Chapter One

Basic Definitions

Before we can give the definition of a quadrangular algebra (in 1.17 below), we need to review a few standard notions.

Definition 1.1. A *quadratic space* is a triple (K, L, q), where K is a (commutative) field, L is a vector space over K and q is a quadratic form on L, that is, a map from L to K such that

(i) $q(u+v) = q(u) + q(v) + f(u,v)$ and
(ii) $q(tu) = t^2 q(u)$

for all $u, v \in L$ and all $t \in K$, where f is a bilinear form on L (i.e. a symmetric bilinear map from $L \times L$ to K). A quadratic space (K, L, q) is called *anisotropic* if

$$q(u) = 0 \text{ if and only if } u = 0.$$

A *basepoint* of a quadratic space (K, L, q) is an element 1 of L such that

$$q(1) = 1.$$

A *pointed quadratic space* is a quadratic space with a distinguished basepoint. Suppose that $(K, L, q, 1)$ is a pointed quadratic space (with basepoint 1). The *standard involution* of $(K, L, q, 1)$ is the map σ from L to itself given by

$$u^\sigma = f(1,u)1 - u \tag{1.2}$$

for all $u \in L$, where f is as in (i) above. We set

$$v^{-1} = v^\sigma / q(v) \tag{1.3}$$

for all v such that $q(v) \neq 0$. We will always identify K with its image under the map $t \mapsto t \cdot 1$ from K to L.

Note that if q, f and σ are as in 1.1 then $\sigma^2 = 1$ and $q(u^\sigma) = q(u)$ for all $u \in L$. It follows that

$$f(u^\sigma, v^\sigma) = f(u,v) \tag{1.4}$$

for all $u, v \in L$ as well as

$$q(v^{-1}) = q(v)^{-1} \tag{1.5}$$

and

$$(v^{-1})^{-1} = v \tag{1.6}$$

for all $v \in L^*$.

Definition 1.7. Let L be an arbitrary ring. An *involution* of L is an anti-automorphism σ of L such that $\sigma^2 = 1$. For each involution σ of L, we set

$$L_\sigma = \{u + u^\sigma \mid u \in L\}.$$

The elements of L_σ are called *traces* (with respect to σ).

Note that in 1.2 we have called σ the standard involution of the pointed quadratic space $(K, L, q, 1)$ even though it is not, according to 1.7, really an involution (since there is no multiplication on L). The next two definitions will make it clear why we have done this.

Definition 1.8. Let E/K be a separable quadratic field extension, let N denote its norm and let σ denote the non-trivial element of $\mathrm{Gal}(E/K)$, so $N(u) = uu^\sigma$ for all $u \in E$. Let $\alpha \in K^*$ and let $M(2, E)$ denote the K-algebra of 2×2 matrices over E. The *quaternion algebra* $(E/K, \alpha)$ is the subalgebra

$$\left\{ \begin{bmatrix} a & \alpha b^\sigma \\ b & a^\sigma \end{bmatrix} \mid a, b \in E \right\}$$

of $M(2, E)$. Let $L = (E/K, \alpha)$.

We identify E (and thus also $K \subset E$) with its image in L under the map

$$u \mapsto \begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix}$$

and set

$$e = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}.$$

Then every element of L can be written uniquely in the form $u+ev$ with $u, v \in E$ and multiplication in L is determined by associativity, the distributive laws and the identities $e^2 = \alpha$ and $ue = eu^\sigma$ for all $u \in E$. The extension of $\sigma \in \mathrm{Gal}(E/K)$ to the map from L to itself (which we also denote by σ) given by

$$(u + ev)^\sigma = u^\sigma - ev \tag{1.9}$$

for all $u, v \in E$ is an involution of L (as defined in 1.7) called the *standard involution* of L. Both the center $Z(L)$ of L and the set of traces L_σ (as defined in 1.7) equal K and $ww^\sigma \in K$ for all $w \in L$. The extension of the norm N to a map from L to K (which we also denote by N) given by

$$N(w) = ww^\sigma$$

for all $w \in L$ is a quadratic form on L (as a vector space over K) called the *reduced norm* of L. An element w of L is invertible if and only if $N(w) \neq 0$, in which case

$$w^{-1} = w^\sigma / N(w). \tag{1.10}$$

In particular, L is a division algebra if and only if the reduced norm N is anisotropic (as defined in 1.1). Since

$$N(u + ev) = N(u) - \alpha N(v) \tag{1.11}$$

for all $u, v \in E$, it follows that L is a division algebra (i.e. a skew field) if and only if $\alpha \notin N(E)$.

Definition 1.12. Let L be a skew field, let σ be an involution of L and let $K = L_\sigma$ (as defined in 1.7). We will say that the pair (L, σ) is *quadratic* if either L is commutative and $\sigma \neq 1$ (in which case K is a subfield of L, L/K is a separable quadratic extension and σ is the unique non-trivial element in $\mathrm{Gal}(L/K)$[1]) or L is quaternion and σ is its standard involution as defined in 1.9 (in which case $K = Z(L)$). Suppose that (L, σ) is quadratic and let

$$q(u) = uu^\sigma \tag{1.13}$$

for all $u \in L$ (so q is the norm of the extension L/K if L is commutative and q is the reduced norm of L if L is quaternion). Then $(K, L, q, 1)$ is a pointed anisotropic quadratic space and

$$f(u, v) = uv^\sigma + vu^\sigma \tag{1.14}$$

for all $u, v \in L$, where f is as in 1.1.i. By 1.2, it follows that σ is the standard involution of $(K, L, q, 1)$ and by 1.10, the element u^{-1} defined in 1.3 is, in fact, the inverse of u in the skew field L for all $u \in L^*$.

The following classical result is attributed (but only in characteristic different from two) in [5] (page 187) to J. Dieudonné. The general case can be found in Theorem 2.1.10 of [4]. We mention this result only to indicate that it is well known that among skew fields with involution, those singled out in 1.12 are exceptional.

Theorem 1.15. *Let L be a skew field with a non-trivial involution σ and let L_σ be as in 1.7. Then either L is generated by L_σ as a ring or the pair (L, σ) is quadratic as defined in 1.12.*

Definition 1.16. A (skew-hermitian) *pseudo-quadratic space* is a set

$$(L, \sigma, X, h, \pi),$$

where L is a skew field, σ is an involution of L (as defined in 1.7), X is a right vector space over L, h is a skew-hermitian form on X (i.e. h is a bi-additive map from $X \times X$ to L such that

(i) $h(a, bu) = h(a, b)u$ and
(ii) $h(a, b)^\sigma = -h(b, a)$

for all $a, b \in X$ and all $u \in L$) and π is a map from X to L such that

(iii) $\pi(a + b) \equiv \pi(a) + \pi(b) + h(a, b) \pmod{L_\sigma}$ and
(iv) $\pi(au) \equiv u^\sigma \pi(a) u \pmod{L_\sigma}$

for all $a, b \in X$ and all $u \in L$, where L_σ is as in 1.7. A pseudo-quadratic space

$$(L, \sigma, X, h, \pi)$$

is called *anisotropic* if

[1] Let $F = \{u \in L \mid u^\sigma = u\}$. Then F is a subfield of L and L/F is a separable quadratic extension. Since the map $x \mapsto x + x^\sigma$ from L to F is linear over F and not the 0-map, it is onto. Thus $K = F$.

(v) $\pi(a) \equiv 0 \pmod{L_\sigma}$ only if $a = 0$.

and *standard* if

(iv′) $\pi(au) = u^\sigma \pi(a) u$

for all $a \in X$ and all $u \in L$.

Let

$$\Pi = (L, \sigma, X, h, \pi)$$

be an arbitrary pseudo-quadratic space. By (11.28) and (11.31) of [12], there exists a map $\hat{\pi}$ from X to L such that

(i) $\hat{\pi}(a) \equiv \pi(a) \pmod{L_\sigma}$ for all $a \in X$ (so

$$\hat{\Pi} = (L, \sigma, X, h, \hat{\pi})$$

is also a pseudo-quadratic space which is, in an obvious sense, equivalent to Π) and

(ii) $\hat{\Pi}$ is standard (as defined in 1.16.iv′).

Here now is the main definition of this monograph.

Definition 1.17. A *quadrangular algebra* is a set

$$(K, L, q, 1, X, \cdot, h, \theta),$$

where $(K, L, q, 1)$ is a pointed anisotropic quadratic space as defined in 1.1, X is a non-trivial vector space over K, $(a, v) \mapsto a \cdot v$ is a map from $X \times L$ to X (which is denoted below, and, in general, simply by juxtaposition), h is a map from $X \times X$ to L and θ a map from $X \times L$ to L satisfying the following twelve axioms (in which f is as in 1.1.i, σ is as in 1.2 and v^{-1} is as in 1.3):

(A1) The map $\cdot$ is bilinear (over K).
(A2) $a \cdot 1 = a$ for all $a \in X$.
(A3) $(av)v^{-1} = a$ for all $a \in X$ and all $v \in L^*$.

(B1) h is bilinear (over K).
(B2) $h(a, bv) = h(b, av) + f(h(a, b), 1)v$ for all $a, b \in X$ and all $v \in L$.
(B3) $f(h(av, b), 1) = f(h(a, b), v)$ for all $a, b \in X$ and all $v \in L$.

(C1) For each $a \in X$, the map $v \mapsto \theta(a, v)$ is linear (over K).
(C2) $\theta(ta, v) = t^2\theta(a, v)$ for all $t \in K$, all $a \in X$ and all $v \in L$.
(C3) There exists a function g from $X \times X$ to K such that

$$\theta(a + b, v) = \theta(a, v) + \theta(b, v) + h(a, bv) - g(a, b)v$$

for all $a, b \in X$ and all $v \in L$.

(C4) There exists a function ϕ from $X \times L$ to K such that

$$\begin{aligned}\theta(av, w) = \theta(a, w^\sigma)^\sigma q(v) - f(w, v^\sigma)\theta(a, v)^\sigma \\ + f(\theta(a, v), w^\sigma)v^\sigma + \phi(a, v)w\end{aligned}$$

for all $a \in X$ and $v, w \in L$.

(D1) Let $\pi(a) = \theta(a, 1)$ for all $a \in X$. Then
$$a\theta(a, v) = (a\pi(a))v$$
for all $a \in X$ and all $v \in L$ and

(D2) $\pi(a) \equiv 0 \pmod{K}$ if and only if $a = 0$ (where K has been identified with its image under the map $t \mapsto t \cdot 1$ from K to L).

As was indicated in the introduction, the definition of a quadrangular algebra is derived from the definition of an anisotropic pseudo-quadratic space defined over a skew field with involution (L, σ) which is quadratic as defined in 1.12. Roughly speaking, the axioms A1–D2 capture the structure which remains when the multiplication on L is discarded. We make these comments more precise with the following result:

Proposition 1.18. *Let*
$$\Pi = (L, \sigma, X, h, \pi)$$
be a standard anisotropic pseudo-quadratic space as defined in 1.16, suppose that the pair (L, σ) is quadratic as defined in 1.12 and let $K = L_\sigma$. Let $\cdot$ denote the scalar multiplication from $X \times L$ to X (so X is a vector space over K with respect to the restriction of $\cdot$ to $X \times K$), let q be as in 1.12 (i.e. q is the norm of L/K if L is commutative and q is the reduced norm of L if L is quaternion) and let
$$\theta(a, u) = \pi(a)u$$
for all $a \in X$ and all $u \in L$. Then
$$(K, L, q, 1, X, \cdot, h, \theta)$$
is a quadrangular algebra with ϕ identically zero (where ϕ is as in axiom C4).

Proof. As observed in 1.12, $(K, L, q, 1)$ is an anisotropic pointed quadratic space and σ is its standard involution (as defined in 1.1). Axioms A1–A3 hold since X is a right vector space over L, $K \subset Z(L)$ and $v^{-1} = v^\sigma/q(v)$ in the inverse of v in the skew field L for all $v \in L^*$.

Let f denote the bilinear form associated with q. Since $q(u^\sigma) = q(u)$ for all $u \in L$, we have $q(u) = uu^\sigma = u^\sigma u$ and thus (by 1.1.i)
$$f(u, v) = u^\sigma v + v^\sigma u = uv^\sigma + vu^\sigma \tag{1.19}$$
for all $u, v \in L$. By 1.16.i and 1.16.ii (and 1.7), we have
$$\begin{aligned} h(av, b) = -h(b, av)^\sigma &= -(h(b, a)v)^\sigma \\ &= -v^\sigma h(b, a)^\sigma = v^\sigma h(a, b) \end{aligned} \tag{1.20}$$
for all $a, b \in X$ and all $v \in L$. Thus B1 holds by 1.16.i, 1.20 (since σ acts trivially on $K \subset L$) and the assumption in 1.16 that h is bi-additive. Moreover,
$$\begin{aligned} h(a, bv) - h(b, av) &= (h(a, b) - h(b, a))v && \text{by 1.16.i} \\ &= (h(a, b) + h(a, b)^\sigma)v && \text{by 1.16.ii} \\ &= f(h(a, b), 1)v && \text{by 1.19} \end{aligned}$$

and

$$\begin{aligned} f(h(av,b),1) &= h(av,b) + h(av,b)^\sigma && \text{by 1.19} \\ &= v^\sigma h(a,b) + (v^\sigma h(a,b))^\sigma && \text{by 1.20} \\ &= v^\sigma h(a,b) + h(a,b)^\sigma v && \text{by 1.7} \\ &= f(h(a,b),v) && \text{by 1.19} \end{aligned}$$

for all $a, b \in X$ and all $v \in L$. Thus axioms B2 and B3 hold. Axioms C1 and D1 are obvious. Axiom C2 holds by 1.16.iv$'$ (and the assumption that Π is standard) since $K \subset Z(L)$ and σ acts trivially on K. Axiom C3 holds by 1.16.iii and axiom D2 holds by 1.16.v. It remains only to prove that C4 holds. Choose $a \in X$ and $v, w \in L$. We first observe that

$$\begin{aligned} f(\theta(a,w^\sigma),v) &= f(\pi(a)w^\sigma, v) \\ &= \pi(a)w^\sigma v^\sigma + vw\pi(a)^\sigma \end{aligned}$$

and

$$\begin{aligned} f(\theta(a,v),w^\sigma) &= f(\pi(a)v, w^\sigma) \\ &= \pi(a)vw + w^\sigma v^\sigma \pi(a)^\sigma \end{aligned}$$

by 1.7 and 1.19. Thus

$$\begin{aligned} &f(\theta(a,w^\sigma),v) + f(\theta(a,v),w^\sigma) \\ &\qquad = \pi(a)(w^\sigma v^\sigma + vw) + (w^\sigma v^\sigma + vw)\pi(a)^\sigma. \end{aligned}$$

Since

$$w^\sigma v^\sigma + vw \in L_\sigma \subset Z(L),$$

it follows (by 1.19) that

$$\begin{aligned} f(\theta(a,w^\sigma),v) + f(\theta(a,v),w^\sigma) &= (\pi(a) + \pi(a)^\sigma)(w^\sigma v^\sigma + vw) \\ &= f(\pi(a),1) \cdot f(v,w^\sigma). \end{aligned} \tag{1.21}$$

Therefore

$$\begin{aligned} \theta(av,w) &= \pi(av)w \\ &= v^\sigma \pi(a)vw && \text{by 1.16.iv}' \\ &= -v^\sigma \pi(a)w^\sigma v^\sigma + v^\sigma \pi(a) f(v^\sigma, w) && \text{by 1.19} \\ &= -v^\sigma \theta(a,w^\sigma)v^\sigma + v^\sigma \pi(a) f(v^\sigma, w) \\ &= v^\sigma v \theta(a,w^\sigma)^\sigma - f(\theta(a,w^\sigma),v)v^\sigma \\ &\qquad + v^\sigma \pi(a) f(v^\sigma, w) && \text{by 1.19} \\ &= \theta(a,w^\sigma)^\sigma q(v) + f(\theta(a,v),w^\sigma)v^\sigma \\ &\qquad - f(\pi(a),1) f(v,w^\sigma)v^\sigma \\ &\qquad\quad + f(v^\sigma,w)v^\sigma \pi(a) && \text{by 1.13 and 1.21} \\ &= \theta(a,w^\sigma)^\sigma q(v) + f(\theta(a,v),w^\sigma)v^\sigma \\ &\qquad - f(v^\sigma,w)v^\sigma \pi(a)^\sigma && \text{by 1.2 and 1.4} \\ &= \theta(a,w^\sigma)^\sigma q(v) - f(w,v^\sigma)\theta(a,v)^\sigma \\ &\qquad + f(\theta(a,v),w^\sigma)v^\sigma && \text{by 1.7.} \end{aligned}$$

Thus C4 holds with ϕ identically zero. □

In Chapter 5 we will characterize the quadrangular algebras which arise from anisotropic pseudo-quadratic spaces as in 1.18; see 1.28 and 3.1.

Here are some further comments about the definition 1.17 of a quadrangular algebra:

(i) We will, in general, refer to the axioms in 1.17 by their names—A1, A2, etc.—without referring explicitly to 1.17.

(ii) We have put the axioms of a quadrangular algebra into four groups in the hope of making them easier to recall. Axioms A1–A3 establish properties of the "scalar multiplication" $(a, v) \mapsto av$ (those properties which remain after the multiplication on L is taken away). Axioms B1–B3 establish properties of the "skew-hermitian form" h. Axioms C1–C4 and D1–D2 establish properties of the "quasi-pseudo-quadratic form" π and its companion θ. Axioms D1 and D2 are somehow different from the others and have been listed separately. The "polarization" of axiom D1 given in 3.22 below will play an especially important role in the classification of quadrangular algebras.

(iii) Among the axioms of a quadrangular algebra, C4 is conspicuously less "natural"-looking than the others. In the proof of 1.18, however, we saw that C4 is just 1.16.iv$'$ re-written in a way that avoids the multiplication on L.

(iv) We emphasize that there is no multiplication on L in 1.17. Thus if $a \in X$ and $u, v \in L$, then auv can only mean $(au)v$, and we will almost always, in fact, omit the parentheses in such an expression. Similarly, $auvw$ can only mean $((au)v)w$, etc.

(v) We will write sometimes at (for $a \in X$ and $t \in K$) and sometimes ta to denote the scalar multiple of a by t. Similarly, we will write sometimes ut and sometimes tu to denote the scalar multiple of an element $u \in L$ by an element $t \in K$.

(vi) Since $\sigma^2 = 1$, axiom A3 can (with the help of A1) be rewritten to say that $auu^\sigma = au^\sigma u = aq(u)$ for all $a \in X$ and all $u \in L^*$.

(vii) Note that there are no assumptions about $\dim_K L$ or $\dim_K X$ in 1.17. In particular, these dimensions are allowed to be infinite.

(viii) The map g in C3 and the map called g in Chapter 13 of [12] (which we call, for the moment, g') are related by the formula $g(a, b) = g'(b, a)$ for all $a, b \in X$. This choice makes axiom C3 look a little nicer (but at the expense of making 11.10 below look a little less nice). The basepoint 1 is called ϵ and the vector spaces X and L are called X_0 and L_0 in [12]. We have tried to make all the other notation used in this monograph conform with that in [12].

We turn now to the question of how to define an isomorphism of quadrangular algebras. We begin with a notion we call equivalence:

Definition 1.22. Two quadrangular algebras

$$(K, L, q, 1, X, \cdot, h, \theta) \quad \text{and} \quad (\hat{K}, \hat{L}, \hat{q}, \hat{1}, \hat{X}, \hat{\cdot}, \hat{h}, \hat{\theta})$$

are *equivalent* if

$$(K, L, \ldots, \cdot) = (\hat{K}, \hat{L}, \ldots, \hat{\cdot})$$

and there exists an $\omega \in K^*$ such that

$$\hat{h}(a, b) = \omega h(a, b)$$

and

$$\hat{\theta}(a, u) \equiv \omega\theta(a, u) \pmod{\langle u \rangle}$$

for all $a, b \in X$ and all $u \in L$ (where $\langle u \rangle$ denotes the subspace spanned by u over K).

Proposition 1.23. *If $(K, \ldots, \theta)$ and $(\hat{K}, \ldots, \hat{\theta})$ are equivalent quadrangular algebras, then there exist an element $\omega \in K^*$ and a map $p\colon X \to K$ such that*

(i) $p(ta) = t^2 p(a)$ *and*
(ii) $\hat{\theta}(a, u) = \omega\theta(a, u) + p(a)u$

for all $a \in X$, all $u \in L$ and all $t \in K$.

Proof. By 1.22, there exists a map $r\colon X \times L \to K$ such that

$$\hat{\theta}(a, u) = \omega\theta(a, u) + r(a, u)u$$

for all $a \in X$ and all $u \in L$. Let $p(a) = r(a, 1)$ for all $a \in X$. By A1, A2 and D1,

$$\begin{aligned} a\big(\omega\theta(a, u) + r(a, u)u\big) &= a\hat{\theta}(a, u) \\ &= a\hat{\pi}(a)u \\ &= a\big(\omega\pi(a) + p(a) \cdot 1\big)u \\ &= \omega a\pi(a)u + aup(a) \\ &= \omega a\theta(a, u) + aup(a) \end{aligned}$$

(where $\hat{\pi}(a) = \hat{\theta}(a, 1)$ is as in D1) for all $a \in X$ and all $u \in L$. Thus $p(a)au = r(a, u)au$ for all $a \in X$ and all $u \in L$, again by A1. By A3, $au \neq 0$ and hence $p(a) = r(a, u)$ for all $a \in X^*$ and all $u \in L^*$. By C2, $\theta(0, u) = \hat{\theta}(0, u)$ for all $u \in L$. Hence $r(0, u) = p(0) = 0$ for all $u \in L^*$. Thus (ii) holds. By C2 again, it follows that (i) holds. □

Proposition 1.24. *Let $\Xi = (K, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra, let $\omega \in K^*$ and let p be a map from X to K such that $p(ta) = t^2 p(a)$ for all $a \in X$ and all $t \in K$. Then $\hat{\Xi} = (\hat{K}, \ldots, \hat{\cdot}, \hat{h}, \hat{\theta})$ is also a quadrangular algebra, where*

$$\begin{aligned} (\hat{K}, \ldots, \hat{X}, \hat{\cdot}) &= (K, \ldots, X, \cdot), \\ \hat{h}(a, b) &= \omega h(a, b), \\ \hat{\theta}(a, u) &= \omega\theta(a, u) + p(a)u \end{aligned}$$

for all $a, b \in X$ and all $u \in L$. Moreover,

$$\hat{g}(a,b) = \omega g(a,b) + p(a) + p(b) - p(a+b),$$
$$\hat{\phi}(a,u) = \omega\phi(a,u) + p(av) - p(a)q(v)$$

for all $a, b \in X$ and all $u \in L$, where g and $\hat{g}$ are as in C3 and ϕ and $\hat{\phi}$ is as in C4 (with respect to Ξ and $\hat{\Xi}$).

Proof. By 1.4, $\hat{\Xi}$ satisfies C4 with $\hat{\phi}$ as indicated. It is clear that $\hat{\Xi}$ satisfies all the other axioms (with $\hat{g}$ as indicated). □

Definition 1.25. An *isomorphism* from one pointed quadratic space

$$(K, L, q, 1)$$

to another $(\hat{K}, \hat{L}, \hat{q}, \hat{1})$ is a pair of maps (ψ_0, ψ_1) such that ψ_0 is an isomorphism from K to $\hat{K}$, ψ_1 is a ψ_0-linear[2] isomorphism from L to $\hat{L}$, $\hat{q}(\psi_1(u)) = \psi_0(q(u))$ for all $u \in L$ and $\psi_1(1) = \hat{1}$. An *isomorphism* from one quadrangular algebra $(K, \ldots, \theta)$ to another $(\hat{K}, \ldots, \hat{\theta})$ is a triple of maps (ψ_0, ψ_1, ψ_2) such that (ψ_0, ψ_1) is an isomorphism from $(K, L, q, 1)$ to $(\hat{K}, \hat{L}, \hat{q}, \hat{1})$ and ψ_2 is a ψ_0-linear isomorphism from X to $\hat{X}$ satisfying, for some $\hat{\omega} \in \hat{K}^*$,

(i) $\psi_2(a \cdot u) = \psi_2(a) \hat{\cdot} \psi_1(u)$,
(ii) $\psi_1(h(a,b)) = \hat{\omega}\hat{h}(\psi_2(a), \psi_2(b))$,
(iii) $\psi_1(\theta(a,u)) \equiv \hat{\omega}\hat{\theta}(\psi_2(a), \psi_1(u)) \pmod{\langle \psi_1(u) \rangle}$

for all $a, b \in X$ and all $u \in L$. If $K = \hat{K}$ and ψ_0 is the identity map, then we will say that the isomorphism is *K-linear* and we will write (ψ_1, ψ_2) in place of (ψ_0, ψ_1, ψ_2). Two pointed quadratic spaces (respectively, quadrangular algebras) are *isomorphic* if there is an isomorphism from one to the other.

Remark 1.26. The composition of isomorphisms of quadrangular algebras is again an isomorphism. Also the inverse of an isomorphism is an isomorphism. Note, too, that if $\Xi = (K, \ldots, \theta)$ and $\hat{\Xi} = (\hat{K}, \ldots, \hat{\theta})$ are equivalent quadrangular algebras as defined in 1.22, the triple consisting of three identity maps from K to itself, from L to itself and from X to itself is an isomorphism from Ξ to $\hat{\Xi}$.

Definition 1.27. Let $\Xi = (K, \ldots, \theta)$ be a quadrangular algebra and let f and σ be as in 1.17. Then Ξ is *proper* if $\sigma \neq 1$; Ξ is *regular* if f is nondegenerate (in which case Ξ, by 3.14 below, is automatically proper); and Ξ is *defective* if it is not regular.

In 9.2 below, we will also introduce the notion of an *improper* quadrangular algebra. (Improper is not quite simply the opposite of proper.)

Definition 1.28. A quadrangular algebra is *special* if it is isomorphic (as defined in 1.25) to a quadrangular algebra coming from a standard anisotropic pseudo-quadratic space as described in 1.18. A quadrangular algebra is *exceptional* if it is proper but not special.

[2] i.e. ψ_1 is additive and $\psi_1(tu) = \psi_0(t)\psi_1(u)$ for all $t \in K$ and all $u \in L$.

If U is a ring, a vector space or an additive group, we will denote by U^* the set of non-zero elements of U. If U is a multiplicative group, we will denote by U^* the set of non-identity elements of U.