

CHAPTER 1

Growth Model in Excel

Most economists are familiar with the spreadsheet and even with the database capabilities of the Excel software, but fewer are aware that Excel also contains powerful procedures for solving both linear and nonlinear programming problems. As the Excel interface is such a familiar one and the specification of programming problems in Excel is relatively straightforward, there are times when it is the software of choice for solving certain types of optimization problems. In particular, when the models are small enough that the set-driven nature of GAMS does not give it an advantage over Excel, it may be advantageous to solve optimization problems in Excel. To illustrate this we use a one-sector growth model of the type that abounds in the economics literature—the famous Ramsey model of economic growth. In particular, we follow the versions developed by Chakravarty (1962) and Taylor and Uhlig (1990). We employ a finite horizon version with a terminal capital stock constraint.

The model is first introduced in a mathematical form and then in a computational form.¹ The essential economics of the simple growth model used in this chapter is a trade-off between consumption and investment. More consumption in a time period means more utility in that time period but less investment and therefore less capital stock and less production in future time periods. Thus the key elements of the model are the production function with capital being used to produce output, the capital accumulation relationship with investment creating new capital, and the utility function with consumption resulting in utility.

1. Most models used in this book cannot be solved analytically, so numerical methods are required. However, even when analytical solutions can be obtained, as shown later in Appendix J, it is still useful to get numerical solutions so that the code can be checked on simple models. Then the numerical methods can be used with more confidence when they are applied to more complex models that cannot be solved analytically.

1.1. MATHEMATICAL FORM

The production side of the economy is specified in a stylized form by means of an aggregate production function

$$Y_t = \theta K_t^\alpha \quad (1)$$

where

Y_t = output in period t

θ = a technology parameter

K_t = the capital stock in period t

α = exponent of capital in the production function

This is the widely used Cobb-Douglas form of a production function except that function usually includes both capital and labor inputs. However, for the sake of simplicity, the production function in this model includes only capital.

Consider next the capital accumulation constraint

$$K_{t+1} = K_t + Y_t - C_t \quad (2)$$

where

C_t = consumption in period t

which says that the capital stock next period will be the same as this period plus the difference between output and consumption, which is saving or investment. For the moment, depreciation of the capital stock is ignored though you might want to add that to the model in an experiment.

The production function (1) can be substituted into the capital accumulation equation (2) to obtain the equation

$$K_{t+1} = K_t + \theta K_t^\alpha - C_t \quad (3)$$

In addition, the model has an initial condition that specifies the size of the capital stock in the initial period,

$$K_0 \text{ given} \quad (4)$$

and includes a terminal condition that fixes a minimum amount of capital that must be left to the next generation after the time horizon covered by the model,

$$K_N \geq K^* \quad (5)$$

where

K^* = a lower bound on the amount of capital required in the terminal period N .

Finally, the model has a criterion function that is the discounted value of the utility that is obtained from consumption over all of the periods covered by the model. It is written in two steps. First the utility in each period is defined as

$$U(C_t) = \frac{1}{(1 - \tau)} C_t^{(1-\tau)} \quad (6)$$

where

$U(C_t)$ = the utility in period t as a function of consumption in that period
 τ = a parameter in the utility function²

Then the sum of the discounted utilities is specified as

$$J = \sum_{t=0}^{N-1} \beta^t U(C_t) \quad (7)$$

where

J = the criterion value
 β = the discount factor = $1/(1 + \rho)$
 ρ = the discount rate

and the substitution of Eq. (6) into Eq. (7) yields the criterion function

$$J = \sum_{t=0}^{N-1} \beta^t \frac{1}{(1 - \tau)} C_t^{(1-\tau)} \quad (8)$$

In summary, the model consists of the criterion function (8), the capital accumulation equation (3), and the initial and terminal conditions (4) and (5), and can be stated as find $(C_0, C_1, \dots, C_{N-1})$ to maximize

$$J = \sum_{t=0}^{N-1} \beta^t \frac{1}{(1 - \tau)} C_t^{(1-\tau)} \quad [\text{Eq. (8)}]$$

subject to

$$K_{t+1} = K_t + \theta K_t^\alpha - C_t \quad [\text{Eq. (3)}]$$

$$K_0 \text{ given} \quad [\text{Eq. (4)}]$$

$$K_N \geq K^* \quad [\text{Eq. (5)}]$$

The essential problem, then, is to choose those levels of consumption over the time periods covered by the model that strike the right balance between consumption

2. This is a popular form of the utility function that is known as the “constant elasticity of intertemporal substitution” function. Roughly speaking, think of the elasticity of intertemporal substitution as measuring the degree of substitutability between consumption “today” and “tomorrow” or, in geometric terms, measuring the curvature of the indifference curves corresponding to consumption at any two points in time. For this function, the elasticity of substitution is constant and equal to $1/\tau$.

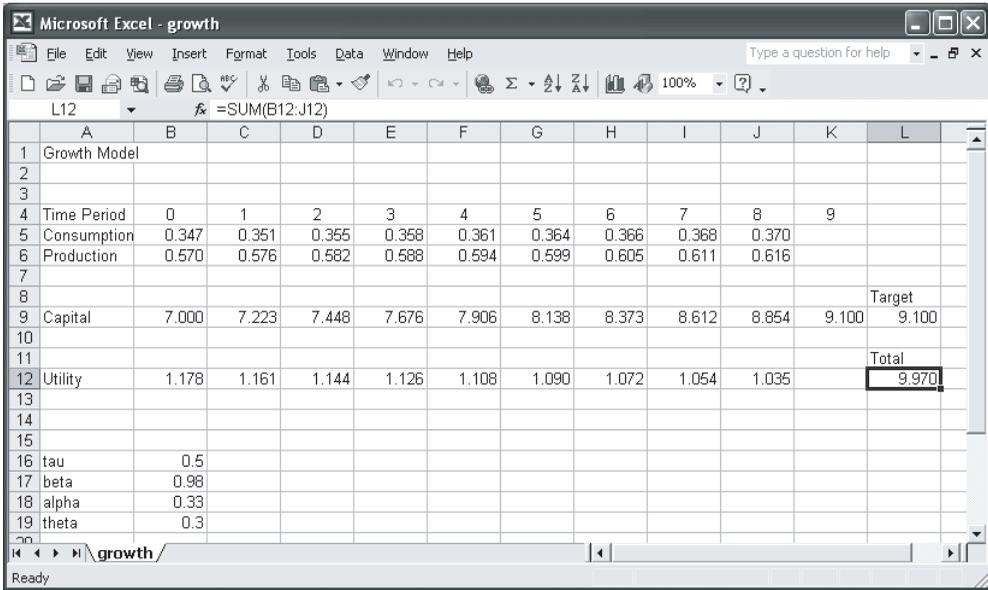


Figure 1.1. Growth model in Excel with total utility highlighted.

and investment. Lower consumption in any given period means less utility in that period but more savings and therefore larger capital stocks and more production in future years.

This growth model is a nonlinear programming problem because of the nonlinearities in the criterion function (8) and the capital accumulation equation (3). It can be stated and solved rather nicely in Excel, as is discussed in the next section.

1.2. COMPUTATIONAL FORM

Consider first a spreadsheet layout of the model as shown in Figure 1.1. The corresponding Excel file is on the book web page. Note first that the model horizon covers time periods numbered from zero through nine so that zero represents the initial period and nine the terminal period. The rows below the time periods display the consumption, C_t ; production, Y_t ; capital stock, K_t ; and utility, $U(C_t)$ in each time period. All of these values are calculated when the model is solved and we show shortly how the calculations are structured. However, for now look only at the cell below the Total label, that is, cell L12, which is highlighted in the bottom right-hand corner of the spreadsheet. It contains the value 9.97; however, we are not so much interested in that value as in how it is obtained. Look at the formula bar at the top of the spreadsheet, which contains the expression

$$\text{SUM}(B12:J12)$$

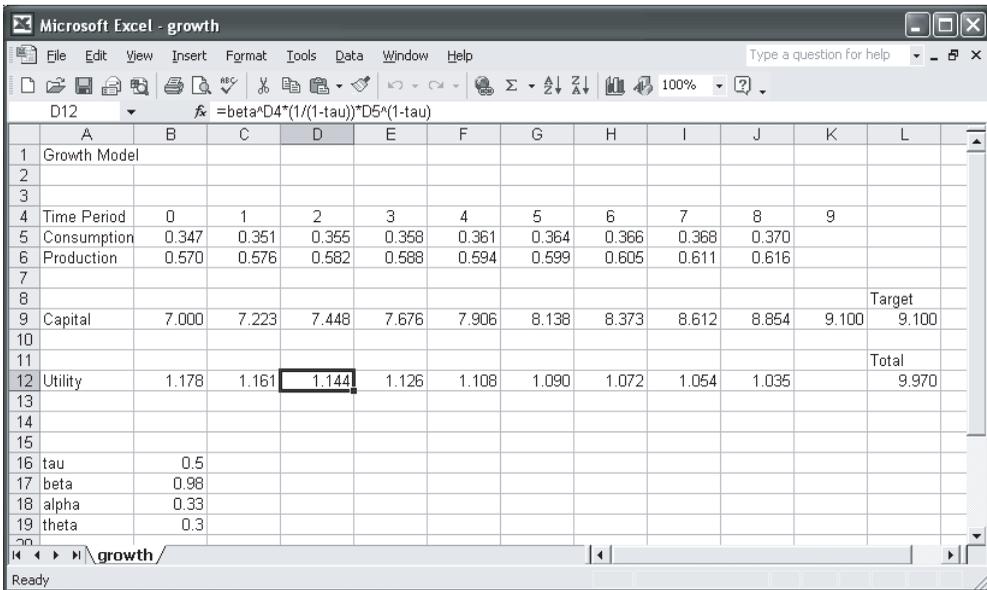


Figure 1.2. The calculation of discounted utility in each period.

This indicates that this cell contains the sum of the utility values for periods zero through eight that are contained in cells B12 through J12.

Actually, the value in each of the cells B12 through J12 is not, strictly speaking, the utility but rather the *discounted* utility for each period. This is illustrated in Figure 1.2. The cell D12 in the utility row is highlighted and the expression that is used to calculate the value in that cell is displayed in the formula bar as

$$= \text{beta}^{\text{D4}} * (1 / (1 - \text{tau})) * \text{D5}^{(1 - \text{tau})}$$

This is complicated so let us consider it one piece at a time. Begin with

$$\text{beta}^{\text{D4}}$$

which means that beta is raised to the power of the number in cell D4. This makes use of the “naming” capability for constants in Excel and is equivalent to B17^{D4} . The number in cell D4 is 2 so this term becomes β^2 , which is the discount factor squared. Beta is defined in line 17 of the spreadsheet as 0.98. Moreover, since

$$\beta = \frac{1}{1 + \rho}$$

we can infer that the discount rate ρ is equal to about 0.02.

Next consider the term

$$(1 / (1 - \text{tau})) * \text{D5}^{(1 - \text{tau})}$$

which can be rewritten as

$$\frac{1}{1 - \tau} D5^{1-\tau}$$

Since the cell D5 contains consumption we can further rewrite this expression as

$$\frac{1}{1 - \tau} C_t^{1-\tau}$$

which is the same as the utility function in Eq. (6). Thus, cell D12 contains the mathematics

$$\beta^t \frac{1}{1 - \tau} C_t^{1-\tau}$$

which is the discounted utility for period t . Also, the parameter τ of the utility function is defined in line 16 of the spreadsheet as being equal to 0.5.

In summary, line 12 of the spreadsheet is used to calculate the discounted utility in each period and then to sum those values so as to obtain the total discounted utility in cell L12. Thus the criterion function for the model is contained in line 12.

Next consider the constraints of the model. Begin with the expression for production, which is illustrated in Figure 1.3. In this figure cell D6 is highlighted and the formula bar contains the expression

$$\text{theta} * \text{D9}^{\text{alpha}}$$

which is the same as Eq. (1) for production, that is,

$$Y_t = \theta K_t^\alpha$$

since cell D9 contains the capital stock for period t ; θ is defined near the bottom of the spreadsheet in line 19 as being equal to 0.3 and α is defined in line 18 as being equal to 0.33.

Next consider the expression for the capital accumulation constraint, shown in Figure 1.4, where cell D9 is highlighted. The expression in the formula bar this time,

$$C9 + \text{theta} * C9^{\text{alpha}} - C5$$

contains at its core the expression for production that we just developed, that is,

$$\text{theta} * C9^{\text{alpha}}$$

Thus we can translate the entire expression as

$$K_{t-1} + Y_{t-1} - C_{t-1}$$

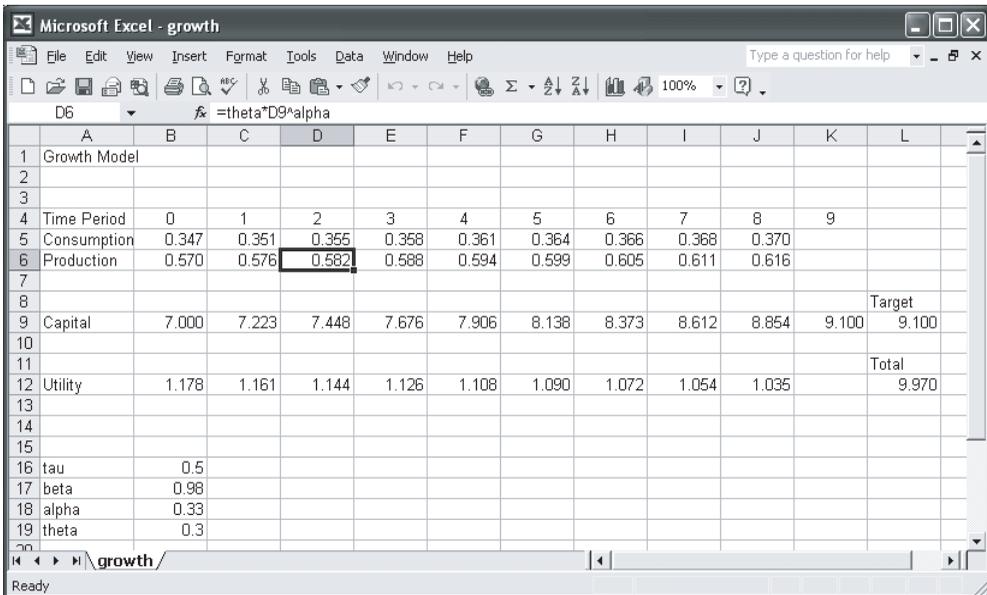


Figure 1.3. The calculation of production in each period.

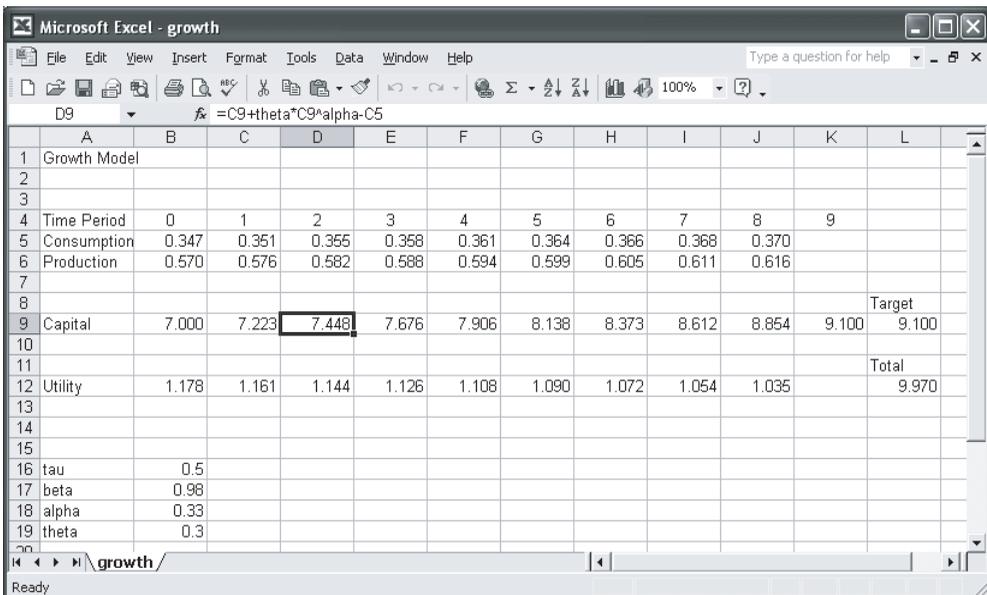


Figure 1.4. The capital accumulation constraint.

	A	B	C	D	E	F	G	H	I	J	K	L
1	Growth Model											
2												
3												
4	Time Period	0	1	2	3	4	5	6	7	8	9	
5	Consumption	0.347	0.351	0.355	0.358	0.361	0.364	0.366	0.368	0.370		
6	Production	0.570	0.576	0.582	0.588	0.594	0.599	0.605	0.611	0.616		
7												
8												
9	Capital	7.000	7.223	7.448	7.676	7.906	8.138	8.373	8.612	8.854	9.100	Target 9.100
10												
11												Total 9.970
12	Utility	1.178	1.161	1.144	1.126	1.108	1.090	1.072	1.054	1.035		
13												
14												
15												
16	tau	0.5										
17	beta	0.98										
18	alpha	0.33										
19	theta	0.3										

Figure 1.5. The initial capital stock.

since row 9 contains the capital stock figures and row 5 contains the consumption figures. As you can see by comparing the expression above to the capital accumulation constraint in Eq. (2) with the time periods each decreased by one period, that is,

$$K_t = K_{t-1} + Y_{t-1} - C_{t-1}$$

In addition, note that at the end of row 9 in the spreadsheet there is a target capital stock. We discuss this in detail when we describe how the model is actually solved in Excel. However, before we do that it is necessary to indicate how the initial condition for capital stock is specified. This is shown in Figure 1.5, where cell B9 is highlighted. When cell B9 is highlighted the fomula bar does not show a mathematical expression like those in the other cells in line 9, but rather just the number 7. This is the initial capital stock that was specified in the mathematical statement of the models in Eq. (4) as

$$K_0 \text{ given}$$

So the initial capital stock is given and it has been specified as equal to 7 in this version of the model.

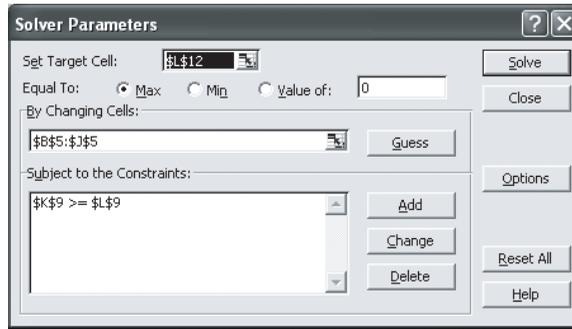


Figure 1.6. Solver dialog box.

The tour of the model in Excel we have just given is slightly confusing because it discusses both the data elements that the user must provide and the variables that are calculated when the spreadsheet is solved. Now let us separate the two by looking again at Figure 1.5. The user must supply the time period numbers in row 4, the initial capital stock in cell B9, and the parameter values τ , β , α , and θ in cells B16 through B19, as well as the terminal capital stock target in cell L9. Excel computes all the rest. Then why are all those other numbers shown in Figure 1.5? Those other numbers were all computed the last time the model was solved and are updated if you alter one of the inputs just mentioned and then solve the model again.

Let us consider next how the model is solved. This is accomplished by selecting the **Tools** menu and the **Solver** option from that menu. When you do this the dialog box in Figure 1.6 appears.³ Consider first the top line in this box in the section called **Set Target Cell**. The **Edit** box to the right of this caption indicates that cell L12 has been chosen, and this corresponds to the total discounted utility on the right-hand side of the utility line in the spreadsheet. Just beneath this the user can specify whether the value in the cell is to be maximized or minimized. In the growth model at hand we are seeking to maximize the total discounted utility so **Max** is selected.

The next line is used to specify cells to be changed while searching for the solution to the model. In the growth model we are solving for the values of consumption in each period that provide the best trade-off between utility in that period and saving that becomes future capital stocks and permits more production later. Therefore, we specify here that the variables to be used in searching for this optimum are those in cells B5 to J5, which are the consumption values.

Next consider the box labeled **Subject to the Constraints**, in which appears the constraint

$$K9 \geq L9$$

3. In case the dialog box does not appear, see Appendix C.

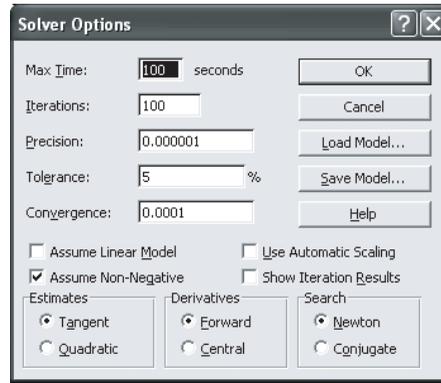


Figure 1.7. Solver options dialog box.

As cell K_9 contains the capital stock for period 9 and cell L_9 contains the target capital stock, this constraint requires that the terminal period capital stock computed by the model be greater than or equal to the user-specified target, which in this case is set to 9.1, that is, 30 percent higher than the initial capital stock. This corresponds to the mathematical constraint in Eq. (5), that is,

$$K_N \geq K^*$$

where K_N is the capital stock in the terminal period and K^* is the target capital stock.

Note that it is not necessary in the Solver dialog box to specify all of the capital accumulation constraints in line 9 of the spreadsheet as constraints. Rather they are effectively linked together by the mathematical expressions so it is necessary to include only cell K_9 when specifying the constraints.

To solve the model one selects the Solve button in the Solver dialog box in Figure 1.6. What happens behind the scenes in Excel next is the solution of the nonlinear programming model that is represented by the growth model. A Newton method or a conjugate gradient method can be used in Excel to solve the model. A brief discussion of nonlinear optimization methods is provided in Appendix F at the end of the book.

Clicking on the Options button in the Solver dialog box displays the Solver Options dialog box shown in Figure 1.7. In this box you are able to change different parameters—for example, maximum time, number of iterations, precision, tolerance, and convergence—that allow you to control the performance of the nonlinear optimization method used by Excel. Note that the Assume Non-Negative option has been selected to constrain the solution values of the model to nonnegative values.

1.3. RESULTS

When solving the growth model with the Excel Solver it is useful to remember the essential trade-off in the model. More consumption today means more utility today. However, less consumption today means more saving and more investment today, which means more capital stock in the future and thus more output and more consumption possibilities in the future.

Therefore, the problem is to find just the right level of consumption in each time period given the parameters of the model. The key parameters are

β , beta	discount factor	0.98
K^*	target capital stock	9.1
θ , theta	production function parameter	0.30
α , alpha	production function exponent	0.33
K_0	initial capital stock	7
τ , tau	utility function parameter	0.50

The discount factor is the most intuitive of these parameters. Recall that it is equal to

$$\beta = \frac{1}{1 + \rho} \quad (9)$$

Solving Eq. (9) for the discount rate ρ yields

$$\rho = \frac{1}{\beta} - 1 \quad (10)$$

Thus when $\beta = 0.98$

$$\rho = \frac{1}{0.98} - 1 = 1.02 - 1.00 = 0.02 \quad (11)$$

and when $\beta = 0.95$

$$\rho = \frac{1}{0.95} - 1 = 1.052 - 1.00 = 0.052 \quad (12)$$

So to a reasonable approximation in the range of interest

$$\beta \approx 1.00 - \rho \quad (13)$$

Thus a discount rate of 6 percent or 0.06 implies a discount factor of 0.94.

Table 1.1. Values of Discount Term

ρ	β	Values of β^t corresponding to each time period					
		Time periods					
		0	1	2	3	4	5
0.02	0.98	1.00	0.98	0.96	0.94	0.92	0.90
0.05	0.95	1.00	0.95	0.90	0.86	0.81	0.77

Next, remember that the criterion function in Eq. (8) includes the discount factor, β , raised to the power t , that is,

$$J = \sum_{t=0}^{N-1} \beta^t \frac{1}{(1-\tau)} C_t^{(1-\tau)}$$

and consider how β^t varies with β and t as shown in Table 1.1.

You can see that when the discount rate is 5 percent the term β^t becomes smaller much faster as the time period increases than it does when the discount rate is 2 percent. Thus when the discount rate is higher, future utility is “discounted” more heavily, that is, given less weight in the criterion function, so if your discount rate is 2 percent you have relatively more interest in your consumption in future years than if your discount rate is 5 percent.

Therefore, altering β is one of the interesting experiments to do with this model. As you increase the discount rate (and thus decrease the discount factor β) you should expect to see more consumption early in the time horizon covered by the model. An illustration of this result is shown in Figure 1.8, which contains plot lines for three experiments corresponding to three different values of β .

A second key parameter of the model that also plays a role in this matter of time preference of consumption and may affect your results in the experiments just described is the target capital stock, K^* . The relevant constraint of the model is Eq. (5), that is,

$$K_N \geq K^*$$

which requires that the capital stock in the terminal period be greater than or equal to the target. This can be thought of as a constraint that represents the interest of the next generation. Without such a constraint, the optimal solution to the growth model is to invest little or nothing in the last years covered by the model and to make consumption very high in those periods. Thus a constraint of this sort is normally added to numerical growth models.

There can be an interplay between the choice of discount rate and the choice of the target capital stock. If you choose a high-target capital stock, then changes in

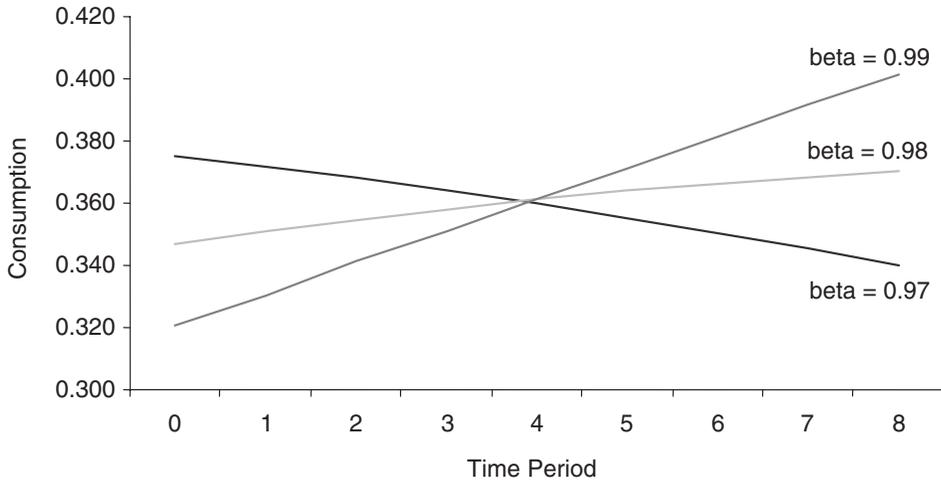


Figure 1.8. Consumption paths for different discount factors.

the discount rate may not have much effect on the pattern of consumption over time since consumption must in any event be very low in order to ensure that there is enough investment to meet the target capital stock in the terminal period.

One of the most straightforward experiments with the model is to increase the initial capital stock. This has the effect of permitting more consumption with less investment, and one would expect to see higher levels of both output and consumption in the model solution.

If you alter the θ parameter in the production function in Eq. (1), that is,

$$Y_t = \theta K_t^\alpha$$

you are modifying the efficiency of the production process. For example, if you increase θ , more output can be produced with the same capital stock and you should find higher levels of both output and consumption in the model solution. Similarly altering the α parameter affects the efficiency of the production process.

The last parameter that can be modified is τ , a parameter in the utility function. Intuition here is a little hard to come by, but as τ approaches zero the utility function becomes linear and as τ approaches one it becomes logarithmic, so it may be useful to think of τ as a parameter that affects the curvature of the utility function or the degree of diminishing marginal utility.

Note that when you perform these experiments, if the changes you make in the parameter values are relatively small, the Excel Solver easily converges to a new solution. This may not be the case for significant changes. Thus in those model runs you may have to “guess” and provide new values for the sequence of consumption values to be used by the Solver as new starting values, or you may have to play with different Solver options to control the Solver performance.

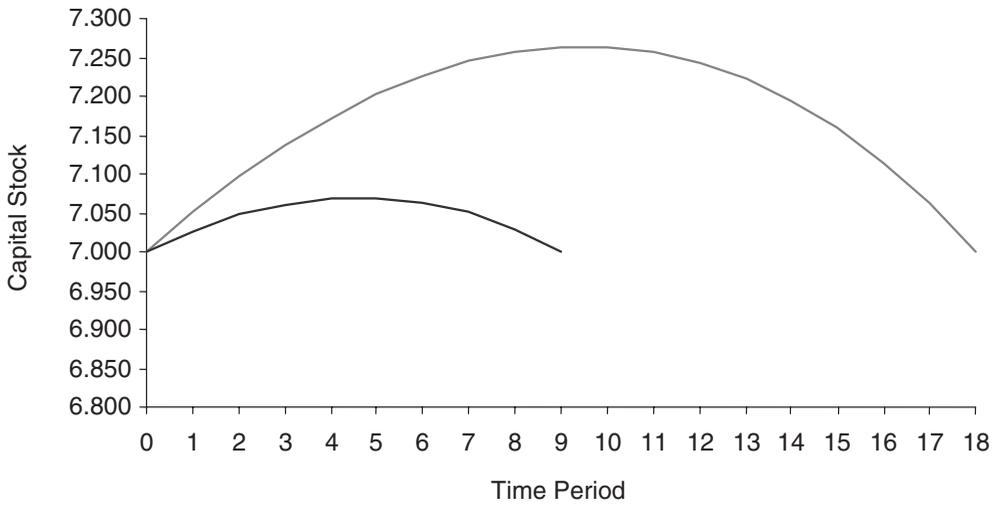


Figure 1.9. Capital stock paths.

In contrast to numerical growth models, theoretical growth models are usually solved for infinite horizons and do not have a terminal capital stock target. As an approximation to this, some numerical growth models are solved for much longer time horizons than the period of interest and the solution is used only for a shorter period. Thus if one is interested in a 20-year period the model might be solved for 40 or 60 years so that the end conditions do not have much effect on the solution paths for the first 20 years. When extending the time horizon, make sure that as you insert more columns in the Excel spreadsheet the equations of the model are copied in a proper manner, that the cell containing the sum of utilities is properly updated to cover the new range, and that the specifications of the target cell, the changing cell, and the constraint are properly updated in the Solver dialog box.

An interesting experiment is to impose a terminal capital stock equal to the initial capital stock and solve the model for different time horizons. The optimal capital stock path for an experiment like this is shown in Figure 1.9.

We can see that optimal values for the capital stock first increase then decrease. If we keep extending the time horizon, we generate a sequence of even higher arches whose top parts become flatter as they get closer to an upper limit value of about 10.5. This behavior is known as the “turnpike property.” To understand this, we have to point out that a model like the one presented in this chapter has a steady-state solution—a solution that, given enough time, the consumption and capital stock levels would converge to and stay at forever. It can be shown (see Appendix J) that for this model the steady-state capital stock is

$$K_{ss} = \left(\frac{1 - \beta}{\beta \alpha \theta} \right)^{\frac{1}{\alpha - 1}} \quad (14)$$

Substituting the corresponding parameter values we obtain $K_{ss} = 10.559$. (To confirm that this is indeed a steady-state solution, you may want to impose this value as the initial and target capital stock values and solve the model.) Thus, any finite optimal path tends to reach the steady-state value, stay there or close to it as long as possible, and then leave it to go back to the target capital stock.

1.4. EXPERIMENTS

Computational economics is not a subject that is easy to learn with the traditional lecture and exam style of teaching. Rather the crucial learning process is to first solve the models that other scholars have used, then to repeatedly make minor modifications and solve the model again in order to gain a clear understanding of how the model works and its strengths and weaknesses. At a later stage substantial structural changes can be made to the model so that it is more applicable to an economic situation of interest to the student.

Perform a series of experiments by modifying one of the parameters discussed earlier and observing the effects on the paths for the capital stock, output, and consumption. Though it might be interesting to change several parameters at one time it is usually better when you are first studying a model to only change one parameter at a time. Save your results from one run to the next so that you can use Excel to plot the results across runs as in Figures 1.8 and 1.9.

A more challenging experiment that you may want to undertake (or may not want to undertake at this stage) is to treat the technology parameter, θ , as stochastic. For example, you can define it as having a uniform distribution. To do so, you can use the Excel function `RAND`, which generates random numbers uniformly distributed between zero and one. Be aware that you should generate a random number for each time period.

1.5. FURTHER READING

Jones (1998) provides a systematic introduction to growth models. Azariadis (1993) and Barro and Sala-i-Martin (1995), at a more advanced level, present a variety of optimal growth Ramsey-type models similar to the one developed in this chapter. Aghion and Howitt (1997) present a systematic treatment of endogenous growth models. Ros (2001) develops a presentation of growth models for developing countries. Mercado, Lin, and Kendrick (2003) present a GAMS version of a single-sector growth model like the one used in this chapter and a multisector optimal growth model in GAMS that is an extension of the Kendrick and Taylor (1971) model. See Judd (1998: Ch. 13) for perturbation methods for solving growth models.

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