left\|\frac{1}{n}Su_n\right\| \leq 2\|f\|. \tag{1.21}$$

The first inequality follows form the accretivity of $A + \frac{1}{n}S$, since

$$\begin{aligned}(u_n; u_n) &\leq \Re\left(u_n + Au_n + \frac{1}{n}Su_n; u_n\right)\\ &= \Re(f; u_n)\\ &\leq \|f\|\|u_n\|.\end{aligned}$$

The second inequality follows from the first one and the following estimates

$$\begin{aligned}\|\frac{1}{n}Su_n\|^2 &\leq \Re\left(Au_n + \frac{1}{n}Su_n; \frac{1}{n}Su_n\right)\\ &= \Re\left(f - u_n; \frac{1}{n}Su_n\right)\\ &\leq (\|f\| + \|u_n\|)\left\|\frac{1}{n}Su_n\right\|.\end{aligned}$$

Now we prove that $\overline{A}$ is m-accretive. By Lemma 1.48, it suffices to prove that $I + \overline{A}$ has dense range. Actually, we will show that the operator $I + A$ with domain $D(S)$ has dense range. Let $f \in H$ be such that

$$(f; u + Au) = 0 \text{ for all } u \in D(S). \tag{1.22}$$

Let $(u_n)_n \in D(S)$ be a sequence satisfying (1.20). By (1.22) we have

$$(f; f) = \left(u_n + Au_n + \frac{1}{n}Su_n; f\right) = \left(\frac{1}{n}Su_n; f\right). \quad (1.23)$$

On the other hand, since S is m-accretive, it follows that the adjoint operator S^* is densely defined.[3] Consider a sequence $(f_k)_k \in D(S^*)$ which converges to f in H. We have

$$\left|\left(\frac{1}{n}Su_n; f\right) - \left(\frac{1}{n}Su_n; f_k\right)\right| \le \left\|\frac{1}{n}Su_n\right\| \|f - f_k\|$$
$$\le 2\|f\|\|f - f_k\|.$$

It follows that $|(\frac{1}{n}Su_n; f) - (\frac{1}{n}Su_n; f_k)|$ converges to 0 as $k \to \infty$, uniformly with respect to n. Thus, using the fact that

$$\left|\left(\frac{1}{n}Su_n; f_k\right)\right| = \frac{1}{n}|(u_n; S^* f_k)| \le \frac{1}{n}\|f\|\|S^* f_k\|$$

(which follows from (1.21)), we obtain

$$\left(\frac{1}{n}Su_n; f\right) \to 0 \text{ as } n \to \infty.$$

We conclude from (1.23) that $f = 0$. Thus, we have proved that the closure B of the restriction of A to $D(S)$ is m-accretive. Since $I + \overline{A}$ is an extension of $I + B$, it follows that $(I + \overline{A})D(\overline{A}) = H$. Thus, $\overline{A}$ is m-accretive.

Finally, it remains to prove the equality $B = \overline{A}$ and conclude that $D(S)$ is a core of $\overline{A}$. If $u \in D(\overline{A})$, there exists $v \in D(B)$ such that

$$u + \overline{A}u = v + Bv = v + \overline{A}v.$$

It follows from the fact that $I + \overline{A}$ is injective that $u = v \in D(B)$. □

Remark. Let S and A be two operators acting in a Banach space X, with norm $\|.\|$. We assume that S is closed, A is closable and $D(S) \subseteq D(A)$. Then there exist two constants a and b such that

$$\|Au\| \le a\|Su\| + b\|u\| \text{ for all } u \in D(S).$$

In order to prove this, we first observe that $D(S)$, endowed with the graph norm $\|.\| + \|S.\|$, is a Banach space and the restriction A_S of A to $D(S)$

[3] It is easily seen that $I + S^*$ is invertible with inverse $(I + S^*)^{-1} = ((I + S)^{-1})^*$. Now, if $u \in H$ is such that $(u; v) = 0$ for all $v \in D(S^*)$, we write $u = (I + S)\phi$ for some $\phi \in D(S)$ and obtain $(\phi; (I + S^*)v) = 0$. Since this is true for all $v \in D(S^*)$ and $R(I + S^*) = H$, we obtain $\phi = 0$ and then $u = 0$.

can be seen as an operator defined from $D(S)$ into X. Let $(x_n, A_S x_n)$ be a sequence in the graph $G(A_S)$ of A_S. Assume that $(x_n, A_S x_n)$ converges in the Banach space $D(S) \times X$ to (x, y) (here $D(S)$ is endowed with its graph norm). If $\overline{A}$ denotes the closure of A, then $x \in D(\overline{A})$ and $y = \overline{A}x$. But $x \in D(S)$ and hence $y = A_S x$. Consequently, $G(A_S)$ is closed in $D(S) \times X$. We conclude by the closed graph theorem that A_S is a continuous operator from $(D(S), \|.\| + \|S.\|)$ into X, and this implies the desired inequality.

On the basis of this remark, we can add to the conclusions of Theorem 1.50 that every core of S is a core of $\overline{A}$.

1.4 SEMIGROUPS ASSOCIATED WITH SESQUILINEAR FORMS

1.4.1 The semigroup on the Hilbert space H.

In this section we use the same notation as in Section 1.2. Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed sesquilinear form on a Hilbert space H (see (1.2)–(1.5)). Denote by A the operator associated with $\mathfrak{a}$. Clearly, A is an accretive operator since $\mathfrak{a}$ is an accretive form. As a consequence of Proposition 1.22 and Theorem 1.49, we have

PROPOSITION 1.51 *The operator $-A$ is the generator of a strongly continuous contraction semigroup on H.*

In the next result, we show that the semigroup generated by $-A$ is holomorphic. More precisely,

THEOREM 1.52 *Suppose that H is a complex Hilbert space. Denote by $(e^{-tA})_{t\geq 0}$ the semigroup generated by $-A$ on H. Then $(e^{-tA})_{t\geq 0}$ is a holomorphic semigroup on the sector $\Sigma(\frac{\pi}{2} - \arctan M)$ where M is the constant in the continuity assumption (1.4). In addition, for every $\varepsilon \in (0,1]$, $e^{-\varepsilon z}e^{-zA}$ is a contraction operator on H for all $z \in \Sigma(\frac{\pi}{2} - \arctan \frac{M}{\varepsilon}) = \Sigma(\arctan \frac{\varepsilon}{M})$.*

Proof. The continuity assumption (1.4) implies that for every $\varepsilon \in (0,1]$

$$\begin{aligned} |\Im(Au;u)| &\leq M[\Re(Au;u) + (u;u)] \\ &\leq \frac{M}{\varepsilon}[\Re(Au;u) + \varepsilon(u;u)] \text{ for all } u \in D(A). \end{aligned}$$

Thus, if we set $B := \varepsilon I + A$, the above inequality shows that B is sectorial and

$$|\Im(Bu;u)| \leq \frac{M}{\varepsilon}\Re(Bu;u) \text{ for all } u \in D(B).$$

By Theorem 1.53 or Theorem 1.54 below, we conclude that $-B$ is the generator of a semigroup which is holomorphic on the sector $\Sigma(\frac{\pi}{2} - \arctan \frac{M}{\varepsilon})$. In addition, e^{-zB} is a contraction operator for $z \in \Sigma(\frac{\pi}{2} - \arctan \frac{M}{\varepsilon})$. Since $e^{-tB} = e^{-\varepsilon t} e^{-tA}$ for $t > 0$, we obtain the theorem. □

Observe that if one could write (1.4) as

$$|\mathfrak{a}(u,v)| \leq M'[\Re\mathfrak{a}(u,u) + w\|u\|^2]^{1/2}[\Re\mathfrak{a}(v,v) + w\|v\|^2]^{1/2}$$

with a constant $M' < M$ (at the cost of enlarging the constant w), then it follows that $(e^{-tA})_{t\geq 0}$ is holomorphic on the larger sector $\Sigma(\frac{\pi}{2} - \arctan M')$. In addition, $e^{-wz}e^{-zA}$ is a contraction operator on H for every z in that sector.

Recall that every densely defined accretive operator B is closable (see Lemma 1.47). We denote by $\overline{B}$ its closure.

Theorem 1.53 *Let B be a densely defined operator on a complex Hilbert space H. Assume that both B and B^* are sectorial, that is, there exists a non-negative constant C such that for every $u \in D(B)$ and $v \in D(B^*)$,*

$$|\Im(Bu;u)| \leq C\Re(Bu;u) \text{ and } |\Im(B^*v;v)| \leq C\Re(B^*v;v). \tag{1.24}$$

Then $-\overline{B}$ generates a strongly continuous semigroup $(e^{-t\overline{B}})_{t\geq 0}$ on H. This semigroup is holomorphic on the sector $\Sigma(\frac{\pi}{2} - \arctan C)$ and $e^{-z\overline{B}}$ is a contraction operator on H for every $z \in \Sigma(\frac{\pi}{2} - \arctan C)$.

Proof. Using the definition of $\overline{B}$, we see that the first inequality in (1.24) extends to all $u \in D(\overline{B})$. On the other hand, it follows from the definition of the adjoint operator that B^* is an extension of $(\overline{B})^*$. Thus, (1.24) holds for all $v \in D((\overline{B})^*)$.

Now let $u \in D(\overline{B})$ be such that $\|u\| = 1$ and let $\lambda \in \mathbb{C}$. Denote by dist the usual distance in $\mathbb{C}$. We have

$$\begin{aligned}\|(\lambda I - \overline{B})u\| &\geq |(\lambda u - \overline{B}u;u)| \\ &= |\lambda - (\overline{B}u;u)| \\ &\geq \operatorname{dist}(\lambda, \Sigma(\arctan C)).\end{aligned}$$

Hence, we have

$$\|(\lambda I - \overline{B})u\| \geq \operatorname{dist}(\lambda, \Sigma(\arctan C))\|u\| \text{ for all } u \in D(\overline{B}). \tag{1.25}$$

It follows that for $\lambda \notin \overline{\Sigma(\arctan C)}$ (where the latter denotes the closed sector), the operator $\lambda I - \overline{B}$ is injective. Moreover, $\lambda I - \overline{B}$ has closed range $R(\lambda I - \overline{B})$ for all $\lambda \notin \overline{\Sigma(\arctan C)}$. To see this, let $v_k = \lambda u_k - \overline{B}u_k$ be a convergent sequence with limit $v \in H$ and apply (1.25) to obtain that $(u_k)_k$

is a Cauchy sequence. Let u be the limit of $(u_k)_k$. The fact that the operator $\overline{B}$ is closed implies that $u \in D(\overline{B})$ and $v = (\lambda I - \overline{B})u \in R(\lambda I - \overline{B})$.

Let us show that $R(\lambda I - \overline{B})$ is dense for $\lambda \notin \overline{\Sigma(\arctan C)}$. If $g \in H$ is such that

$$(g; \lambda u - \overline{B}u) = 0 \text{ for all } u \in D(\overline{B}),$$

then $g \in D((\overline{B})^*)$ and $\overline{\lambda} g - (\overline{B})^* g = 0$. But $\overline{\lambda}$ and the adjoint $(\overline{B})^*$ satisfy the same properties as λ and $\overline{B}$. In particular, $\overline{\lambda} I - (\overline{B})^*$ is injective. This implies that $g = 0$ and thus $R(\lambda I - \overline{B})$ is dense. It follows that $(\lambda I - \overline{B})$ is invertible for all $\lambda \notin \overline{\Sigma(\arctan C)}$. In addition, (1.25) gives

$$\|(\lambda I - \overline{B})^{-1}u\| \leq \frac{1}{\operatorname{dist}(\lambda, \Sigma(\arctan C))}\|u\| \text{ for all } u \in H. \tag{1.26}$$

Fix now $\theta \in (\arctan C, \pi)$. We have for every $\lambda \notin \Sigma(\theta)$,

$$\sin(\theta - \arctan C) \leq \frac{\operatorname{dist}(\lambda, \Sigma(\arctan C))}{|\lambda|}.$$

It follows from this and (1.26) that for all $u \in H$ and $\lambda \notin \Sigma(\theta)$,

$$\|\lambda(\lambda I - \overline{B})^{-1}u\| \leq \frac{1}{\sin(\theta - \arctan C)}\|u\|.$$

Theorem 1.45 allows us to conclude that $-\overline{B}$ generates a bounded holomorphic semigroup on the sector $\Sigma(\frac{\pi}{2} - \arctan C)$.

It remains to show that $e^{-z\overline{B}}$ is a contraction operator for every $z \in \Sigma(\frac{\pi}{2} - \arctan C)$. Fix $\theta \in (0, \frac{\pi}{2} - \arctan C)$ and consider the semigroup $(e^{-te^{i\theta}\overline{B}})_{t\geq 0}$. For every $u \in H$ and $t > 0$

$$\begin{aligned}
&\frac{d}{dt}\|e^{-te^{i\theta}\overline{B}}u\|^2 \\
&= -2\Re(e^{i\theta}\overline{B}e^{-te^{i\theta}\overline{B}}u; e^{-te^{i\theta}\overline{B}}u) \\
&= -[\Re(\overline{B}e^{-te^{i\theta}\overline{B}}u; e^{-te^{i\theta}\overline{B}}u)\cos\theta - \Im(\overline{B}e^{-te^{i\theta}\overline{B}}u; e^{-te^{i\theta}\overline{B}}u)\sin\theta] \\
&\leq -\left(\frac{\cos\theta}{C} - \sin\theta\right)|\Im(\overline{B}e^{-te^{i\theta}\overline{B}}u; e^{-te^{i\theta}\overline{B}}u)|.
\end{aligned}$$

Since $\theta \in (0, \frac{\pi}{2} - \arctan C)$, it follows that $\frac{\cos\theta}{C} - \sin\theta \geq 0$. This shows that $\|e^{-te^{i\theta}B}u\|^2$ is non-increasing (as a function of t). Thus,

$$\|e^{-te^{i\theta}\overline{B}}u\| \leq \|u\| \text{ for all } t > 0$$

and this finishes the proof. □

The sectoriality assumption of the adjoint operator B^* was only used to prove that $\lambda I - \overline{B}$ has dense range. If we assume that there exists $\lambda_0 \in \rho(\overline{B})$ with $\mathrm{dist}(\lambda_0, \Sigma(\arctan C)) > 0$, then the theorem holds without assuming sectoriality of B^*. In order to prove this, we only have to show that $\lambda \in \rho(\overline{B})$ for all λ such that $\mathrm{dist}(\lambda, \Sigma(\arctan C)) > 0$ and argue as in the previous proof. Fix now λ such that $\mathrm{dist}(\lambda, \Sigma(\arctan C)) > 0$ and write

$$\begin{aligned} \lambda I - \overline{B} &= \lambda_0 I - \overline{B} + \lambda I - \lambda_0 I \\ &= (\lambda_0 I - \overline{B})[I + (\lambda - \lambda_0)(\lambda_0 I - \overline{B})^{-1}]. \end{aligned}$$

Using (1.26) for λ_0, we see that $\lambda I - \overline{B}$ is invertible for all λ such that $|\lambda - \lambda_0| < \mathrm{dist}(\lambda_0, \Sigma(\arctan C))$. Repeating this procedure, we obtain $\lambda \in \rho(\overline{B})$ for all λ such that $\mathrm{dist}(\lambda, \Sigma(\arctan C)) > 0$. The estimate (1.26) holds for such λ since it is only based on the sectoriality of B. Finally, recall that B is a closed operator whenever its resolvent set $\rho(B)$ is not empty (see Proposition 1.35). Thus, we have proved the following

THEOREM 1.54 *Let B be a densely defined operator on a complex Hilbert space H. Assume that B is sectorial, that is,*

$$|\Im(Bu; u)| \leq C\Re(Bu; u) \ \text{for all} \ u \in D(B), \tag{1.27}$$

where $C \geq 0$ is a constant. Assume also that there exists $\lambda_0 \in \rho(B)$ with $\mathrm{dist}(\lambda_0, \Sigma(\arctan C)) > 0$. Then $-B$ generates a strongly continuous semigroup which is holomorphic on the sector $\Sigma(\frac{\pi}{2} - \arctan C)$ and such that e^{-zB} is a contraction operator on H for every $z \in \Sigma(\frac{\pi}{2} - \arctan C)$.

Remark. The study of the holomorphy of the semigroup associated with a form $\mathfrak{a}$ requires that the Hilbert space H is complex. In the case where H is real, one uses the following complexification procedure.

Let $H_{\mathbb{C}} := H + iH$ and define the form

$$\tilde{\mathfrak{a}}(u + iv, g + ih) := \mathfrak{a}(u, g) + \mathfrak{a}(v, h) + i[\mathfrak{a}(v, g) - \mathfrak{a}(u, h)] \tag{1.28}$$

for all $u, v, g, h \in D(\mathfrak{a})$. The domain of the form $\tilde{\mathfrak{a}}$ is given by $D(\mathfrak{a}) + iD(\mathfrak{a})$.

One checks easily that the assumptions (1.2)–(1.5) carry over from $\mathfrak{a}$ to $\tilde{\mathfrak{a}}$. The semigroup associated with $\tilde{\mathfrak{a}}$ is given by

$$T(t)(u + iv) := e^{-tA}u + ie^{-tA}v.$$

This is the complexification of the semigroup $(e^{-tA})_{t\geq 0}$. The semigroup $(T(t))_{t\geq 0}$ is holomorphic on $H_{\mathbb{C}}$. From this, one obtains several interesting consequences for the semigroup $(e^{-tA})_{t\geq 0}$ on H. In particular, $e^{-tA}H \subseteq D(A) \subseteq D(\mathfrak{a})$ for all $t > 0$.

1.4.2 Extrapolation to the (anti-) dual space $D(\mathfrak{a})'$

As previously, we denote by $\mathfrak{a}$ a densely defined, accretive, continuous, and closed sesquilinear form on a Hilbert space H and by A the operator associated with $\mathfrak{a}$. The semigroup $(e^{-tA})_{t\geq 0}$ is defined on H and enjoys several interesting properties. In this section, we extend this semigroup to a larger space and prove similar properties for the extension. Our interest in doing this lies in the fact that it is sometimes more flexible to work with the semigroup on the larger space. For example, the function $t \to e^{-tA}u$ has a derivative in H at $t = 0$ only when $u \in D(A)$, whereas in the larger space the derivative may exist for $u \notin D(A)$. This gives a new point of view on the semigroup $(e^{-tA})_{t\geq 0}$ itself, and will allow us to prove other properties of this semigroup and its generator.

Denote by $D(\mathfrak{a})'$ the anti-dual space of $D(\mathfrak{a})$, that is, the space of continuous functionals ϕ such that

$$\phi(u+v) = \phi(u) + \phi(v),\ \phi(\alpha u) = \overline{\alpha}\phi(u) \ \text{ for all } \alpha \in \mathbb{K}, u, v \in D(\mathfrak{a}).$$

When H is real, $D(\mathfrak{a})'$ is of course the dual space of $D(\mathfrak{a})$.

Identifying H' with H yields

$$D(\mathfrak{a}) \subset H \subset D(\mathfrak{a})' \tag{1.29}$$

with continuous and dense imbedding. The dualization between $D(\mathfrak{a})'$ and $D(\mathfrak{a})$ is denoted by $< .,. >$ (i.e., $< \phi, u >$ denotes the value of ϕ at u for $u \in D(\mathfrak{a})$ and $\phi \in D(\mathfrak{a})'$). We note that if $\phi \in H$ and $u \in D(\mathfrak{a})$, then $< \phi, u >= (\phi; u)$, the inner product in H.

Fix $u \in D(\mathfrak{a})$ and consider the functional

$$\phi(v) := \mathfrak{a}(u,v),\ \ v \in D(\mathfrak{a}).$$

It follows from the continuity assumption (1.4) that ϕ is continuous on $D(\mathfrak{a})$, and hence $\phi \in D(\mathfrak{a})'$. Thus, it can be represented as $\phi(v) = < \mathcal{A}u, v >$, where $\mathcal{A}u \in D(\mathfrak{a})'$ depends on u. Using the fact that $\mathfrak{a}$ is sesquilinear, we see that $\mathcal{A}$ is a linear operator which maps $D(\mathfrak{a})$ into $D(\mathfrak{a})'$. In addition, using again the continuity assumption (1.4), we have

$$\begin{aligned}\|\mathcal{A}u\|_{D(\mathfrak{a})'} &= \sup_{\|v\|_{\mathfrak{a}}\leq 1} | < \mathcal{A}u, v > | \\ &= \sup_{\|v\|_{\mathfrak{a}}\leq 1} |\mathfrak{a}(u,v)| \\ &\leq M\|u\|_{\mathfrak{a}}.\end{aligned}$$

Thus, $\mathcal{A}$ is a continuous operator from $D(\mathfrak{a})$ (endowed with the norm $\|.\|_{\mathfrak{a}}$) into $D(\mathfrak{a})'$. The operator $\mathcal{A}$ can also be seen as an unbounded operator on

$D(\mathfrak{a})'$, with domain $D(\mathcal{A}) = D(\mathfrak{a})$, and such that

$$\mathfrak{a}(u, v) = < \mathcal{A}u, v > \text{ for all } u, v \in D(\mathfrak{a}). \tag{1.30}$$

Now let A be the operator associated with $\mathfrak{a}$ (defined in Section 1.2.3). Using the fact that $D(\mathfrak{a})$ is dense in H and the definition of A, we see that A is precisely the part of $\mathcal{A}$ in H. That is,

$$D(A) = \{u \in D(\mathcal{A}); \mathcal{A}u \in H\} \text{ and } Au = \mathcal{A}u \text{ for } u \in D(A).$$

The following result shows that $(e^{-tA})_{t\geq 0}$ extends from H to the larger space $D(\mathfrak{a})'$.

THEOREM 1.55 *Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed sesquilinear form on a Hilbert space H. Then the operator $-\mathcal{A}$, with domain $D(\mathcal{A}) = D(\mathfrak{a})$, generates a strongly continuous semigroup $(e^{-t\mathcal{A}})_{t\geq 0}$ on $D(\mathfrak{a})'$. Moreover,*

$$e^{-t\mathcal{A}} f = e^{-tA} f \text{ for every } f \in H \text{ and } t \geq 0. \tag{1.31}$$

If H is complex, the semigroup $(e^{-t\mathcal{A}})_{t\geq 0}$ is holomorphic (on $D(\mathfrak{a})'$) on the sector $\Sigma(\frac{\pi}{2} - \arctan M)$, where M is the constant in (1.4).

Proof. We first assume that the Hilbert space H is complex. Given $u \in D(\mathfrak{a})$, let $\phi = (\lambda + 1 + \mathcal{A})u$. Clearly,

$$\begin{aligned} < \phi, u > &= \lambda(u; u) + (u; u) + < \mathcal{A}u, u > \\ &= \lambda(u; u) + (u; u) + \mathfrak{a}(u, u). \end{aligned}$$

Hence,

$$\|u\|_{\mathfrak{a}}^2 \leq \|\phi\|_{D(\mathfrak{a})'} \|u\|_{\mathfrak{a}} + |\lambda| \|u\|^2. \tag{1.32}$$

On the other hand, by Lemma 1.9 we have for every $u \in D(\mathfrak{a})$ with $u \neq 0$,

$$\mathfrak{a}\left(\frac{u}{\|u\|}, \frac{u}{\|u\|}\right) + \left(\frac{u}{\|u\|}; \frac{u}{\|u\|}\right) \in \Sigma(\arctan M).$$

Thus,

$$\begin{aligned} \|u\|_{\mathfrak{a}} \|\phi\|_{D(\mathfrak{a})'} &= \|u\|_{\mathfrak{a}} \|(\lambda + 1 + \mathcal{A})u\|_{D(\mathfrak{a})'} \\ &\geq | < (I + \lambda I + \mathcal{A})u, u > | \\ &= |\lambda + \mathfrak{a}\left(\frac{u}{\|u\|}, \frac{u}{\|u\|}\right) + \left(\frac{u}{\|u\|}; \frac{u}{\|u\|}\right) | \|u\|^2 \\ &\geq \operatorname{dist}(\lambda, -\Sigma(\arctan M)) \|u\|^2. \end{aligned}$$

We have proved that

$$\|u\|_{\mathfrak{a}}\|\phi\|_{D(\mathfrak{a})'} \geq \operatorname{dist}(\lambda, -\Sigma(\arctan M))\|u\|^2. \tag{1.33}$$

Fix $\theta \in (\arctan M, \pi)$. We have, as in the proof of Theorem 1.53,

$$\operatorname{dist}(\lambda, -\Sigma(\arctan M)) \geq |\lambda| \sin(\theta - \arctan M)$$

for all $\lambda \notin -\Sigma(\theta)$. Inserting this in (1.33) gives

$$\|u\|_{\mathfrak{a}}\|\phi\|_{D(\mathfrak{a})'} \geq c|\lambda|\|u\|^2 \text{ for all } \lambda \notin -\Sigma(\theta), \tag{1.34}$$

where c is a positive constant. The last estimate together with (1.32) gives for all $\lambda \notin -\Sigma(\theta)$

$$\|u\|_{\mathfrak{a}} \leq \left(1 + \frac{1}{c}\right)\|\phi\|_{D(\mathfrak{a})'} = C\|(\lambda I + I + \mathcal{A})u\|_{D(\mathfrak{a})'}. \tag{1.35}$$

The estimate (1.35) shows that $\lambda I + I + \mathcal{A}$ is invertible on $D(\mathfrak{a})'$ for all $\lambda \notin -\Sigma(\theta)$. Indeed, it is clear that $\lambda I + I + \mathcal{A}$ is injective. It has dense range because H is dense in $D(\mathfrak{a})'$ and

$$(\lambda I + I + \mathcal{A})D(\mathfrak{a}) \supseteq (\lambda I + I + A)D(A) = H,$$

where the last equality follows from the fact that $\lambda \in \rho(-A)$ (see Theorems 1.52 and 1.45). Finally, if $(\lambda I + I + \mathcal{A})u_n$ is a convergent sequence in $D(\mathfrak{a})'$, then we obtain from (1.35) that (u_n) is a Cauchy sequence in $D(\mathfrak{a})$. It is then convergent in $D(\mathfrak{a})$. From the continuity of $\mathcal{A}$, as an operator from $D(\mathfrak{a})$ into $D(\mathfrak{a})'$, we obtain that that $\lambda I + I + \mathcal{A}$ has closed range. Thus $\lambda I + I + \mathcal{A}$ is invertible in $D(\mathfrak{a})'$ for $\lambda \notin -\Sigma(\theta)$.

Now let $v \in D(\mathfrak{a})$. We have

$$\begin{aligned} |\lambda||<u,v>| &= |<\phi,v> -(u;v) - \mathfrak{a}(u,v)| \\ &\leq \|\phi\|_{D(\mathfrak{a})'}\|v\|_{\mathfrak{a}} + (M+1)\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}}. \end{aligned}$$

Taking the supremum over $\|v\|_{\mathfrak{a}} \leq 1$ and using (1.35), we obtain

$$\begin{aligned} |\lambda|\|u\|_{D(\mathfrak{a})'} &\leq \|\phi\|_{D(\mathfrak{a})'} + (M+1)\|u\|_{\mathfrak{a}} \\ &\leq \|\phi\|_{D(\mathfrak{a})'} + (M+1)C\|\phi\|_{D(\mathfrak{a})'} \\ &= C'\|(\lambda I + I + \mathcal{A})u\|_{D(\mathfrak{a})'}. \end{aligned}$$

We have proved that $\lambda I + I + \mathcal{A}$ is invertible on $D(\mathfrak{a})'$ for $\lambda \notin -\Sigma(\theta)$ and

$$\sup_{\lambda \notin -\Sigma(\theta)} \|\lambda(\lambda I + I + \mathcal{A})^{-1}\|_{\mathcal{L}(D(\mathfrak{a})')} < \infty.$$

Theorem 1.45 ensures that $-(\mathcal{A}+I)$ generates on $D(\mathfrak{a})'$ a semigroup which is bounded holomorphic on the sector $\Sigma(\frac{\pi}{2} - \arctan M)$.

Now we prove (1.31). Let $f \in H$ and $u(t) := e^{-t\mathcal{A}}f - e^{-tA}f$. From the embedding $H \subset D(\mathfrak{a})'$, it follows that $\frac{d}{dt}e^{-tA}f$ exists (at each $t > 0$) in $D(\mathfrak{a})'$ and equals $-Ae^{-tA}f = -\mathcal{A}e^{-tA}f$. This gives

$$\frac{d}{dt}u(t) = -\mathcal{A}u(t),\ t > 0.$$

Thus, $u(t) = e^{-t\mathcal{A}}u(0) = 0$ and the desired equality holds.

We have proved the theorem in the case where H is complex. Now, if H is real, we use the complexification argument described in the Remark at the end of the previous section. One obtains then a holomorphic semigroup $(T(t))_{t\geq 0}$ on $(D(\mathfrak{a}) + iD(\mathfrak{a}))'$ whose generator is (minus) the operator associated with the form defined in (1.28)). If $\phi \in D(\mathfrak{a})'$, then it can be written as the limit (in $D(\mathfrak{a})'$) of a sequence $(u_n) \in H$ and hence $T(t)\phi$ is the limit of $T(t)u_n = e^{-tA}u_n \in H$. Consequently, $T(t)\phi \in D(\mathfrak{a})'$ for every $t \geq 0$. Therefore, $T(t)D(\mathfrak{a})' \subseteq D(\mathfrak{a})'$ for all $t \geq 0$. Thus, the restriction of $(T(t))_{t\geq 0}$ to $D(\mathfrak{a})'$ is then a strongly continuous semigroup whose generator is $-\mathcal{A}$. □

It is shown in this proof that if H is complex, the semigroup satisfies

$$\sup_{z\in\Sigma(\psi)} \|e^{-z}e^{-z\mathcal{A}}\|_{\mathcal{L}(D(\mathfrak{a})')} < \infty$$

for all $0 \leq \psi < \frac{\pi}{2} - \arctan M$. For the same reasons as in Theorem 1.52, we have

$$\sup_{z\in\Sigma(\psi)} \|e^{-\varepsilon z}e^{-z\mathcal{A}}\|_{\mathcal{L}(D(\mathfrak{a})')} < \infty$$

for all $0 \leq \psi < \frac{\pi}{2} - \arctan \frac{M}{\varepsilon}$ and all $\varepsilon \in (0, 1]$.

If the form $\mathfrak{a}$ is sectorial, i.e.,

$$|\Im\mathfrak{a}(u,u)|| \leq M\Re\mathfrak{a}(u,u) \text{ for all } u \in D(\mathfrak{a}), \tag{1.36}$$

then

$$\sup_{z\in\Sigma(\psi)} \|e^{-z\mathcal{A}}\|_{\mathcal{L}(D(\mathfrak{a})')} < \infty$$

for all $0 \leq \psi < \frac{\pi}{2} - \arctan M$. The proof is the same as the previous one, replacing (1.4) by (1.36). That is, we can replace $I + \mathcal{A}$ by $\mathcal{A}$ in the previous proof.

Note also that these estimates hold in $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$. More precisely, for every $\varepsilon \in (0, 1]$,

$$\sup_{z\in\Sigma(\psi)} \|e^{-\varepsilon z}e^{-zA}\|_{\mathcal{L}(D(\mathfrak{a}))} < \infty \tag{1.37}$$

for all $0 \le \psi < \frac{\pi}{2} - \arctan \frac{M}{\varepsilon}$. Indeed, let $u \in D(\mathfrak{a})$ and write

$$\begin{aligned}\|e^{-zA}u\|_{\mathfrak{a}}^2 &= \Re < e^{-z\mathcal{A}}\mathcal{A}u, e^{-zA}u > + \|e^{-zA}u\|^2\\ &\le \|e^{-z\mathcal{A}}\mathcal{A}u\|_{D(\mathfrak{a})'}\|e^{-zA}u\|_{\mathfrak{a}} + \|e^{-zA}u\|^2.\end{aligned}$$

Hence,

$$\|e^{-zA}u\|_{\mathfrak{a}}^2 \le \|e^{-z\mathcal{A}}\mathcal{A}u\|_{D(\mathfrak{a})'}^2 + 2\|e^{-zA}u\|^2. \tag{1.38}$$

It follows from Theorem 1.52 and the above observations that both terms $\|e^{-z\varepsilon}e^{-z\mathcal{A}}\|_{D(\mathfrak{a})'}$ and $\|e^{-z\varepsilon}e^{-zA}\|_{\mathcal{L}(H)}$ are uniformly bounded on $\Sigma(\psi)$ for $0 \le \psi < \frac{\pi}{2} - \arctan \frac{M}{\varepsilon}$. Using this and the fact that $\mathcal{A}$ is a bounded operator from $D(\mathfrak{a})$ into $D(\mathfrak{a})'$, we see that (1.37) follows from (1.38).

Again, if we assume the stronger condition (1.36), we obtain

$$\sup_{z\in\Sigma(\psi)} \|e^{-zA}\|_{\mathcal{L}(D(\mathfrak{a}))} < \infty. \tag{1.39}$$

Proof of Lemma 1.25. First, $e^{-tA}H \subseteq D(A)$ for all $t > 0$. Indeed, if H is complex the semigroup generated by $-A$ is holomorphic on H (cf. Theorem 1.52) and this implies trivially the above inclusion. Now if H is real, we use again the complexification argument to obtain a holomorphic semigroup $(e^{-t(A+iA)})_{t\ge 0}$ on $H + iH$, from which we obtain $e^{-tA}H \subseteq D(A)$ for $t > 0$. We prove that every $u \in D(\mathfrak{a})$ can be approximated in $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$ by $e^{-tA}u$. We have

$$\begin{aligned}\|e^{-tA}u - u\|_{\mathfrak{a}}^2 &= \Re < e^{-t\mathcal{A}}\mathcal{A}u - \mathcal{A}u, e^{-tA}u - u > + \|e^{-tA}u - u\|^2\\ &\le \|e^{-t\mathcal{A}}\mathcal{A}u - \mathcal{A}u\|_{D(\mathfrak{a})'}\|e^{-tA}u - u\|_{\mathfrak{a}} + \|e^{-tA}u - u\|^2.\end{aligned}$$

Hence

$$\|e^{-tA}u - u\|_{\mathfrak{a}}^2 \le \|e^{-t\mathcal{A}}\mathcal{A}u - \mathcal{A}u\|_{D(\mathfrak{a})'}^2 + 2\|e^{-tA}u - u\|^2.$$

The strong continuity of $(e^{-t\mathcal{A}})_{t\ge 0}$ on $D(\mathfrak{a})'$ and of $(e^{-tA})_{t\ge 0}$ on H imply

$$\|e^{-tA}u - u\|_{\mathfrak{a}} \to 0 \text{ as } t \to 0.$$

This proves the lemma. □

It is seen in this proof that $e^{-tA}H \subseteq D(A) \subseteq D(\mathfrak{a})$ for $t > 0$ and that the restriction of $(e^{-tA})_{t\ge 0}$ is a strongly continuous semigroup on $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$.

If the Hilbert space H is complex, then Theorem 1.55 and the same arguments as in the above proof show that

$$\|e^{-zA}u - u\|_{\mathfrak{a}} \to 0 \text{ as } z \to 0, z \in \Sigma(\psi)$$

for every $u \in D(\mathfrak{a})$ and every $\psi \in [0, \frac{\pi}{2} - \arctan M)$.

If for each $t \geq 0$, $T(t)$ denotes the restriction of e^{-tA} to $D(\mathfrak{a})$, then $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$. It is also holomorphic on the sector $\Sigma(\frac{\pi}{2} - \arctan M)$ when the space H is complex. If $-B$ denotes the generator of $T(t)_{t\geq 0}$, then B is the part of A in $D(\mathfrak{a})$, that is,

$$D(B) = \{u \in D(A), Au \in D(\mathfrak{a})\}, \quad Bu = Au \text{ for all } u \in D(B).$$

1.5 CORRESPONDENCE BETWEEN FORMS, OPERATORS, AND SEMIGROUPS

Let $\mathfrak{a}$ be a sesquilinear form on H which satisfies the standard assumptions (1.2)–(1.5). One associates with $\mathfrak{a}$ an operator A and a semigroup $(e^{-tA})_{t\geq 0}$. In this section we show that there is a unique correspondence between sesquilinear forms and a class of operators and semigroups.

The first result shows that $\mathfrak{a}$ can be described completely by its associated semigroup $(e^{-tA})_{t\geq 0}$.

LEMMA 1.56 *Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed sesquilinear form. Let $u \in H$. Then $u \in D(\mathfrak{a})$ if and only if*

$$\sup_{t>0} \frac{1}{t} \Re(u - e^{-tA}u; u) < \infty.$$

In addition, for every $u, v \in D(\mathfrak{a})$

$$\mathfrak{a}(u,v) = \lim_{t\downarrow 0} \frac{1}{t}(u - e^{-tA}u; v).$$

Proof. Let $u, v \in D(\mathfrak{a})$. By Theorem 1.55 we have

$$\frac{1}{t}(u - e^{-tA}u; v) = \frac{1}{t} < u - e^{-t\mathcal{A}}u, v > .$$

Since $u \in D(\mathfrak{a}) = D(\mathcal{A})$, we have

$$\frac{1}{t} < u - e^{-t\mathcal{A}}u, v > \to < \mathcal{A}u, v > = \mathfrak{a}(u,v) \text{ as } t \to 0.$$

This proves the last assertion of the lemma. In particular,

$$\frac{1}{t}(u - e^{-tA}u; u) \to \mathfrak{a}(u, u) \text{ for all } u \in D(\mathfrak{a}).$$

Assume now that $u \in H$ is such that $\sup_{t>0} \frac{1}{t}\Re(u - e^{-tA}u; u) < \infty$. For $\lambda > 0$, we write for simplicity $(\lambda I + A)^{-1} = R(\lambda)$. We have

$$\begin{aligned}
\Re\mathfrak{a}(\lambda R(\lambda)u; \lambda R(\lambda)u) &= \Re\lambda(AR(\lambda)u; \lambda R(\lambda)u) \\
&= \Re\lambda(u - \lambda R(\lambda)u; \lambda R(\lambda)u) \\
&\le \Re\lambda(u - \lambda R(\lambda)u; u) \\
&= \Re \int_0^\infty \lambda^2 e^{-\lambda t}(u - e^{-tA}u; u)dt \\
&\le \sup_{t>0} \frac{1}{t}\Re(u - e^{-tA}u; u) \int_0^\infty t\lambda^2 e^{-\lambda t}dt \\
&= \sup_{t>0} \frac{1}{t}\Re(u - e^{-tA}u; u) \int_0^\infty s e^{-s} ds.
\end{aligned}$$

It follows now that $\lambda R(\lambda)u$ is bounded uniformly with respect to λ in $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$ (recall that $\lambda R(\lambda)$ is a contraction operator on H by Proposition 1.22). In addition, $\lambda R(\lambda)$ converges strongly to the identity operator in H as $\lambda \to +\infty$ (this can be seen by again using Proposition 1.22 and $\|u - \lambda R(\lambda)u\| = \|R(\lambda)Au\| \le \lambda^{-1}\|Au\|$ for $u \in D(A)$. The desired convergence then follows by the density of $D(A)$ in H). By Lemma 1.32, we deduce that $u \in D(\mathfrak{a})$. □

A natural question is how to recognize in Hilbert spaces those operators or semigroups that are associated with sesquilinear forms. In the next results, we describe such operators and semigroups.

If A is the operator associated with a densely defined, accretive, continuous, and closed sesquilinear form $\mathfrak{a}$, then $I + A$ is sectorial (cf. Lemma 1.9) and A is m-accretive (cf. Proposition 1.22). The next result shows that these properties characterize operators that are associated with sesquilinear forms.

Theorem 1.57 *Let A be an m-accretive operator on a complex Hilbert space H. Assume that $I+A$ is sectorial. Then there exists a unique sesquilinear form $\mathfrak{a}$ which is densely defined, accretive, continuous, and closed and such that A is the operator associated with $\mathfrak{a}$.*

Proof. Define the form

$$\mathfrak{b}(u, v) := (Au; v), \ D(\mathfrak{b}) = D(A).$$

By Lemma 1.29, the form $\mathfrak{b}$ is closable. Let A_1 be the operator associated with the closure $\mathfrak{a} := \overline{\mathfrak{b}}$ of $\mathfrak{b}$.[4] By assumption, $I + A$ is invertible. By Proposition 1.22, $I + A_1$ is invertible, too. Thus, $I + A_1$ is an extension of $I + A$ and both operators are invertible. This implies that $A = A_1$ and hence A is the operator associated with the form $\mathfrak{a}$.

The uniqueness of $\mathfrak{a}$ follows from Lemma 1.25. □

If we assume in the previous theorem that A is sectorial, then the associated form $\mathfrak{a}$ is sectorial. This follows from the above proof, since then $\mathfrak{b}$ is sectorial and this also holds for the closure $\mathfrak{a} = \overline{\mathfrak{b}}$.

Note also that if A is self-adjoint, then $\mathfrak{a}$ is a symmetric form.

Theorem 1.53 asserts that the semigroup generated by (minus) the operator associated with a sesquilinear form is holomorphic and $e^{-z}e^{-zA}$ is a contraction operator for every z in some sector. Here we give a converse to that result.

THEOREM 1.58 *Let $(T(t))_{t\geq 0}$ be a contraction semigroup acting on a complex Hilbert space H. Assume that this semigroup is holomorphic on the sector $\Sigma(\psi)$ (for some $\psi \in (0, \frac{\pi}{2})$) and such that for every $z \in \Sigma(\psi)$, $e^{-z}T(z)$ is a contraction operator on H. Then the generator of $(T(t))_{t\geq 0}$ is (minus) the operator associated with a densely defined, accretive, continuous, and closed sesquilinear form.*

Proof. Denote by $-A$ the generator of the contraction semigroup $(T(t))_{t\geq 0}$. The operator A is accretive by Theorem 1.49. Now fix $\theta \in (0, \psi)$. The semigroup $(T(te^{i\theta})e^{-te^{i\theta}})_{t\geq 0}$ is contractive on H and its generator is $-e^{i\theta}(I + A)$. Thus, $e^{i\theta}(I+A)$ is accretive and hence $\Re(e^{i\theta}(I+A)u; u) \geq 0$ for every $u \in D(A)$. This gives $\Im(Au; u) \leq \frac{1}{\tan\theta}\Re((I + A)u; u)$. For similar reasons, $\Re(e^{-i\theta}(I+A)u; u) \geq 0$ and thus $-\Im(Au; u) \leq \frac{1}{\tan\theta}\Re((I+A)u; u)$. It follows that $I+A$ is a sectorial operator. The proof is finished by applying Theorem 1.57. □

Remark. If we assume that $e^{-\alpha z}T(z)$ is a contraction operator for every $z \in \Sigma(\psi)$, then we obtain an operator A such that $\alpha I + A$ is sectorial. In particular, if $T(z)$ is a contraction for all $z \in \Sigma(\psi)$, then A is sectorial.

Applying Lemma 1.56, we can reformulate the above theorem as follows.

THEOREM 1.59 *Let $(T(t))_{t\geq 0}$ be a contraction semigroup acting on a complex Hilbert space H. Assume that this semigroup is holomorphic on the sector $\Sigma(\psi)$ (for some $\psi \in (0, \frac{\pi}{2})$) and such that for every $z \in \Sigma(\psi)$, $e^{-z}T(z)$*

[4] A_1 is the Friedrich extension of A.

is a contraction operator on H. Then the form given by

$$\mathfrak{a}(u,v) := \lim_{t\downarrow 0} \frac{1}{t}(u - T(t)u; v),$$
$$D(\mathfrak{a}) := \left\{ u \in H, \sup_{t>0} \frac{1}{t} \Re(u - e^{-tA}u; u) < \infty \right\},$$

is densely defined, accretive, continuous, and closed and $(T(t))_{t\geq 0}$ is its associated semigroup.

Notes

The material of this chapter is known. See Davies [Dav80], Kato [Kat80], Lions [Lio61], Reed-Simon [ReSi80] or Tanabe [Tan79]. Our presentation is however different and some proofs are simplified by using semigroups from the beginning. One of the aims of this chapter is to give a systematic account of the interplay between forms, operators, and semigroups.

Sections 1.1 and 1.2. Bounded sesquilinear forms on Hilbert spaces can be found in many textbooks on Functional Analysis. For the Lax-Milgram lemma see, e.g., Brezis [Bre92], Lions [Lio61], Yosida [Yos65].

An exhaustive study of sectorial forms can be found in [Kat80]. In [ReSi80] closed sectorial forms are called strictly accretive. Note also that the notion of a sectorial form in [ReSi80] is slightly different from ours. Operators associated with forms are called regularly accretive in [Tan79].

Proposition 1.13 is sometimes considered as the definition of a closable form. Example 1.2.1 of a symmetric form which is not closable is borrowed from [Kat80]. At this point, we mention the following remarkable result proved by Simon [Sim78] (see also Reed-Simon [ReSi80]).

Theorem. *Let $\mathfrak{a}$ be a symmetric non-negative form on a Hilbert space H. Then, there exists a largest closable symmetric form $\mathfrak{a}_r$ that is smaller than $\mathfrak{a}$.*

In this theorem, a symmetric form $\mathfrak{b}$ is said to be smaller than $\mathfrak{a}$ if $D(\mathfrak{a}) \subseteq D(\mathfrak{b})$ and $\mathfrak{b}(u,u) \leq \mathfrak{a}(u,u)$ for all $u \in D(\mathfrak{a})$.

Proposition 1.18 and related results can also be found in [Kat80]. Theorem 1.19 is often called the KLMN theorem. The version given here can be found in [Kat80] (where it is formulated for sectorial forms). This theorem was proved in various versions by Kato [Kat55], Lions [Lio61], Lax-Milgram [LaMi54]. See also Nelson [Nel64] and Reed-Simon [ReSi75].

Section 1.3. Semigroup theory and its various applications is a well documented subject; hence we make only brief comments and give some more references. The fundamental Hille-Yosida generation theorem was proved in 1948 and became the starting point of the subsequent theory of semigroups. Because of applications to equations of different types and because of interactions with other fields of analysis and probability, the theory has seen important developments. Several textbooks on semigroups are now available. Systematic treatments of semigroups as well as applications to many equations can be found in Arendt et al. [ABHN01], Clement et al. [CHADP87], Davies [Dav80], Engel-Nagel [EnNa99], Fattorini [Fat83], Goldstein [Gol85], Hille-Phillips [HiPh57], Kato [Kat80], Nagel et al. [Nag86], Pazy [Paz83], Robinson [Rob96], Yosida [Yos65]. For holomorphic semigroups and applications to parabolic problems, see Amann [Ama95] and Lunardi [Lun95]. Applications to Volterra equations are given in Engel-Nagel [EnNa99] and Prüss [Prü93].

We gave the definition of accretive operators only in the Hilbert space case. The reason why we considered only operators on Hilbert spaces is to make connection with operators that are associated with accretive forms. The definition makes sense for operators acting on Banach spaces. This was introduced by Phillips [Phi59] (he used the terminology of dissipative operator; which means that $-A$ is accretive). A linear operator A acting on a Banach space E is called accretive (or $-A$ is dissipative) if $\Re < Au, u^* > \geq 0$ for all $u \in D(A)$ and some u^* in the subdifferential of the norm $\|.\|_E$ of E at x. That is, u^* is in the dual space E' and such that $\|u^*\|_{E'} \leq 1$ and $< u, u^* > = \|u\|_E$, where $< ., . >$ denotes the pairing between E and E'.

Theorem 1.49 is the Hilbert space version of the well-known Lumer-Phillips theorem proved in [LuPh61]. The latter holds for accretive operators on any Banach space. Note also that similar theorems hold for nonlinear accretive operators, see Bénilan-Crandall-Pazy [BCP90].

Theorem 1.50 is due to Kato and is taken from Okazawa [Oka80], where Banach space versions are also given.

Sections 1.4 and 1.5. Related results to Theorems 1.53 and 1.54 can be found in Goldstein [Gol85] and Kato [Kat80]. Theorem 1.55 can be found in a different form in Tanabe [Tan79], but here we give a more precise angle of holomorphy in $D(\mathfrak{a})'$. Lemma 1.56 is an extension to the nonsymmetric case of a well-known result for symmetric forms (in the latter case, it is usually proved by using the spectral theorem for self-adjoint operators). The ”if” part of this lemma is shown in Albeverio, Röckner, and Stannat [ARS95] and the ”only if” part in Ouhabaz [Ouh92a] (see also [Ouh96]). Finally, Theorems 1.57 and 1.58 are implicit in [Kat80] and [Tan79].