Chapter One

Introduction

We will consider infinite matrices indexed by $\mathbf{Z}$ (or $\mathbf{Z}^b$) associated to a dynamical system in the sense that

$$H = \big(H(x)_{m,n}\big)_{m,n\in\mathbf{Z}}$$

satisfies

$$H(x)_{m+1,n+1} = H(Tx)_{m,n}$$

where $x \in \Omega$, and T is an ergodic measure-preserving transformation of Ω. Typical settings considered here are

$$\begin{array}{lll}
\Omega = \mathbf{T} & Tx = x + \omega & (1 - \text{frequency shift}) \\
\Omega = \mathbf{T}^d & Tx = x + \omega & (d - \text{frequency shift}) \\
\Omega = \mathbf{T}^2 & Tx = (x_1 + x_2, x_2 + \omega) & \text{(skewshift)} \\
\Omega = \mathbf{T}^2 & Tx = Ax, \text{ where } A \in SL_2(\mathbf{Z}), \text{hyperbolic} &
\end{array}$$

Thus

$$H(x)_{m,n} = \phi_{m-n}(T^m x) \tag{1.0}$$

where the ϕ_k are functions on Ω.

We will usually assume that $H(x)$ is self-adjoint, although many parts of our analysis are independent of this fact. Define

$$H_N = R_{[1,N]} H R_{[1,N]}$$

where $R_{[1,N]} =$ coordinate restriction to $[1, N] \subset \mathbf{Z}$, and the associated Green's functions are

$$G_N(E) = (H_N - E)^{-1}$$

(if $H_N - E$ is an invertible $N \times N$ matrix).

One of our concerns will be to obtain a 'good bound' on $G_N(E, x)$, except for x in a "small" exceptional set. A typical statement would be the following:

$$\|G_N(E, x)\| < e^{N^{1-\delta}} \tag{1.1}$$

and

$$|G_N(E, x)(m, n)| < e^{-c|m-n|} \text{ for } |m - n| > \frac{N}{10} \tag{1.2}$$

for all x outside a set of measure $< e^{-N^\sigma}$. Here $\delta, \sigma > 0$ are some constants. The exceptional set in x does depend on E, of course. Such estimates are of importance in the following problems, for instance.

1. Spectral problems for lattice Schrödinger operators

Description of the spectrum Spec $H(x)$ and eigenstates of $H(x)$ (i.e., point spectrum, continuous (absolutely continuous or singular continuous) spectrum, localization, extended states, etc.)

2. Long-time behavior of linear time-dependent Schrödinger operators

$$i\frac{\partial u}{\partial t} + \Delta u + V(x,t)u = 0 \tag{1.3}$$

The spatial variable $x \in \mathbf{T}^d$ (i.e., periodic bc).

The potential V depends on time. It is well known that if V is periodic in time (say, 1-periodic), we are led to study the monodromy operator

$$Wu(t) = u(t+1)$$

(which is unitary).

Again, the nature of spectrum and localization of eigenfunctions are key issues.

A well known example is the so-called kicked rotor problem

$$i\frac{\partial u}{\partial t} + a\frac{\partial^2 u}{\partial x^2} + ib\frac{\partial u}{\partial x} + \kappa\left[\cos x \sum_{n\in\mathbf{Z}} \delta(t-n)\right]u = 0 \tag{1.4}$$

involving periodic "kicks" in time introduced as a model in quantum chaos. Here V is discontinuous in time.

We assume V real. We will also assume $V(\cdot, t)$ smooth in $x \in \mathbf{T}^d$ for all time. By the reality of V, there is conservation of the L^2-norm.

If $u_0 = u(0) \in H^s(\mathbf{T}^d)$, then

$$u(t) \in H^s \text{ for all time}$$

Problem. Possible growth of $\|u(t)\|_{H^s}$.

Remark 1. It turns out that in (1.4) with typical values of a, b there is *almost-periodicity* in the following sense: Assume u_0 sufficiently smooth (depending on s). Then $u(t)$ is almost as periodic as an H^s-valued function and, in particular, $\sup_t \|u(t)\|_{H^s} < \infty$.

Remark 2. If in (1.3) we take V also to be t-periodic, $u(t)$ is well known to be almost periodic in time as an L^2-valued function. But there are examples where V is smooth in x and t and such that for some smooth initial data u_0

$$\sup_t \|u(t)\|_{H^s} = \infty \text{ for all } s > 0$$

3. KAM-theory via the Nash-Moser method

We refer here to a method developed by W. Craig, G. Wayne, and myself to construct quasi-periodic solutions of nonlinear Hamiltonian PDEs. This approach was used originally as a substitute of the usual KAM-scheme (as used in this context by S. Kuksin) in situations involving multiplicities or near-multiplicities of normal frequencies. These always appear, except in 1D problems with Dirichlet boundary conditions. It was realized later that this technique is also of interest in the "classical context" involving finite-dimensional phase space (leading, for instance, to a Melnikov-type result with the "right" nonresonance assumptions) and applies in certain non-Hamiltonian settings.

If we follow a Newton-type iteration scheme, the basic difficulty is the inversion of nondiagonal operators obtained by linearizing the (nonlinear) PDE.

Consider, for instance, the Schrödinger case

$$iu_t + \Delta u + \varepsilon F(u, \bar{u}) = 0 \tag{1.5}$$

The linearized operator expressed in Fourier modes then becomes

$$T = D + \varepsilon S$$

where D is diagonal with diagonal elements of the form

$$D_{k,n} = k.\omega + \mu_n = k.\omega + |n|^2 + 0(1) \quad (k \in \mathbf{Z}^b, n \in \mathbf{Z}^d) \tag{1.6}$$

and S is a Toeplitz-type matrix with (very) smooth symbol, i.e.,

$$S\big((k,n),(k',n')\big) = \hat{\varphi}(k-k', n-n')$$

where $\hat{\varphi}(\xi)$ decays rapidly for $|\xi| \to \infty$.

In (1.6), $b =$ dimension of invariant tori, and $\omega \in \mathbf{R}^b$ is the frequency vector. The matrix T is finite (depending on the iteration step), and we seek appropriate bounds on T^{-1}. The problem again involves small-divisor issues and is treated by multiscale analysis.

Returning to $H(x)$, one important special case is given by

$$H(x) = \lambda.v(T^n x)\delta_{nn'} + \Delta \tag{1.7}$$

where Δ is the usual lattice Laplacian

$$\begin{aligned} \Delta(n,n') &= 1 \text{ if } |n-n'| = 1 \\ &= 0 \text{ otherwise} \end{aligned}$$

Letting $v(x) = \cos x$ on $\mathbf{T}$, $Tx = x + \omega =$ shift, we obtain the Almost Mathieu operator

$$H_\lambda(x) = \lambda \cos(x + n\omega) + \Delta \tag{1.8}$$

introduced by Peierls and Hofstadter in the study of a Bloch electron in a magnetic field and studied extensively afterwards by many authors.

For (1.8), there is basically a complete understanding of the nature of the spectrum. Assume that ω satisfies a diophantine condition

$$\text{dist}\,(k\omega, 2\pi\mathbf{Z}) = \|k.\omega\| > c|k|^{-C} \text{for } k \in \mathbf{Z}\backslash\{0\}$$

Then, for a.e. x,

(i) $\underline{\lambda > 2}$: $H(x)$ has p.p. spectrum

(ii) $\underline{\lambda = 2}$: $H(x)$ has purely s.c. spectrum

(iii) $\underline{\lambda < 2}$: $H(x)$ has purely a.c. spectrum

Thus there is a phase transition at $\lambda = 2$.

This model has a special and remarkable self-duality property (wrt Fourier transform)

$$\begin{aligned} \cos &\rightarrow \tfrac{1}{2}\Delta \\ \Delta &\rightarrow 2\cos \end{aligned}$$

observed and exploited first by Aubry. One of its implications is that

$$\text{Spec}\, H_\lambda = \text{Spec}\, H_{\frac{4}{\lambda}}$$

(referring to the "topological spectrum" that is independent of x).

In more general situations involving shifts,

$$\lambda v(x + n\omega)\delta_{nn'} + \Delta \tag{1.9}$$

with v real analytic on $\mathbf{T}^d$, a rough picture is the following:

λ **large**: p.p. spectrum with Anderson localization

λ **small**: purely a.c. spectrum

λ **intermediate**: possible coexistence of different spectral types

Recall that Anderson localization means the following:

Assume ψ an extended state, i.e.,

$$H\psi = E\psi \ \text{ and } \ |\psi_n| \lesssim |n|^C$$

Then $\psi \in \ell^2$ and

$$|\psi_n| < e^{-c|n|} \text{ for } |n| \rightarrow \infty$$

(in particular, E is an eigenvalue).

Related to possible coexistence of different spectral types (in various energy regions), one may prove the following:

Consider

$$H = (\lambda \cos n\omega_1 + \tau \cos n\omega_2)\delta_{nn'} + \Delta \tag{1.10}$$

where $\lambda < 2$, and τ is small. Then, for $\omega = (\omega_1, \omega_1)$ in a set of positive measure, H has both point spectrum and a.c. spectrum.

Remark. If in (1.9) we replace the shift by the skew shift, one expects a different spectral behavior with localization for all $\lambda > 0$ (as is the case of a random potential).

This problem is open at this time. It is known that for all $\lambda > 0$ and ω in a set of positive measure

$$H = \lambda\left(\cos \frac{n(n-1)}{2}\omega\right)\delta_{nn'} + \Delta$$

has some p.p. spectrum.

This text originates from lectures given at the University of California, Irvine, in 2000 and UCLA in 2001. The first 17 chapters deal mainly with localization problems for quasi-periodic lattice Schrödinger operators. Part of this material is borrowed from the original research papers. However, we did revise the proofs in order to present them in a concise form with emphasis on the key analytical points. The main interest, independent of style, is that we give an overview of a large body of research presently scattered in the literature. The results in Chapter 8 on regularity properties of the Lyapounov exponent and Integrated Density of states (IDS) are new. They refine the work from [G-S] described in Chapter 7. (Nonperturbative quasi-periodic localization is discussed in Chapter 10. We follow the paper [B-G] but also treat the general multifrequency case (in 1D). In [B-G], only the case of two frequencies was considered. Our presentation here uses the full theory of semialgebraic sets and in particular the Yomdin-Gromov uniformization theorem. This material is discussed in Chapter 9.

Chapters 18, 19, and 20 deal with the problem of constructing quasi-periodic solutions for infinite-dimensional Hamiltonian systems given by nonlinear Schrödinger (NLS) or nonlinear wave equations (NLW). Earlier research, mainly due to C. Wayne, S. Kuksin, W. Craig, and myself (see [C] for a review), left open a number of problems. Roughly, only 1D models and the 2D NLS could be treated.

In this work we develop a method to deal with this problem in general. Thus we consider NLS and NLW (with periodic boundary conditions) given by a smooth Hamiltonian perturbation of a linear equation with parameters and proof persistency of a large family of smooth quasi-periodic solutions of the linear equation. This is achieved in arbitrary dimension. Compared with earlier works, such as [C-W] and [B1], we do rely here on more powerful methods to control Green's functions. These methods were developed initially to study quasi-periodic localization problems. Thus the material in Chapters 18 to 20 is also new.

We want to emphasize that it is our only purpose here to convey a number of recent developments in the general area of quasi-periodic localization and the many remaining problems. This is an ongoing area of research, and our understanding of most issues is still far from fully satisfactory. The material discussed, moreover, covers only a portion of these developments (for instance, we don't discuss at all renormalization methods, as initiated by B. Hellfer and J. Sjostrand). We have largely ignored the historical perspective. Nevertheless, it should be pointed out that this field to a large extent owes its existence to the seminal work of Y. Sinai and his collaborators (in particular, the papers [Si], [C-S], and [D-S]), as well as the paper [F-S-W] by Frohlich, Spencer, and Wittwer. One of the significant differences, however, between these works (and some later developments such as [E]) and ours on the technological side is the fact that we don't rely on eigenvalue parametrization methods, which seem, in particular, very hard to pursue in multidimensional problems (such as considered in [B-G-S], for instance). It turns out that, as mentioned earlier, lots of the analysis is independent of self-adjointness and has potential applications to non-self-adjoint problems. We rely heavily in both perturbative and nonperturbative settings on methods from subharmonic function theory and the theory of semianalytic sets, which somehow turn out to be more "robust" than eigenvalue techniques (the results obtained are a bit weaker in the sense that "good" frequencies are not always characterized by diophantine conditions, as in [Si], [F-S-W], [E], or [J]). Jitomirskaya's paper [J] certainly underlies much of this recent research. Besides settling the spectral picture for the Almost Mathieu operator and the phase transition mentioned earlier, it initiated the nonperturbative approach with emphasis on the Lyapounov exponent and transfer matrix. Some parts of the analysis were restricted to the cosine potential, and the extension to general polynomial or real analytic potentials (see [B-G]) lies at the root of the material presented in these notes.

Next, a bit more detailed discussion of the content of the different chapters. Chapters 2 through 11 are closely related to the papers [B-G] and [G-S] on nonperturbative localization for quasi-periodic lattice Schrödinger operators of the form

$$H_x = \lambda v(x + n\omega) + \Delta \tag{1.11}$$

where v is a real analytic potential on $\mathbf{T}^d$ ($d = 1$ or $d > 1$), and Δ denotes the lattice Laplacian on $\mathbf{Z}$. We are mainly concerned with the issues of pure point spectrum, Anderson localization, dynamical localization, and regularity properties of the IDS. A key ingredient is the positivity of the Lyapounov exponent for sufficiently large λ. The results are nonperturbative in the sense that the condition $\lambda > \lambda_0(v)$ depends on v only and not on the arithmetical properties of the rotation vector ω (provided we assume ω to satisfy some diophantine condition).

Here and throughout this exposition, extensive use is made of subharmonic function techniques and the theory of semialgebraic sets. A summary of certain basic results in semialgebraic set theory appears in Chapter 9. The basic localization theorem is proven in Chapter 10, and some extensions of the method to more general operators are given in Chapter 11.

In Chapter 12 we recall some elements from Kotani's theory for later use. But this is far from a complete treatment of this topic, and several other results and aspects are not mentioned.

In Chapter 13 we exhibit point spectrum in certain two-frequency models of the form (0.11) with small λ. This fact shows that, contrary to the localization theory, the nonperturbative results on absolutely continuous spectrum, as obtained in [B-J] for one-frequency models, fail in the multifrequency case. Equivalently, invoking the Aubry duality, the quasi-periodic localization results on the $\mathbf{Z}^2$-lattice (as discussed in Chapter 17) are only perturbative.

In Chapter 14 we develop a general perturbative method to control Green's functions of certain lattice Schrödinger operators. The main result is in some way an "analogue" of Cartan's theorem in analytic function theory for holomorphic matrix-valued functions.

This approach has a wide range of applications. First, it allows us to control Green's functions for general Jacobi operators of the form (1.0) associated to a dynamical system given by a skew shift (Chapter 15). As an application, we prove the almost periodicity of smooth solutions of the kicked rotor equation (1.4) with small κ and typical parameter values a, b (Chapter 16). Next, an extension of Chapter 14 to a 2D setting permits us to establish Anderson localization for operators of the form (1.11) on the $\mathbf{Z}^2$-lattice. The statement is perturbative, i.e., $\lambda > \lambda_0(v, \omega)$. However, as indicated earlier, a nonperturbative result may be false in this situation. In fact, considering the multifrequency generalizations of the Almost-Mathieu operator

$$H_x = \lambda\big(\cos(x_1 + n\omega_1) + \cos(x_2 + n\omega_2)\big) + \Delta \quad (\text{on } \mathbf{Z}) \tag{1.12}$$

and its "dual"

$$\widetilde{H}_\theta = \cos(\theta + n_1\omega_1 + n_2\omega_2) + \frac{\lambda}{4}\Delta \quad (\text{on } \mathbf{Z}^2) \tag{1.13}$$

it turns out that for arbitrary $\lambda > 0$, there is a set of frequencies $\Omega = \Omega_\lambda \subset \mathbf{T}^2$ of small but positive measure such that for $\omega \in \Omega$ and x in a set of positive measure, we have

$$\text{mes}\left(\sum_{pp} H_x\right) > 0$$

(in fact, there may be coexistence of different spectral types here). Hence $\widetilde{H}_\theta$ has true (i.e., not ℓ^2) extended states for almost all θ. ($\mathbf{Z}^\ell$-operators of the form (1.13) were first studied in [C-D].)

Finally, the method from Chapter 14 enable us to treat KAM-type problems via the Lyapounov-Schmidt approach (see [C-W]) in a number of situations that, due to large sets of resonances, seemed untractable previously. (Typical issues left open here from the earlier works are the NLS in space dimension $D \geq 3$ and the NLW in space dimension $D \geq 2$).

In Chapter 18 we give a new proof of Melnikov's theorem on persistency of b-dimensional tori in (finite-dimensional) phase space of dimension $> 2b$ (for Hamiltonian perturbations of a linear system, assuming the Hamiltonian given by a polynomial.) The spirit of the argument is closely related to earlier discussion on perturbative localization. In particular, semialgebraic set theory is used again to restrict the parameter space.

In Chapters 19 and 20 we then apply this scheme to obtain quasi-periodic solutions for nonlinear PDE (with periodic bc), thus involving an infinite-dimensional phase space. Chapter 19 deals with NLS and Chapter 20 with NLW. Compared with the finite-dimensional phase space setting discussed in Chapter 18, there are some additional difficulties (due to large sets of resonant normal modes). But the method is sufficiently robust to deal with them. An additional ingredient involved here is a "separated cluster structure" for the near-resonant sets (noticed first by T. Spencer in a 2D-Schrödinger context).

As mentioned earlier, results from Chapters 18 to 20 treat only perturbations of linear systems with parameters. Starting from a genuine nonlinear problem, this format may often be reached through the theory of normal forms and amplitude-frequency modulation (see [K-P] and [B2]). This is a different aspect of the general problem, however, that is not addressed here.

In the Appendix we consider lattice Schrödinger operators associated to strongly mixing dynamical systems. We mainly summarize results from [C-S] and [B-S] based on the Figotin-Pastur approach. So far, this method to evaluate Lyapounov exponent has succeeded only in a strongly mixing context.