

Chapter One

Some Fundamental Principles

1.1 UNITS AND CHARACTERISTIC LENGTHS, TIMES, ENERGIES, ETC.

In the measurement of quantities by laboratory instruments, both the c.g.s. and m.k.s. units are convenient. However, for the description of particle and atomic processes, the c.g.s. system is preferable in that equations and formulas are sometimes simpler in form; for this reason, the c.g.s. system will be employed throughout this book. At the same time, it is often useful to express quantities in dimensionless units in terms of certain “fundamental” values defined in terms of the fundamental physical constants. Different fundamental quantities—for example, a characteristic length—can be formed from different combinations of physical constants, and the particular choice appropriate for the description of some process is dictated by the nature of the process.

Concerning the physical constants themselves, the most fundamental one is perhaps the “velocity of light” (c). The constant is of more general significance than the name given to it, since it is the characteristic parameter of spacetime, and its value is relevant to all dynamical processes in physics. Our fundamental theory of spacetime is special relativity, and we shall review certain basic features of the theory in the following section. Considerations of some general consequences of special relativity are extremely powerful, in particular, as a guide in formulating the fundamental equations of physics.

After c , the most fundamental physical constant is probably Planck’s constant (\hbar). Loosely put, this constant might be designated as a “quantization parameter,” but this is probably not a good description. Another try at description might be to call it the “fundamental indeterminacy parameter,” but it is questionable whether the “uncertainty relations” deserve the title of *principle*, since they follow from the superposition principle (which really *is* a principle). Given that discrete particle motion is to be described in terms of an associated wave or propagation vector \mathbf{k} and frequency ω , Planck’s constant is then the proportionality factor between \mathbf{k} and the particle momentum:

$$\mathbf{p} = \hbar\mathbf{k}. \quad (1.1)$$

The uncertainty relations for an individual particle follow from this relation and the superposition principle. If momentum is to be regarded as a particle dynamical property and the wave propagation vector a kinematical variable, we might designate \hbar more descriptively as a parameter of particle dynamics. However, we shall, as usual, refer to \hbar simply as Planck’s constant like everyone else.

The third most fundamental physical constant may be the “electronic charge” (e), since it seems to be a fundamental unit common to the various charged elementary particles. That is, although there is a spectrum of masses for the particles, except for the fractionally charged “quarks,” the particle charges are multiples of e .

From the three physical constants c , \hbar , and e , it is not possible to construct a fundamental length by various combinations of products. From e and \hbar it is possible to form a characteristic velocity

$$v_0 = e^2/\hbar, \quad (1.2)$$

and this velocity is of significance for particle processes. Combining the fundamental physical constants, a dimensionless number

$$\alpha = e^2/\hbar c \approx 1/137 \quad (1.3)$$

can be formed that is of great importance, especially for electromagnetic processes. This number is called the “fine structure constant” because of its role in determining the magnitude of the small relativistic level shifts in atomic hydrogen; it can also be regarded as a dimensionless coupling constant for electromagnetic processes. Because of its small value, these processes can be calculated well by perturbation theory.

The masses of the various elementary particles play a major role in particle processes. The electron (and positron) mass (m), being the smallest of all, is of great importance because the particle is easily perturbed by an electromagnetic field. In particular, a variety of radiative (photon-producing) processes are associated with the electron and its interactions. A description of these processes is the principal task of this book. Almost all of our knowledge about the world outside our solar system comes from the analysis of the spectral distribution of radiation from distant sources. Our understanding of the details of the microscopic photon-producing processes allows us to interpret these source spectra and learn something of the nature of the sources. Fortunately, the electromagnetic processes are very well understood, and they can be calculated to high accuracy by perturbation theory.

The nucleon mass (M)—say, the mass of the proton, which is stable—is significant in that it is much larger (about $1836m$) than that of the electron. Along with their corresponding antiparticles, the electron and proton are the only stable “particles.” In fact, there is now good evidence, from inelastic scattering of very high energy electrons off protons, that the latter are not “elementary” or “fundamental” particles. Instead, protons are thought to be composites, built from quarks, and they have, as a consequence, *structure*. For example, protons have a characteristic size and charge distribution that can be measured. Pions (and also kaons) are also quark composites, and the pion is especially important as the least massive of the strongly interacting species. In the older theory of strong interactions, the pion was treated as a fundamental particle and its mass (m_π) determined the characteristic range of the interaction. These ideas are still useful in understanding certain particle and nuclear processes.

The masses of the elementary particles determine the various fundamental or characteristic lengths, all of which are inversely proportional to the mass value. There are different kinds of lengths, each having a different physical meaning and playing

a different role in determining the characteristic magnitude of importance of various processes. Along with \hbar and e , the electron mass determines the characteristic atomic size

$$a_0 = \hbar^2/m_e^2. \quad (1.4)$$

This is the Bohr radius, and it is one of the triumphs of quantum mechanics that the atomic radius ($\sim a_0 \sim 10^{-8}$ cm) is explained by physical principles. Classical physics had no explanation for the characteristic size of atoms as determined in the last century. The basic physical meaning of the characteristic length a_0 can be indicated through considerations of atomic binding. The classical total energy of an electron of momentum p in the neighborhood of a proton is

$$E_{\text{cl}} = p^2/2m - e^2/r. \quad (1.5)$$

In a quantum-mechanical description, the spectra of position and momentum values are such that there is a spread in each, determined by the uncertainty relation. Setting $pr \sim \hbar$ as a constraint condition added to Equation (1.5), we see that E_{cl} is minimized at a value

$$(E_{\text{cl}})_{\text{min}} \sim -e^2/2a_0 \equiv -Ry \quad (1.6)$$

for the r -value

$$r_{\text{min}} \sim a_0. \quad (1.7)$$

This little analysis shows, very simply, why atoms have a ground state or state of minimum energy. In a classical model with $\hbar \rightarrow 0$ the electron “orbit” size could be infinitesimally small and the energy would be infinitely negative.

A characteristic length that does not involve \hbar is the “classical electron radius” r_0 . If the electron mass is attributed to its electrostatic self-energy ($\sim e^2/r_0 \sim mc^2$), the result is

$$r_0 = e^2/mc^2. \quad (1.8)$$

This is a very small distance ($\sim 3 \times 10^{-13}$ cm), and the quantity really has no physical meaning, because the classical self-energy considerations are not valid. However, the combination e^2/mc^2 appears often to various powers in expressions for parameters for electromagnetic quantities. Thus, it is still designated r_0 and called by its original name.

The erroneous nature of the classical model for electromagnetic self-energy is clear through considerations that introduce another characteristic length. If we attempt to localize an electron to a very small distance, of necessity we introduce a spectrum of momentum states extending to high values. For $p \sim mc$, the energy values become large enough to produce e^\pm pairs, which affect and limit the localization. The uncertainty relation then suggests a minimum localization distance

$$r_{\text{loc}} \sim \hbar/mc \equiv \Lambda. \quad (1.9)$$

Again for historical reasons, the quantity Λ is called the *electron Compton wavelength*. It appears often as a factor in formulas for cross sections for electromagnetic processes and, in general, in many equations describing phenomena involving electrons.

The three lengths a_0 , r_0 , and Λ are related through a linear equation with the fine-structure constant as a proportionality factor:

$$r_0 = \alpha \Lambda = \alpha^2 a_0. \quad (1.10)$$

Although the three lengths are connected by means of the factor α , only a_0 and Λ have a useful physical meaning, and most formulas given throughout this work will not be expressed in terms of r_0 .

It might be noted that each of the lengths a_0 , Λ , and r_0 is inversely proportional to the electron mass. For some problems it is convenient to consider corresponding lengths involving masses of other particles. While a_0 determines the characteristic (electron) atomic unit of length, and $E_0 = e^2/a_0 (= 2\text{Ry})$ the atomic unit of energy, the electron mass can be replaced by the nucleon (proton) mass M to introduce a “nucleon atomic unit” of distance

$$a_M = (m/M)a_0 \quad (1.11)$$

and a characteristic “nucleon Rydberg energy”

$$\text{Ry}_M = (M/m)\text{Ry}. \quad (1.12)$$

These units are convenient, for example, in the treatment of proton-proton scattering; in that problem, in which the nuclear and Coulomb forces contribute, the Coulomb force plays the major role (except at very high energy).

Another important characteristic distance is the particle Compton wavelength associated with the least massive of the strongly interacting particles (i.e., the pion). The quantity

$$\Lambda_\pi = \hbar/m_\pi c \quad (1.13)$$

determines the range of the strong interaction and the magnitude of characteristic cross sections associated purely with this interaction. The cross section is

$$\sigma_s \sim \Lambda_\pi^2 \sim 20 \text{ mb}, \quad (1.14)$$

where $1 \text{ mb} = 10^{-3} \text{ b}$, the *barn* (b), defined as 10^{-24} cm^2 , being a cross section unit common in nuclear and strong-interaction physics (1 barn is a large cross section for nuclear processes: “as big as a barn”).

The choice of units for the description of some particular phenomenon is dictated not just by considerations of characteristic numerical values of relevant quantities. Depending on the type of units chosen, the equations describing a process take on slightly different form. When formulated in terms of the “most natural” units, the equations are more transparent in exhibiting the nature of the physics involved. In problems of atomic structure or in the description of the scattering of non-relativistic electrons by atomic systems or by a pure Coulomb field, the so-called atomic or “hartree” units are natural. In these units e , \hbar , and m are each set equal to unity, and lengths are in units of the Bohr radius a_0 , cross sections are in units of a_0^2 , and energies are in units of $e^2/a_0 = 2\text{Ry}$. The atomic units are, however, not as convenient in problems involving relativistic particles; then the more useful choice is $\hbar = c = 1$ for which $e^2 = \alpha$ is fixed by the dimensionless fine-structure constant [Equation (1.3)]. These units are particularly useful in describing electromagnetic

phenomena. Further, if the process involves electrons or positrons, the rest energy $m(c^2)$ is a natural characteristic energy. Throughout this book, certain important results will often be expressed in forms that exhibit dimensions clearly by collecting products of factors that are dimensionless ratios. For example, if a cross section for some electromagnetic process at energy E is expressed in terms of a factor Λ^2 , a function of E/mc^2 , and a factor α^n , we immediately identify n as the “order” of the process. Higher order electromagnetic processes have cross sections down by powers of α . Equations expressed in this manner are preferable to those in which numerical values of physical constants are substituted in.

1.2 RELATIVISTIC COVARIANCE AND RELATIVISTIC INVARIANTS

Ideas of covariance are extremely powerful as a guide in formulating basic physical laws and in the derivation of results in mathematical descriptions of certain physical processes. Considerations of covariance can even provide a path to the discovery of new fundamental laws and then to the development of these new areas of physics. In the description of physical processes, it is often possible to simplify derivations by imposing conditions of relativistic covariance as a trick to arrive at formulas of general validity. We shall often make use of this kind of device.

1.2.1 Spacetime Transformation

The basic laws of physics are generally expressed as differential equations with space and time coordinates as independent variables. The spacetime coordinates refer, in some cases, to “events” such as the position (or possible position) of a particle or of a particle process. Further, the properties of spacetime are described in terms of its “structure” or its transformation properties, and this is the theory of special relativity. For spacetime reference frames K and K' whose spatial coordinate axes are moving with constant relative velocity, the relationship between the coordinates of events in the two frames is the Lorentz transformation

$$x'_\mu = \sum_\nu a_{\mu\nu} x_\nu. \quad (1.15)$$

Here, x_ν , with $\nu = 0, 1, 2, 3$, represents the time ($\nu = 0$) and space ($\nu = 1, 2, 3$) coordinates. Because of the fundamental isotropy of space, it is convenient to choose Cartesian coordinates ($x_1, x_2, x_3 = x, y, z$) for the spatial coordinate description. These are thus “natural” or “preferred” coordinates for formulating the basic equations of physics. It is, in one sense, convenient to choose an imaginary component $x_0 = ict$ for the time variable. This is because the fundamental property of spacetime can then be described by, in addition to the property (1.15), the equation

$$ds^2 \equiv \sum_\mu dx'_\mu dx'_\mu = \sum_\nu dx_\nu dx_\nu = \text{invariant (inv)}, \quad (1.16)$$

in which dx_μ are the differential coordinate separations between two spacetime events.

Because of the choice of an imaginary time component, it has not been necessary to introduce a “metric” or metric tensor. The spacetime is essentially four-dimensional cartesian, and the metric tensor ($g_{\mu\nu}$) is identical to the Kronecker δ -function

$$\delta_{\mu\nu} = \begin{cases} 1 & \text{for } \mu = \nu \\ 0 & \text{otherwise.} \end{cases} \quad (1.17)$$

It is somewhat a matter of taste whether this notation procedure is adopted. Actually, the trend in physics seems to be away from the use of the imaginary zero component of spacetime and other four-vectors, at the expense of the introduction of a metric tensor. There is then also the necessity of the introduction of “covariant” and “contravariant” vectors and tensors, etc. with component indices appearing as superscripts and subscripts. Perhaps the resurgence of interest in general relativity during the past forty years has led to this fashion, and it is a necessity in that subject. However, for the treatment of physical processes occurring in a localized region, even if a strong gravitational field is present, coordinates can be chosen corresponding to a “flat” or Minkowskian spacetime. We can then do without a metric and reduce the notation complexity by employing $x_0 = ict$ and indices that only appear as subscripts.

The summation convention notation henceforth is adopted, in which summation over an index is always implied if it appears twice either on a single symbol or in a product of factors with subscripts. For example, in Equations (1.15) and (1.16), the summation sign could simply be left off. This procedure saves space and is employed extensively with no confusion or difficulty.

It is a fundamental assumption that all inertial systems are equally good for formulating a mathematical description of physical phenomena, and in terms of coordinates and other quantities (momenta, fields, etc.), the equations must have the same form whether expressed in terms of “unprimed” or “primed” quantities. Ideas like this—a *Principle of Relativity*—really go back to Newton’s time. The principle would imply that, if we transform from the primed to the unprimed coordinates, the transformation in the inverse direction with an inverse matrix should have the same form as Equation (1.15):

$$x_\nu = a_{\nu\mu}^{-1} x'_\mu. \quad (1.18)$$

The invariance of the quantity ds^2 implies the existence of sets of orthogonality relations for the transformation coefficients:

$$a_{\mu\rho} a_{\mu\lambda} = a_{\nu\rho}^{-1} a_{\nu\lambda}^{-1} = \delta_{\rho\lambda}. \quad (1.19)$$

A double transformation from the primed to unprimed to primed coordinates (or vice versa) can also be considered, leading to the set of relations

$$a_{\rho\mu}^{-1} a_{\mu\lambda} = \delta_{\rho\lambda}, \quad (1.20)$$

and, through comparison with (1.19), to the identity

$$a_{\rho\mu}^{-1} = a_{\mu\rho}; \quad (1.21)$$

that is, the inverse matrix is equal to the transposed matrix.

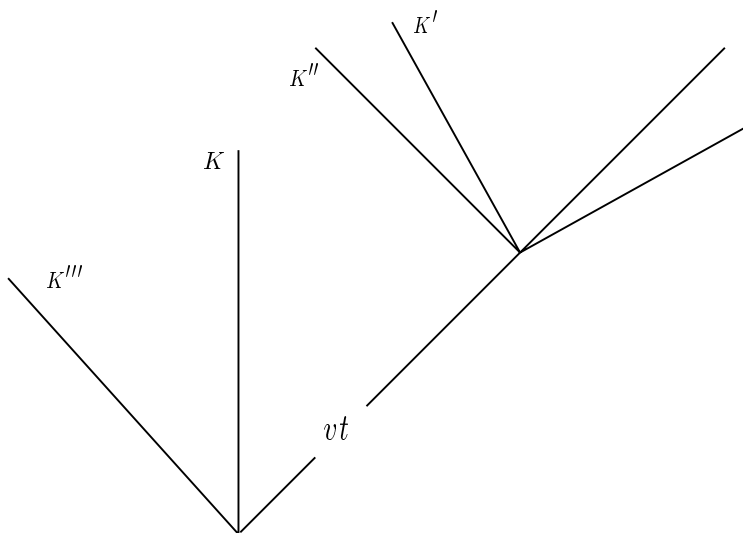


Figure 1.1 The reference frames K , K' , K'' , and K''' .

The spatial axes of K and K' can have any orientation with respect to the direction of relative motion. However, the transformation coefficients are particularly simple for the case where¹ the x - and x' -axes are aligned and the relative motion of the two frames is along this axis. Since there is no relative motion in the y and z directions, these coordinates must transform as $y' = y$ and $z' = z$, and there can be no dependence of x' and t' on y and z . The Lorentz transformation matrix must then be of the form

$$a_{\mu\nu} = \begin{pmatrix} a_{00} & a_{01} & 0 & 0 \\ a_{10} & a_{11} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.22)$$

If $x'_1 = x' = 0$ represents the coordinate of the origin of K' , which moves with a velocity $v = \Delta x_1 / \Delta t = ic \Delta x_1 / \Delta x_0 = \beta c$ with respect to K , the $\mu = 1$ component of Equation (1.15) then yields the result $a_{10} / a_{11} = i\beta$, and the identities (1.19) and (1.21) can be employed to obtain the solutions

$$\begin{aligned} a_{00} &= a_{11} = \gamma, \\ a_{10} &= -a_{01} = i\beta\gamma, \end{aligned} \quad (1.23)$$

where

$$\gamma = (1 - \beta^2)^{-1/2} = (1 - v^2/c^2)^{-1/2}. \quad (1.24)$$

¹If K and K' are not oriented this way we can consider two other frames K'' and K''' that are and such that, say, K'' is not in motion with respect to K' and has the same origin as K' . Then the $K - K''$ and $K' - K'''$ transformations are simple spatial rotation operations and the $K'' - K'''$ transformation would be an ordinary Lorentz transformation with relative velocity along, say, x'' and x''' . See Figure 1.1.

1.2.2 Other Four-Vectors and Tensors—Covariance

Because the basic equations of physics correspond to a description of phenomena in space and time, other quantities appearing as symbols in the equations must have transformation properties that are determined by those of the spacetime four-vector x_μ . In fact, the equations are such that other quantities should have the *same* transformation properties as that of x_μ . That is, a general four-vector transforms as

$$V'_\mu = a_{\mu\nu} V_\nu, \quad (1.25)$$

with the $a_{\mu\nu}$ as the same transformation coefficients as in Equation (1.16). A product of two four-vectors would transform as

$$V'_\mu U'_\nu = a_{\mu\lambda} a_{\nu\rho} V_\lambda U_\rho, \quad (1.26)$$

where, again, the transformation coefficients are the same as that for x_μ . A tensor² is a double index quantity that transforms in the same manner as the product of two four-vectors:

$$T'_{\mu\nu} = a_{\mu\lambda} a_{\nu\rho} T_{\lambda\rho}. \quad (1.27)$$

The Kronecker $\delta_{\mu\nu}$, although it does not represent a physical quantity, can be considered, mathematically, a tensor, because its components, which are the same in all Lorentz frames, satisfy the transformation law (1.27).

Entities with more than two indices can be introduced, defined in terms of a transformation law that is an obvious generalization of Equations (1.25) and (1.27). However, almost all physical quantities are either scalars, vectors, or tensors, depending on whether they possess 0, 1, or 2 indices, respectively. Scalars are numbers, equal to the same value when evaluated in any Lorentz frame. The clever use of certain scalars can often simplify the derivation of results, and we shall see many examples of this throughout our treatment of physical processes. The invariant differential quantity (1.16) is a scalar, and can be employed to introduce an “invariant proper time” defined by³

$$d\tau^2 = -ds^2/c^2 = dt^2 - dx_j dx_j/c^2, \quad (1.28)$$

where the index j on the spatial Cartesian coordinates runs from 1 to 3. If the spacetime four-vector x_μ refers to events designated by the coordinates of a particle moving with velocity $v_j = dx_j/dt$, then $d\tau = dt/\gamma$, with

$$\gamma^{-2} = 1 - v_j v_j/c^2 = 1 - v^2/c^2. \quad (1.29)$$

The particle four-vector velocity is then defined as

$$v_\mu = dx_\mu/d\tau = \gamma(ic, \mathbf{v}), \quad (1.30)$$

²Here the designation “tensor” is simply that for a two-index quantity satisfying the transformation law (1.27) and is not to be confused with the more general entity introduced in differential geometry (and general relativity) and called by the same name.

³This quantity is often defined with a sign difference. The definition (1.28) is convenient in that, as $c \rightarrow \infty$, $d\tau \rightarrow dt$.

and we can also introduce a momentum four-vector by multiplying by the particle mass⁴:

$$p_\mu = mv_\mu = \gamma m(ic, \mathbf{v}). \quad (1.31)$$

In Section 1.2.3 we shall consider a number of important scalars or invariants. One very useful scalar is formed from the four-dimensional “dot product” of two four-vectors. From the orthogonality relations (1.19) for the transformation coefficients we find that

$$A'_\mu B'_\mu = a_{\mu\nu} a_{\mu\lambda} A_\nu B_\lambda = \delta_{\nu\lambda} A_\nu B_\lambda = A_\nu B_\nu = \text{inv.} \quad (1.32)$$

Two correlaries of this result are that (i) the sum of the squares of the components of any four-vector is an invariant ($A_\mu A_\mu = \text{inv}$) and (ii) the trace of a tensor is an invariant:

$$\text{Tr } T_{\mu\nu} = T_{\mu\mu} = \text{inv.} \quad (1.33)$$

By the general *Principle of Covariance*, all equations of physics must be covariant in that they must have the same form in any Lorentz frame. This principle is extremely powerful as a guide to formulating a description of basic physical laws as well as in the calculation of particular processes. For example, if a physical law cannot be expressed mathematically in a form where it is manifestly covariant, the law cannot be correct. The equations must always be expressed in terms of four-vectors, four-tensors, scalars, etc., such that their covariant nature is evident. That is, every term in an equation must have the same transformation property. Often it is possible to guess the correct relativistic law or expression for some quantity by constructing a covariant combination of factors such that the Newtonian formula is obtained in the non-relativistic limit. This is a convenient procedure, since the Newtonian limit is a valid asymptotic domain of physics for which certain physical laws were first established. It is significant that the more general covariant equations, of more general validity, always have a more simple and compact form. The mathematical simplicity of the covariant equations can be considered as strong evidence for their fundamentally correct nature.

Considerations of covariance can even be useful as a guide to formulating basic physical laws where even the non-relativistic limit had not been established. Quantum mechanics is a good example. The description of a propagating plane wave is always in terms of a Fourier component proportional to an amplitude of the form

$$\text{amplitude} \propto e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \quad (1.34)$$

Through the introduction of the propagation four-vector

$$k_\mu = (i\omega/c, \mathbf{k}), \quad (1.35)$$

the amplitude (1.34) can be written in covariant form in terms of an invariant phase:

$$\text{amplitude} \propto e^{ik_\mu x_\mu}. \quad (1.36)$$

⁴By mass we mean “rest mass,” which is an invariant parameter of a particle. To then define γm as a “relativistic mass,” which varies with velocity—as is done in many textbooks—is inappropriate and misleading in that it amounts to introducing an unnecessary concept that can only lead to confusion and misunderstanding. There is no reason why quantities and equations in relativistic mechanics have to have a form identical to those in Newtonian mechanics.

The most critical step in quantum mechanics was made by de Broglie in 1924, when he wrote down the relationship between the particle propagation vector and momentum:

$$\mathbf{p} = \hbar\mathbf{k}, \quad (1.37)$$

that is, a relationship between a mechanical property (\mathbf{p}) and a wave kinematic property (\mathbf{k}). But if we write the de Broglie relation in the covariant form

$$p_\mu = \hbar k_\mu, \quad (1.38)$$

since $p_0 = iE/c$, we realize that the zero component of the relation ($E = \hbar\omega$) was actually written down for photons by Einstein in 1905 in his description of the photoelectric effect. One of the properties of covariant laws is that if they are valid for one component (of μ), they are valid for the other components. Thus, the de Broglie relation is a natural one, and the proportionality constant (\hbar) is fixed, essentially by the Einstein relation. The generalization of the de Broglie relation to applicability for any kind of particle would follow from the introduction of the concept of the photon, since it had already been clear that electrons, photons, etc. were fundamental particles. The basic germs of both special relativity and modern quantum mechanics were introduced in 1905—and by the same man.⁵

Special relativity can also be given *too much* credit for explaining something. For example, it is stated in many textbooks that “the existence of the spin- $\frac{1}{2}$ value is a consequence of a relativistic quantum-mechanical description of the electron” (or words to that effect). This is nonsense. Spin, including spin- $\frac{1}{2}$, has nothing to do with relativity; it is associated with the fundamental isotropy of space—and *not* relativity. Another misconception is that the value of the electron gyromagnetic ratio (or intrinsic magnetic moment) needs relativistic quantum mechanics for an understanding. It does not, and the literature is filled with wrong statements on this question.⁶

1.2.3 Some Useful and Important Invariants

We have already noted the important invariant (1.32), of which a special example is that associated with the momentum four-vector:

$$p_\mu p_\mu = \text{inv.} \quad (1.39)$$

However, there are a number of important invariants associated with particle distribution functions and volume elements. A very important invariant is the four-dimensional spacetime volume element

$$dX = \prod_{\mu=0}^3 dx_\mu. \quad (1.40)$$

It is easy to prove that dX is an invariant. Consider two reference frames K and K' with the origin of K' moving with a velocity \mathbf{v} with respect to that of K (see Figure 1.1). The axes of K and K' are not necessarily aligned, but they do not

⁵Although “quantization” was first introduced in 1900 by Planck, the particle or “corpuscular” character of light was suggested first by Einstein. Quantization effects are imposed as a result of applications of the Schrödinger equation, which is a natural extension of the Einstein–de Broglie work.

⁶*cf.* R. J. Gould, *Am. J. Phys.* **64**, 597 (1996); **64**(E), 1202 (1996).

rotate since K and K' are inertial frames. We introduce two other frames K'' and K''' , whose x -axes are aligned along \mathbf{v} and which are just orientations of K' and K , respectively. Since there is no motion between K and K''' and between K' and K'' ,

$$dX = dX'''; \quad dX' = dX''. \quad (1.41)$$

The K'' – K''' transformation is the special type with coefficients given by Equations (1.22)–(1.24), and the volume elements are related by the Jacobian of the transformation:

$$\begin{aligned} dX'' &= J \left(\begin{array}{c} x_0'', x_1'' \\ x_0''', x_1''' \end{array} \right) dX''' \\ &= \begin{vmatrix} \gamma & -i\beta\gamma \\ i\beta\gamma & \gamma \end{vmatrix} dX''' = dX''', \end{aligned} \quad (1.42)$$

since the Jacobian is unity. Then, by Equation (1.41),

$$dX = dX' = \text{inv.} \quad (1.43)$$

In an exactly similar way, it can be shown that the four-dimensional momentum space volume element is an invariant:

$$dP = \prod_{\mu=0}^3 dp_{\mu} = \text{inv.} \quad (1.44)$$

A four-vector three-dimensional “surface element” can be introduced, defined by

$$d\sigma_{\mu} = dX/dx_{\mu} = (dx_1 dx_2 dx_3, \quad dx_0 dx_2 dx_3, \dots) \quad (1.45)$$

That this quantity is a four-vector can be seen from the identity

$$d\sigma_{\mu} dx_{\mu} = 4 dX = \text{inv.} \quad (1.46)$$

From what is sometimes called the “quotient rule” in general tensor analysis,⁷ since the result (1.46) is true for arbitrary dx_{μ} , it follows that $d\sigma_{\mu}$ is a four-vector with, in fact, components proportional to those of the “missing” differential dx_{μ} .

An important and useful invariant is the “invariant phase-space⁸ element”

$$d^3 \mathbf{p}/E = \text{inv.}, \quad (1.47)$$

which is the ratio of the ordinary momentum-space volume element to the particle energy corresponding to the momentum value defined by the element $d^3 \mathbf{p}$. That is, if the differential dp defines the magnitude of $p = |\mathbf{p}|$ and the solid angle element $d\Omega$ defines the direction of \mathbf{p} , then $d^3 \mathbf{p} = p^2 dp d\Omega$ and $E = (p^2 c^2 + m^2 c^4)^{1/2} = E(p)$.

⁷If A_{μ} is a four-vector and it is established that

$$A_{\nu} B_{\nu} = \text{inv} = A'_{\mu} B'_{\mu} = a_{\mu\nu} A_{\nu} B_{\mu},$$

then

$$(a_{\mu\nu} B'_{\mu} - B_{\nu}) A_{\nu} = 0.$$

For this relation to be true for A_{ν} with arbitrary components, the parentheses must be identically zero; then B_{ν} has the correct four-vector transformation properties.

⁸The element $d^3 \mathbf{p}$ is only the momentum part of what is meant, strictly speaking, by *phase space*, which is a product of pairs of differential Hamiltonian coordinates ($\prod dp dq$). In considerations of particle processes “phase space” generally refers to momentum space volume, which is a measure of the number of single-particle quantum states available.

We shall prove a more general theorem for which the result (1.47) would be a special case for the space of the momentum four-vector.

As with the four-dimensional momentum space element (1.44) we can consider the space element

$$dU = \prod_{\mu=0}^3 dU_{\mu} = dU_0 d^3\mathbf{U} \quad (1.48)$$

associated with an arbitrary four-vector U_{μ} . Since it is a four-vector, its squared “length” is an invariant: $U_{\nu}U_{\nu} = C = \text{inv}$. We can then form the manifestly invariant quantity

$$\int dU \delta(U_{\nu}U_{\nu} - C) = \int dU_0 d^3\mathbf{U} \delta(U_0^2 + U_j U_j - C) = \text{inv}, \quad (1.49)$$

where $U_j U_j$ is the sum over the three “space” components ($j = 1, 2, 3$). Integration over dU_0 with the δ -function yields the result

$$\frac{d^3\mathbf{U}}{2(C - U_j U_j)^{1/2}} = \frac{d^3\mathbf{U}}{2U_0} = \text{inv}, \quad (1.50)$$

for which the invariant (1.47) is a special case for $U_{\mu} = p_{\mu}$. Further, by the relation (1.39), $p dp = E dE/c^2$ and $p/E = v/c^2$, where v is the particle velocity. The invariant (1.47) can then also be expressed in the following terms:

$$(p^2/E) dp d\Omega = \text{inv}, \quad pv dp d\Omega = \text{inv}, \quad p dE d\Omega = \text{inv}. \quad (1.51)$$

Another useful invariant associated with distributions of particles involves the differential number density of the particles having a particular energy E and moving in a particular direction. For example, suppose that dn is the differential number density of particles having energies within dE and moving in a direction defined by the solid angle element $d\Omega$. Since this density refers to a given (invariant) number dN of particles, $dn = dN/d^3\mathbf{r} = (dN/dX)dx_0$ is the zero component of a four-vector.⁹ Now consider the transformation properties of dn/E , that is, of dn/p_0 . This quantity transforms as

$$\frac{dx'_0}{p'_0} = \frac{a_{0\mu} dx_{\mu}}{a_{0\mu} p_{\mu}} = \frac{dx_0 a_{00} + a_{0j} dx_j/dx_0}{p_0 a_{00} + a_{0j} p_j/p_0}. \quad (1.52)$$

But when the spacetime coordinates refer to the particle of given p_{μ} , both p_j/p_0 and dx_j/dx_0 are equal to $-i\beta_j$ and it follows that $dx_0/p_0 = \text{inv}$. Since dn can be written as proportional to dx_0 (see above), we then have

$$dn/E = \text{inv}. \quad (1.53)$$

A related invariant is the *occupation number* describing a distribution of particles. If dn_s is the differential density per polarization (spin) state, the occupation number is (defined as)

$$\bar{n} = (2\pi\hbar)^3 dn_s/d^3\mathbf{p}. \quad (1.54)$$

By virtue of the invariance of the differentials (1.47) and (1.53), it follows that

$$\bar{n} = \text{inv}. \quad (1.55)$$

⁹The four-vector is the current density $j_{\mu} = (icn, \mathbf{j})$, where \mathbf{j} is the (three-dimensional) current density. The continuity equation in covariant form is then $\partial j_{\mu}/\partial x_{\mu} = \text{Div } j_{\mu} = 0$.

1.2.4 Covariant Mechanics and Electrodynamics

Because Newtonian mechanics is covariant under Galilean spacetime transformations, it is inconsistent with Lorentz-Einstein spacetime. Electromagnetism is, however, Lorentz covariant when written in terms of a four-dimensional spacetime formulation, and it can be helpful as a guide in reformulating mechanics. Because of the validity of the Newtonian limit, to every basic equation in non-relativistic mechanics there must be a corresponding relativistic covariant one having, in fact, a similar form. Actually, the relativistic equations are more compact and elegant, and it is not difficult to guess their form on the basis of their non-relativistic limits. For example, we expect the covariant form of Hamilton's Principle to be

$$\delta \int L(x_\mu, v_\mu, \tau) d\tau = 0, \quad (1.56)$$

where $v_\mu = dx_\mu/d\tau$. The Lagrangian must be an invariant function of the four-vectors x_μ and v_μ and of other field parameters, and the resulting Euler-Lagrange equation will be

$$\frac{d}{d\tau} \frac{\partial L}{\partial v_\mu} - \frac{\partial L}{\partial x_\mu} = 0. \quad (1.57)$$

The correct relativistic Lagrangian for a charged¹⁰ particle in an electromagnetic field may be obtained easily. For fields derived from a vector potential \mathbf{A} and scalar potential Φ , the non-relativistic Lagrangian for the particle motion is

$$L_{\text{NR}} = \frac{1}{2}mv_j v_j + (q/c)A_j v_j - q\Phi. \quad (1.58)$$

If we define the quantity $A_\mu = (i\Phi, \mathbf{A})$, we can easily establish its four-vector character. The Lorentz gauge condition is then simply $\text{Div } A_\mu = 0$, which is manifestly covariant. Also, the transformation relation to other gauges is covariant: $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$, where Λ is an arbitrary scalar function.¹¹ Thus, we are immediately led to suggest the covariant relativistic Lagrangian

$$L = \frac{1}{2}mv_\mu v_\mu + (q/c)A_\mu v_\mu. \quad (1.59)$$

The Lagrangian itself has, basically, no special physical meaning; rather it is a *function*, which, when employed in Equation (1.57), yields the correct equation of motion. For example, from the definition of v_μ , the factor $v_\mu v_\mu$ in the Lagrangian (1.59) actually equals $-c^2$, but we do not write it that way, since the important character of the Lagrangian is its functional form.

With the Lagrangian (1.59), the equation of motion (1.57) is

$$\frac{d}{d\tau}(mv_\mu) = \frac{q}{c} \left[\frac{\partial}{\partial x_\mu}(A_\nu v_\nu) - \frac{dA_\mu}{d\tau} \right], \quad (1.60)$$

and the right-hand side of this equation is clearly some kind of four-vector force. We expect a relativistic equation of motion to be of the form

$$dp_\mu/d\tau = K_\mu, \quad (1.61)$$

¹⁰Effects of the interaction of the particle's magnetic moment (if it has one) with the field are neglected.

¹¹The convenient compact notation $\partial_\mu = \partial/\partial x_\mu$ is introduced here.

where $K_\mu = (K_0, \mathbf{K})$, and the space part (\mathbf{K}) of this four-vector must be related to the Newtonian notion of the force (\mathbf{F}). In fact, this relationship can be discovered very easily through a comparison with the result (1.60) for the special case where $A_\mu = (i\Phi, 0)$, corresponding to $\mathbf{A} = 0$ and there is only the presence of an electrical component of the Lorentz force $\mathbf{F} \rightarrow q\mathbf{E} = -q\nabla\Phi$, for which $A_\nu v_\nu \rightarrow -\gamma c\Phi$. Since $dt = \gamma d\tau$, the space part of the covariant equation (1.61) can then be written (in non-covariant form)

$$d(\gamma m \mathbf{v})/dt = \mathbf{F} = \mathbf{K}/\gamma, \quad (1.62)$$

which also allows the identification of \mathbf{K} in terms of \mathbf{F} .

The meaning of the zero or “time” component of K_μ becomes clear when we form the invariant

$$K_\mu v_\mu = m v_\mu dv_\mu/d\tau = \frac{1}{2} m d(v_\mu v_\mu)/d\tau. \quad (1.63)$$

But $v_\mu v_\mu = -c^2$, so that

$$K_\mu v_\mu = 0, \quad (1.64)$$

which must hold for all types of four-vector forces. Written explicitly in terms of the components, the identity (1.64) is

$$i\gamma c K_0 + \gamma^2 \mathbf{F} \cdot \mathbf{v} = 0. \quad (1.65)$$

Maintaining the same concept of energy in relativistic mechanics as in Newtonian mechanics,

$$\mathbf{F} \cdot \mathbf{v} = dT/dt, \quad (1.66)$$

where T is the particle kinetic energy. Then

$$K_0 = (i/c)\gamma dT/dt = (i/c) dT/dt, \quad (1.67)$$

and the zero component of the equation of motion (1.61) is a relation for the particle energy. The relation also allows the identification of the zero component of the momentum four-vector p_μ ; that is,

$$p_0 = m v_0 = i m \gamma c = i E/c, \quad (1.68)$$

where E is the particle energy. The identity $p_\mu p_\mu = m^2 v_\mu v_\mu = -m^2 c^2$ then gives a relativistic relation between energy and momentum:

$$E^2 = p^2 c^2 + m^2 c^4, \quad (1.69)$$

where p is the magnitude of the (relativistic) momentum.

The equations of classical electrodynamics are elegant when expressed in covariant form. The relationships between the vector and scalar potentials and the sources of the fields (charges and currents) are very simple:

$$\square^2 A_\mu = -(4\pi/c) j_\mu, \quad (1.70)$$

where $\square^2 = \partial_\mu \partial_\mu$ is the invariant D’Alembertian operator. In terms of the anti-symmetric electromagnetic field tensor,

$$F_{\mu\nu} = \partial A_\nu / \partial x_\mu - \partial A_\mu / \partial x_\nu = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (1.71)$$

the components of the electric and magnetic fields can be written

$$\begin{aligned} E_j &= iF_{j0}, \\ B_j &= \varepsilon_{jkl}\partial_k A_l = \frac{1}{2}\varepsilon_{jkl}(\partial_k A_l - \partial_l A_k), \\ &= \frac{1}{2}\varepsilon_{jkl}F_{lk}. \end{aligned} \quad (1.72)$$

Here, ε_{jkl} is the Levi-Civita or Ricci symbol

$$\varepsilon_{jkl} = \begin{cases} +1 & jkl = \text{even permutation of } 123 \\ -1 & jkl = \text{odd permutation of } 123 \\ 0 & \text{otherwise.} \end{cases} \quad (1.73)$$

The four Maxwell equations in non-covariant form are condensed into two covariant equations:

$$\partial_\nu F_{\mu\nu} = (4\pi/c)j_\mu, \quad (1.74)$$

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \quad (\lambda \neq \mu \neq \nu). \quad (1.75)$$

There are four relations associated with each of the above equations; four are from the four values of μ in the inhomogeneous equation (1.74), and there are four possible values for the index not appearing in the cyclic permutation in the homogeneous equation (1.75). Without the radiation reaction force term, the covariant equation of motion for a particle in an electromagnetic field is

$$dp_\mu/d\tau = (q/mc)F_{\mu\nu}p_\nu. \quad (1.76)$$

In treating the various electromagnetic processes, we shall make extensive use of covariance considerations.

1.3 KINEMATIC EFFECTS

Certain important aspects of particle processes have nothing to do with the nature of the interactions involved but are, rather, a consequence of the kinematics. That is, the effects are associated with energy and momentum conservation or with the availability of continuum states for the process (“phase-space effects”). The kinematic effects often dominate the character of a process, and a few examples will be mentioned briefly in this section. The examples will be of a general nature and have applications to a variety of processes.

1.3.1 Threshold Energies in Non-Relativistic and Relativistic Processes

In descriptions of many inelastic processes, it is convenient to consider the processes in a center-of-mass (c.m.) frame for which the total momentum is zero. For example, in a collision of two particles, it may be possible to produce another particle or to excite a higher bound energy level in one of the colliding particles or systems. The minimum energy for such a phenomenon would correspond to the case where the outgoing particles have zero kinetic energy in the c.m. frame.

Let us consider first the case of the collision of two non-relativistic particles or systems (such as an atom) of mass m_1 and m_2 and initial velocity \mathbf{v}_1 and \mathbf{v}_2 in the lab frame (K). Then the c.m. frame (K') moves with velocity

$$\mathbf{V} = (m_1\mathbf{v}_1 + m_2\mathbf{v}_2)/M, \quad (1.77)$$

with

$$M = m_1 + m_2. \quad (1.78)$$

In K' the particle velocities are $\mathbf{v}' = \mathbf{v} - \mathbf{V}$, each particle having a magnitude of momentum

$$p' = |\mathbf{p}_1| = |\mathbf{p}_2| = \mu v_r, \quad (1.79)$$

where

$$\mu = m_1 m_2 / M \quad (1.80)$$

is the reduced mass and

$$v_r = |\mathbf{v}_1 - \mathbf{v}_2| \quad (1.81)$$

is the magnitude of the relative velocity in the *lab frame* (K). The total kinetic energy in the c.m. frame (K') is

$$E' = p'^2/2m_1 + p'^2/2m_2 = p'^2/2\mu = \frac{1}{2}\mu v_r^2. \quad (1.82)$$

Suppose that we are interested in some inelastic process whereby the energy E' is to be used in the excitation of some level (which may be in the continuum) or in producing some new particle. If the c.m. energy required for this excitation is χ , setting $E' \geq \chi$ yields, by Equation (1.82), a condition on the relative velocity v_r :

$$\frac{1}{2}\mu v_r^2 \geq \chi. \quad (1.83)$$

For the case, say, $\mathbf{v}_2 = 0$ (one particle initially at rest in the lab frame), we have a condition on the required kinetic energy of the incident particle:

$$E_1 = \frac{1}{2}m_1 v_1^2 = (m_1/\mu)\chi. \quad (1.84)$$

But $\mu < m_1$, and we see that the kinetic energy required is always *greater than* the excitation energy χ needed in K' . The physical reason for this result is very simple; to satisfy momentum conservation, the products of the collision process must, at threshold, all be moving in the direction of the incident particle at a velocity $\mathbf{V} = (m_1/M)\mathbf{v}_1$. The kinetic energy (in K) of the outgoing M is just $E_M = (m_1/M)E_1$ at threshold, and the difference ($E_1 - E_M$) equals χ .

The kinematics of relativistic particle collisions can be handled with about the same degree of simplicity as the non-relativistic case. Instead of transforming the particle energy and momentum between the lab frame (K) and the c.m. frame (K') by means of the four-vector Lorentz transformation, it is more convenient to employ the invariant (1.39). Also, in manipulations involving relativistic kinematics, to simplify the algebra somewhat, we can set $c = 1$; if necessary, in final formulas the factors of c to various powers can be reinserted through considerations of dimensionality.

The invariant

$$E^2 - \mathbf{p}^2 = \text{inv} \quad (1.85)$$

can refer to either the components of the total energy and momentum or those of an individual particle, for which $E_1^2 - \mathbf{p}_1^2 = m_1^2$. When E and \mathbf{p} refer to the total energy and momentum of a two-particle system, since the individual energy and momentum are related by $\mathbf{p}_1 = E_1 \mathbf{v}_1$, application of the invariant (1.85) yields

$$m_1^2 + m_2^2 + 2E_1 E_2 (1 - \mathbf{v}_1 \cdot \mathbf{v}_2) = E'^2 \quad (1.86)$$

where E' is the total energy in the c.m. frame. This simple and general formula has many important applications. In the calculation of binary-collision processes, generally the cross section is most conveniently represented in terms of the energy E' , and the expression (1.86) provides the explicit relationship to the lab-frame quantities.¹² Equation (1.86) can also be applied to determine (lab-frame) threshold energies as in the non-relativistic limit. If $m_1 = m_2 = m$ and $\mathbf{v}_2 = 0$, for a process corresponding to a c.m. threshold energy $E' = \chi$, the lab-frame kinetic energy required is

$$T_1 = E_1 - m = \chi^2/2m - 2m. \quad (1.87)$$

For example, in producing particle-antiparticle pairs of the same type, the c.m. threshold energy would be $\chi = 4m$ and the lab-frame value would be $T_1 = 6m$. As in our non-relativistic problem, because of the necessity of satisfying momentum conservation, much more lab-frame energy is required than in the c.m. frame.

In the case of photon-photon collisions or for collisions of highly relativistic particles, the relation (1.86) is further simplified:

$$2E_1 E_2 (1 - \cos \theta_{12}) = E'^2, \quad (1.88)$$

where θ_{12} is the angle between the (massless) particle directions of motion. For example, in the process of pair production in photon-photon collisions, E' could be set equal to $2E'_e$, where E'_e is the c.m. energy of the electron or positron. For a head-on photon-photon collision ($\theta_{12} = \pi$), the threshold condition would be, very simply, $E_1 E_2 = m^2$.

1.3.2 Transformations of Angular Distributions

In the evaluation of particle processes, it is often necessary to consider the relationship between the direction of particle motion in different reference frames. For relativistic particle processes, there is an especially important and interesting phenomenon associated with the directions of motion of other particles (with which it interacts) in the lab frame and in the rest frame of the relativistic particle.

Consider the motion of a particle (P_v) of velocity \mathbf{v} in the lab frame (K). Let the axes of K be such that $\mathbf{v} = v(1, 0, 0)$; that is, \mathbf{v} is along the x -axis. Now consider the motion of another particle (P_u) of velocity \mathbf{u} ; let this velocity lie in K 's x - y -plane so that $\mathbf{u} = u(\cos \theta, \sin \theta, 0)$. In P_v 's rest frame (K'), P_u will be moving at an angle θ' with respect to the x - and x' -axes. From the velocity transformation,

$$\tan \theta' = u'_y/u'_x = u_y/\gamma(u_x - v) = \sin \theta/\gamma(\cos \theta - v/u), \quad (1.89)$$

¹²Often it is necessary to integrate over a spectrum of lab-frame energies and directions of motion.

where $\gamma = (1 - v^2/c^2)^{-1/2}$. The transformation from K' to K is, of course, of the same form except that the sign of v_x ($= v$) is reversed. For the case where P_u is non-relativistic and $v/u \gg \cos \theta$, $\tan \theta' \rightarrow -(u/\gamma v) \sin \theta$, for which $\pi/2 < \theta' < \pi$. In fact, if $v \rightarrow c$ ($\gamma \gg 1$), θ' will be very close to π . This means that in P_v 's frame (K') every other particle is moving toward it head on. Even if the P_u 's are photons ($u \rightarrow c$), this will be true as long as $\theta \gtrsim \gamma^{-1}$. A highly relativistic particle moving through a gas “sees” particles incident head on (like a beam).

We could also consider some process involving a highly relativistic P_v in which, in K' , there is some outgoing particle P_u (for example, a scattered photon) moving at an angle θ' . Then in the lab frame (K) the corresponding angle would be θ with

$$\tan \theta = \sin \theta' / \gamma (\cos \theta' + v/u'); \quad (1.90)$$

unless θ' is very close to π , θ' will be very small: $\theta' \rightarrow (u/c\gamma) \sin \theta' \ll 1$. In other words, the process involves a beam of associated particles (scattered or produced, for example) in the forward direction.

1.4 BINARY COLLISION RATES

An important relativistic kinematic effect is associated with the formula for collision rates for two particles. The relativistic features come in when we consider rates associated with a gas of particles and when we do covariant calculations of cross sections for various processes. The collision rate formula can be derived¹³ through the use of kinematic invariants and relativistic transformations for quantities in convenient Lorentz frames. First, suppose we consider the non-relativistic expression for the collision rate for a particle of type a passing through a gas of particles of type b in which both types of particle have finite mass. The collision rate, per particle a , is

$$(dN/dt)_a = \iint |\mathbf{v}_a - \mathbf{v}_b| d\sigma dn_b \quad (\text{non-relativistic}), \quad (1.91)$$

where the double integration is over the differential cross section $d\sigma$ and the distribution of particles of number density dn_b . The relative velocity $|\mathbf{v}_a - \mathbf{v}_b|$ multiplied by the differential density dn_b is the flux of particles of type b incident on particle a . If we have a gas of particles of type a the number of collisions per unit volume per unit time is

$$(dN/dVdt)_{ab} = (1 + \delta_{ab})^{-1} \iiint |\mathbf{v}_a - \mathbf{v}_b| d\sigma dn_a dn_b, \quad (1.92)$$

with the factor involving δ_{ab} correcting for counting the same type of collision twice if the particles a and b are identical.

We want the relativistic generalizations of the above formulas, including the class of collision in which either or both of the particles have, like photons, zero mass. In particular, the relative velocity factor must be replaced by a more general relativistic expression, and it is not hard to derive the correct formula. First, we note that dN and $dVdt$ are relativistic invariants. The simplest case to consider is that where particle

¹³See, for example, R. J. Gould, *Am. J. Phys.* **39**, 911 (1971).

a has a finite mass and particles b have zero mass, with the real life application being the case of electron-photon collisions. For the rate dN_a/dt we go from the “lab” frame (K) to the rest frame (K') of particle a :

$$dN_a/dt = (dN_a/dt')(dt'/dt). \quad (1.93)$$

Since a is at rest in K' , $dt = \gamma dt'$, in terms of the particle’s Lorentz γ in K . In K' ,

$$(dN/dt)' = \iint c d\sigma dn'_b, \quad (1.94)$$

where $c dn'_b$ is just the incident photon flux (in K'). The result can be written in terms of the lab-frame (K) distribution dn_b by making use of the invariance of dn/E and by writing

$$E' = \gamma E(1 - \beta \cos \theta), \quad (1.95)$$

where θ is the lab-frame angle between the directions of motion of a and b . We then have

$$dN_a/dt = c \iint (1 - \beta \cos \theta) d\sigma dn_b \quad (b \text{ massless}). \quad (1.96)$$

The factor $c(1 - \beta \cos \theta)$ is *not* the relative velocity; instead it is the projection of the relative velocity along the direction of motion of the zero-mass particle b .

When both particles a and b are massless, we cannot go to one or the other’s rest frame and the lab-frame rate must be derived by going, instead, to the c. m. frame. Since the cross section $d\sigma$ is an area perpendicular to the direction of relative motion between the rest frame (for a finite-mass particle) and the c. m. frame, it is an invariant for transformations between these two frames; that is, $d\sigma' = d\sigma$. We can take the limit $\beta \rightarrow 1$ in the above formula to get the result for the collision rate for a massless particle a traversing a gas of massless particles b :

$$dN_a/dt = c \iint (1 - \cos \theta) d\sigma dn_b \quad (a \text{ and } b \text{ massless}). \quad (1.97)$$

Similarly, the collision rate for the two gasses, if one is massless, would be

$$(dN/dVdt)_{ab} = (1 + \delta_{ab})^{-1} c \times \iiint (1 - \beta \cos \theta) d\sigma dn_a dn_b \quad (a \text{ or } b \text{ massless}), \quad (1.98)$$

and we can obtain the result when both particles are massless by taking the limit $\beta \rightarrow 1$.

Now we look for the relativistic generalization of the non-relativistic results (1.91) and (1.92), that is, the case where both particles have finite mass. The derivation of the results starts from the collision rate for particle a in the lab frame (K), relating it to the rate in a ’s rest frame (K') as is done in (1.93). In K' , the rate from the incident flux of particles of type b is

$$dN_a/dt' = dN'_a/dt' = \iint d\sigma |\mathbf{v}'_b| dn'_b. \quad (1.99)$$

The factors $|\mathbf{v}'_b|$ and dn'_b are then written in terms of lab-frame quantities. To do this it is convenient to take an orientation of the frame K such that \mathbf{v}_a is along

the x -axis and \mathbf{v}_b is in the x - y -plane. When the final result is obtained we can recognize rotational invariants equal to these expressions for the special case of this orientation of K 's axes, yielding a formula for general direction of motion for \mathbf{v}_a and \mathbf{v}_b . The number densities dn_a and dn_b are differential in the velocity or energy interval and differential in, say, a solid angle element designating direction. That is, the magnitude and direction of the velocities are specified. We then write

$$dn'_b = dn_b(E'_b/E_b) \quad (1.100)$$

The relative velocity for the frames K and K' is just v_{ax} , and

$$E'_b/E_b = \gamma(1 - v_{ax}v_{bx}/c^2), \quad (1.101)$$

with

$$1/\gamma^2 = 1 - v_{ax}^2/c^2. \quad (1.102)$$

Since the position coordinates of particle a are constant in K' , the time intervals are related by $dt = \gamma dt'$. The flux velocity $|\mathbf{v}'_b|$ in (1.99) is given by

$$|\mathbf{v}'_b| = (v_{bx}^2 + v_{by}^2)^{1/2}, \quad (1.103)$$

and the two velocities here can be related to their lab-frame variables by simple Lorentz transformations. Specifically, we have

$$v'_{bx} = \frac{v_{bx} - v_{ax}}{1 - v_{ax}v_{bx}/c^2}, \quad v'_{by} = \frac{v_{by}}{\gamma(1 - v_{ax}v_{bx}/c^2)}. \quad (1.104)$$

The result for the velocity $|\mathbf{v}'_b|$ can be put in general form by recognizing that terms from components for the particular choice of orientation of axes can be written as special cases of a formula that is a rotational (in velocity space) scalar. We have

$$v_{bx}^2 + v_{by}^2 = \frac{(\mathbf{v}_a - \mathbf{v}_b)^2 - (\mathbf{v}_a \times \mathbf{v}_b)^2/c^2}{(1 - \mathbf{v}_a \cdot \mathbf{v}_b/c^2)^2}, \quad (1.105)$$

where the expression on the right is symmetric in a and b and can be regarded as the squared magnitude of the velocity of either a or b in the rest frame (K') of the other. The differential densities in K' and in the general lab frame K are related by

$$dn'_b = \gamma(1 - \mathbf{v}_a \cdot \mathbf{v}_b/c^2)dn_b, \quad (1.106)$$

$$dn'_a dn'_b = (1 - \mathbf{v}_a \cdot \mathbf{v}_b/c^2)dn_a dn_b. \quad (1.107)$$

Further, we can introduce a fundamental velocity

$$v_0 = [(\mathbf{v}_a - \mathbf{v}_b)^2 - (\mathbf{v}_a \times \mathbf{v}_b)^2/c^2]^{1/2}, \quad (1.108)$$

which is a relativistic generalization of the non-relativistic expression $|\mathbf{v}_a - \mathbf{v}_b|$. We then have

$$dN_a/dt = \iint v_0 d\sigma dn_b, \quad (1.109)$$

$$(dN/dV dt)_{ab} = (1 + \delta_{ab})^{-1} \iiint v_0 d\sigma dn_a dn_b. \quad (1.110)$$

These formulas apply, in fact, to the cases where one or both particles are massless. In those cases we would take the limits $m_a \rightarrow 0$ and/or $m_b \rightarrow 0$ for which $v_a \rightarrow c$ and/or $v_b \rightarrow c$. Note how the factors γ and $(1 - \mathbf{v}_a \cdot \mathbf{v}_b/c^2)$ cancel to give formulas for the rates only in terms of the kinematic factor v_0 .

1.5 PHASE-SPACE FACTORS

1.5.1 Introduction

There are in physics many phenomena or processes that are essentially “phase-space effects.” By this is meant that the magnitude of the effect is proportional to the available number of states. Actually, the factor that usually comes in is the number of states per unit energy available, and this can be seen from the Fermi Golden Rule formula. From time-dependent perturbation theory (see Chapter 3), the transition rate from an initial state (0) to a final continuum state (f) is given by

$$\frac{\Delta W_{f0}}{\Delta t} = \frac{2\pi}{\hbar} \left(\sum_f \right) |H'_{f0}|^2 \left(\frac{dN}{dE} \right)_f = \frac{2\pi}{\hbar} \left(\sum_f \right) |H'_{f0}|^2 \delta(E_f - E_0); \quad (1.111)$$

here H'_{f0} is the matrix element of the perturbation Hamiltonian that causes the transition, and there is usually a sum over final states. The δ -function expresses energy conservation; that is, the final states are restricted to the “energy shell” determined by the total energy ($E_0 = E$) available to the various final states. For processes between initial and final free-particle (continuum) states there is also conservation of momentum:

$$\sum_f \mathbf{p}_f = \sum_0 \mathbf{p}_0 \equiv \mathbf{P}, \quad (1.112)$$

and $\mathbf{P} = 0$ in the c.m. frame. The condition (1.112) also follows from the evaluation of H'_{f0} in the position representation, employing plane-wave states for the particle wave functions in the initial and final states (see Chapter 3).

Aside from spin states, the number of momentum (\mathbf{p}) or wave-vector (\mathbf{k}) states available in a spatial volume L^3 and momentum-space volume $d^3\mathbf{p}$ is¹⁴

$$dN = L^3 d^3\mathbf{p} / (2\pi\hbar)^3 = L^3 d^3\mathbf{k} / (2\pi)^3. \quad (1.113)$$

In the description of particle processes the factors involving L^3 always cancel, since L^3 appears in inverse powers in the matrix element H'_{f0} as normalization factors in the individual one-particle wave functions. As remarked earlier in this chapter, although $L^3 d^3\mathbf{p}$ is, strictly speaking, what is known as phase space, in the description of particle processes we often refer to the momentum-space part as the phase-space factor. If N particles are being described, the associated phase-space volume, with both energy and momentum-space conservation, is then, in the c.m. frame,

$$\Phi^{(N)} = \int \cdots \int \prod_{j=1}^N d^3\mathbf{p}_j \delta^{(3)}\left(\sum_j \mathbf{p}_j\right) \delta\left(\sum_j E_j - E\right). \quad (1.114)$$

For many processes this factor determines the magnitude of the transition rate or, for example, the cross section for the process. Of special importance is the dependence of $\Phi^{(N)}$ on the available energy E , which would determine the behavior of, say, a cross section in the neighborhood of the threshold energy for the process. This

¹⁴See Equation (1.54) and any book on quantum mechanics treating the elementary “particle-in-a-box” problem.

dependence can be derived very readily, the result being that $\Phi^{(N)}$ has a power-law form

$$\Phi^{(N)} \propto E^q, \quad (1.115)$$

for certain energy domains [outgoing particles non-relativistic (NR) or extreme relativistic (ER)], with q increasing with N . We shall derive these results, obtaining general formulas for $\Phi^{(N)}$ in the NR and ER limits. In the derivation of the general results, the mathematical techniques and tricks are interesting in themselves, and several such methods will be introduced or, at least, mentioned. Although very general formulas can be obtained, usually the most important applications are for $N \sim 2$ or 3, say, and these specific expressions will be given first, since the mathematics is simple.

There are many important applications or uses for the expressions for the $\Phi^{(N)}$. They are fundamental for general processes in which particles are produced, and often the energies involved are relativistic. Unfortunately, the problem for (general) outgoing particle energies $E_j \sim m_j c^2$ is complex except for small N and, in fact, in this case the invariant expression

$$\phi^{(N)} = \int \cdots \int \prod_{j=1}^N (d^3 \mathbf{p}/2E)_j \delta^{(4)}\left(\sum_j (p_u)_j - P_\mu\right) \quad (1.116)$$

is sometimes more useful. Considerations of particle processes in terms of invariant phase-space factors are employed extensively in high-energy physics. However, a covariant description¹⁵ is not always most appropriate, especially if the outgoing particles are non-relativistic. This is usually the case in applications to problems involving nuclear reactions ($E_0 \sim \text{MeV}$ energies), and always in the area of chemical kinetics where the reaction products are atoms, molecules, ions, and electrons, at $\sim \text{eV}$ energies. In some processes there might be outgoing particles both NR and ER, the latter type being photons, for example, as in some nuclear reactions; thus, it is useful to have expressions for the phase-space factor in these “mixed” cases. To describe the cases we shall employ the notation N and N' for the total number of NR and ER particles, respectively, and the indices j and k to designate the particles among the N and N' . We shall see that it is possible to obtain exact formulas for Φ for arbitrary N and N' in this limit, where

$$\begin{cases} E_j \rightarrow \mathbf{p}_j^2/2m_j & (j = 1 \text{ to } N), \\ E_k = p_k c & (k = 1 \text{ to } N'). \end{cases} \quad (1.117)$$

There is an important theorem involving the phase-space volume (1.114) for the case where one of the outgoing particles (say, m_N) has a large mass. In this limit its energy would have to be non-relativistic and the arguments of the δ -functions in $\Phi^{(N)}$ can be written

$$\sum_{j=1}^N \mathbf{p}_j = \mathbf{p}_N + \sum_{j=1}^{N-1} \mathbf{p}_j, \quad (1.118)$$

$$\sum_{j=1}^N E_j - E = \mathbf{p}_N^2/2m_N + \sum_{j=1}^{N-1} E_j - E. \quad (1.119)$$

¹⁵In Section 1.5.6 we shall return to the use of the invariant expression (1.116). We shall also briefly discuss the relative merits of the covariant and non-covariant formulations and their applications.

For very large m_N the first term on the right of Equation (1.119) is negligible. The δ -function $\delta^{(3)}$ in $\Phi^{(N)}$ can then be eliminated by integrating over $d^3 \mathbf{p}_N$ using the argument (1.118), and we have the result

$$\Phi^{(N)} \xrightarrow{m_N \text{ large}} \Psi^{(N-1)}, \quad (1.120)$$

where Ψ is a phase-space volume like $\Phi^{(N)}$ but *without* the momentum-conservation δ -function:

$$\Psi^{(N)} = \int \cdots \int \prod_{j=1}^N d^3 \mathbf{p}_j \delta\left(\sum_j E_j - E\right). \quad (1.121)$$

That is, when one of the outgoing particles is very massive, it performs the function of carrying away momentum but not energy. The energy-conservation restriction is still imposed for the other particles and, for the validity of the theorem (1.120), no assumption has been made about the characteristics or energies of the other $N - 1$ particles. Incidentally, the form $\Psi^{(N)}$ is the one generally introduced in classical statistical mechanics in which the basic assumption is that equal phase-space volumes within the energy shell have equal weighting or probability. In that subject it is inherently assumed that there is momentum exchange to some (heavy) containment vessel or thermalizing agent.

Similar remarks can be made concerning angular momentum. No restriction on total angular momentum is made in the integrals (1.114) and (1.121). It is assumed that there is some mechanism, perhaps associated with the interaction H' , in which angular momentum is conserved without affecting the momentum-space integrations $\Phi^{(N)}$ or $\Psi^{(N)}$. Actually, the angular momentum constraints are quite complex, since the spatial integrations over the $d^3 \mathbf{r}_j$ are involved as well as particle spin states. Other complicating factors will also be omitted here, such as the spin degeneracy factor

$$g_{\text{spin}} = \prod_j g_j. \quad (1.122)$$

When some of the outgoing particles are identical there is also a “symmetry factor” $1/\sigma$ that must multiply $\Phi^{(N)}$ or $\Psi^{(N)}$ to avoid counting equivalent states more than once.¹⁶ There is also a restriction associated with baryon and lepton conservation that must be applied in considerations of possible combinations of outgoing particles. All of these effects will be omitted here and only formulas for the classical kinematic factors $\Phi^{(N)}$ and $\Psi^{(N)}$ will be given.

1.5.2 Simple Examples

Let us take the very simplest example, which will immediately yield an interesting and instructive result. We consider some process in which the product is a very massive particle (like a nucleus) plus a single additional particle (perhaps a pion or a photon). The relevant phase-space factor would then be

$$\Psi^{(1)} = \int d^3 \mathbf{p}_1 \delta(E_1 - E) = 4\pi \int p_1^2 dp_1 \delta(E_1 - E). \quad (1.123)$$

¹⁶The factor is, in fact, given by $\sigma = N!$ for N identical particles or, for the case where there are several kinds (α) of identical outgoing particles, $\sigma = \prod_{\alpha} (N_{\alpha})!$.

Employing the relation (1.69), we obtain

$$\Psi^{(1)} = (4\pi/c^3)E(E^2 - m^2c^4)^{1/2} \quad (1.124)$$

for a particle of arbitrary mass. If the particle is massless or is highly relativistic,

$$\Psi^{(1)} \rightarrow (4\pi/c^3)E^2 \quad (\text{ER}). \quad (1.125)$$

On the other hand, if $E = mc^2 + T$, with $T \ll mc^2$, corresponding to an outgoing low-energy particle with mass,

$$\Psi^{(1)} \rightarrow 4\sqrt{2}\pi m^{3/2}T^{1/2} \quad (\text{NR}). \quad (1.126)$$

In the limit $T \ll mc^2$, the *ratio* of the two formulas above is

$$\frac{(\text{NR})}{(\text{ER})} \rightarrow \left(\frac{2T}{mc^2}\right)^{1/2} \ll 1. \quad (1.127)$$

The result tells us, for example, that phase-space favors photon production over particle (with mass) production in this application.

Now consider the case of two outgoing particles accompanying a heavy one, so that the relevant factor is

$$\Psi^{(2)} = \iint d^3\mathbf{p}_1 d^3\mathbf{p}_2 \delta(E_1 + E_2 - E). \quad (1.128)$$

When both particles are NR or both are ER, the integration is elementary and yields

$$\Psi^{(2)} \rightarrow \begin{cases} 4\pi^3(m_1m_2)^{3/2}E^2 & (\text{NR}), \\ (8\pi^2/15c^6)E^5 & (\text{ER}). \end{cases} \quad (1.129)$$

The case where one particle is NR and the other is ER is also easy to evaluate:

$$\Psi^{(2)} \rightarrow \frac{256\sqrt{2}\pi^2 m_1^{3/2}}{105 c^3} E^{7/2} \quad (1 \text{ NR}, 1 \text{ ER}). \quad (1.130)$$

In the result (1.129) the energy E is the (maximum) *kinetic* energy available to the NR particles; that is, it is the available energy over and above the (rest) energies necessary to produce the particles. The different dependence on E in the above three results should, in particular, be noted.

For the case of arbitrary energy and finite m_1 and m_2 , the integral $\Psi^{(2)}$ is fairly complicated. Setting $d^3\mathbf{p}_2 = (4\pi/c^2)p_2E_2dE_2$, integration over dE_2 with the δ -function is performed, giving

$$\begin{aligned} \Psi^{(2)} &= (4\pi/c^2) \int d^3\mathbf{p}_1 p_2 (E - E_1) \\ &= (4\pi/c^2)^2 \int p_1 p_2 E_1 (E - E_1) dE_1, \end{aligned} \quad (1.131)$$

with

$$p_1c = (E_1^2 - m_1^2c^4)^{1/2}, \quad p_2c = [(E - E_1)^2 - m_2^2c^4]^{1/2}. \quad (1.132)$$

The lower and upper limits on the integral (1.131) are, respectively, m_1c^2 and $E - m_2c^2$, in which E ($> m_1c^2 + m_2c^2$) is the total energy. The integral is difficult to evaluate,¹⁷ except for $m_2 = 0$; in that case, the result is

$$\Psi^{(2)}(m_2 = 0) = (4\pi c^2)^2 m_1^5 I, \quad (1.133)$$

where I is the integral

$$\begin{aligned} I &= \int_0^{\eta_0} \eta^2 [(1 + \eta_0^2)^{1/2} - (1 + \eta^2)^{1/2}]^2 d\eta \\ &= -\frac{1}{4}\eta_0 - \frac{1}{12}\eta_0^3 + \frac{1}{30}\eta_0^5 + \frac{1}{4}(1 + \eta_0^2)^{1/2} \ln[(1 + \eta_0^2)^{1/2} + \eta_0], \end{aligned} \quad (1.134)$$

η_0 being related to E by

$$\eta_0 = [(E/m_1c^2)^2 - 1]^{1/2}. \quad (1.135)$$

The result has an important application in nuclear β -decay, essentially determining the radioactive lifetime of a nucleus. In that application, particle 2 would be the (massless) neutrino and particle 1 an electron or positron; the heavy residual nucleus carries away momentum, yielding $\Psi^{(2)}$ rather than $\Phi^{(3)}$ as the relevant phase-space volume.

The evaluation of $\Phi^{(N)}$ in simple cases is readily accomplished. With $N = 2$ the expression (1.114) yields, in the NR and ER limits,

$$\Phi^{(2)} \rightarrow \begin{cases} 4\sqrt{2}\pi(m_1m_2/M)^{3/2}E^{1/2} & \text{(NR),} \\ (\pi/2c^3)E^2 & \text{(ER),} \end{cases} \quad (1.136)$$

where $M = m_1 + m_2$. In the mixed case,

$$\Phi^{(2)} \rightarrow (4\pi/c^3)E^2 \quad (1 \text{ NR, } 1 \text{ ER}), \quad (1.137)$$

which is identical to the result (1.125); in this case, the massless particle carries off negligible momentum, an important result to keep in mind.

For general energy and general m_1, m_2 , the relativistically correct expression for $\Phi^{(2)}$ is obtained:

$$\begin{aligned} \Phi^{(2)} &= \frac{\pi}{2c^3 E^2} [(E^2 - m_1^2c^4 - m_2^2c^4)^2 - 4m_1^2m_2^2c^8]^{1/2} \\ &\quad \times \frac{E^4 - (m_2^2 - m_1^2)^2c^8}{E^2}, \end{aligned} \quad (1.138)$$

in which $E > (m_1 + m_2)c^2$ is the total energy including rest energy. The result (1.138), which contains the expressions (1.136) and (1.137) in the corresponding limits, has been written in the above form for comparison with the covariant expression ($\phi^{(2)}$), which will be given later.

It is obvious from, for example, the complexity of the formula (1.138), that the relativistic problem gets complicated for $N > 2$. However, it is possible to derive expressions for $\Psi^{(N)}$ and $\Phi^{(N)}$ for arbitrary N when the outgoing particles are NR or ER or a mixture of each. The result ($M = m_1 + m_2 + m_3$)

$$\Phi^{(3)} \rightarrow 4\pi^3(m_1m_2m_3/M)^{3/2}E^2 \quad \text{(NR)} \quad (1.139)$$

¹⁷The integrand itself provides the distribution ($d\Psi^{(2)}/dE_1$) in outgoing energies for particle 1 (see Section 4.5).

can be obtained directly through some cumbersome algebra. About as much time is spent deriving the expression (NR limit) for general N , and we now turn to this task. Incidentally, for $m_3 \gg m_1$ and m_2 , we see that the formula (1.139) reduces to the expression (1.129) for $\Psi^{(2)}$ —a specific application of the theorem (1.120). In fact, we shall see that in the NR limit the result for $\Phi^{(N)}$ is very closely related to $\Psi^{(N-1)}$ for general m_1, m_2, \dots

1.5.3 General Theorems—Formulation

Some features of the formulas for $\Phi^{(N)}$ and $\Psi^{(N)}$, such as the very important dependence of E , can be derived without an evaluation of the multiple integrals that are involved in the formulas. This can be achieved through convenient and natural changes of variables to dimensionless form, such that the resulting integrals are functions only of N . The formulation given here is for the special cases where the particles are either NR or ER or a mixture of the two. That is, we consider these three cases with the third obviously containing the first two in a general treatment. To designate the individual and total number of NR and ER particles the notation (1.117) will be employed for their index and total:

$$\begin{aligned} \text{NR: } j &= 1, 2, \dots, N, \\ \text{ER: } k &= 1, 2, \dots, N'. \end{aligned}$$

The phase-space integrals will be given first in the c.m. system ($\mathbf{P} = 0$); later some generalization for finite \mathbf{P} will be given.

In the NR limit, $E_j = \mathbf{p}_j^2/2m_j$, and the integral (1.121) for $\Psi^{(N)}$ can be transformed through the variable change

$$\mathbf{p}_j = (2Em_j)^{1/2} \mathbf{x}_j. \quad (1.140)$$

The integral is then given by

$$\Psi^{(N)} = 2^{3N/2} \left(\prod_j m_j \right)^{3/2} E^{3N/2-1} J_N, \quad (1.141)$$

where

$$\begin{aligned} J_N &= \int \cdots \int \prod_{j=1}^N d^3 \mathbf{x}_j \delta \left(\sum_{j=1}^N \mathbf{x}_j^2 - 1 \right) \\ &= \int \cdots \int \prod_{j=1}^{3N} dx_j \delta \left(\sum_{j=1}^{3N} x_j^2 - 1 \right) \end{aligned} \quad (1.142)$$

is a function of $3N$. This integral occurs in classical statistical mechanics and there are various ways to evaluate it; these methods will be outlined later in this section. However, even without evaluating the integral, we see [Equation (1.141)] that the energy dependence of $\Psi^{(N)}$ has been obtained.

In the ER limit the particle energy is $E_k = p_k c$, and it is convenient to make the variable change

$$\mathbf{p}_k c = E \mathbf{y}_k. \quad (1.143)$$

The Ψ -integral becomes

$$\Psi^{(N')} = c^{-3N'} E^{3N'-1} J_{N'}, \quad (1.144)$$

where

$$J_{N'} = \int \cdots \int \prod_{k=1}^{N'} d^3 \mathbf{y}_k \delta\left(\sum_{k=1}^{N'} y_k - 1\right), \quad (1.145)$$

in which $y_k = |\mathbf{y}_k|$. This integral (1.145) will be evaluated shortly, but we again see that the energy dependence of $\Psi^{(N')}$ has already been obtained in Equation (1.144).

If we have N outgoing NR particles and N' ER particles, the above variable changes yield the general result

$$\Psi^{(NN')} = 2^{3N/2} c^{-3N'} \left(\prod_j m_j\right)^{3/2} E^{3N/2+3N'-1} J_{NN'}, \quad (1.146)$$

where

$$J_{NN'} = \int \cdots \int \prod_{j=1}^N d^3 \mathbf{x}_j \prod_{k=1}^{N'} d^3 \mathbf{y}_k \delta\left(\sum_{j=1}^N x_j^2 + \sum_{k=1}^{N'} y_k - 1\right) \quad (1.147)$$

Evaluation of this general expression contains the results for J_N and $J_{N'}$ for, respectively, N' and N equal to 0.

Now we turn to the phase-space integrals Φ that involve both energy and momentum conservation. For NR particles it is again convenient to make the variable change (1.140) and, in addition, to express the individual masses m_j as fractions of the total mass M :

$$m_j = v_j^2 M, \quad (1.148)$$

$$\sum_{j=1}^N v_j^2 = 1. \quad (1.149)$$

We then have

$$\Phi^{(N)} = (2M)^{3(N-1)/2} \left(\prod_{j=1}^N v_j\right)^3 E^{3(N-1)/2-1} I_N, \quad (1.150)$$

where the dimensionless integral is now

$$I_N = \int \cdots \int \prod_{j=1}^N d^3 \mathbf{x}_j \delta^{(3)}\left(\sum_{j=1}^N v_j \mathbf{x}_j\right) \delta\left(\sum_{j=1}^N x_j^2 - 1\right). \quad (1.151)$$

Although the integral appears to be a function of the parameters v_j , we shall show that it is not, and that, in fact, $I_N = J_{N-1}$. Inspection of the formulas (1.150) and (1.151) and comparison with the results (1.141) and (1.142) shows that the limit (1.120) is satisfied.

For ER particles the transformation (1.143) casts the corresponding $\Phi^{(N')}$ formula into the form

$$\Phi^{(N')} = c^{-(3N'-3)} E^{3N'-4} I_{N'}, \quad (1.152)$$

with

$$I_{N'} = \int \cdots \int \prod_{k=1}^{N'} d^3 \mathbf{y}_k \delta^{(3)}\left(\sum_{k=1}^{N'} \mathbf{y}_k\right) \delta\left(\sum_{k=1}^{N'} y_k - 1\right). \quad (1.153)$$

The integral (1.153) is a little difficult to evaluate, but the technique is outlined in the following subsection.

In the mixed case the corresponding expression for $\Phi^{(NN')}$ can be simplified¹⁸ because most of the momentum is carried by the NR particles. This is easily seen, since, if $E_j = \mathbf{p}_j^2/2m_j \sim E_k = p_k c$, then $p_k/p_j \sim p_j/m_j c \ll 1$. Then in the momentum-conservation δ -function only the contribution from the NR particles must be included. The result is then

$$\Phi^{(NN')} = (2M)^{3(N-1)/2} c^{-3N'} \left(\prod_{j=1}^N v_j \right)^3 E^{3(N-1)/2+3N'-1} I_{NN'}, \quad (1.154)$$

with

$$\begin{aligned} I_{NN'} = & \int \cdots \int \prod_{j=1}^N d^3 \mathbf{x}_j \prod_{k=1}^N d^3 \mathbf{y}_k \\ & \times \delta^{(3)} \left(\sum_{j=1}^N v_j \mathbf{x}_j \right) \delta \left(\sum_{j=1}^N x_j^2 + \sum_{k=1}^{N'} y_k - 1 \right). \end{aligned} \quad (1.155)$$

We shall show that $I_{NN'} = J_{N-1, N'}$.

1.5.4 General Formulas—Evaluation of Multiple Integrals

Here the various J - and I -integrals will be evaluated. There are different ways of calculating these quantities and, to illustrate the methods, we employ the simplest ones for the individual cases. When possible, short cuts will be taken to arrive at the desired result, and the significance of these simplifications will be discussed. In some of the cases, the mathematical methods are very similar, so that most of the details of certain derivations need not be given.

The most elementary general method¹⁹ of evaluating the integral (1.142) obtains a recursion relation. If we write $3N = \nu$, the integral to be evaluated is ($J_N = \mathcal{J}_\nu$)

$$\mathcal{J}_\nu = \int \cdots \int dx_1 \cdots dx_\nu \delta(x_1^2 + \cdots + x_\nu^2 - 1). \quad (1.156)$$

The recursion relation is obtained by expressing the integral in terms of the integral over the “last” variable x_1 from -1 to 1 and by making the variable changes

$$x_2 = (1 - x_1^2)^{1/2} \xi_2, \quad x_3 = (1 - x_1^2)^{1/2} \xi_3, \quad \text{etc.} \quad (1.157)$$

Further, if write $u = x_1^2$, the integral can be written

$$\begin{aligned} \mathcal{J}_\nu &= \int_0^1 u^{-1/2} (1-u)^{(\nu-3)/2} du \int \cdots \int d\xi_2 \cdots d\xi_\nu \\ &\quad \times \delta(\xi_2^2 + \cdots + \xi_\nu^2 - 1) \\ &= B\left(\frac{1}{2}, \frac{\nu-1}{2}\right) \mathcal{J}_{\nu-1}, \end{aligned} \quad (1.158)$$

¹⁸See E. Fermi, *Prog. Theor. Phys.* **5**, 570 (1950).

¹⁹Actually, there is a simpler method described in a number of books on statistical mechanics: cf. C. Kittel, *Elementary Statistical Physics*, p. 37, New York: John Wiley and Sons, Inc., 1958.

in terms of the beta function²⁰

$$B(m, n) = \int_0^1 u^{m-1}(1-u)^{n-1} du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \quad (1.159)$$

Since $\Gamma(\frac{1}{2}) = \pi^{1/2}$, we have, by the results (1.158) and (1.159),

$$\Gamma\left(\frac{\nu}{2}\right) \mathcal{J}_\nu = \pi^{1/2} \Gamma\left(\frac{\nu-1}{2}\right) \mathcal{J}_{\nu-1}, \quad (1.160)$$

or

$$\mathcal{J}_\nu \propto \pi^{\nu/2} / \Gamma(\nu/2) \quad (1.161)$$

Direct calculation of the case $\nu = 1$ yields unity for the proportionality constant, so that

$$\mathcal{J}_\nu = \pi^{\nu/2} / \Gamma(\nu/2) = J_N = \pi^{3N/2} / \Gamma(3N/2). \quad (1.162)$$

The calculation of the ER integral (1.145) for $J_{N'}$ can be carried out in a very similar way. Writing $d^3 \mathbf{y}_k = 4\pi y_k^2 dy_k$, with the last integral over dy_1 from 0 to 1, we are then led to make the transformation

$$y_2 = (1 - y_1)\xi_2, \quad y_3 = (1 - y_1)\xi_3, \quad \text{etc.} \quad (1.163)$$

A recursion relation is again obtained, involving another B -function integral of the type (1.159), and we find

$$J_{N'} = 2^{3N'} \pi^{N'} / \Gamma(3N'). \quad (1.164)$$

The evaluation of the more general integral (1.147) for $J_{NN'}$ can be accomplished in a similar manner, by making the transformations (1.157) and (1.163), yielding a two-index recursion relation and then the result

$$J_{NN'} = \frac{2^{3N'} \pi^{3N/2+N'}}{\Gamma(3N/2 + 3N')}. \quad (1.165)$$

This expression obviously contains the special cases (1.162) and (1.164). We can, moreover, easily check that formulas (1.125), (1.126), (1.129), and (1.130) are recovered in this general result. However, let us now introduce another method for evaluating the phase-space integrals; this procedure is more powerful and direct, and makes use of the integral representation of the δ -function:

$$\delta(\omega) = (2\pi)^{-1} \int d\alpha e^{i\alpha\omega}. \quad (1.166)$$

Here the integration is over all values of α ($-\infty$ to ∞) and, for the purposes of evaluating subsequent integrals, it is convenient to give α a small imaginary part $i\epsilon$. This makes the integral well defined and dictates the way of evaluation of the subsequent integrals in terms of contour integration. With the form (1.166) for the δ -function, the energy-shell restriction is imposed in a convenient way; the integrations over the $d^3 \mathbf{x}_j$ and $d^3 \mathbf{y}_k$ are performed first, followed by the final integration

²⁰For an extensive discussion of the beta and gamma functions, including the identity (1.159), see E. T. Whittaker and G. N. Watson, *Modern Analysis*, 4th ed., Cambridge, UK: Cambridge University Press, 1927.

over α . The method works well for the former integrations, essentially because in the exponential term in the integrand (1.166) we can write

$$\exp\left[i\alpha\left(\sum_j x_j^2 + \sum_k y_k - 1\right)\right] = e^{-i\alpha} \prod_j \exp(i\alpha x_j^2) \prod_k \exp(i\alpha y_k). \quad (1.167)$$

There results a product of individual $d^3 x_j$ and $d^3 y_k$ integrations, each equal to, respectively,

$$I_j = 4\pi \int_0^\infty x^2 e^{i\alpha x^2} dx = e^{-i\pi/4} (\pi/\alpha)^{3/2}, \quad (1.168)$$

$$I_k = 4\pi \int_0^\infty y^2 e^{i\alpha y} dy = 8\pi e^{-i\pi/2} \alpha^{-3}. \quad (1.169)$$

These are then raised to the powers N and N' , respectively, because of the product, and there remains a final integration over α of the form

$$I_\alpha = \int d\alpha \frac{e^{-i\alpha}}{\alpha^{P+1}}. \quad (1.170)$$

This can be evaluated by contour integration, making use of the integral representation of the Γ -function,²¹ giving

$$I_\alpha = 2\pi i^{3P-1} / \Gamma(P+1). \quad (1.171)$$

The general result (1.165) is then obtained for $J_{NN'}$.

Now let us turn to the problem of evaluating the general phase-space integrals (1.151), (1.153), and (1.155) containing both the energy- and momentum-conservation δ -functions. First we consider the NR integral (1.151) and the relationship (identity) to J_{N-1} , alluded to earlier. The identity can be established easily and directly for the case $N = 2$ and 3 , but it is of interest to prove it for general N . Basically, the identity follows as a result of the relation (1.149) involving the dimensionless mass parameters. We consider the indices $j = 1$ to N , labeling the particles, as designating an N -dimensional space. If \hat{e}_j is a unit vector along the j th axis in this space, the axes are taken as orthogonal such that $\hat{e}_j \cdot \hat{e}_k = \delta_{jk}$. Because of the relation (1.149), the collection of parameters

$$\hat{v} = (v_1, v_2, \dots, v_N) \quad (1.172)$$

can be described as a vector of unit length in the space:

$$\hat{v} \cdot \hat{v} = \sum_{j=1}^N v_j^2 = 1. \quad (1.173)$$

The integral (1.151) can then be written

$$I_N = \int d^3 \hat{x} \delta(\hat{x}^2 - 1) \delta^{(3)}(\hat{v} \cdot \hat{x}), \quad (1.174)$$

where

$$\hat{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N), \quad (1.175)$$

²¹See Whittaker and Watson (loc. cit.), p. 245.

and

$$d^3 \hat{\mathbf{x}} = dx_1 \cdots dx_N dy_1 \cdots dy_N dz_1 \cdots dz_N. \quad (1.176)$$

The terms

$$\hat{\mathbf{x}}^2 = x_1^2 + \cdots + x_N^2 + y_1^2 + \cdots + y_N^2 + z_1^2 + \cdots + z_N^2 \quad (1.177)$$

and

$$\hat{\mathbf{v}} \cdot \hat{\mathbf{x}} = \sum_{j=1}^N v_j x_j \quad (1.178)$$

are invariant under spatial rotation²² of the axes of the space, as would be the volume (1.176). If the result is invariant to an orientation of these axes, it is convenient to take an orientation such that, say, $\hat{\mathbf{v}} = (0, 0, \dots, 1)$, corresponding, mathematically, to the case where all masses except m_N are zero. The δ -function $\delta^{(3)}$ is then simply $\delta^{(3)}(\mathbf{x}_N)$, which can be eliminated by integration over $d^3 \mathbf{x}_N$. The resulting integral is then just J_{N-1} , and we have

$$I_N = J_{N-1}. \quad (1.179)$$

A more conventional method of proving the identity (1.179) and of evaluating other general phase-space integrals employs the integral form (1.166) for both the energy- and momentum-conservation δ -functions. That is, for the latter in Equation (1.151),

$$\delta^{(3)}\left(\sum_{j=1}^N v_j \mathbf{x}_j\right) = (2\pi)^{-3} \int d^3 \boldsymbol{\xi} \exp\left(i \sum_{j=1}^N v_j \mathbf{x}_j \cdot \boldsymbol{\xi}\right). \quad (1.180)$$

The individual $d^3 \mathbf{x}_j$ integrations of the exponentials from the δ and $\delta^{(3)}$ functions are now [see Equation (1.168)]

$$I_j = 2\pi \int_0^\infty x^2 dx \int_0^\pi \sin \theta d\theta \exp[i(\alpha x^2 + \xi v_j x \cos \theta)]. \quad (1.181)$$

After convenient changes of variables, the integral (1.181) is evaluated; it is found, for example, that the v_j dependence is

$$I_j \propto \exp[-i(\xi^2/4\alpha)v_j^2]. \quad (1.182)$$

The independence of the integral I_N on the distribution of v_j is then seen when the product of the I_j is taken:

$$\prod_j I_j \propto \exp\left[-i(\xi^2/4\alpha) \sum_{j=1}^N v_j^2\right] = \exp[-i(\xi^2/4\alpha)]. \quad (1.183)$$

This proves directly the independence of the orientation of the vector (1.172), allowing the simplified derivation of the theorem (1.179).

The evaluation of the integral (1.153) for massless particles proceeds in the above manner by taking integral representations for the two δ -functions. Integration over $d^3 \boldsymbol{\xi}$ and $d\alpha$ is carried out by contour integration. The result is²³

$$I_{N'} = \left(\frac{\pi}{2}\right)^{N'-1} \frac{(4N' - 4)!}{(2N' - 1)!(2N' - 2)!(3N' - 4)!}. \quad (1.184)$$

²²The \hat{e}_j would transform as $\hat{e}'_k = \gamma_{kj} \hat{e}_j$, a unitary transformation that can be considered as a way of relabeling the particles.

²³Note the correction to the formula given by J. V. Lepore and R. N. Stuart, *Phys. Rev.* **94**, 1724 (1954), also noted by R. H. Milburn, *Rev. Mod. Phys.* **27**, 1 (1955).

For the derivation of the result for the integral (1.155), we can employ the tedious method involving integral representations of the δ -functions, or we can employ the same trick that led to the establishment of the theorem (1.179). The latter method is much simpler and leads directly to the result

$$I_{NN'} = J_{N-1, N'}. \quad (1.185)$$

That is, the $I_{NN'}$ formula can be evaluated directly from the expression (1.165) with N replaced by $N - 1$.

Finally, concerning general formulas, we might ask about the case where the motion is in a space of arbitrary dimensions $n \neq 3$. For example, reactions and phase space-effects on a solid surface ($n = 2$) could be of interest. If, for example,

$$\Psi_n^{(N)} = \int \cdots \int \prod_j (dp_1 \cdots dp_n)_j \delta\left(\sum_{j=1}^N E_j - E\right), \quad (1.186)$$

with

$$E_j = (p_1^2 + \cdots + p_n^2)_j / 2m_j, \quad (1.187)$$

we have

$$\Psi_n^{(N)} = 2^{nN/2} \left(\prod_{j=1}^N m_j\right)^{n/2} E^{nN/2-1} J_{N(n)}, \quad (1.188)$$

where

$$J_{N(n)} = \int \cdots \int \prod_{j=1}^N (dx_1 \cdots dx_n)_j \delta\left(\sum_{j=1}^N (x_1^2 + \cdots + x_n^2)_j - 1\right). \quad (1.189)$$

We obtain

$$J_{N(n)} = \frac{\pi^{nN/2}}{\Gamma(nN/2)}. \quad (1.190)$$

1.5.5 One-Particle Distributions

Sometimes the angular or energy distribution of one particle (or one type of particle) is of interest for a process involving a number of outgoing particles. Although the interaction H'_{f_0} can play some role in this, it is principally the phase-space factor that determines the distribution. For some simple cases the distributions can be obtained from the corresponding explicit formulas already introduced. For example, if we have two outgoing particles with no momentum-conservation restriction (third heavy particle involved), the formula (1.128) can be employed, integrating over $d^3 \mathbf{p}_2 = (4\pi/c^2) p_2 E_2 dE_2$:

$$d\Psi^{(2)}/d^3 \mathbf{p}_1 = (4\pi/c^2) p_2 E_2 \quad (1.191)$$

or

$$d\Psi^{(2)}/dp_1 = (4\pi/c)^2 p_1^2 p_2 E_2, \quad (1.192)$$

with

$$E_2 = E - E_1, \quad p_2 c = (E_2^2 - m_2^2 c^4)^{1/2}. \quad (1.193)$$

Depending on the nature of the particles involved, the distribution has different forms:

$$d\Psi^{(2)}/dp_1 = 16\pi^2(m_2^{3/2}/m_1^{1/2})p_1^2(2m_1E - p_1^2)^{1/2} \quad (\text{NR}), \quad (1.194)$$

$$d\Psi^{(2)}/dE_1 = (4\pi/c^3)^2 E_1^2 (E - E_1)^2 \quad (\text{ER}). \quad (1.195)$$

Some general formulas can be obtained easily. Without momentum conservation, from

$$\Psi^{(N+1)} = \int d^3\mathbf{p}_1 \int \cdots \int \prod_{j=2}^{N+1} d^3\mathbf{p}_j \delta\left(\sum_{j=2}^{N+1} E_j + E_1 - E\right), \quad (1.196)$$

we have

$$d\Psi^{(N+1)}/d^3\mathbf{p}_1 = \Psi^{(N)}(E \rightarrow E - E_1). \quad (1.197)$$

In other words, the one-particle distribution is determined by the phase-space volume available to the *other* particles but with the total energy available being reduced to $E - E_1$. The formula (1.197) is very general²⁴ and not restricted to NR or ER particles. Another general result is

$$d\Phi^{(N,N'+1)}/d^3\mathbf{p}_1 = \Phi^{(NN')}(E \rightarrow E - E_1), \quad (1.198)$$

when \mathbf{p}_1 refers to a massless particle; as we have seen, momentum is carried by the NR particles in this case.

When \mathbf{p}_1 is an NR particle and momentum conservation is applied, we must know the expression for Φ for arbitrary total momentum \mathbf{P} , that is, not just for the c.m. frame. The general formula for $\Phi^{(N)}$ for all NR particles is

$$\Phi^{(N)}(E, \mathbf{P}) = \int \cdots \int \prod_{j=1}^N d^3\mathbf{p}_j \delta^{(3)}\left(\sum_{j=1}^N \mathbf{p}_j - \mathbf{P}\right) \delta\left(\sum_{j=1}^N E_j - E\right). \quad (1.199)$$

Now we can transform the integration to c.m. frame variables (\mathbf{p}'_j):

$$\mathbf{p}_j = \mathbf{p}'_j + (m_j/M)\mathbf{P}, \quad (1.200)$$

$$\sum_j E_j = \sum_j \mathbf{p}'_j{}^2/2m_j + \sum_j E'_j + \mathbf{P}^2/2M. \quad (1.201)$$

Thus, we see that

$$\Phi^{(N)}(E, \mathbf{P}) = \Phi^{(N)}(E - \mathbf{P}^2/2M, 0). \quad (1.202)$$

Similarly,

$$\Phi^{(NN')}(E, \mathbf{P}) = \Phi^{(NN')}(E - \mathbf{P}^2/2M, 0). \quad (1.203)$$

²⁴The formula yields, in statistical mechanics, for example, the Maxwellian distribution for $N \gg 1$ and more general such distributions for arbitrary N . The Maxwellian result follows immediately from Equations (1.141) and (1.197) if we set $E = (3N/2)kT$ and make use of the limit $(1-x)^{1/x} \rightarrow 1/e$ for $x \rightarrow 0$.

We then find, in the c.m. frame,

$$\begin{aligned} d\Phi^{(N+1)}/d^3\mathbf{p}_1 &= \int \cdots \int \prod_{j=2}^{N+1} d^3\mathbf{p}_j \\ &\times \delta^{(3)}\left(\sum_{j=2}^{N+1} \mathbf{p}_j + \mathbf{p}_1\right) \delta\left(\sum_{j=2}^{N+1} E_j + E_1 - E\right) \\ &= \Phi^{(N)}(E - E_1, -\mathbf{p}) = \Phi^{(N)}(E - E_r, 0), \end{aligned} \quad (1.204)$$

where the reduced energy is

$$E_r = E_1 - \mathbf{p}_1^2/2M = \mathbf{p}_1^2/2\mu, \quad (1.205)$$

with

$$\mu = m_1 M / (m_1 + M), \quad M = \sum_{j=2}^{N+1} m_j. \quad (1.206)$$

1.5.6 Invariant Phase Space

We have introduced the invariant phase-space factor $\phi^{(N)}$ in Equation (1.116). In modern high-energy physics, it is this quantity that is employed much more extensively than the non-covariant expression $\Phi^{(N)}$; we have examined the latter in more detail in this section because of the relative lack of emphasis it has received in the past few decades. There is no doubt that, in relativistic formulations valid for arbitrary energy, the invariant $\phi^{(N)}$ is more appropriate and convenient. However, it is not possible to derive explicit expressions for arbitrary N as we have done in the NR and ER limits for $\Phi^{(N)}$ and $\Psi^{(N)}$. Whether $\phi^{(N)}$ or $\Phi^{(N)}$ is inherently more fundamental and useful would depend on, for example, the formulation and results for the matrix element H'_{f0} . Sometimes there are reasons for expressing this quantity in terms of certain kinematic invariants,²⁵ since we expect that these are “natural” quantities classifying the kinematics of a process; the invariant $\phi^{(N)}$ is then clearly more appropriate in measuring the phase space for outgoing particles. On the other hand, the non-covariant $\Phi^{(N)}$ is directly proportional to the number of final states (per unit energy) and, in this sense, it is more fundamental than $\phi^{(N)}$. We shall be very brief here,²⁶ working out the simplest case ($N = 2$); we now also set $c = 1$, although in final formulas it is clear where factors of c must be inserted.

The two-particle invariant phase-space factor is, when computed in the c.m. frame,

$$\begin{aligned} \phi^{(2)} &= \iint (d^3\mathbf{p}_1/2E_1) d^3\mathbf{p}_2/2E_2 \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2) \delta(E_1 + E_2 - E) \\ &= \int d^3\mathbf{p}_1 [4E_1 E_2(p_1)]^{-1} \delta[E_1 + E_2(p_1) - E], \end{aligned} \quad (1.207)$$

where

$$E_1 = (p_1^2 + m_1^2)^{1/2}, \quad E_2(p_1) = (p_1^2 + m_2^2)^{1/2}. \quad (1.208)$$

²⁵See Section 3.6.3.

²⁶For a more complete discussion of invariant phase space, including applications and use of “Dalitz plots,” see the article by P. Nyborg and O. Skjeggstad in *Kinematics and Multiparticle Systems*, M. Nikolic, Ed., New York: Gordon and Breach, 1968.

The evaluation of the integral (1.207) yields the extremely simple result

$$\phi^{(2)} = \pi p_1(E)/E \quad (1.209)$$

where $p_1(E)$ is the solution of the equation $E_1 + E_2(p_1) = E$, or

$$p_1(E) = [(E^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2]^{1/2} / 2E. \quad (1.210)$$

The phase-space factor is then determined by the total c.m. energy E and the masses:

$$\phi^{(2)} = \phi^{(2)}(E; m_1, m_2). \quad (1.211)$$

Since $\phi^{(2)}$ is an invariant, it is then also equal to the same value in another frame where the total momentum is $\mathbf{P}' \neq 0$ and the energy is E' , with these quantities related to E by

$$E^2 = E'^2 - \mathbf{P}'^2. \quad (1.212)$$

The result can also be expressed explicitly in terms of Lorentz-invariant quantities.²⁷ The only kinematic invariant involved is $s = -E^2$ so that $\phi^{(2)} = \phi^{(2)}(s)$.

Finally, let us compare the invariant phase-space factor with the corresponding non-invariant expression (1.138); the two are related by

$$\Phi^{(2)} = 4E_1 E_2(p_1) \phi^{(2)}, \quad (1.213)$$

where

$$4E_1 E_2(p_1) = [E^4 - (m_2^2 - m_1^2)^2] / E^2. \quad (1.214)$$

The non-invariant expression is more complicated, precisely by the factor (1.214). Further, it should be noted that the formula (1.213) relates only the $\phi^{(2)}$ and $\Phi^{(2)}$ in the c.m. frame. On the other hand, the invariant $\phi^{(2)}$ is easily evaluated for any frame with arbitrary E' and \mathbf{P}' because of the simple relation (1.212).

²⁷See Section 3.6.3.

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