chapter 1

Introduction

Most of us are comfortable in the conviction, with Mr. Smith, that

$$2 + 2 = 4.$$

And some of us might be sufficiently unconcerned with math to be amused by people who, by contrast, wonder about two plus two making three, or five, or all numbers at once. But when we need to construct an airplane, or have an employer pay us for hours of work, we have no doubt that two plus two must be four. So we are happy with this simple arithmetical proposition.

But what about some other mathematical propositions? Is it really true that

$$-4 \times -4 = 16?$$

What does this mean physically? Without using this rule, can we possibly build an airplane that will fly safely, by using instead, say,

$$-4 \times -4 = -16,$$

or something else? How close is the connection, really, between physical experience and the standard rules for operations on negative numbers?

Think of your school days. When you were taught basic arithmetic, teachers used many examples to explain problems and operations in terms of relationships among quantities of physical things. The procedures and results usually made sense. The same was true with the introduction to elementary algebra and geometry. But as you advanced to higher courses, gradually you had to learn rules and procedures that were harder to understand in terms of ordinary experience. Despite your confusion, teachers encouraged you to learn the new rules and to apply them consistently, assuring you that the more advanced mathematics was all a necessary consequence of the basic rules. They claimed that it all had very many uses in physics, just like the basic rules of arithmetic. But eventually, if you continued your mathematical studies far enough, you had to accept symbolic rules and propositions without any explanatory reference to physical experience. By then, teachers probably encouraged you to think that mathematics is true irrespective of any correspondence to the physical world. Along the way, if you paid close attention, you may have noticed that not only was the notion of mathematical truth divorced from that of physical experience, but the very notion of truth, in and of itself, was gradually neglected. Instead of regarding results and propositions as true, maybe you learned to characterize them as "correct," "valid," or merely "consistent" with the premises.

Notice the transition. At first, you became convinced of the meaning and truth of mathematical propositions by virtue of practical explanations. But finally you abandoned experience as the justification. Meanwhile, if you also studied physics, it all seemed to depend on mathematics. Hence, you might wonder: how can mathematics be essential for the description of physical phenomena if the rules of mathematics do not need to be based on experience?

The answer lies in the history of mathematics. Even a slight acquaintance with history demonstrates that originally some mathematical rules were based on physical procedures and results, while other rules stemmed from more abstract considerations. Hence, some parts of mathematics correspond to ordinary experience and other parts do not. Some mathematical operations have a close similarity to physical operations, others do not. Accordingly, it is possible to identify those rules that more closely serve to represent observed relations among things. And, by understanding the extent to which other rules do not correspond to observed relations, it is possible to develop new rules that do so to a greater degree.

So we can devise new mathematical rules. Yet how much creative freedom do we have? Can we construct a system in which, say,

$$-4 \times -4 = -16?$$

Actually, yes we can. Mathematical logicians know that one can devise unusual rules of signs and explore their consequences systematically. But this subject is rarely conveyed to general readers. This kind of playful deviance is usually not taught. It's a bit of a secret. We don't want to confuse students who already are busy enough learning minus times minus is plus, along with much else. At universities, once in a while, it is usually allowed that in a backroom basement of a tall ivory tower, some beady-eyed specialist may carry out unnatural experiments with signs. But tentative experiments with the elements of math are not the sort of thing that is usually given press; they are more along the lines of "kids, don't try this at home." The present book, however, shows how to carry out experimental variations of the ways we usually operate with numbers and signs. It illustrates also how to devise such alterations with the aim of describing ordinary perceptions.

Perhaps it is not difficult to convince the novice that we can develop new systems of symbols following specific rules that do not correspond to physical facts. But it is difficult to convince someone that we can modify rules of ordinary numerical algebra to devise new rules that better correspond with matters of everyday experience. To many people it appears impossible to change the "laws of mathematics." But in truth this *can* be done, and it has been done in one way or another, at different points throughout history. Since the mid-1800s, in particular, mathematicians realized clearly that they could invent new geometries and new algebras having principles and results that diverge from those of traditional mathematics. So nowadays there is not just *one* mathematics, really, there are many. There exist a great variety of geometries and algebras, and they are not all equivalent to one another. For instance, some are more useful than others for describing physical processes.

Someone acquainted with such relatively new geometries and algebras might prefer to say that such developments do not constitute alterations in the known laws of mathematics, but instead the creation or discovery of distinct mathematical systems. But nevertheless, it is useful to speak about "changing" the laws of mathematics because it is essentially by modifying established rules that new sorts of mathematics are devised.

Traditional mathematics has been enormously useful and valuable, so we should have good reasons to tamper with it. Granted, one reason is the need to demonstrate that the principles of mathematics are not unique and immutable as they might appear. But the more important reason is to demonstrate that traditional mathematics involves principles that do not correspond clearly or directly to our perceptions. Simple examples can be found in the concepts of number.

Many students of elementary math are puzzled initially by irrational, imaginary, and even negative numbers. But teachers usually convince them to accept such numbers as just as valid as ordinary numbers. Students are taught to not be misled by the names of such numbers into supposing that they are in any way less real or logical than ordinary numbers. Yet there *are* differences. There are many physical situations where such numbers correspond to operations that are simply impossible. For example, given a box with five apples, you cannot physically "subtract," that is, remove, apples in a way that will leave a negative quantity of apples in the box. The physical world prevents you from taking more than those five. Likewise, there is no physical experiment, no measuring device, that when made to obtain a numerical measurement, under whatever circumstance, will yield an imaginary number. The same is true for irrational numbers. For example, no matter what seemingly circular physical object you choose to measure, by trying to fit its diameter several times into its circumference, you *never* obtain π as the result. To be sure, you may obtain a number resembling π, such as 3.14158, but you do not obtain π. This is not to say, of course, that certain machines cannot convey approximations of irrational numbers. Moreover, with or without the aid of calculating machines, such numbers are commonly used in the theories of physical science. They serve to obtain results that do relate to experimental measurements. But in these cases, such numbers are never the results of direct measurements. By contrast, the quantity five can result directly from measurements. Accordingly, we may distinguish how elements of mathematics, such as the different sorts of numbers, correspond either exactly or only approximately to the results of practical operations such as counting and measuring.

The main goals of this book are the following. First, to remind us that some aspects of traditional numerical algebra do not correspond to our everyday experiences. Second, to show that we can modify traditional rules and hence devise new mathematics. And third, to show how we can produce new mathematics that serves to describe aspects of the physical world.

Each of these claims is true not only of arithmetic and algebra but also of other branches of mathematics, such as geometry and calculus. For example, nowadays many physicists believe that the structure of the universe does not quite correspond to the principles of ordinary ancient geometry. Indeed, anyone acquainted with questions of physical meaning in mathematics might immediately think of physical geometry as the main field of interest. Hence, many books and articles have been written about physical geometry. Likewise, many writers have discussed the physical significance of the calculus.

But here we will not discuss such matters, and instead will focus on elementary numerical and algebraic concepts. Just simple numbers and signs. Why? For three reasons: (1) Because basic important questions about the physical meaning of mathematical concepts are also present in the history and subject of numerical algebra. (2) Because such issues began to transpire in the development of the algebra of signed numbers before similar questions arose in calculus and modern geometry. (3) Because the subject of abstract numbers and elementary algebra is more easily accessible to general readers.

We will concentrate attention on negative numbers. Why? Because they are unassuming but fun. Simple paradoxical gems of the practical imagination. The long-neglected negatives stand as just about the only kind of numbers about which a book has not been written. And they suit the study of creative mathematics well because they lie precisely between the obviously meaningful and the physically meaningless. Thus we think about negative temperatures, but not about a negative width. By first studying interpretive disputes in the history of signed numbers, we will place ourselves on that borderline between the meaningful and the meaningless, and we will gradually pry open the old dusty crate of innovative mathematical representation.

Furthermore, whereas the subject of physical geometry has been investigated at length, by contrast, the analogous case for arithmetic and algebra remains obscure. Even nowadays. Do traditional quantitative methods serve to represent all physical magnitudes and relations exactly?

But wait, this is not a trick question. To answer it we will not turn to the abstruse or the technical. For example, philosophically minded readers might search in vain through these pages for terms such as intuitionism, idealism, physicalism, fictionalism, and so forth. Of course, any text can be given a philosophical reading, but the present book is not composed of philosophical subjects or opinions. Moreover, it does not discuss, nor even mention, not at all, seemingly intimidating physical concepts and subjects, like potentials, differentiable manifolds, isospin, negative energy, entanglement, gauge fields, phase space, or fiber bundles. Instead, we will deal with such very basic primitive concepts that it might seem as if we are scarcely even talking about physics at all. The examples discussed and the puzzles laid out will deal only with seemingly simple notions, such as quantity, direction, position, length, area, displacement, symmetry, and speed.

Accordingly, the present book deals mainly with the elementary treatment of numbers and variables as a simple way of introducing the study of custom-made mathematics. We will analyze old ambiguities in traditional rules that otherwise have remained neglected for a long time. We will develop new solutions to basic problems that were solved or dismissed long ago.

Now, of course, it is not a popular practice to work on questions that are not widely acknowledged as problems. Some people might shudder at the thought of seriously reviving discussions that are seldom encountered except in old books and history books. Certain sorts of superior persons believe that they were born *after* history. For example, they believe, for all practical purposes, that the elementary foundations of mathematics were discovered long ago, and established so securely that today we need not bother to question the validity of the simplest operations. They presume that if anything remained ambiguous or problematic in the elements of mathematics, then it would have been solved already by some clever fellow back in the nineteenth century.

But there are various reasons why problems cease to concern people. Oftentimes, yes, a problem is superseded because someone finds a solution that is generally satisfactory. Other times, however, a problem is not solved directly; people just find ways to circumvent it. And sometimes a particular problem ceases to be a focus of attention for more mundane reasons: individuals grow tired of the problem, their attention is diverted to other subjects, or they become satisfied with catchphrases and rhetoric that dismiss it as illusory or trivial. Once a problem has been discussed for a while, proposals for its solution or dismissal may be sufficiently attractive to convince newcomers that they need not pursue it in detail nor attempt to elucidate the matter any further. In the meantime, those who were originally attempting to find or refine a solution, or those who at least openly recognized the significance and persistence of the problem, eventually die. Without newcomers continuing to analyze the problem, the impression can emerge that the problem was solved or was merely apparent. But with any such vague impression, we may ask, exactly who solved the problem, when, where, and *how*? If such questions are met merely with a dismissive shrug of the shoulders or with the innuendo that there merely *seemed* to be a problem only to people of the past who were rather stupid or confused, then we may suspect that the problem actually remains unsolved. Perhaps people just learned to live with it. And even if we are told how the problem was allegedly solved oh-so-long-ago, we may nonetheless analyze the solution to see if it is really satisfactory *to us*.

It is well known that to advance science and mathematics researchers should carefully study contemporary and recent works in their field. By contrast, the study of the distant past rarely receives close attention, except from historians. But old works and ideas are not always irrelevant to contemporary discussions. We should not assume that all that was written long ago ever received its proper share of attention. We cannot assume that everything of value in old books has effectively been carried into the present in subsequent works. Any acquaintance with history can show that sometimes fertile ideas are forgotten or neglected. Depending on the subject of investigation, we may find that more analyses and commentaries were produced in the distant past rather than the recent past. Hopefully, the present study will demonstrate that studying the distant past can serve also for the advancement of scientific understanding.

So we will reconsider old questions and controversies buried in the history of mathematics. Happily, the discussions and historical excursions that follow focus on basic mathematical procedures familiar to most people. Otherwise, nonspecialists who begin reading books on mathematics or its history often lose interest quickly when such books abruptly discuss complicated subjects entirely alien to them. Furthermore, the historical portion of this book, unlike many works in the history of mathematics, is not a history of success. It is not an account of how individuals solved one perplexing problem after another. Instead, the historical excursions are meant to convey a sense of the ambiguities inherent in ordinary mathematical rules, ambiguities that have been made invisible by generations of textbook writers.

Finally, a word should be said about the style and format of this book. It could have been unreadable. But what for? Already plenty of seemingly incomprehensible books sit on library bookshelves, enough to last anyone a lifetime. Big words in small print, like thick brick walls separating the expert from the innocent; so many technical books. So let this one read easily instead. And if, along the way, mathematically expert readers complain (as they should) that the book lacks thoroughness because it doesn't trace out the intricate consequences of certain propositions, then they can well pursue such questions further. Mathematicians have the skills to read many texts that for other people are virtually inaccessible. They can make sense of this one as well. Meanwhile, it should be clear enough for anyone who knows at least some elementary mathematics.