Chapter One

On Strichartz's Inequalities and the Nonlinear Schrödinger Equation on Irrational Tori

J. Bourgain

1.0 INTRODUCTION

Strichartz's inequalities and the Cauchy problem for the nonlinear Schrödinger equation are considerably less understood when the spatial domain is a compact manifold M, compared with the Euclidean situation $M = \mathbb{R}^d$. In the latter case, at least the theory of Strichartz inequalities (i.e., moment inequalities for the linear evolution, of the form $\|e^{it\Delta}\phi\|_{L^p_{x,t}} \le C\|\phi\|_{L^2_x}$) is basically completely understood and is closely related to the theory of oscillatory integral operators. Let $M = \mathbb{T}^d$ be a flat torus. If M is the usual torus, i.e.,

$$(e^{it\Delta}\phi)(x) = \sum_{n\in\mathbb{Z}^d} \hat{\phi}(n) e^{2\pi i(nx+|n|^2 t)} \quad (|n|^2 = n_1^2 + \cdots + n_d^2), \tag{1.0.1}$$

a partial Strichartz theory was developed in [B1], leading to the almost exact counterparts of the Euclidean case for $d = 1, 2$ (the exact analogues of the $p = 6$ inequality for $d = 1$ and $p = 4$ inequality for $d = 2$ are *false* with periodic boundary conditions). Thus, assuming $\operatorname{supp}\hat{\phi} \subset B(0, N)$,

$$\|e^{it\Delta}\phi\|_{L^6_{([0,1]\times[0,1])}} \ll N^{\varepsilon}\|\phi\|_2 \quad \text{for } d = 1 \tag{1.0.2}$$

and

$$\|e^{it\Delta}\phi|_{L^4_{([0,1]^2\times[0,1])}} \ll N^{\varepsilon}\|\phi\|_2 \quad \text{for } (d = 2). \tag{1.0.3}$$

For $d = 3$, we have the inequality

$$\|e^{it\Delta}\phi\|_{L^4([0,1]^3\times[0,1])} \ll N^{\frac{1}{4}+\varepsilon}\|\phi\|_2 \quad (d = 3), \tag{1.0.4}$$

but the issue:

Problem. *Does one have an inequality*

$$\|e^{it\Delta}\phi\|_{L^{10/3}([0,1]^3\times[0,1))} \ll N^{\varepsilon}\|\phi\|_2 \quad (d = 3)$$

for all $\varepsilon > 0$ and $\operatorname{supp}\hat{\phi} \subset B(0, N)$?

is still unanswered.

There are two kinds of techniques involved in [B1]. The first kind are arithmetical, more specifically the bound

$$\#\{(n_1, n_2) \in \mathbb{Z}^2 \big| \, |n_1| + |n_2| \le N \quad \text{and} \quad |n_1^2 + n_2^2 - A| \le 1\} \ll N^{\varepsilon}, \tag{1.0.5}$$

which is a simple consequence of the divisor function bound in the ring of Gaussian integers. Inequalities (1.0.2), (1.0.3), (1.0.4) are derived from that type of result.

The second technique used in [B1] to prove Strichartz inequalities is a combination of the Hardy-Littlewood circle method together with the Fourier-analytical approach from the Euclidean case (a typical example is the proof of the Stein-Tomas L^2-restriction theorem for the sphere). This approach performs better for larger dimension d although the known results at this point still leave a significant gap with the likely truth.

In any event, (1.0.2)–(1.0.4) permit us to recover most of the classical results for NLS

$$iu_t + \Delta u - u|u|^{p-2} = 0,$$

with $u(0) \in H^1(\mathbb{T}^d)$, $d \le 3$ and assuming $p < 6$ (subcriticality) if $d = 3$.

Instead of considering the usual torus, we may define more generally

$$\Delta\phi(x) = \sum_{n \in \mathbb{Z}^d} Q(n)\hat{\phi}(n)e^{2\pi i n.x}, \tag{1.0.6}$$

with $Q(n) = \theta_1 n_1^2 + \cdots \theta_d n_d^2$ and, say, $\frac{1}{C} \le \theta_i < C$ $(1 \le 1 \le d)$ arbitrary (what we refer to as "(irrational torus)."

In general, we do not have an analogue of (1.0.5), replacing $n_1^2 + n_2^2$ by $\theta_1 n_1^2 + \theta_2 n_2^2$. It is an interesting question what the optimal bounds are in N for

$$\#\{(n_1, n_2) \in \mathbb{Z}^2 | \, |n_1| + |n_2| \le N \quad \text{and} \quad |\theta_1 n_1^2 + \theta_2 n_2^2 - A| \le 1\} \tag{1.0.7}$$

and

$$\begin{aligned} \#\{(n_1, n_2, n_3) \in \mathbb{Z}^3 \big| \, & |n_1| + |n_2| + |n_3| \le N \\ & \text{and } |\theta_1 n_1^2 + \theta_2 n_2^2 + \theta_3 n_3^2 - A| \le 1\} \end{aligned} \tag{1.0.8}$$

valid for all $\frac{1}{2} < \theta_i < 2$ and A.

Nontrivial estimates may be derived from geometric methods such as Jarnick's bound (cf. [Ja], [B-P]) for the number of lattice points on a strictly convex curve. Likely stronger results are true, however, and almost certainly better results may be obtained in a certain averaged sense when A ranges in a set of values (which is the relevant situation in the Strichartz problem). Possibly the assumption of specific diophantine properties (or genericity) for the θ_i may be of relevance.

In this paper, we consider the case of space dimension $d = 3$ (the techniques used have a counterpart for $d = 2$ but are not explored here).

Taking $\frac{1}{C} < \theta_i < C$ arbitrary and defining Δ as in (1.0.6), we establish the following:

Proposition 1.1 *Let supp $\hat{\phi} \subset B(0, N)$. Then for $p > \frac{16}{3}$,*

$$\|e^{it\Delta}\phi\|_{L_t^p L_x^4} \le CN^{\frac{3}{4} - \frac{2}{p}} \|f\|_2, \tag{1.0.9}$$

where L_t^p refers to $L_{[0,1]}^p(dt)$.

Proposition 1.3′. *Let supp $\hat{\phi} \subset B(0, N)$. Then*

$$\|e^{it\Delta}\phi\|_{L^4_{x,t}} < C_\varepsilon N^{\frac{1}{3}+\varepsilon}\|\phi\|_2 \text{ for all } \varepsilon > 0. \tag{1.0.10}$$

The analytical ingredient involved in the proof of (1.0.9) is the well-known inequality for the squares

$$\left\| \sum_{j=1}^{N} e^{2\pi i j^2\theta} \right\|_{L^q(\mathbb{T})} < CN^{1-\frac{2}{q}} \quad \text{for } q > 4. \tag{1.0.11}$$

The proof of (1.0.10) is more involved and relies on a geometrical approach to the lattice point counting problems, in the spirit of Jarnick's estimate mentioned earlier. Some of our analysis may be of independent interest. Let us point out that both (1.0.9), (1.0.10) are weaker than (1.0.4). Thus,

Problem. *Does* (1.0.4) *hold in the context of* (1.0.6)*?*

Using similar methods as in [B1, 2] (in particular $X_{s,b}$-spaces), the following statements for the Cauchy problem for NLS on a 3D irrational torus may be derived.

Proposition 1.2 *Let Δ be as in* (1.0.6). *Then the 3D defocusing NLS*

$$iu_t + \Delta u - u|u|^{p-2} = 0$$

is globally wellposed for $4 \leq p < 6$ and H^1-data.

Proposition 1.4′. *Let Δ be as in* (1.0.6). *Then the 3D defocusing cubic NLS*

$$iu_t + \Delta u \pm u|u|^2 = 0$$

is locally wellposed for data $u(0) \in H^s(\mathbb{T}^3)$, $s > \frac{2}{3}$.

This work originates from discussion with P. Gerard (March, 04) and some problems left open in his joint paper [B-G-T] about NLS on general compact manifolds. The issues in the particular case of irrational tori, explored here for the first time, we believe, unquestionably deserve to be studied more. Undoubtedly, further progress can be made on the underlying number theoretic problems.

1.1 AN INEQUALITY IN 3D

$$Q(n) = \theta_1 n_1^2 + \theta_2 n_2^2 + \theta_3 n_3^2, \tag{1.1.1}$$

where the θ_i are arbitrary, θ_i and θ_i^{-1} assumed bounded. Write

$$(e^{it\Delta} f)(x) = \sum_{n\in\mathbb{Z}^3} \hat{f}(n) e^{2\pi i(n.x+Q(n)t)}. \tag{1.1.2}$$

Proposition 1.1 *For $p > \frac{16}{3}$, we have*

$$\|e^{it\Delta} f\|_{L_t^p L_x^4} \leq C_p N^{\frac{3}{4}-\frac{2}{p}} \|f\|_2 \tag{1.1.3}$$

assuming $\operatorname{supp} \hat{f} \subset B(0, N)$. *Here* L_t^p *denotes* $L_t^p(\text{loc})$.

Remark. Taking $f(x) = N^{-3/2} \sum_{|n|<N} e^{inx}$, we see that (1.1.3) is optimal.

Proof of Proposition 1.1.

$$\begin{aligned}\|e^{it\Delta} f\|^2_{L_t^p L_x^4} &= \|(e^{it\Delta} f)^2\|_{L_t^{p/2} L_x^2} \\ &= \left\| \left[\sum_{a\in\mathbb{Z}^3} \left| \sum_n \hat{f}(n) \hat{f}(a-n) e^{i[Q(n)+Q(a-n)]t} \right|^2 \right]^{1/2} \right\|_{L_t^{p/2}} \\ &\leq \left[\sum_{a\in\mathbb{Z}^3} \left\| \sum_n \hat{f}(n) \hat{f}(a-n) e^{i[Q(n)+Q(a-n)]t} \right\|^2_{L_t^{p/2}} \right]^{1/2} \end{aligned} \tag{1.1.4}$$

since $p \geq 4$.

Denote $c_n = |\hat{f}(n)|$. Applying Hausdorff-Young,

$$\|\cdots\|_{L_t^{p/2}} \lesssim \left[\sum_{k\in\mathbb{Z}} \left| \sum_{|Q(n)+Q(a-n)-k|\leq \frac{1}{2}} c_n c_{a-n} \right|^{\frac{p}{p-2}} \right]^{\frac{p-2}{p}}. \tag{1.1.5}$$

Rewrite $|Q(n) + Q(a-n) - k| \leq \frac{1}{2}$ as $|Q(2n-a) + Q(a) - 2k| \leq 1$ and hence

$$2n \in a + \mathfrak{S}_\ell,$$

where

$$\ell = 2k - Q(a) \text{ and } \mathfrak{S}_\ell = \{m \in \mathbb{Z}^3 |\, |Q(m) - \ell| \leq 1\}. \tag{1.1.6}$$

Clearly (1.1.5) may be replaced by

$$\left[\sum_{\ell\in\mathbb{Z}} \left| \sum_{2n\in a+\mathfrak{S}_\ell} c_n c_{a-n} \right|^{\frac{p}{p-2}} \right]^{\frac{p-2}{p}} \tag{1.1.6$'$}$$

and an application of Hölder's inequality yields

$$\begin{aligned} &\left[\sum_\ell |\mathfrak{S}_\ell|^{\frac{p}{2(p-2)}} \left(\sum_{2n\in a+\mathfrak{S}_\ell} c_n^2 c_{a-n}^2 \right)^{\frac{p}{2(p-2)}} \right]^{\frac{p-2}{p}} \\ &\lesssim \left(\sum_\ell |\mathfrak{S}_\ell|^{\frac{p}{p-4}} \right)^{\frac{p-4}{2p}} \left[\sum_n c_n^2 c_{a-n}^2 \right]^{1/2} \end{aligned} \tag{1.1.7}$$

(since the $\mathfrak{S}_\ell$ are essentially disjoint).

Substitution of (1.1.7) in (1.1.4) gives the bound

$$\|e^{it\Delta} f\|_{L_t^p L_x^4} \le C\Big(\sum_\ell |\mathfrak{S}_\ell|^{\frac{p}{p-4}}\Big)^{\frac{p-4}{4p}} \|f\|_2. \tag{1.1.8}$$

Next, write

$$|\mathfrak{S}_\ell| \le \int \Big[\sum_{|m|\le N} e^{iQ(m)t}\Big] e^{-i\ell t}\varphi(t)dt, \tag{1.1.8$'$}$$

where φ is compactly supported and $\hat\varphi \ge 0$, $\hat\varphi \ge 1$ on $[-1, 1]$.

Assume $p \le 8$, so that $\frac{p}{p-4} \ge 2$ and from the Hausdorff-Young inequality again

$$\Big(\sum |\mathfrak{S}_\ell|^{\frac{p}{p-4}}\Big)^{\frac{p-4}{p}} \lesssim \Big[\int_{\text{loc}} \prod_{j=1}^3 \Big|\sum_{0\le m\le N} e^{i\theta_j m^2 t}\Big|^{\frac{p}{4}} dt\Big]^{\frac{4}{p}}$$

$$\lesssim \Big[\int_{\text{loc}} \Big|\sum_{0\le m\le N} e^{im^2 t}\Big|^{\frac{3p}{4}} dt\Big]^{\frac{4}{p}}. \tag{1.1.9}$$

Since $p > \frac{16}{3}$, $q = \frac{3p}{4} > 4$ and

$$\int_{\text{loc}} \Big|\sum_{0\le m\le N} e^{im^2 t}\Big|^q dt \sim N^{q-2} \tag{1.1.10}$$

(immediate from Hardy-Littlewood).

Therefore,

$$(1.1.9) \lesssim N^{3-\frac{8}{p}},$$

and substituting in (1.1.8), we obtain (1.1.3)

$$\|e^{it\Delta} f\|_{L_t^p L_x^4} \le C N^{\frac{3}{4}-\frac{2}{p}} \|f\|_2$$

for $p \le 8$. For $p > 8$, the result simply follows from

$$\|e^{it\Delta} f\|_{L_t^p L_x^4} \le N^{2(\frac{1}{8}-\frac{1}{p})} \|e^{it\Delta} f\|_{L_t^8 L_x^4}. \tag{1.1.11}$$

This proves Proposition 1.

Remarks.

1. For $p = \frac{16}{3}$, we have the inequality

$$\|e^{it\Delta} f\|_{L_t^{16/3} L_x^4} \le N^{\frac{3}{8}+} \|f\|_2 \tag{1.1.12}$$

assuming $\operatorname{supp}\hat f \subset B(0, N)$.

2. Inequalities (1.1.3) and (1.1.12) remain valid if $\operatorname{supp}\hat f \subset B(a, N)$ with $a \in \mathbb{Z}^3$ arbitrary.

Indeed,

$$|e^{it\Delta} f| = \left| \sum_{|m|\leq N} \hat{f}(a+m) e^{i[(x+2(\theta_1 a_1+\theta_2 a_2+\theta_3 a_3)t).m+Q(m)t]} \right|$$

so that

$$\|e^{it\Delta} f\|_{L_t^p L_x^4} = \left\| \sum_{|m|\leq N} \hat{f}(a+m) e^{i(x.m+Q(m)t)} \right\|_{L_t^p L_x^4}.$$

1.2 APPLICATION TO THE 3D NLS

Consider the defocusing 3D NLS

$$iu_t + \Delta u - u|u|^{p-2} = 0 \tag{1.2.1}$$

on $\mathbb{T}^3$ and with Δ as in (1.1.2).

Assume $4 \leq p < 6$.

Proposition 1.2 (1.2.1) *is locally and globally wellposed in H^1 for $p < 6$.*

Sketch of Proof. Using $X_{s,b}$-spaces (see [B1]), the issue of bounding the nonlinearity reduces to an estimate on an expression

$$\| \, |e^{it\Delta}\phi_1| \, |e^{it\Delta}\phi_2| \, |e^{it\Delta}\psi|^{p-2}\|_1,$$

with $\|\phi_1\|_2, \|\phi_2\|_2 \leq 1$ and $\|\psi\|_{H^1} \leq 1$.

Thus we need to estimate

$$\| \, |e^{it\Delta}\phi_1| \, |e^{it\Delta}\psi|^{\frac{p-2}{2}}\|_2. \tag{1.2.2}$$

By dyadic restriction of the Fourier transform, we assume further

$$\operatorname{supp} \hat{\phi}_1 \subset B(0, 2M)\backslash B(0, M) \tag{1.2.3}$$

$$\operatorname{supp} \hat{\psi} \subset B(0, 2N)\backslash B(0, N) \tag{1.2.4}$$

for some dyadic $M, N > 1$.

Write

$$(1.2.2) \leq \|[e^{it\Delta}\phi_1][e^{it\Delta}\psi](1+|e^{it\Delta}\psi|^2)^{\frac{p}{4}-1}\|_2, \tag{1.2.5}$$

where $(1+|z|^2)^{\frac{p}{4}-1}$ is a smooth function of z.

If in (1.2.3), (1.2.4), $M > N$, partition $\mathbb{Z}^3$ in boxes I of size N and write

$$\phi_1 = \sum_I P_I \phi_1,$$

and by almost orthogonality

$$(1.2.5) \lesssim \left[\sum_I \| \, |e^{it\Delta} P_I \phi_1| \, |e^{it\Delta}\psi|(1+|e^{it\Delta}\psi|^2)^{\frac{p}{4}-1}\|_2^2 \right]^{1/2}. \tag{1.2.6}$$

For fixed I, estimate

$$\| \, |e^{it\Delta} P_I \phi_1| \, |e^{it\Delta}\psi|(1+|e^{it\Delta}\psi|^2)^{\frac{p}{4}-1} \|_2$$
$$\leq \|e^{it\Delta} P_I \phi_1\|_{L_t^{16/3} L_x^4} \|e^{it\Delta}\psi\|_{L_t^{16/3} L_x^4} \Big(1 + \|e^{it\Delta}\psi\|_{L_t^{8(\frac{p}{2}-2)} L_x^\infty}\Big)^{\frac{p}{2}-2}, \quad (1.2.7)$$

and in view of (1.1.12) and Remarks (1), (2) above and (1.2.4),

$$\|e^{it\Delta} P_I \phi_1\|_{L_t^{16/3} L_x^4} \leq N^{\frac{3}{8}+} \|P_I \phi_1\|_2 \quad (1.2.8)$$

$$\|e^{it\Delta}\psi\|_{L_t^{16/3} L_x^4} \leq N^{\frac{3}{8}+} N^{-1} \|\psi\|_{H^1} < N^{-\frac{5}{8}+}. \quad (1.2.9)$$

To bound the last factor in (1.2.7), distinguish the cases

(A) $\underline{4 \leq p \leq \frac{16}{3}}$

Then $8(\frac{p}{2}-2) \leq \frac{16}{3}$ and by (1.2.9)

$$\|e^{it\Delta}\psi\|_{L_t^{8(\frac{p}{2}-2)} L_x^\infty} \leq \|e^{it\Delta}\psi\|_{L_t^{16/3} L_x^\infty} \leq N^{3/4} \|e^{it\Delta}\psi\|_{L_t^{16/3} L_x^4} < N^{1/8+}. \quad (1.2.10)$$

Substitution of (1.2.8)–(1.2.10) in (1.2.7) gives

$$N^{-\frac{1}{4}+} N^{\frac{1}{8}(\frac{p}{2}-2)+} \|P_I \phi_1\|_2 \leq N^{-\frac{1}{6}+} \|P_I \phi_1\|, \quad (1.2.11)$$

hence

$$(1.2.6) < N^{-\frac{1}{6}+}.$$

(B) $\underline{\frac{16}{3} < p < 6}$

$$\|e^{it\Delta}\psi\|_{L_t^{8(\frac{p}{2}-2)} L_x^\infty} \leq N^{\frac{3}{8}-\frac{1}{2p-8}+\frac{3}{4}} \|e^{it\Delta}\psi\|_{L_t^{16/3} L_x^4} < N^{\frac{1}{2}-\frac{1}{2(p-4)}+} \quad (1.2.12)$$

and

$$(1.2.7) \leq N^{\frac{p}{4}-\frac{3}{2}+} \|P_I \phi_1\|_2 \quad (1.2.13)$$

$$(1.2.6) < N^{\frac{p}{4}-\frac{3}{2}+}.$$

This proves Proposition 1.2.

1.3 IMPROVED L^4-BOUND

It follows from (1.1.12) that

$$\|e^{it\Delta} f\|_{L_{t,x}^4} \leq N^{\frac{3}{8}+} \|f\|_2 \text{ if } \operatorname{supp} \hat{f} \subset B(0, N). \quad (1.3.1)$$

In this section, we will obtain the following first improvement:

$$\|e^{it\Delta} f\|_{L^4_{t,x}} \leq N^{\frac{7}{20}} \|f\|_2 \text{ for supp } \hat{f} \subset B(0, N). \tag{1.3.2}$$

Restrict $\hat{f}$ to a one level set, thus

$$\hat{f} = \hat{f} \chi_{\Omega_\mu} \tag{1.3.3}$$

with

$$\Omega_\mu = \{n \in [-N, N]^3 | \, |\hat{f}(n)| \sim \mu\}$$

$$|\Omega_\mu| \leq \mu^{-2}. \tag{1.3.4}$$

In what follows, we assume f of the form (1.3.3).

Lemma 1.1

$$\|e^{it\Delta} f\|_{L^4_{x,t}} < \mu^{1/6} N^{\frac{1}{2}+} \tag{1.3.5}$$

Proof. From estimates (1.1.4) and (1.1.5′) with $p = 4$ and letting

$$c_n = \begin{cases} \mu & \text{if } n \in \Omega\mu \\ 0 & \text{otherwise} \end{cases}$$

we get the following bound on $\|e^{it\Delta} f\|_4^2$:

$$\mu^2 \Bigg[\sum_{a \in \mathbb{Z}^3} \sum_{\ell \in \mathbb{Z}} |(a + \mathfrak{S}_\ell) \cap (2\Omega_\mu) \cap (2a - 2\Omega_\mu)|^2 \Bigg]^{1/2}. \tag{1.3.6}$$

Recall also estimate (1.1.9) for $p = \frac{16}{3}$,

$$\Big(\sum |\mathfrak{S}_\ell|^4 \Big)^{1/4} < N^{\frac{3}{2}+}. \tag{1.3.7}$$

Hence, if we denote for $L \geq 1$ (a dyadic integer)

$$\mathcal{L}_L = \{\ell \in \mathbb{Z} \big| \, |\mathfrak{S}_\ell| \sim N^{\frac{3}{2}+} L^{-1/4}\}, \tag{1.3.8}$$

it follows that

$$|\mathcal{L}_L| < L. \tag{1.3.9}$$

Estimate (1.3.6) by

$$\mu^2 \Bigg[\sum_{\ell \in \mathbb{Z}} |\mathfrak{S}_\ell| \sum_a |(a + \mathfrak{S}_\ell) \cap (2\Omega_\mu) \cap (2a - 2\Omega_\mu)| \Bigg]^{1/2} \tag{1.3.10}$$

and restrict in (1.3.10) the ℓ-summation to $\mathcal{L}_L$.

There are the following two bounds:

$$\mu^2\left[\sum_{\ell\in\mathcal{L}_L}|\mathfrak{S}_\ell|\sum_a|(a+\mathfrak{S}_\ell)\cap(2\Omega_\mu)|\right]^{1/2}\le\mu^2\left[\sum_{\ell\in\mathcal{L}_L}|\mathfrak{S}_\ell|^2|\Omega_\mu|\right]^{1/2} < \mu N^{\frac{3}{2}+}L^{1/4} \tag{1.3.11}$$

and also

$$\mu^2N^{\frac{3}{4}+}L^{-1/8}\left[\sum_{\ell,a}|(a+\mathfrak{S}_\ell)\cap(2\Omega_\mu)\cap(2a-2\Omega_\mu)|\right]^{1/2} < \mu^2N^{\frac{3}{4}+}L^{-1/8}|\Omega_\mu| < N^{\frac{3}{4}+}L^{-1/8}. \tag{1.3.12}$$

Taking the minimum of (1.3.11), (1.3.12), we obtain $\mu^{1/3}N^{1+}$. Summing over dyadic values of $L\lesssim N^2$, the estimate follows.

Next, we need a discrete maximal inequality of independent interest.

Lemma 1.2 *Consider the following maximal function on* $\mathbb{Z}^3$

$$F^*(x)=\max_{1<\ell<N^2}\sum_{|Q(y)-\ell|\le 1}F(x+y). \tag{1.3.13}$$

For

$$\lambda > N^{\frac{1}{2}}\|F\|_2 \tag{1.3.14}$$

we have

$$|[F^*>\lambda]| < N^{\frac{3}{2}+}\|F\|_2^2\,\lambda^{-2}. \tag{1.3.15}$$

($\|F\|_2$ *denotes* $(\sum_{x\in\mathbb{Z}^3}|F(x)|^2)^{1/2}$).

Proof. Let $A=[F^*>\lambda]\subset\mathbb{Z}^3$. Thus for $x\in A$, there is ℓ_x s.t.

$$\langle F,\chi_{x+\mathfrak{S}_{\ell_x}}\rangle>\lambda.$$

Estimate as usual

$$\begin{aligned}\lambda.|A| &\le\left\langle F,\sum_{x\in A}\chi_{x+\mathfrak{S}_{\ell_x}}\right\rangle\\ &\le\|F\|_2\left\|\sum_{x\in A}\chi_{x+\mathfrak{S}_{\ell_x}}\right\|_2\\ &=\|F\|_2[|A|\max_\ell|\mathfrak{S}_\ell|+|A|^2\max_{x\neq y}|(x+\mathfrak{S}_x)\cap(y+\mathfrak{S}_y)|]^{1/2}.\end{aligned} \tag{1.3.16}$$

Use the crude bound $|\mathfrak{S}_\ell|<N^{\frac{3}{2}+}$ from (1.3.7) and denote

$$K=\max_{x,y\in\mathbb{Z}^3,x\neq y}|(x+\mathfrak{S}_x)\cap(y+\mathfrak{S}_y)|. \tag{1.3.17}$$

From (1.3.16), we conclude that

$$|A| < N^{\frac{3}{2}+} \|F\|_2^2 \lambda^{-2} \tag{1.3.18}$$

if

$$\lambda > \|F\|_2 K^{1/2}. \tag{1.3.19}$$

It remains to evaluate K.

If $n \in \mathbb{Z}^3$ lies in $(x + \mathfrak{S}_{\ell_x}) \cap (y + \mathfrak{S}_{\ell_y})$, then

$$|Q(x-n) - \ell_x| \le 1$$
$$|Q(y-n) - \ell_y| \le 1,$$

and subtracting

$$\begin{aligned} &|2\theta_1(x_1 - y_1)n_1 + 2\theta_2(x_2 - y_2)n_2 + 2\theta_3(x_3 - y_3)n_3 \\ &\quad -Q(x) + Q(y) + \ell_x - \ell_y| \le 2. \end{aligned} \tag{1.3.20}$$

Since $x \neq y$ in $\mathbb{Z}^3$, $|x - y| \ge 1$ and (1.3.20) restricts n to a 1-neighborhood $\prod_{(1)}$ of some plane $\prod$. Therefore (fig. 1.1.),

$$|(x + \mathfrak{S}_x) \cap (y + \mathfrak{S}_y)\Big| < \max_{\ell, \prod} |\mathfrak{S}_\ell \cap \prod_{(1)}\Big| \tag{1.3.21}$$

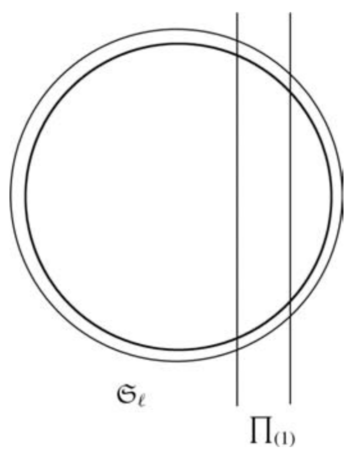

Fig. 1.1.

Recall that $\mathfrak{S}_\ell$ is a $\frac{1}{\sqrt{\ell}}$-neighborhood of a "regular" ellipsoid $\mathcal{E}$ of size $\sqrt{\ell}$. Estimate the number of lattice points $|\mathfrak{S}_\ell \cap \prod_{(1)}|$ in $\mathfrak{S}_\ell \cap \prod_{(1)}$ by the area of $\mathcal{E}$ inside $\prod_{(1)}$. By affine transformation, we may assume $\mathcal{E}$ a sphere of radius at most N. A simple calculation shows that this area is at most $\sim N$. Hence $K \lesssim N$ and (1.3.18) holds if (1.3.14).

This proves Lemma 1.2.

Remark. The number K in (1.3.17) allows more refined estimates that will be pointed out later.

Taking in Lemma 1.2 $F = \chi_{2\Omega_\mu}$, we get

Corollary 1.3 *If* $\lambda > N^{\frac{1}{2}}\mu^{-1}$, *then*

$$|\{a \in \mathbb{Z}^3 | \max_{\ell \lesssim N^2} |(a + \mathfrak{S}_\ell) \cap 2\Omega_\mu| > \lambda\}| < N^{\frac{3}{2}+}(\mu\lambda)^{-2}. \tag{1.3.22}$$

Now we establish

Lemma 1.4

$$\|e^{it\Delta} f\|_4 < N^{\frac{3}{16}+} + N^{\frac{1}{8}}\mu^{-\frac{1}{4}}. \tag{1.3.23}$$

Proof. We return to (1.3.6).

Denote for dyadic λ

$$A_\lambda = \{a \in \mathbb{Z}^3 \cap [-N, N]^3 | \max_\ell |(a + \mathfrak{S}_\ell) \cap (2\Omega_\mu)| \sim \lambda\}.$$

For $a \in A_\lambda$, there are at most $\mu^{-2}\lambda^{-1}$ values of $\ell \in \mathbb{Z}$ s.t.

$$|(a + \mathfrak{S}_\ell) \cap (2\Omega_\mu)| > \lambda \tag{1.3.24}$$

(since the $\mathfrak{S}_\ell$ are disjoint).

We estimate

$$\sum_{a \in A_\lambda} \sum_{\substack{\ell \in \mathbb{Z} \\ (1.3.24)}} |(a + \mathfrak{S}_\ell) \cap (2\Omega_\mu) \cap (2a - 2\Omega_\mu)|^2 \tag{1.3.25}$$

distinguishing the following two cases:

Case 1.1 $\lambda \leq N^{\frac{1}{2}}\mu^{-1}$.

Write

$$\begin{aligned} (1.3.25) &\leq \lambda \sum_a \sum_{\ell \in \mathbb{Z}} |(a + \mathfrak{S}_\ell) \cap (2\Omega_\mu) \cap (2a - 2\Omega_\mu)| \\ &< \lambda |\Omega_\mu|^2 < N^{\frac{1}{2}}\mu^{-5}. \end{aligned} \tag{1.3.26}$$

Case 1.2 $\lambda > N^{\frac{1}{2}}\mu^{-1}$.

Then (1.3.22) *applies and* $|A_\lambda| < N^{\frac{3}{2}+}(\mu\lambda)^{-2}$. *Hence*

$$(1.3.25) < |A_\lambda|\mu^{-2}\lambda < N^{\frac{3}{2}+}\mu^{-4}\lambda^{-1}. \tag{1.3.27}$$

Since there is also the obvious bound given by (1.3.26)

$$(1.3.25) < \lambda\mu^{-4}, \tag{1.3.28}$$

we obtain

$$(1.3.25) < N^{\frac{3}{4}+}\mu^{-4}. \tag{1.3.29}$$

Substitution of (1.3.26), (1.3.29) *implies*

$$\|e^{it\Delta} f\|_4^2 \leq (1.3.6) < N^{\frac{1}{4}}\mu^{-\frac{1}{2}} + N^{\frac{3}{8}+}.$$

Proposition 1.3 $\|e^{it\Delta} f\|_{L^4_{x,t}} \leq N^{\frac{7}{20}+}\|f\|_2$ *if* $\operatorname{supp}\hat{f} \subset B(0, N)$.

Proof. With f as above, it follows from Lemma 1.1 and 1.4 that

$$\|e^{it\Delta} f\|_4 < N^{\frac{3}{16}+} + \min(\mu^{\frac{1}{6}} N^{\frac{1}{2}+}, N^{\frac{1}{8}}\mu^{-\frac{1}{4}}) < N^{\frac{7}{20}+}.$$

As a corollary of Proposition 1.3, we get the following wellposedness result for cubic NLS in 3D.

Proposition 1.4 *Consider* $iu_t + \Delta u \pm u|u|^2 = 0$ *on* $\mathbb{T}^3$ *and with* Δ *as above. There is local wellposedness for* $u(0) \in H^s(\mathbb{T}^3)$, $s > \frac{7}{10}$.

1.4 A REFINEMENT OF PROPOSITION 3

Our purpose is to improve upon Lemma 1.2 by a better estimate on the quantity K in (1.3.17), thus

$$|(x + \mathcal{E}_\varepsilon) \cap (x' + \mathcal{E}'_\varepsilon) \cap \mathbb{Z}^3|, \tag{1.4.1}$$

where $\mathcal{E}, \mathcal{E}'$ are nondegenerated ellipsoids centered at 0 of size $\sim R < N$ and $\varepsilon = \frac{1}{R}$ refers to an ε'-neighborhood, $x \neq x'$ in $\mathbb{Z}^3$.

The main ingredients are versions of the the standard Jarnick argument to estimate the number of lattice points on a curve (cf. [Ja]). Here we will have to deal with neighborhoods.

We start with a 2-dimensional result.

Lemma 1.5 *Let* $\mathcal{E}$ *be a "regular" oval in* $\mathbb{R}^2$ *of size* R. *Then*

$$\max_a |B(a, R^{1/3}) \cap \mathcal{E}_{\frac{1}{R}} \cap \mathbb{Z}^2| < C. \tag{1.4.2}$$

In particular

$$|\mathcal{E}_{\frac{1}{R}} \cap \mathbb{Z}^2| < CR^{2/3} \tag{1.4.3}$$

and for all $\rho > 1$

$$|B(a, \rho) \cap \mathcal{E}_{\frac{1}{R}} \cap \mathbb{Z}^2| < C\rho^{2/3} \tag{1.4.4}$$

($\mathcal{E}_{\frac{1}{R}}$ *denotes a* $\frac{1}{R}$*-neighborhood of* $\mathcal{E}$).

Proof. Let P_1, P_2, P_3 be noncolinear points in $B(a, cR^{1/3}) \cap \mathcal{E}_{\frac{1}{R}} \cap \mathbb{Z}^2$, letting c be a sufficiently small constant. Following Jarnick's argument,

$$0 \neq \text{ area triangle } (P_1, P_2, P_3) = \frac{1}{2} \left| \begin{vmatrix} 1 & 1 & 1 \\ P_1 & P_2 & P_3 \end{vmatrix} \right| \in \frac{1}{2}\mathbb{Z}_+$$

and hence

$$\text{area } (P_1, P_2, P_3) \geq \frac{1}{2}. \tag{1.4.5}$$

Take $P_1', P_2', P_3' \in \mathcal{E}$ so that $|P_j - P_j'| < \frac{1}{R}$. Clearly,

$$\left| \begin{vmatrix} 1 & 1 & 1 \\ P_1 & P_2 & P_3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ P_1' & P_2' & P_3' \end{vmatrix} \right| < R^{\frac{1}{3}} R^{-1} \ll 1$$

so that

$$\text{area } (P_1', P_2', P_3') > \frac{1}{4}. \tag{1.4.6}$$

On the other hand, obviously

$$\text{area } (P_1', P_2', P_3') \leq cR^{1/3} \frac{R^{2/3}}{R} \ll 1 \tag{1.4.7}$$

a contradiction. This proves (1.4.2), observing that if Λ is a line, then clearly $\Lambda \cap \mathcal{E}_{\frac{1}{R}}$ is at most of bounded length. Hence $|\mathcal{E}_{\frac{1}{R}} \cap \Lambda \cap \mathbb{Z}^2| < C$.

Partitioning $\mathcal{E}_{\frac{1}{R}}$ in sets of size $cR^{1/3}$ (1.4.3) follows.

Finally, estimate (1.4.4) by $\min(1 + \rho R^{-1/3}, R^{2/3}) \lesssim \rho^{2/3}$.

Remark. Projecting on one of the coordinate planes, Lemma 1.5 applies equally well to a regular oval $\mathcal{E}$ in a 2-plane $\prod$ in $\mathbb{R}^3$ and

$$\max_a |B(a, R^{1/3}) \cap \mathcal{E}_{\frac{1}{R}} \cap \mathbb{Z}^3| < C \tag{1.4.8}$$

and

$$\max_a |B(a, \rho) \cap \mathcal{E}_{\frac{1}{R}} \cap \mathbb{Z}^3| < C\rho^{2/3}, \tag{1.4.9}$$

where $\mathcal{E}$ is of size R and $\mathcal{E}_{\frac{1}{R}}$ denotes an $\frac{1}{R}$-neighborhood of $\mathcal{E}$.

There is an obvious extension of (1.4.2) in dimension 3. One has

Lemma 1.6 *Let $\mathcal{E}$ be a 2-dim regular oval in $\mathbb{R}^3$ of size R. Then, for all $a \in \mathbb{R}^3$ and appropriate c, $B(a, cR^{1/4}) \cap \mathcal{E}_{\frac{1}{R}} \cap \mathbb{Z}^3$ does not contain 4 noncoplanar points.*

Proof. If P_1, P_2, P_3, P_4 are noncoplanar points in $B(a, cR^{1/4}) \cap \mathcal{E}_{\frac{1}{R}} \cap \mathbb{Z}^3$ and $|P_j - P_j'| < \frac{1}{R}$, $P_j' \in \mathcal{E}$, write

$$\frac{1}{6}\mathbb{Z}_+ \ni \text{ Vol } (P_1, P_2, P_3, P_4) = \text{Vol } (P_1', P_2', P_3', P_4') + 0(R^{1/2} R^{-1}),$$

and hence

$$\text{Vol}\,(P_1', P_2', P_3', P_4') > \frac{1}{7}. \tag{1.4.10}$$

On the other hand, this volume may be estimated by the volume of the cap obtained as convex hull conv$(\mathcal{E} \cap B(a, cR^{1/4})$ bounded by $(cR^{1/4})^2 \frac{R^{\frac{1}{2}}}{R} \ll 1$. This proves Lemma 1.6.

We now return to (1.3.21) and estimate $|\mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)} \cap \mathbb{Z}^3|$, where $\prod_{(1)}$ is a 1-neighborhood of a plane $\prod$ in $\mathbb{R}^3$. Our purpose is to show

Lemma 1.7

$$\left|\mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)} \cap \mathbb{Z}^3\right| < R^{2/3+}. \tag{1.4.11}$$

Proof. (see fig. 1.2.).

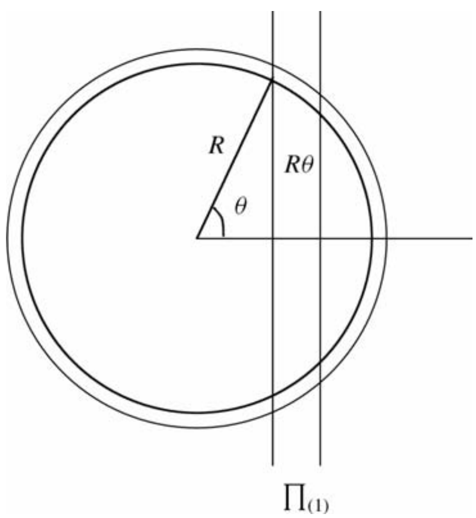

Fig. 1.2.

Thus $\mathcal{E} \cap \prod_{(1)}$ is a truncated conical region of base-size $R\theta$, slope θ and height 1, for some $\theta > \frac{1}{R}$.

Consider first the case $\theta < R^{-1/4}$. Partition $\mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)}$ in $\sim R^{-\frac{1}{2}}(R\theta)\frac{1}{\theta}$ regions $\mathcal{D}$ of size $cR^{1/4}$. According to Lemma 1.6, $\mathcal{D} \cap \mathbb{Z}^3$ consists of coplanar points, therefore lying in some plane $\mathcal{P} \subset \mathbb{R}^3$ and

$$\mathcal{D} \cap \mathbb{Z} = \mathcal{D} \cap \mathcal{P} \cap \mathcal{E}_{\frac{1}{R}} \cap \mathbb{Z}^3. \tag{1.4.12}$$

$\mathcal{P} \cap \mathcal{E}$ is an ellipse $\mathcal{E}'$ of size r (we may assume $r \gg 1$) and we claim that $\mathcal{P} \cap \mathcal{E}_{\frac{1}{R}} \subset \mathcal{E}'_{\frac{1}{r}} = \frac{1}{r}$-neighborhood of $\mathcal{E}'$. To see this, we may by affine transformation assume $\mathcal{E}$ to be a sphere of radius R, in which case it is a straightforward calculation.

From (1.4.12) and the preceding,

$$\begin{aligned} |\mathcal{D} \cap \mathbb{Z}^3| &\leq |\mathcal{D} \cap \mathcal{E}'_{\frac{1}{r}} \cap \mathbb{Z}^3| \\ &< C(\operatorname{diam} \mathcal{D})^{2/3} \\ &< C R^{1/6}, \end{aligned} \tag{1.4.13}$$

applying (1.4.9) to $\mathcal{E}'_{\frac{1}{r}}$ in the plane $\mathcal{P}$.

We conclude that for $\theta < R^{-1/4}$,

$$\left| \mathcal{E}_{\frac{1}{R}} \cap \prod\nolimits_{(1)} \cap \mathbb{Z}^3 \right| < C R^{\frac{1}{2}} R^{\frac{1}{6}} < C R^{\frac{2}{3}} \tag{1.4.14}$$

and hence (1.4.12).

Assume next that $\theta > R^{-1/4}$.

Let $D > 1$ be such that $B(a, D) \cap \mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)} \cap \mathbb{Z}^3$ (for some $a \in \mathbb{R}^3$) contains 4 noncoplanar points P_1, P_2, P_3, P_4. Assume

$$D < (\theta R)^{1/2}. \tag{1.4.15}$$

Repeating the argument in Lemma 1.6, let $|P_j - P'_j| \leq \frac{1}{R}$, $P'_j \in B(a, D+1) \cap \mathcal{E} \cap \prod_{(2)}$.

By (1.4.15),

$$\operatorname{Vol}(P'_1, P'_2, P'_3, P'_4) > \frac{1}{6} - 0(D^2 R^{-1}) > \frac{1}{7}. \tag{1.4.16}$$

Considering sections parallel to $\prod$, write an upper bound on the left side of (1.4.16) by

$$\operatorname{Vol}\left(\operatorname{conv}(B(a, 2D) \cap \mathcal{E} \cap \prod\nolimits_{(2)}\right) \leq D \frac{D^2}{R\theta}. \tag{1.4.17}$$

Together with (1.4.16), (1.4.17) implies

$$D \gtrsim (R\theta)^{1/3} \gtrsim \frac{1}{\theta}, \tag{1.4.18}$$

which therefore holds independently from assumption (1.4.15).

Next, we consider a cover of $\mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)}$ by essentially disjoint balls $B(a_\alpha, D_\alpha)$ chosen in such a way that the following properties hold:

1. (1.4.19) All elements of $B(a_\alpha, D_\alpha) \cap \mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)} \cap \mathbb{Z}^3$ are coplanar.
2. (1.4.20) $B(a_\alpha, 2D_\alpha) \cap \mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)} \cap \mathbb{Z}^3$ contains 4 noncoplanar points.

By (1.4.18), $D_\alpha > \frac{1}{\theta}$. Fixing a dyadic size $\theta R > D > \frac{1}{\theta}$ and considering α's such that

$$D_\alpha \sim D, \tag{1.4.21}$$

their number is at most

$$\frac{R\theta}{D}. \tag{1.4.22}$$

Proceeding as earlier, let $\mathcal{P}$ be a plane containing the elements of

$$B(a_\alpha, D_\alpha) \cap \mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)} \cap \mathbb{Z}^3 \tag{1.4.23}$$

and $\mathcal{E}'$ an ellipse of size r in $\mathcal{P}$ such that $\mathcal{E}' = \mathcal{P} \cap \mathcal{E}, \mathcal{E}'_{\frac{1}{r}} \supset \mathcal{E}_{\frac{1}{R}} \cap \mathcal{P}$.

Let P_1 be any point in (1.4.23) and denote τ the tangent plane to $\mathcal{E}$ at P_1, ψ the angle of τ and P. Thus (fig. 1.3.),

$$r \sim R\psi$$

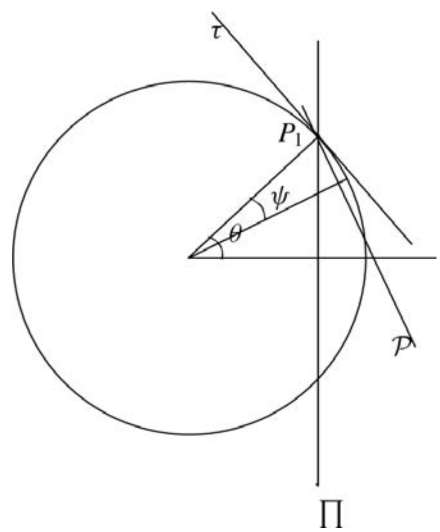

Fig. 1.3.

If $\psi \gtrsim \theta$, then $r \sim R\psi \gtrsim R\theta$ and we estimate

$$\left| B(P_1, 2D) \cap \prod_{(1)} \cap \mathcal{E}'_{\frac{1}{r}} \cap \mathbb{Z}^3 \right| \lesssim D.r^{-1/3}. \tag{1.4.24}$$

The corresponding contribution to $\mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)} \cap \mathbb{Z}^3$ is at most

$$\frac{R\theta}{D} D(R\theta)^{-1/3} \lesssim R^{2/3}. \tag{1.4.25}$$

Assume thus $\psi \ll \theta$, in which case $\theta \approx$ angle $(\prod, \mathcal{P})$ and

$$\text{diam}\left(\mathcal{E}' \cap \prod_{(1)}\right) \sim \sqrt{\frac{r}{\theta}}. \tag{1.4.26}$$

Estimate

$$\left|B(P_1, 2D) \cap \prod_{(1)} \cap \mathcal{E}'_{\frac{1}{r}} \cap \mathbb{Z}^3\right| \lesssim \min\left(D, \sqrt{\frac{r}{\theta}}\right) r^{-1/3} < \theta^{-1/3} D^{1/3}, \quad (1.4.27)$$

which collected contribution to $\mathcal{E} \cap \prod_{(1)} \cap \mathbb{Z}$ is bounded by

$$\frac{R\theta}{D} \theta^{-1/3} D^{1/3} = \frac{R\theta^{2/3}}{D^{2/3}} \leq \frac{R}{D^{2/3}}. \quad (1.4.28)$$

We may thus assume $D < R^{1/2}$.

Assume (1.4.23) contains K points and denote $d < D$ the diameter of (1.4.23). Hence, from (1.4.26),

$$\theta d^2 \lesssim r. \quad (1.4.29)$$

Partitioning $\mathcal{E}'$ in arcs of size $\frac{d}{K}$, we get thus a set $\mathcal{E}'_{\frac{1}{r}} \cap B(P_1, \frac{d}{K}) \cap \mathbb{Z}^3$ containing 3 noncollinear points P_1, P_2, P_3 from (1.4.23). Recalling assumption (1.4.20) there is $P \in B(P_1, 2D) \cap \mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)} \cap \mathbb{Z}^3$ such that P_1, P_2, P_3, P are noncoplanar and therefore

$$\text{Vol}\,(P_1, P_2, P_3, P) \geq \frac{1}{6}. \quad (1.4.30)$$

Estimate from above (since $P_1, P_2, P_3 \in \mathcal{P}$)

$$\begin{aligned}\text{Vol}\,(P_1, P_2, P_3, P) &\leq \text{ area}\,(P_1, P_2, P_3)\, \text{dist}\,(P, \mathcal{P}) \\ &\lesssim \frac{1}{r}\left(\frac{d}{K}\right)^3 \text{dist}\,(P, \mathcal{P}). \quad (1.4.31)\end{aligned}$$

(We use here the fact that $P_1, P_2, P_3 \in \mathcal{E}'_{1/r}$ and $\text{diam}\{P_1, P_2, P_3\} < \frac{d}{K}$.)

It remains to estimate $\text{dist}\,(P, \mathcal{P})$.

Letting τ be again the tangent plane at P_1, write

$$\text{dist}\,(P, \mathcal{P}) \leq |P - \overline{P}| + \text{dist}\,(\overline{P}, \mathcal{P}), \quad (1.4.32)$$

where $\bar{P} \in \tau$

$$|P - \overline{P}| = \text{dist}\,(P, \tau) \lesssim \frac{D^2}{R} < 1, \quad (1.4.33)$$

and hence

$$\overline{P} \in \tau \cap \prod_{(2)} \cap B(P_1, 2D + 1).$$

We may assume $P_1 \in \prod$. Denote ℓ_0 the line

$$\ell_0 = \prod \cap \tau \quad (1.4.34)$$

and ℓ_1 the line

$$\ell_1 = \mathcal{P} \cap \tau. \quad (1.4.35)$$

Thus $P_1 \in \ell_0 \cap \ell_1$

$$\begin{aligned}\text{dist}(\overline{P}, \mathcal{P}) = \text{dist}(\overline{P}, \ell_1)\ \text{angle}(\tau, \mathcal{P}) &= \text{dist}(\overline{P}, \ell_1)\psi \\ &\sim \frac{r}{R}\ \text{dist}(\overline{P}, \ell_1) \\ &\sim \frac{r|\bar{P} - P_1|}{R}\ \text{angle}([P_1, \overline{P}], \ell_1). \end{aligned} \tag{1.4.36}$$

By assumption, there is a point $P_4 \in$ (1.4.23) s.t.

$$\frac{d}{2} < |P_1 - P_4| \le d. \tag{1.4.37}$$

Thus $P_4 \in \mathcal{P} \cap \prod_{(1)}$. Estimate

$$\begin{aligned}\text{angle}([P_1, \overline{P}], \ell_1) &\le \text{angle}([P_1, P_4], \ell_1) + \text{angle}([P_1, \overline{P}], [P_1, P_4]) \\ &= (1.4.38) + (1.4.39).\end{aligned}$$

Since $\text{dist}(P_4, \tau) \sim \frac{d^2}{R}$, we have

$$\frac{d^2}{R} \sim \text{dist}(P_4, \ell_1).\psi$$

and

$$(1.4.38) \sim \frac{1}{d}\text{dist}(P_4, \ell_1) \sim \frac{d}{R\psi} \sim \frac{d}{r}. \tag{1.4.40}$$

Estimate

$$(1.4.39) \le \text{angle}([P_1, \overline{P}], \ell_0) + \text{angle}([P_1, P_4], \ell_0). \tag{1.4.41}$$

Since $\text{dist}(P_4, \prod) \le 1$ and $\text{dist}(P_4, \tau) \sim \frac{d^2}{R} < 1$, we get

$$2 \ge \text{dist}(P_4, \ell_0)\ \text{angle}\left(\tau, \prod\right) = \theta.\ \text{dist}(P_4, \ell_0)$$

$$\text{angle}([P_1, P_4], \ell_0) \lesssim \frac{1}{\theta d}. \tag{1.4.42}$$

Similarly,

$$\text{angle}([P_1, \overline{P}], \ell_0) \lesssim \frac{1}{\theta|P_1 - \bar{P}|}, \tag{1.4.43}$$

and hence

$$(1.4.39) \lesssim \frac{1}{\theta d} + \frac{1}{\theta|P_1 - \bar{P}|}. \tag{1.4.44}$$

It follows that

$$\text{angle}([P_1, \overline{P}], \ell_1) \lesssim \frac{d}{r} + \frac{1}{\theta d} + \frac{1}{\theta|P_1 - \bar{P}|} \overset{(1.4.29)}{<} \frac{1}{\theta d} + \frac{1}{\theta|P_1 - \bar{P}|}. \tag{1.4.45}$$

Recalling (1.4.36),

$$\text{dist}\,(\overline{P}, \mathcal{P}) \lesssim \frac{rD}{R\theta d} + \frac{r}{R\theta} \lesssim \frac{rD}{R\theta d} \tag{1.4.46}$$

and, by (1.4.32),

$$\text{dist}\,(P, \mathcal{P}) \lesssim \frac{D^2}{R} + \frac{rD}{Rd\theta}. \tag{1.4.47}$$

Substituting in (1.4.31) gives by (1.4.30)

$$1 \lesssim \frac{d^3 D^2}{rRK^3} + \frac{d^2 D}{\theta R K^3} \overset{(1.4.29)}{\lesssim} \frac{dD^2}{\theta R K^3}$$

$$K \lesssim \frac{D}{(\theta R)^{1/3}}. \tag{1.4.48}$$

Multiplying with (1.4.22), we obtain again

$$\frac{R\theta}{D} \frac{D}{(\theta R)^{1/3}} \leq R^{2/3} \tag{1.4.49}$$

as a bound on $|\mathcal{E}_{\frac{1}{R}} \cap \prod_{(1)} \cap \mathbb{Z}^3|$.

This proves Lemma 1.7.

Lemma 1.7 allows for the following improvement of Lemma 1.2.

Lemma 1.2′. *Let F^* be the discrete maximal function* (1.3.13). *Then*

$$|[F^* > \lambda]| < N^{\frac{3}{2}+} \|F\|_2^2 \lambda^{-2} \tag{1.4.50}$$

provided

$$\lambda > N^{\frac{1}{3}+} \|F\|_2. \tag{1.4.51}$$

Proof. Returning to the proof of Lemma 1.2, Lemma 1.7 implies the bound on K introduced in (1.3.17)

$$K < N^{\frac{2}{3}+}, \tag{1.4.52}$$

and (1.3.19) becomes (1.4.51) instead of (1.3.14).

Hence (1.3.32) in Corollary 1.1 holds under the assumption

$$\lambda > N^{\frac{1}{3}+} \mu^{-1}, \tag{1.4.53}$$

which leads to the following improved Lemma 1.3 and Propositions 1.5, 1.6.

Lemma 1.4′.

$$\|e^{it\Delta} f\|_4 < N^{\frac{3}{16}+} + N^{\frac{1}{12}+} \mu^{-\frac{1}{4}}. \tag{1.4.54}$$

Proposition 1.3′.

$$\|e^{it\Delta} f\|_4 \leq N^{\frac{1}{3}+} \|f\|_2 \text{ if } \operatorname{supp} \hat{f} \subset B(0, N). \tag{1.4.55}$$

Proposition 1.4′. *The 3D cubic NLS* $iu_t + \Delta u \pm u|u|^2 = 0$ *on* $\mathbb{T}^3$ *with* Δ *as in* (1.1.2) *is locally well-posed for* $u(0) \in H^s(\mathbb{T}^3)$, $s > \frac{2}{3}$.

REFERENCES

[B1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations*, GAFA 3(1993), no. 2, 107–156.

[B2] J. Bourgain, *Exponential sums and nonlinear Schrödinger equations*, GAFA 3 (1993), no. 2, 157–178.

[B-G-T] N. Burq, P. Gerard, N. Tzvetkov, *Strichartz' inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math. 126 (2004), no. 3, 569–605.

[B-P] E. Bombieri, J. Pila, *The number of integral points on arcs and ovals*, Duke Math. J. 59 (1989), no. 2, 337–357.

[Ja] V. Jarnick, *Über die Gitterpunkte auf konvexen Curven*, Math. Z. 24 (1926), 500–518.