Chapter One

Introduction

1.1 GENERAL COMMENTS

In this work we shall define the tangent spaces

$$TZ^n(X)$$

and

$$TZ^1(X)$$

to the spaces of 0-cycles and of divisors on a smooth, n-dimensional complex algebraic variety X. We think it may be possible to use similar methods to define $TZ^p(X)$ for all codimensions, but we have not been able to do this because of one significant technical point. Although the final definitions, as given in sections 7 and 8 below, are algebraic and formal, the motivation behind them is quite geometric. This is explained in the earlier sections; we have chosen to present the exposition in the monograph following the evolution of our geometric understanding of what the tangent spaces should be rather than beginning with the formal definition and then retracing the steps leading to the geometry.

Briefly, for 0-cycles an *arc* in $Z^n(X)$ is given by a $\mathbb{Z}$-linear combination of arcs in the symmetric products $X^{(d)}$, where such an arc is given by a smooth algebraic curve B together with a regular map $B \to X^{(d)}$. If t is a local uniformizing parameter on B we shall use the notation $t \to x_1(t)+\cdots+x_d(t)$ for the arc in $X^{(d)}$. Arcs in $Z^n(X)$ will be denoted by $z(t)$. We set $|z(t)| =$ support of $z(t)$, and if $0 \in B$ is a reference point we denote by $Z^n_{\{x\}}(X)$ the subgroup of arcs in $Z^n(X)$ with $\lim_{t\to 0} |z(t)| = x$. The tangent space will then be defined to be

$$TZ^n(X) = \left\{\text{arcs in } Z^n(X)\right\} / \equiv_{1^{\text{st}}},$$

where $\equiv_{1^{\text{st}}}$ is an equivalence relation. Although we think it should be possible to define $\equiv_{1^{\text{st}}}$ axiomatically, as in differential geometry, we have only been able to do this in special cases.

Among the main points uncovered in our study we mention the following:

(a) *The tangent space to the space of algebraic cycles is quite different from—and in some ways richer than—the tangent space to Hilbert schemes.*

This reflects the group structure on $Z^p(X)$ and properties such as

$$\begin{cases} (z(t)+\widetilde{z}(t))' = z'(t)+\widetilde{z}'(t), \\ (-z(t))' = -z'(t), \end{cases} \tag{1.1}$$

where $z(t)$ and $\widetilde{z}(t)$ are arcs in $Z^p(X)$ with respective tangents $z'(t)$ and $\widetilde{z}'(t)$. As a simple illustration, on a surface X an irreducible curve Y with a normal vector field ν may be obstructed in $\mathrm{Hilb}^1(X)$—e.g., the first order variation of Y in X given by ν may not be extendable to second order. However, considering Y in $Z^1(X)$ as a codimension-1 cycle the first order variation given by ν extends to second order. In fact, it can be shown that both $TZ^1(X)$ and $TZ^n(X)$ are smooth, in the sense that for $p = 1, n$ every map $\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \to Z^p(X)$ is tangent to a geometric arc in $Z^p(X)$.

For the second point, it is well known that algebraic cycles in codimension $p \geqq 2$ behave quite differently from the classical case $p = 1$ of divisors. It turns out that infinitesimally this difference is reflected in a very geometric and computable fashion. In particular,

(b) *The differentials $\Omega^k_{X/\mathbb{C}}$ for all degrees k with $1 \leqq k \leqq n$ necessarily enter into the definition of $TZ^n(X)$.*

Remark that a tangent to the Hilbert scheme at a smooth point is uniquely determined by evaluating 1-forms on the corresponding normal vector field to the subscheme. However, for $Z^n(X)$ the forms of all degrees are required when we want to evaluate on a tangent vector, and it is in this sense that again the tangent space to the space of 0-cycles has a richer structure than the Hilbert scheme. Moreover, we see in (b) that the geometry of higher codimensional algebraic cycles is fundamentally different from that of divisors.

A third point is the following: For an algebraic curve one may give the definition of $TZ^1(X)$ either complex-analytically or algebro-geometrically with equivalent end results. However, it turns out that

(c) *For $n \geqq 2$, even if one is only interested in the complex geometry of X the field of definition of an arc $z(t)$ in $Z^n(X)$ necessarily enters into the description of $z'(0)$.*

Thus, although one may formally define $TZ^n(X)$ in the analytic category, it is only in the algebraic setting that the definition is satisfactory from a geometric perspective. One reason is the following: Any reasonable set of axiomatic properties on first order equivalence of arcs in $Z^n(X)$—including (1.1) above—leads for $n \geqq 2$ to the defining relations for absolute Kähler differentials (cf. section 6.2 below). However, only in the algebraic setting is it the case that the sheaf of Kähler differentials over $\mathbb{C}$ coincides with the sheaf of sections of the cotangent bundle (essentially, one cannot differentiate an infinite series term by term using Kähler differentials). For subtle geometric reasons, (b) and (c) turn out to be closely related.

(d) *A fourth significant difference between divisors and higher codimensional cycles is the following: For divisors it is the case that*

$$\textit{If } z_{u_k} \equiv_{\mathrm{rat}} 0 \textit{ for a sequence } u_k \textit{ tending to } 0, \textit{ then } z_0 \equiv_{\mathrm{rat}} 0.$$

For higher codimension this is false; rational equivalence has an intrinsic "graininess" in codimension $\geqq 2$. If one enhances rational equivalence by closing it up under this property, one obtains the kernel of the Abel-Jacobi map. As will be seen in the text, this graininess in codimension $\geqq 2$ manifests itself in the tangent spaces to cycles in that absolute differentials appear. This is related to the spread construction referred to later in this introduction.

(e) *Although creation/annihilation arcs are present for divisors on curves, they play a relatively inessential role. However, for $n \geqq 2$ it is crucial to understand the infinitesimal behavior of creation/annihilation arcs as these represent the tangencies to "irrelevant" rational equivalences which, in some sense, are the key new aspects in the study of higher codimensional cycles.*

One may of course quite reasonably ask:

Why should one **want** *to define $TZ^p(X)$?*

One reason is that we wanted to understand if there is geometric significance to Spencer Bloch's expression for the formal tangent space to the higher Chow groups, in which absolute differentials mysteriously appear. One of our main results is a response to this question, given by Theorem (8.47) in section 8.3 below. A perhaps deeper reason is the following: The basic Hodge-theoretic invariants of an algebraic cycle are expressed by integrals which are generally transcendental functions of the algebraic parameters describing the cycle. Some of the most satisfactory studies of these integrals have been when they satisfy some sort of functional equation, as is the situation for elliptic functions. However, this will not be the case in general. The other most fruitful approach has been by infinitesimal methods, such as the Picard-Fuchs differential equations and infinitesimal period relations (including the infinitesimal form of functional equations), both of which are of an algebraic character. Just as the infinitesimal period relations for variation of Hodge structure are expressed in terms of the tangent spaces to moduli, it seemed to us desirable to be able to express the infinitesimal Hodge-theoretic invariants of an algebraic cycle—especially those beyond the usual Abel-Jacobi images—in terms of the tangent spaces to cycles. In this monograph we will give such an expression for 0-cycles on a surface.

In the remainder of this introduction we shall summarize the different parts of this book and in so doing explain in more detail the above points.

In chapter 2 we begin by defining $TZ^1(X)$ when X is a smooth algebraic curve, a case that is both suggestive and misleading. Intuitively, we consider arcs $z(t)$ in the space $Z^1(X)$ of 0-cycles on X, and we want to define an equivalence relation $\equiv_{1^{\text{st}}}$ on such arcs so that

$$TZ^1(X) = \{\text{set of arcs in } Z^1(X)\}/\equiv_{1^{\text{st}}} .$$

The considerations are clearly local,[1] and locally we may take

$$z(t) = \operatorname{div} f(t)$$

where $f(t)$ is an arc in $\mathbb{C}(X)^*$. We set $|z(t)| =$ "support of $z(t)$" and assume that $\lim_{t\to 0} |z(t)| = x$. Writing $f(t) = f + tg + \cdots$ elementary geometric considerations suggest that, with the obvious notation, we should define

$$\operatorname{div} f(t) \equiv_{1^{\text{st}}} \operatorname{div} \widetilde{f}(t) \Leftrightarrow [g/f]_x = [\widetilde{g}/\widetilde{f}]_x$$

[1]Throughout this work, unless stated otherwise we will use the Zariski topology. We also denote by $\mathbb{C}(X)$ the field of rational functions on X, with $\mathbb{C}(X)^*$ being the multiplicative group of nonzero functions.

where $[h]_x$ denotes the principal part of the rational function h at the point $x \in X$. Thus, letting $T_{\{x\}}Z^1(X) := TZ^1_{\{x\}}(X)$ be the tangents to arcs $z(t)$ with $\lim_{t\to 0} |z(t)| = x$, we have as a provisional description

$$T_{\{x\}}Z^1(X) = \mathcal{PP}_{X,x}. \tag{1.2}$$

where $\mathcal{PP}_{X,x} = \underline{\underline{\mathbb{C}}}(X)_x/\mathcal{O}_{X,x}$ is the stalk at x of the sheaf of principal parts.

Another possible description of $T_{\{x\}}Z^1(X)$ is suggested by the classical theory of abelian sums. Namely, working in a neighborhood of $x \in X$ and writing

$$z(t) = \sum_i n_i x_i(t),$$

for $\omega \in \Omega^1_{X/\mathbb{C},x}$ we set

$$I(z,\omega) = \frac{d}{dt}\left(\sum_i n_i \int_x^{x_i(t)} \omega\right)_{t=0}.$$

Then $I(z,\omega)$ should depend only on the equivalence class of $z(t)$, and in fact we show that

$$I(z,\omega) = \operatorname{Res}_x(z'\omega)$$

where $z' \in \mathcal{PP}_{X,x}$ is the tangent to $z(t)$ using the description (1.2). This leads to a nondegenerate pairing

$$T_{\{x\}}Z^1(X) \otimes_{\mathbb{C}} \Omega^1_{X/\mathbb{C},x} \to \mathbb{C}$$

so that with either of the above descriptions we have

$$T_{\{x\}}Z^1(X) \cong \operatorname{Hom}^c_{\mathbb{C}}(\Omega^1_{X/\mathbb{C},x}, \mathbb{C}), \tag{1.3}$$

where $\operatorname{Hom}^c_{\mathbb{C}}(\Omega^1_{X/\mathbb{C},x}, \mathbb{C})$ are the continuous homomorphisms in the $\mathfrak{m}_x$-adic topology.

Now (1.3) suggests "duality," and indeed it is easy to see that a third possible description

$$T_{\{x\}}Z^1(X) \cong \lim_{i\to\infty} \mathcal{E}xt^1_{\mathcal{O}_{X,x}}(\mathcal{O}_X/\mathfrak{m}^i_x, \mathcal{O}_X) \tag{1.4}$$

is valid. Of the three descriptions (1.2)–(1.4) of $T_{\{x\}}Z^1(X)$, it will turn out that (1.3) and (1.4) suggest the correct extensions to the case of 0-cycles on n-dimensional varieties. However, the extension is not straightforward. For example, one might suspect that similar consideration of abelian sums would lead to the description (1.3) using 1-forms in general. For interesting geometric reasons, this turns out not to be correct, since, as was suggested above and will be explained below, the correct notion of abelian sums will involve integrals of differential forms of all degrees. Thus, on a smooth variety of dimension n the analogue of the right-hand side of (1.3) will give only part of the tangent space.

As will be explained below, (1.4) also extends but again not in the obvious way. The correct extension which gives the *formal definitions* of the tangent spaces $T_{\{x\}}Z^n(X)$ and tangent sheaf $\underline{\underline{T}}Z^n(X)$ is

$$T_{\{x\}}Z^n(X) := \lim_{i\to\infty} \mathcal{E}xt^n_{\mathcal{O}_{X,x}}(\mathcal{O}_X/\mathfrak{m}^i_x, \Omega^{n-1}_{X/\mathbb{Q}}) \tag{1.5}$$

and

$$\underline{\underline{T}}Z^n(X) = \bigoplus_{x\in X} \lim_{i\to\infty} \underline{\underline{\mathcal{E}xt}}^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{m}_x^i, \Omega^{n-1}_{X/\mathbb{Q}}).$$

The geometric reasons why absolute differentials appear have to do with the points (b) and (c) above and will be discussed below.

The basic building blocks for 0-cycles on a smooth variety X are the *configuration spaces* consisting of sets of m points x_i on X. As a variety this is just the m^{th} symmetric product $X^{(m)}$, whose points we write as effective 0-cycles

$$z = x_1 + \cdots + x_m.$$

We wish to study the geometry of the $X^{(m)}$ collectively, and for this we are interested in differential forms φ_m on $X^{(m)}$ that have the *hereditary property*

(1.6) $$\varphi_{m+1}\Big|_{X_x^{(m)}} = \varphi_m$$

where for each fixed $x \in X$ the inclusion $X_x^{(m)} \hookrightarrow X^{(m+1)}$ is given by $z \to z + x$. One such collection of differential forms on the various $X^{(m)}$'s is given by the *traces* $\operatorname{Tr}\varphi$ of a form $\varphi \in \Omega^q_{X/\mathbb{C}}$. Here, we come to the first geometric reason why forms of higher degree necessarily enter when $n \geqq 1$:

(1.7) $\Omega^*_{X^{(m)}/\mathbb{C}}$ *is generated over* $\mathcal{O}_{X^{(m)}}$ *by sums of elements of the form*

$$\operatorname{Tr}\omega_1 \wedge \cdots \wedge \operatorname{Tr}\omega_k, \qquad \omega_i \in \Omega^{q_i}_{X/\mathbb{C}}.$$

Moreover, we **must** *add generators* $\operatorname{Tr}\omega$ *where* $\omega \in \Omega^q_{X/\mathbb{C}}$ *for all* q *with* $1 \leqq q \leqq n$ *to reach all of* $\Omega^*_{X^{(m)}/\mathbb{C}}$.

Of course, forms of higher degree are needed only in neighborhoods of singular points on the $X^{(m)}$, and for $n \geqq 2$ the singular locus is exactly the diagonals where two or more points coincide.

Put differently, the structure of point configurations is reflected by the geometry of the $X^{(m)}$. The infinitesimal structure of point configurations is then reflected along the diagonals where two or more points have come together and where the $X^{(m)}$ are singular for $n \geqq 2$. The geometric properties of point configurations are in turn reflected by the regular differential forms on the symmetric products, particularly those having the hereditary property (1.6). There is *new* geometric information measured by the traces of q-forms for each q with $1 \leqq q \leqq n$, and thus *the definition of the tangent space to 0-cycles should involve the differential forms of all degrees*. This is clearly illustrated by the coordinate calculations given in chapter 3.

What is this new geometric information reflected by the differential forms of higher degree? One answer stems from E. Cartan, who taught us that when there are natural parameters in a geometric structure then those parameters should be included as part of that structure. In the present situation, if in terms of local uniformizing parameters on B and on X we represent arcs in the space of 0-cycles as sums of Puiseaux series, then the coefficients of these series provide natural parameters for the space of arcs in $Z^n(X)$. It turns out that for $n \geqq 2$ there is new infinitesimal information in these parameters arising from the higher degree forms on X. This phenomenon occurs only in higher codimension and is an essential ingredient in the geometric understanding of the infinitesimal structure of higher codimensional cycles.

The traces of forms $\omega \in \Omega^q_{X/\mathbb{C}}$ give rise to what are provisionally called universal abelian invariants $\widetilde{I}(z, \omega)$ (cf. chapter 3), which in coordinates are certain expressions in the Puiseaux coefficients and their differentials of degree $q - 1$. In order to define the relation of equivalence of arcs in the space of 0-cycles what is needed is some way to map the $q - 1$ forms in the Puiseaux coefficients to a fixed vector space; i.e., a method of comparing the infinitesimal structure at different arcs. Such a map exists, provided that instead of the usual differential forms we take *absolute differentials*. Recall that for any algebraic or analytic variety Y and any subfield k of the complex numbers we may define the Kähler differentials over k of degree r, denoted $\Omega^r_{\mathcal{O}_Y/k}$. For any subvariety $W \subset Y$ there are restriction maps

$$\Omega^r_{\mathcal{O}_Y/k} \to \Omega^r_{\mathcal{O}_W/k}.$$

Taking W to be a point $y \in Y$ and the field k to be $\mathbb{Q}$, since $\mathcal{O}_y \cong \mathbb{C}$ there is an evaluation map

$$e_y : \Omega^r_{\mathcal{O}_{Y,y}/\mathbb{Q}} \to \Omega^r_{\mathbb{C}/\mathbb{Q}}. \tag{1.9}$$

Applying this when Y is the space of Puiseaux coefficients, for an arc z in the space of 0-cycles and form $\omega \in \Omega^q_{\mathcal{O}_X/\mathbb{Q}}$ we may finally define the *universal abelian invariants*

$$I(z, \omega) = e_z \widetilde{I}(z, \omega).$$

Two arcs z and $\widetilde{z}$ are said to be *geometrically equivalent to first order*, written $z \equiv_{1^{\text{st}}} \widetilde{z}$, if

$$I(z, \omega) = I(\widetilde{z}, \omega)$$

for all $\omega \in \Omega^q_{\mathcal{O}_X/\mathbb{Q}}$ and all q with $1 \leqq q \leqq n$. It turns out that here it is sufficient to consider only $\omega \in \Omega^n_{X/\mathbb{Q}}$. The space is filtered with $Gr^q \Omega^n_{X/\mathbb{Q}} \cong \Omega^{n-q}_{\mathbb{C}/\mathbb{Q}} \otimes \Omega^1_{X/\mathbb{C}}$, and roughly speaking we may think of $\Omega^n_{X/\mathbb{Q}}$ as encoding the information in the $\Omega^q_{X/\mathbb{C}}$'s for $1 \leqq q \leqq n$. Intuitively, $\equiv_{1^{\text{st}}}$ captures the invariant information in the differentials at $t = 0$ of Puiseaux series, where the coefficients are differentiated in the sense of $\Omega^1_{\mathbb{C}/\mathbb{Q}}$. The simplest interesting case is when X is a surface defined over $\mathbb{Q}$, ξ and $\eta \in \mathbb{Q}(X)$ give local uniformizing parameters, and $z(t)$ is an arc in $Z^2(X)$ given by

$$z(t) = z_+(t) + z_-(t)$$

where

$$z_\pm(t) = (\xi_\pm(t), \eta_\pm(t))$$

with

$$\begin{cases} \xi_\pm(t) = \pm a_1 t^{1/2} + a_2 t + \cdots \\ \eta_\pm(t) = \pm b_1 t^{1/2} + b_2 t + \cdots . \end{cases}$$

The information in the universal abelian invariants $I(z, \varphi)$ for $\varphi \in \Omega^1_{X/\mathbb{C}}$ is

$$a_1^2, a_1 b_1, b_1^2; a_2, b_2.$$

The additional information in $I(z, \omega)$ for $\omega \in \Omega^2_{X/\mathbb{C}}$ is

$$a_1 db_1 - b_1 da_1,$$

which is not a consequence of the differentials of the $I(z, \varphi)$'s for $\varphi \in \Omega^1_{X/\mathbb{C}}$.

We then give a geometric description of the tangent space as

$$TZ^n(X) = \{\text{arcs in } Z^n(X)\}/ \equiv_{1^{\text{st}}} .$$

The calculations given in chapter 3 show that this definition is independent of the particular coordinate system used to define the space of Puiseaux coefficients. We emphasize that this is *not* the formal definition of $TZ^n(X)$—that definition is given by (1.5), and as we shall show it is equivalent to the above geometric description.

So far this discussion applies to the analytic as well as to the algebraic category. However, *only* in the algebraic setting is it the case that

$$\Omega^1_{\mathcal{O}_X/\mathbb{C}} \cong \mathcal{O}_X(T^*X); \tag{1.10}$$

that is, only in the algebraic setting is it the case that Kähler differentials over $\mathbb{C}$ give the right geometric object. Thus, the above sleight of hand where we used Kähler differentials to define the universal abelian invariants

$$I(z, \omega) \in \Omega^{q-1}_{\mathbb{C}/\mathbb{Q}}$$

will only give the correct geometric notion in the algebraic category. In chapter 4 we give a heuristic, computational approach to absolute differentials. In particular we explain why (1.10) only works in the algebraic setting. The essential point is that the axioms for Kähler differentials extend to allow term by term differentiation of the power series expansion of an *algebraic* function, but this does not hold for a general *analytic* function.

In the algebraic setting $\Omega^1_{\mathcal{O}_X/\mathbb{Q}} = \Omega^1_{X/\mathbb{Q}}$, and there is an additional geometric interpretation of the "arithmetic part" $\Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes \mathcal{O}_X$ of the absolute differentials $\Omega^1_{X/\mathbb{Q}}$. This deals with the notion of a *spread*, and again in chapter 4 we give a heuristic, geometric discussion of this concept. Given a 0-cycle z on an algebraic variety X, both defined over a field k that is finitely generated over $\mathbb{Q}$, the spread will be a family

$$\begin{array}{ccc} \mathcal{X} & \supset & \mathcal{Z} \\ \downarrow & & \downarrow \\ S & = & S \end{array}$$

$$\mathcal{X} = \{X_s\}_{s\in S},\ \mathcal{Z} = \{z_s\}_{s\in S}$$

where $\mathcal{X}$, $\mathcal{Z}$, and S are all defined over $\mathbb{Q}$ and $\mathbb{Q}(S) \cong k$, and where the fiber over a generic point $s_0 \in S$ is our original X and z. Roughly speaking we may think of spreads as arising from the different embeddings of k into $\mathbb{C}$; thus, for $s \in S$ not lying in a proper subvariety defined over $\mathbb{Q}$ the algebraic properties of X_s and z_s are the same as those of X and z. There is a canonical mapping

$$T^*_{s_0} S \to \Omega^1_{k/\mathbb{Q}}, \tag{1.11}$$

and under this mapping the extension class of

$$0 \to \Omega^1_{k/\mathbb{Q}} \otimes_k \mathcal{O}_{X(k)} \to \Omega^1_{X(k)/\mathbb{Q}} \to \Omega^1_{X(k)/k} \to 0$$

corresponds to the Kodaira-Spencer class of the family $\{X_s\}_{s \in S}$ at s_0. The fact that the spread gives in higher codimension the natural parameters of a cycle and that infinitesimally the spread is expressed in terms of $\Omega^1_{X/\mathbb{Q}}$ are two reasons why absolute differentials necessarily appear.

Using this discussion of absolute differentials and spreads in chapter 4, in chapter 5 we turn to the geometric description of the tangent space $TZ^n(X)$ to the space of 0-cycles or a smooth n-dimensional algebraic variety X. We say "geometric" because the formal algebraic definition of the tangent spaces $TZ^n(X)$ will be given in chapter 7 using an extension of the *Ext* construction discussed above in the $n = 1$ case. This definition will then be proved to coincide with the description using the universal abelian invariants discussed above. In chapter 5 we give an alternate, intrinsic definition of the $I(z, \omega)$'s based on functorial properties of absolute differentials.

In chapter 4 we have introduced absolute differentials as a means of mapping the differentials of the parameters of an arc in $Z^n(X)$ expressed in terms of local uniformizing parameters to a reference object. Geometrically, using (1.11) this construction reflects infinitesimal variation in the spread directions. Algebraically, for an algebraic variety Y and point $y \in Y$, the evaluation mapping (1.9) is for $r = 1$ given by

$$f dg \overset{e_y}{\to} f(y) d(g(y)),$$

where $d = d_{\mathbb{C}/\mathbb{Q}}$ and $f, g \in \mathbb{C}(Y)$ are rational functions on Y that are regular near y. If Y is defined over $\mathbb{Q}$ and $f, g \in \mathbb{Q}(Y)$, then e_y reflects the field of definition of y—it is thus measuring arithmetic information.

One may reasonably ask: *Is there an alternative, purely geometric way of defining* $\equiv_{1^{\text{st}}}$ *for arcs in* $Z^n(X)$ *that leads to absolute differentials?* In other words, even if one is only interested in the complex geometry of the space of 0-cycles, is there a geometric reason why arithmetic considerations enter the picture? Although we have not been able to completely define $\equiv_{1^{\text{st}}}$ axiomatically, we suspect that this can be done and in a number of places we will show geometrically how differentials over $\mathbb{Q}$ necessarily arise.

For example, in section 6.2 we consider the free group F generated by the arcs in $Z^2(\mathbb{C}^2)$ given by differences $z_{\alpha\beta}(t) - z_{1\beta}(t)$ where $z_{\alpha\beta}(t)$ is the 0-cycle given by the equations

$$z_{\alpha\beta}(t) = \begin{cases} x^2 - \alpha y^2 = 0 \qquad \alpha \neq 0 \\ xy - \beta t = 0. \end{cases}$$

There we list a set of "evident" geometric axioms for first order equivalence of arcs in F, and then an elementary but somewhat intricate calculation shows that the map

$$F/\equiv_{1^{\text{st}}} \to \Omega^1_{\mathbb{C}/\mathbb{Q}}$$

given by

$$z_{\alpha\beta}(t) \to \beta \frac{d\alpha}{\alpha} \qquad (d \text{ denotes } d_{\mathbb{C}/\mathbb{Q}})$$

is a well-defined isomorphism. Essentially, the condition that the tangent map be a homomorphism to a vector space that factors through the tangent map to the Hilbert scheme leads directly to the defining relations for absolute Kähler differentials.

Another example, one that will be used elsewhere in the book, begins with the observation that $Z^n(X)$ is the group of global sections of the Zariski sheaf

$$\bigoplus_{x \in X} \underline{\underline{\mathbb{Z}}}_x.$$

Taking X to be a curve we may consider the Zariski sheaf

$$\bigoplus_{x \in X} \underline{\underline{\mathbb{C}}}^*_x$$

whose global sections we denote by $Z^1_1(X)$. This sheaf arises naturally when one localizes the tame symbol mappings T_x that arise in the Weil reciprocity law. In section 6.2 we give a set of geometric axioms on arcs in $Z^1_1(X)$ that define an equivalence relation yielding a description of the sheaf $\underline{\underline{T}} Z^1_1(X)$ as

$$\underline{\underline{T}} Z^1_1(X) \cong \bigoplus_{x \in X} \underline{\underline{\mathrm{Hom}}}^o(\Omega^1_{X/\mathbb{Q},x}, \Omega^1_{\mathbb{C}/\mathbb{Q}});$$

here $\mathrm{Hom}^o(\Omega^1_{X/\mathbb{Q},x}, \Omega^1_{\mathbb{C}/\mathbb{Q}})$ are the continuous $\mathbb{C}$-linear homomorphims $\Omega^1_{X/\mathbb{Q},x} \xrightarrow{\varphi} \Omega^1_{\mathbb{C}/\mathbb{Q}}$ that satisfy

$$\varphi(f\alpha) = \varphi_0(f)\alpha,$$

where $f \in \mathcal{O}_{X,x}$, $\alpha \in \Omega^1_{\mathbb{C}/\mathbb{Q}}$, and $\varphi_0 : \mathcal{O}_{X,x} \to \mathbb{C}$ is a continuous $\mathbb{C}$-linear homomorphism. The point is that again purely geometric considerations lead naturally to differentials over $\mathbb{Q}$. Essentially, the reason again comes down to the assumption that

$$(z \pm \widetilde{z})' = z' \pm \widetilde{z}'$$

i.e., the tangent map should be a homomorphism from arcs in $Z^1_1(X)$ to a vector space.

In (d) at the beginning of the introduction we mentioned different limiting properties of rational equivalence for divisors and higher codimensional cycles. One property that our tangent space construction has is the following: Let $z_u(t)$ be a family of arcs in $Z^p(X)$. Then

$$\textit{If } z'_u(0) = 0 \textit{ for all } u \neq 0, \textit{ then } z'_0(0) = 0.$$

Once again the statement

$$\lim z'_{u_k}(0) = 0 \textit{ for a sequence } u_k \to 0 \textit{ implies that } z'_0(0) = 0$$

is true for divisors but false in higher codimension. The reason is essentially this: Any algebraic construction concerning algebraic cycles survives when we take the spread of the variety together with the cycles over their field of definition. Geometric invariants arising in the spread give invariants of the original cycle. Infinitesimally, related to (b) above there is in higher codimensions new information arising from evaluating q-forms ($q \geqq 2$) on multivectors $v \wedge w_1 \wedge \cdots \wedge w_{q-1}$, where v is the tangent to the arc in the usual "geometric" sense and $w_1, \ldots, w_{q-1}$ are tangents in the spread directions. Thus arithmetic considerations appear at the level of the tangent space to cycles (we did not expect this) and survive in the tangent space to Chow groups where they appear in Bloch's formula.

Above, we mentioned the tame symbol $T_x(f, g) \in \mathbb{C}^*$ of $f, g \in \mathbb{C}(X)^*$. It has the directly verified properties

$$\begin{cases} T_x(f^m, g) = T_x(f, g)^m \text{ for } m \in \mathbb{Z} \\ T_x(f_1 f_2, g) = T_x(f_1, g) T_x(f_2, g) \\ T_x(f, g) = T_x(g, f)^{-1} \\ T_x(f, 1 - f) = 1, \end{cases}$$

which show that the tame symbol gives mappings

$$T_x : K_2\big(\mathbb{C}(X)\big) \to \mathbb{C}^*, \text{ and } T : K_2(\mathbb{C}(X)) \to \bigoplus_x \mathbb{C}^*_x.$$

A natural question related to the definition of $\underline{\underline{T}} Z^1_1(X)$ is

What is the differential of the tame symbol?

According to van der Kallen [12], for any field or local ring F in characteristic zero the formal tangent space to $K_2(F)$ is given by

$$TK_2(F) \cong \Omega^1_{F/\mathbb{Q}}. \tag{1.12}$$

Thus, we are seeking to calculate

$$\Omega^1_{\mathbb{C}(X)/\mathbb{Q}} \xrightarrow{dT_x} \operatorname{Hom}^o(\Omega^1_{X/\mathbb{Q},x}, \Omega^1_{\mathbb{C}/\mathbb{Q}}).$$

In section 6.3 we give this evaluation in terms of residues; this calculation again illustrates the linking of arithmetic and geometry. As an aside, we also show that the infinitesimal form of the Weil and Suslin reciprocity laws follow from the residue theorem.

Beginning with the work of Bloch, Gersten, and Quillen (cf. [5] and [16]) one has understood that there is an intricate relationship between K-theory and higher codimensional algebraic cycles. For X an algebraic curve, the Chow group $CH^1(X)$ is defined as the cokernel of the mapping obtained by taking global sections of the surjective mapping of Zariski sheaves

$$\underline{\underline{\mathbb{C}}}(X)^* \xrightarrow{\text{div}} \bigoplus_{x \in X} \underline{\underline{\mathbb{Z}}}_x \to 0. \tag{1.13}$$

This sheaf sequence completes to the exact sequence

$$0 \to \mathcal{O}^*_X \to \underline{\underline{\mathbb{C}}}(X)^* \to \bigoplus_{x \in X} \underline{\underline{\mathbb{Z}}}_x \to 0\,, \tag{1.14}$$

and the exact cohomology sequence gives the well-known identification

(1.15) $$CH^1(X) \cong H^1(\mathcal{O}_X^*).$$

For X an algebraic surface, the analogue of (1.13) is

(1.16) $$\bigoplus_{\left\{\substack{Y \text{ irred} \\ \text{curve}}\right.} \underline{\underline{\mathbb{C}}}(Y)^* \xrightarrow{\text{div}} \bigoplus_{x \in X} \underline{\underline{\mathbb{Z}}}_x \to 0.$$

Whereas the kernel of the map in (1.13) is evidently $\mathcal{O}_X^*$, for (1.16) it is a nontrivial result that the kernel is the image of the map

$$\underline{\underline{K}}_2(\mathbb{C}(X)) \xrightarrow{T} \bigoplus_{\left\{\substack{Y \text{ irred} \\ \text{curve}}\right.} \underline{\underline{\mathbb{C}}}(Y)^*$$

given by the tame symbol. It is at this juncture that K-theory enters the picture in the study of higher codimension algebraic cycles. The sequence (1.16) then completes to the analogue of (1.14), the Bloch-Gersten-Quillen exact sequence

$$0 \to \mathcal{K}_2(\mathcal{O}_X) \to \underline{\underline{K}}_2(\mathbb{C}(X)) \to \bigoplus_{\left\{\substack{Y \text{ irred} \\ \text{curve}}\right.} \underline{\underline{\mathbb{C}}}(Y)^* \to \bigoplus_{x \in X} \underline{\underline{\mathbb{Z}}}_x \to 0$$

which in turn leads to Bloch's analogue

(1.17) $$CH^2(X) \cong H^2(\mathcal{K}_2(\mathcal{O}_X))$$

of (1.15), which opened up a whole new perspective in the study of algebraic cycles.

The infinitesimal form of (1.17) is also due to Bloch (cf. [4] and [27]) with important amplifications by Stienstra [6]. In this work the van der Kallen result is central. Because it is important for our work to understand in detail the infinitesimal properties of the Steinberg relations that give the (Milnor) K-groups, we have in the appendix to chapter 6 given the calculations that lie behind (1.12). At the end of this appendix we have amplified on the above heuristic argument that shows from a geometric perspective how K-theory and absolute differentials necessarily enter into the study of higher codimensional algebraic cycles.

In chapter 7 we give the formal definition

$$\underline{\underline{T}}Z^2(X) = \lim_{\left\{\substack{Z \text{ codim } 2 \\ \text{subscheme}}\right.} \mathcal{E}xt^2_{\mathcal{O}_X}\left(\mathcal{O}_Z, \Omega^1_{X/\mathbb{Q}}\right)$$

for the tangent sheaf to the sheaf of 0-cycles on a smooth algebraic surface X. We show that this is equivalent to the geometric description discussed above. Then, based on a construction of Angéniol and Lejeune-Jalabert [19], we define a map

$$T\text{Hilb}^2(X) \to TZ^2(X),$$

thereby showing that the tangent to an arc in $Z^2(X)$ given as the image in $Z^2(X)$ of an arc in $\text{Hilb}^2(X)$ depends only on the tangent to that arc in $T\text{Hilb}^2(X)$.

In summary, the geometric description has the advantages:

– it is additive;

– it depends only on $z(t)$ as a cycle;

– it depends only on $z(t)$ up to first order in t;

– it has clear geometric meaning.

It is however not clear that two families of effective cycles that represent the same element of $T\,\mathrm{Hilb}^2(X)$ have the same tangent under the geometric description. The formal definition has the properties:

– it clearly factors through $T\mathrm{Hilb}^2(X)$;

– it is easy to compute in examples.

But additivity does not make sense for arbitrary schemes, and in the formal definition it is not clear that $z'(0)$ depends only on the cycle structure of $z(t)$. For this reason it is important to show their equivalence.

In chapter 8 we give the definitions of some related spaces, beginning with the definition

$$\underline{\underline{T}}Z^1(X) = \lim_{\left\{\substack{Z \text{ codim1} \\ \text{subscheme}}\right.} \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X)$$

for the sheaf of divisors on a smooth algebraic surface X. Actually, this definition contains interesting geometry not present for divisors on curves. In section 8.2 this geometry is discussed both directly and dually using differential forms and residues. As background for this, we review duality with emphasis on how one may use the theory to compute in examples.

In section 8.2 we give the definition

$$\underline{\underline{T}}Z^1_1(X) = \bigoplus_{\left\{\substack{Y \text{ codim 1} \\ Y \text{ irred}}\right.} \underline{\underline{H}}^1_y\left(\Omega^1_{X/\mathbb{Q}}\right)$$

for the tangent sheaf to the Zariski sheaf $\bigoplus_Y \underline{\underline{\mathbb{C}}}(Y)^*$. Underlying this definition is an interesting mix of arithmetic and geometry which is illustrated in a number of examples. With this definition there is a natural map $\underline{\underline{T}}Z^1_1(X) \to \underline{\underline{T}}Z^2(X)$ and passing to global sections we may define the *geometric tangent space* to the Chow group $CH^2(X)$ by

$$T_{\mathrm{geom}}CH^2(X) = TZ^2(X)/\mathrm{image}\left\{TZ^1_1(X) \to TZ^2(X)\right\}.$$

Both the numerator and denominator on the RHS have geometric meaning and are amenable to computation in examples. The main result of this work is then given by the

Theorem: (i) *There is a natural surjective map*

$$\begin{pmatrix} \text{arcs in} \\ \bigoplus_Y \underline{\underline{\mathbb{C}}}(Y)^* \end{pmatrix} \to \underline{\underline{T}}Z^1_1(X).$$

(ii) *Denoting by* $T_{\mathrm{formal}}CH^2(X)$ *the formal tangent space to the Chow group given by Bloch* [4], [27], *there is a natural identification*

$$T_{\mathrm{geom}}CH^2(X) \cong T_{\mathrm{formal}}CH^2(X).$$

Contained in (i) and (ii) in this theorem is a geometric existence result, albeit at the infinitesimal level. The interesting but significant difficulties in "integrating" this result are discussed in section 8.4 and again in chapter 10.

In chapter 9 we give some applications and examples. Classically, on an algebraic curve Abel's differential equations—by which we mean the infinitesimal form of Abel's theorem—express the infinitesimal constraints that a 0-cycle move in a rational equivalence class. An application of our work gives an extension of Abel's differential equations to 0-cycles on an n-dimensional smooth variety X. For X a regular algebraic surface defined over $\mathbb{Q}$ these conditions take the following form: Let $z = \sum_i x_i$ be a 0-cycle, where for simplicity of exposition we assume that the x_i are distinct. Given $\tau_i \in T_{x_i}X$ we ask *when is*

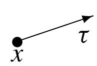
$$\tau = \sum_i (x_i, \tau_i) \in TZ^2(X) \tag{1.18}$$

tangent to a rational equivalence? Here there are several issues that one does not see in the curve case. One is that because of the cancellation phenomenon in higher codimension discussed above it is essential to allow creation/annihilation arcs in $Z^2(X)$, so it is understood that a picture like

$$\begin{array}{c} x_-(t) \\ \leftarrow \cdot \rightarrow \\ x_+(t) \end{array}$$

$$\begin{cases} x(t) = x_+(t) - x_-(t), \\ x_+(0) = x_-(0), \\ x'_+(0) = -x'_-(0) \end{cases}$$

is allowed, and a picture like



could be the tangent to a simple arc $x(t)$ in X with $x(0) = x$ and $x'(0) = \tau$, or it could be the tangent to an arc

$$z(t) = x_1(t) + x_2(t) - x_3(t),$$

where

$$\begin{cases} x_1(0) = x_2(0) = x_3(0) = x, \\ x'_1(0) + x'_2(0) - x'_3(0) = \tau, \end{cases}$$

and so forth.[2]

Second, we can only require that τ be tangent to a first order arc in $Z^2_{\mathrm{rat}}(X)$. Alternatively, we could require (i) that τ be tangent to a formal arc in $Z^2_{\mathrm{rat}}(X)$, or

[2]Of course, for curves one may introduce creation/annihilation arcs, but as noted in point (e) above it is only in higher codimension, due to the presence of "irrelevant" rational equivalences, that they play an essential role.

(ii) that τ be tangent to a geometric arc in $Z^2_{\mathrm{rat}}(X)$. There are heuristic geometric reasons that (i) may be the same as tangent to a first-order arcs, but although (ii) *may* be equivalent for 0-cycles on a surface (essentially Bloch's conjecture), there are Hodge-theoretic reasons why for higher dimensional varieties the analogue of (ii) cannot in general be equivalent to tangency to a first order rational equivalence for general codimension 2 cycles (say, curves on a threefold). In any case, there are natural pairings

$$\langle\ ,\ \rangle : \Omega^2_{X/\mathbb{Q},x} \otimes T_x X \to \Omega^1_{\mathbb{C}/\mathbb{Q}} \tag{1.19}$$

and the condition that (1.18) be tangent to a first order rational equivalence class is

$$\langle \omega, \tau \rangle =: \sum_i \langle \omega, \tau_i \rangle = 0 \qquad \text{in} \quad \Omega^1_{\mathbb{C}/\mathbb{Q}} \tag{1.20}$$

for all $\omega \in H^0(\Omega^2_{X/\mathbb{Q}})$. If the $x_i \in X(k)$ then the pairing (1.19) lies in $\Omega^1_{k/\mathbb{Q}}$.

At one extreme, if $z = \sum_i x_i \in Z^2\big(X(\bar{\mathbb{Q}})\big)$ then all $\langle \omega, \tau_i \rangle = 0$ and the main theorem stated above gives a geometric existence result which is an infinitesimal version of the conjecture of Bloch-Beilinson [22]. At the other extreme, taking the x_i to be independent transcendentals we obtain a quantitative version of the theorem of Mumford-Roitman (cf. [1] and [2]). In between, the behavior of how a 0-cycle moves infinitesimally in a rational equivalence class is very reminiscent of the behavior of divisors on curves where $h^{2,0}(X)$ *together with* tr deg (k) play the role of the genus of the curve.

In section 9.2 we discuss the integration of Abel's differential equations. The exact meaning of "integration" is explained there—roughly it means defining a Hodge-theoretic object $\mathcal{H}$ and map

$$\psi : Z^n(X) \to \mathcal{H} \tag{1.21}$$

whose codifferential factors through the map

$$T^* CH^n(X) \to T^* Z^n(X).$$

For curves, denoting by $Z^1(X)_0$ the divisors of degree zero, the basic classical construction is the pairing

$$H^0(\Omega^1_{X/\mathbb{C}}) \otimes Z^1(X)_0 \to \mathbb{C} \text{ mod periods}$$

given by

$$\omega \otimes z \xrightarrow{\psi} \int_\gamma \omega, \qquad \partial\gamma = z.$$

As z varies along an arc z_t

$$\frac{d}{dt}\big(\psi(\omega \otimes z_t)\big) = \langle \omega, z' \rangle$$

where the right-hand side is the usual pairing

$$H^0(\Omega^1_{X/\mathbb{C}}) \otimes TZ^1(X) \to \mathbb{C}$$

of differential forms on tangent vectors. This of course suggests that the usual abelian sums should serve to integrate Abel's differential equations in the case of curves.

In [32] we discussed the integration of Abel's differential equations in general. Here we consider the first nonclassical case of a regular surface X defined over $\mathbb{Q}$, and we shall explain how the geometric interpretation of (1.20) suggests how one may construct a map (1.21) in this case. What is needed is a pairing

$$H^0(\Omega^2_{X/\mathbb{C}}) \otimes Z^2(X)_0 \to \int_\Gamma \omega \quad \text{mod periods,} \tag{1.22}$$

where Γ is a (real) 2-dimensional chain that is constructed from z using the assumptions that $\deg z = 0$ and that X is regular. If $z \in Z^2\big(X(k)\big)_0$, then using the spread construction together with (1.10) we have a pairing

$$H^0(\Omega^2_{X/\mathbb{Q}}) \otimes T_z Z^2(X)_0 \to T^*_{s_0} S, \tag{1.23}$$

which if we compare it with (1.10) will, according to (1.19) and (1.20), give the conditions that z move infinitesimally in a rational equivalence class. Writing (1.23) as a pairing

$$H^0(\Omega^2_{X/\mathbb{Q}}) \otimes TZ^2(X) \otimes TS \to \mathbb{C}$$

suggests in analogy to the curve case that in (1.21) the 2-chain Γ should be traced out by 1-chains γ_s in X parametrized by a curve λ in S. Choosing γ_s so that $\partial\gamma_s = z_s$ and taking for λ a closed curve in S, we are led to set $\Gamma = \bigcup_{s\in\lambda} \gamma_s$ and define for $\omega \in H^0(\Omega^2_{X/\mathbb{C}})$

$$I(z, \omega, \lambda) = \int_\Gamma \omega \quad \text{mod periods.} \tag{1.24}$$

As is shown in section 9.2 this gives a *differential character* on S that depends only on the k-rational equivalence class of z.[3] If one *assumes* the conjecture of Bloch and Beilinson, then the triviality of $I(z, \cdot, \cdot)$ implies that z is rationally equivalent to zero; this would be an analogue of Abel's theorem for 0-cycles on a surface.

In section 9.3 we give explicit computations for surfaces in $\mathbb{P}^3$ leading to the following results:

Let X be a general surface in $\mathbb{P}^3$ of degree $d \geqq 5$. Then, for any point $p \in X$

$$T_p X \cap TZ^2(X)_{\text{rat}} = 0.$$

If $d \geqq 6$, then for any distinct points $p, q \in X$

$$(T_p X + T_q X) \cap TZ^2(X)_{\text{rat}} = 0.$$

The first statement implies that a general X contains no rational curve—, that is, a g^1_1—which is a well-known result of Clemens. The second statement implies that a general X of degree $\geqq 6$ does not contain a g^1_2. It may well be that the method of proof can be used to show that for each integer k there is a $d(k)$ such that for $d \geqq d(k)$ a general X does not contain a g^1_k.

In section 9.4 we discuss what seems to be the only nonclassical case where the Chow group is explicitly known: the isomorphism

$$Gr^2 CH^2(\mathbb{P}^2, T) \cong K_2(\mathbb{C}) \tag{1.25}$$

---

[3] The regularity of X enters in the rigorous construction and in the uniqueness of the lifting of ω to $H^0(\Omega^2_{X/\mathbb{Q}})$. Also, the construction is only well-defined modulo torsion. Finally, as discussed in section 9.3, one must "enlarge" the construction (1.24) to take into account all the transcendental part $H^2(X)_{\text{tr}}$ of the second cohomology group of X.

due to Bloch and Suslin [26], [21]. We give a proof of (1.25) similar to that of Totaro [9], showing that it is a consequence of the Suslin reciprocity law together with elementary geometric constructions. This example was of particular importance to us as it was one where the infinitesimal picture could be understood explicitly. In particular, we show that if tr deg $k = 1$ so that S is an algebraic curve, the invariant (1.24) coincides with the regulator and the issue of whether it captures rational equivalence, modulo torsion, reduces to an analogue of a well-known conjecture about the injectivity of the regulator.

In the last chapter we discuss briefly some of the larger issues that this study has raised. One is whether or not the space of codimension p cycles $Z^p(X)$ is at least "formally reduced." That is, given a tangent vector $\tau \in TZ^p(X)$, is there a formal arc $z(t)$ in $Z^p(X)$ with tangent τ? If so, is $Z^p(X)$ "actually reduced"; i.e., can we choose $z(t)$ to be a geometric arc? Here we are assuming that a general definition of $TZ^p(X)$ has been given extending that given in this work when $p = 1$ and $p = n$, and that there is a natural map

$$T\mathrm{Hilb}^p(X) \to TZ^p(X).$$

The first part of the following proposition is proved in this book; the second is a result of Ting Fei Ng [39], the idea of whose proof is sketched in chapter 10:

(1.26) $Z^p(X)$ *is reduced for* $p = n, 1$.

What this means is that for $p = n, 1$ every tangent vector in $TZ^p(X)$ is the tangent to a geometric arc in $Z^p(X)$. For $p = n$ this is essentially a local result. However, for $p = 1$ and $n \geqq 2$ it is well known that $\mathrm{Hilb}^1(X)$ may not be reduced. Already when $n = 2$ there exist examples of a smooth curve Y in an algebraic surface and a normal vector field $\nu \in H^0(N_{Y/X})$ which is not tangent to a geometric definition of Y in X; i.e., ν may be obstructed. However, when we consider Y as a codimension one cycle on X the above result implies that there is an arc $Z(t)$ in $Z^1(X)$ with

$$\begin{cases} Z(0) = Y, \\ Z'(0) = \nu; \end{cases}$$

in particular, *allowing Y to deform as a cycle kills the obstructions.*

For Hodge-theoretic reasons, (1.26) cannot be true in general—as discussed in chapter 10, when $p = 2$ and $n = 3$ the result is not true. Essentially there are two possibilities:

(i) $Z^p(X)$ *is not reduced.*

(ii) $Z^p(X)$ *is formally, but not actually, reduced.*

Here we are using "reduced" as if $Z^p(X)$ had a scheme structure, which of course it does not. What is meant is that first an m^{th} order arc is given by a finite linear combination of the map to the space of cycles induced by maps

$$\mathrm{Spec}\left(\mathbb{C}[t]/t^{m+1}\right) \to \mathrm{Hilb}^p(X), \qquad m \geqq 1.$$ [4]

[4]The issue of the equivalence relation on such maps to define the same cycle is non-trivial — cf. section 10.2. In fact, the purpose of chapter 10 is to raise issues that we feel merit further study.

The tangent to such an arc factors as in

$$\begin{array}{ccc} \operatorname{Spec}(\mathbb{C}[t]/t^{m+1}) & \rightarrow & Z^p(X) \\ \downarrow & \searrow & \\ \operatorname{Spec}(\mathbb{C}[t]/t^2) & \rightarrow & TZ^p(X) \end{array}$$

where the top row is the finite linear combination of the above maps, and where the bottom row is surjective. To say that $\tau \in TZ^p(X)$ is *unobstructed to order* m means that it is in the image of the dotted arrow. To say that it is *formally reduced* means that it is unobstructed to order m for all m. To say that it is *actually reduced* means that it comes from a geometric arc

$$B \rightarrow Z^p(X).$$

Another anomaly of the space of cycles is the presence of *null curves* in the Chow group, these being curves $z(t)$ in $CH^p(X)$ that are nonconstant but whose derivative is identically zero. They arise from tangent vectors to rational equivalences that do not arise from actual rational equivalences (nonreduced property of $TZ^n_{\text{rat}}(X) =:$ image $\{TZ^n_1(X) \rightarrow TZ^n(X)\}$—see below for notation). Thus, if one thinks in the language of differential equations:

(1.27) *Because of the presence of null curves, there can be no uniqueness in the integration of Abel's differential equations.*

Thus, the usual existence and uniqueness theorems of differential equations both fail in our context. Heuristic considerations suggest that one must add additional arithmetic considerations to have even the possibility of convergent iterative constructions. The monograph concludes with a discussion of this issue in section 10.4.

We have used classical terminology in discussing the spaces of cycles on an algebraic variety, as if the $Z^p(X)$ were themselves some sort of variety. However, because of properties such as (1.26) and (1.27) the $Z^p(X)$ are decidedly nonclassical objects. This nonclassical behavior is combined Hodge-theoretic and arithmetic in origin, and in our view understanding it presents a deep challenge in the study of algebraic cycles.

To conclude this introduction we shall give some references and discuss the relationship of this material to some other works on the space of cycles on an algebraic variety.

Our original motivation stems from the work of David Mumford and Spencer Bloch some thirty odd years ago. The paper [Rational equivalence of 0-cycles on surfaces., *J. Math. Kyoto Univ.* **9** (1968), 195–204] by Mumford showed that the story for Chow groups in higher codimensions would be completely different from the classical case of divisors. Certainly one of the questions in our minds was whether Mumford's result and the subsequent important extensions by Roitman [Rational equivalence of zero-dimensional cycles (Russian), *Mat. Zametki* **28**(1) (1980), 85–90, 169] and [The torsion of the group of 0-cycles modulo rational equivalence, *Ann. of Math.* **111** (2) (1980), 553–569] could be understood, and perhaps refined, by defining the tangent space to cycles and then passing to the quotient by infinitesimal rational equivalence—this turned out to be the case.

The monograph *Lectures on algebraic cycles* (Duke University Mathematics Series, IV. Duke University, Mathematics Department, Durham, N.C., 1980. 182 pp.) by Bloch was one of the major milestones in the study of Chow groups and provided significant impetus for this work. The initial paper [K_2 and algebraic cycles, *Ann. of Math.* **99**(2) (1974), 349–379] by Bloch its successor [Algebraic cycles and higher K-theory, *Adv. in Math.* **61**(3) (1986), 267–304] together with Quillen's work [16] brought K-theory into the study of cycles, and trying to understand geometrically what is behind this was one principal motivation for this work. We feel that we have been able to do this infinitesimally by giving a geometric understanding of how absolute differentials necessarily enter into the description of the tangent space to the space of 0-cycles on a smooth variety. One hint that this should be the case came from Bloch's early work [*On the tangent space to Quillen K-theory*, *Lecture Notes in Math.* **341** (1974), Springer-Verlag] and summarized in [4] and with important extensions by Stienstra, Balere [On K_2 and K_3 of truncated polynomial rings, Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980), pp. 409–455, Lecture Notes in Math. **854**, Springer, Berlin, 1981].

Another principal motivation for us has been provided by the conjectures of Bloch and Beilinson. These are explained in sections 6 and 8 of [Ramakrishnan, Dinakar, Regulators, algebraic cycles, and values of L-functions, *Contemp. Math.* **83** (1989), 183–310] and in [Jannsen, U., Motivic sheaves and filtrations on Chow groups. Motives (Seattle, WA, 1991), 245–302, *Proc. Sympos. Pure Math.* **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994]. Our work provides a geometric understanding and verification of these conjectures at the infinitesimal level, and it also points out some of the major obstacles to "integrating" these results [8].

In an important work, Blaine Lawson introduced a topology on the space $Z^p(X)$ of codimension p algebraic cycles on a smooth complex projective variety. Briefly, two codimension-p cycles z, z' written as

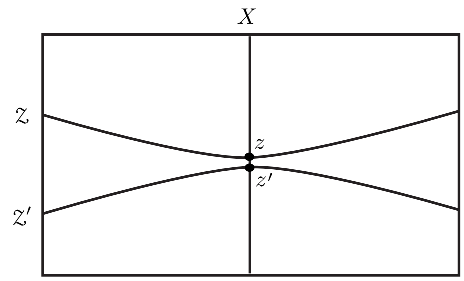
$$\begin{cases} z = z_+ - z_- \\ z' = z'_+ - z'_- \end{cases}$$

where $z_\pm$, $z'_\pm$ and effective cycles are close, if z_+, z'_+ and z_-, z'_- are close in the usual sense of closed subsets of projective space. Lawson then shows that $Z^p(X)$ has the homotopy type of a CW complex, and from this he proceeds to define the Lawson homology of X in terms of the homotopy groups of $Z^p(X)$. His initial work triggered an extensive development, many aspects of which are reported on in his talk at ICM Zürich (cf. [Lawson, Spaces of Algebraic Cycles—Levels of Holomorphic Approximation, *Proc. ICM Zürich*, 574–584] and the references cited therein).

In this monograph, although we do not define a topology on $Z^p(X)$, we do define and work with the concept of a (regular) arc in $Z^p(X)$. Implicit in this is the condition that two cycles z, z' as above should be close: First, there should be a common field of definition for X, z, and z'. This leads to the spreads

$$\begin{array}{c} \mathcal{Z}, \mathcal{Z}' \subset \mathcal{X} \\ \downarrow \\ S \end{array}$$

as discussed in chapter 4 below, and z, z' should be considered close if $\mathcal{Z}$, $\mathcal{Z}' \in Z^p(\mathcal{X})$ are close in the Lawson sense (taking care to say what this means, since the spread is not uniquely defined). As seen in the diagram,



two cycles may be Lawson close without being close in our sense. We do not attempt to formalize this, but rather wish only to point out one relationship between the theory here and that of Lawson and his coworkers.

Finally, we mention that some of the early material in this study has appeared in [23].

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