Chapter One

Introduction

1.1 DEHN FILLING AND THURSTON'S THEOREM

Dehn filling is a basic surgery one can perform on a 3-manifold. Let M be a 3-manifold that is the interior of a compact manifold with boundary $\overline{M}$. We say that a *torus end* of M is a torus boundary component of $\overline{M}$. Let E be such a torus end, and let $\alpha \in H_1(E)$ be a primitive homology element; i.e., α is not a multiple of another $\beta \in H_1(E)$. Let Σ be a solid torus with boundary $\partial\Sigma$. Let $f : E \to \partial\Sigma$ be a homeomorphism such that $f_*(\alpha) = 0$ in $H_1(\Sigma)$. Then the identification space

$$M_\alpha = (M \cup \Sigma)/f \tag{1.1}$$

is called the α-*Dehn filling* of E. The homeomorphism type of M_α only depends on α. When there is some implicit identification of $H_1(E)$ with $\boldsymbol{Z}^2$ carrying α to (p, q), we call M_α the (p, q)-*Dehn filling* of E. Such an identification is called a *marking* of E.

A *hyperbolic* 3-*manifold* M is a 3-manifold equipped with a Riemannian metric locally isometric to hyperbolic 3-space $\boldsymbol{H}^3$. See Section 2.1. We assume that M is oriented, has finite volume, and is metrically complete. When M is not closed, we call M *cusped*. In this case M is the interior of $\overline{M}$, as above. M is the union of a compact set and finitely many ends, each one being the quotient of a horoball by a $\boldsymbol{Z}^2$ subgroup. We call these ends *horocusps*. Each horocusp is homeomorphic to a torus cross a ray and is bounded by a component of $\partial\overline{M}$. Here is Thurston's celebrated hyperbolic Dehn surgery theorem.

Theorem 1.1 (See [**T0**]; cf. [**NZ**], [**PP**], [**R**]): *Suppose M is a cusped hyperbolic* 3*-manifold and E is a horocusp of M. All but finitely many Dehn fillings of E result in another hyperbolic* 3*-manifold.*

Let $\rho : \pi_1(M) \to \mathrm{Isom}(\boldsymbol{H}^3)$ be the representation whose image is the universal covering group of M. A main step in the proof of Theorem 1.1 is the analysis of representations $\widehat{\rho} : \pi_1(M) \to \mathrm{Isom}(\boldsymbol{H}^3)$, which are suitably nearby to ρ. We will describe the HST in such perturbative terms.

1.2 DEFINITION OF A HOROTUBE GROUP

$PU(2, 1)$ is the holomorphic isometry group of $\boldsymbol{CH}^2$, the complex hyperbolic plane. A *parabolic element* $P \in PU(2, 1)$ is one that acts so as to fix a point on the ideal boundary of $\boldsymbol{CH}^2$ but no points in $\boldsymbol{CH}^2$ itself. For instance, one kind of parabolic has the form $P(z,t) = (uz, t+1)$ (where $|u| = 1$) when we normalize so that the ideal boundary of $\boldsymbol{CH}^2$ is identified with $(\boldsymbol{C} \times \boldsymbol{R}) \cup \infty$. See Section 2.3 for details.

The group in the HST, which plays the role analogous to the hyperbolic isometry group $\rho(\pi_1(M))$, is what we call a *horotube group*. Suppose that P is a parabolic element. We say that a *horotube* is a P-invariant open subset $T \subset S^3 - \{p\}$ such that $T/\langle P\rangle$ has a compact complement in $(S^3 - \{p\})/\langle P\rangle$. In the special case mentioned above, the set $\{|z| > 1\} \times \boldsymbol{R}$ is a good example of a horotube.

In general, we call the quotient $T/\langle P\rangle$ a *horocusp*. It is a consequence of Lemma 2.7 that any horocusp E is the union of a compact set and a horocusp E', which is homeomorphic to a torus cross a ray. In other words, after some pruning, a CR horocusp is topologically the same as a hyperbolic horocusp.

Let G be an abstract group, and let $\rho : G \to PU(2,1)$ be a discrete and injective representation. Let $\Gamma = \rho(G)$. Let Λ be the limit set of Γ, and let $\Omega = S^3 - \Lambda$ be the regular set. We say that Γ has *isolated type* if the elliptic elements of Γ have isolated fixed points. We say that Λ is *porous* if there is some $\epsilon_0 > 0$ such that $g(\Omega)$ contains a ball of spherical diameter ϵ_0 for any $g \in PU(2,1)$. See Section 3.5 for more details about these definitions.

Definition: ρ is a *horotube representation* and Γ is a *horotube group* if the following hold: Γ has isolated type, Λ is porous, and Ω/Γ is the union of a compact set together with a finite pairwise disjoint collection of horocusps.

It follows from Lemma 3.4 that Ω/Γ is a manifold when Γ is a horotube group. The following is another basic structural result about horotube groups:

Lemma 10.1 (Structure): *Let $E_1,\ldots,E_n$ be the horocusps of Ω/Γ. There are horotubes $\widetilde{E}_1,\ldots,\widetilde{E}_n$ and elements $\gamma_1,\ldots,\gamma_n \in \Gamma$ such that $E_j = \widetilde{E}_j/\langle\gamma_j\rangle$. Furthermore, every parabolic element of Γ is conjugate to a power of some γ_j. Thus, any maximal $\boldsymbol{Z}$-parabolic subgroup of Γ is conjugate in Γ to some $\langle\gamma_j\rangle$.*

1.3 THE HOROTUBE SURGERY THEOREM

1.3.1 The Result

Let ρ be a horotube representation and let $\Gamma = \rho(G)$ be the associated horotube group. In this case, we have the collection of maximal $\boldsymbol{Z}$ subgroups $\{H_\alpha\}$ of G such that $\rho(H_\alpha)$ is a parabolic group. We call these groups the *peripheral subgroups*. By Lemma 10.1, the conjugacy classes of peripheral subgroups are in natural bijection with the horocusps of Ω/Γ.

Let $\text{Rep}(G)$ be the space of homomorphisms of G into $PU(2,1)$. We say that a sequence $\{\rho_n\}$ *converges nicely* to ρ iff $\rho_n(g)$ converges to $\rho(g)$ for each individual $g \in G$ and $\rho_n(H)$ converges geometrically to $\rho(H)$ for each peripheral subgroup of H. That is, the set $\rho_n(H)$ converges to the set $\rho(H)$ in the Hausdorff topology. See Section 3.1 for a definition of the Hausdorff topology. We make one additional technical requirement: In the case where $\rho_n(H)$ is a finite group, we require that each element of $\rho_n(H)$ acts freely on S^3, which is to say that $\rho_n(H)$ acts with an isolated fixed point in $\boldsymbol{CH}^2$.

In the elliptic case of the nice convergence, it turns out–at least for n large–that there is a preferred generator h of H such that $\rho_n(h)$ is $PU(2,1)$-conjugate to

$$\begin{bmatrix} \exp(2\pi i/m_n) & 0 \\ 0 & \exp(2\pi i k_n/m_n) \end{bmatrix} \tag{1.2}$$

with k_n, m_n relatively prime and $|k_n| < m_n/2$. We say that $\rho_n(H)$ has *type* (m_n, k_n) in this case.

Theorem 1.2 (Horotube Surgery) *Suppose $\rho \in \text{Rep}(G)$ is a horotube representation and $\Gamma = \rho(G)$ has no exceptional cusps. If $\widehat{\rho} \in \text{Rep}(G)$ is sufficiently far along in a sequence of representations converging nicely to ρ, then $\widehat{\Gamma} = \widehat{\rho}(G)$ is discrete and $\widehat{\Omega}/\widehat{\Gamma}$ is obtained from Ω/Γ by performing a Dehn filling on each horocusp E_H of Ω/Γ corresponding to a peripheral subgroup H such that $\widehat{\rho}(H)$ is not parabolic. Relative to a canonical marking, the filling has type $(0,1)$ when $\widehat{\rho}(H)$ is loxodromic and type (m,k) when $\widehat{\rho}(H)$ is elliptic of type (m,k). If at least one cusp is not filled, then $\widehat{\rho}$ is a horotube representation of $\widehat{G} = G/\ker(\widehat{\rho})$.*

We say that a horocusp of Ω/Γ is *exceptional* if the generator of the corresponding group is conjugate to the map $(z,t) \to (-z, t+1)$. When some cusp of Γ is exceptional, there is an ambiguity of sign, and (m,k) needs to be replaced by $(\pm m, k)$. The choice of sign depends on which of two equally canonical markings we choose. See Section 8 for details.

1.3.2 Some Remarks on the HST

Relative to the canonical marking, all the possible filling slopes–i.e., the quantities m/k in the elliptic case of HST–lie in $(-1/2, 1/2)$. Thus, for a given group with flexible cusps, the HST at best gives an open cone's worth of fillings in Dehn filling space. The HST doesn't quite have the same range as Thurston's theorem, which produces essentially all possible filling slopes. One structural reason for the difference is that the elliptic case of the HST involves a transition from $\boldsymbol{Z}/n$ to $\boldsymbol{Z}$, whereas Thurston's theorem involves a transition from $\boldsymbol{Z}$ to $\boldsymbol{Z}^2$.

Our proof of the HST only uses fairly general properties of $\boldsymbol{CH}^2$. The important ingredient is a kind of thick-thin decomposition of the manifold at infinity, where the thick part does not change much with the representation and the thin part undergoes Dehn surgery. This ought to occur in a fairly general setting. Here are some settings related to the HST.

First, some version of the HST should work for complex hyperbolic discrete groups with rank-2 cusps. Actually, in the rank-2 case, one ought to be able to get the kind of transitions that more closely parallel the one in Thurston's theorem,–i.e., a $\boldsymbol{Z}$-loxodromic subgroup limiting to a $\boldsymbol{Z}^2$-parabolic subgroup. Not many groups of this kind have been studied. See [**F**] for some recent examples.

Second, a version of the HST should work in real hyperbolic 4-space $\boldsymbol{H}^4$. Indeed, one might compare the HST with results in [**GLT**]. From a practical standpoint, one difference between $\boldsymbol{H}^4$ and $\boldsymbol{CH}^2$ is that we can easily obtain the kind of convergence we need for the HST just by controlling the traces of certain elements in $PU(2,1)$. This makes it easy to show that particular examples satisfy the hypotheses of the HST. The situation seems more complicated in $\boldsymbol{H}^4$.

Third, there is some connection between the HST and contact topology. A complete spherical CR manifold Ω/Γ has a *symplectically semifillable* contact structure (See [**El**]) because a finite cover Ω/Γ bounds the symplectic orbifold $\boldsymbol{CH}^2/\Gamma$. Thus, the HST gives a geometric way to produce symplectically semifillable (and, hence, tight) contact structures on some hyperbolic 3-manifolds. Compare Eliashberg's Legendrian surgery theorem (see [**El**], [**Go**]).

We don't pursue these various alternate settings because (at least when we wrote this monograph) we had in mind the specific applications listed below. However, the interested reader might want to keep some of the above connections in mind while reading our proof of the HST.

1.4 REFLECTION TRIANGLE GROUPS

The *complex hyperbolic reflection triangle groups* provide our main examples of horotube groups. We will discuss them in more detail in Chapter 4.

Let $\boldsymbol{H}^2$ denote the real hyperbolic plane. Let $\zeta = (\zeta_0, \zeta_1, \zeta_2) \in (\boldsymbol{N} \cup \infty)^3$ satisfy $\sum \zeta_i^{-1} < 1$. Let G'_ζ denote the usual ζ reflection triangle group generated by reflections in a geodesic triangle $T_\zeta \subset \boldsymbol{H}^2$ with internal angles $\pi/\zeta_0,\ \pi/\zeta_1,\ \pi/\zeta_2$. (When $\zeta_i = 0$ the ith vertex of the triangle is an ideal vertex.) Let G_ζ denote the index-2 *even subgroup* consisting of the even-length words in the three generators $\iota_0, \iota_1, \iota_2$ of G'_ζ.

A *complex reflection* is an element of $PU(2,1)$ which is conjugate to the map $(z,w) \to (z,-w)$. See Section 2.2.3. A ζ-*complex reflection triangle group* is a representation of G'_ζ that maps $\iota_0, \iota_1, \iota_2$ to *complex reflections* I_0, I_1, I_2 so that $\iota_i\iota_j$ and I_iI_j have the same order. We call a (∞,∞,∞)-complex reflection triangle group an *ideal complex reflection triangle group*.

Let $\mathrm{Rep}(\zeta)$ denote the set of ζ-complex reflection triangle groups, modulo conjugacy in $\mathrm{Isom}(\boldsymbol{CH}^2)$. It turns out that $\mathrm{Rep}(\zeta)$ is a 1-dimensional halfopen interval. The endpoint of $\mathrm{Rep}(\zeta)$ is the element ρ_0, which stabilizes a totally geodesic slice, like a Fuchsian group. In [**GP**] Goldman and Parker initiated the study of the space $\mathrm{Rep}(\infty, \infty, \infty)$.

In [**S1**] and [**S5**], we proved the Goldman-Parker conjecture (see below) about $\mathrm{Rep}(\infty, \infty, \infty)$. Here is a mild strengthening of our main result from [**S1**].

Theorem 1.3 *An element $\rho \in \mathrm{Rep}\,(\infty, \infty, \infty)$ is a horotube representation, provided that $I_0 I_1 I_2$ is loxodromic. Any parabolic element of $\Gamma = \rho\,(G)$ is conjugate to $(I_i I_j)^a$ for some $a \in \boldsymbol{Z}$.*

There is one element $\rho \in \mathrm{Rep}(\infty, \infty, \infty)$ for which $I_0 I_1 I_2$ is parabolic. ρ is the "endpoint" of the representations considered in Theorem 1.3. We call the corresponding group $\mathbf{\Gamma}'$ the *golden triangle group* because of its special beauty. Let $\mathbf{\Gamma}$ be the even subgroup. Let $\mathbf{\Gamma_3}$ be the group obtained by adjoining the "vertex-cycling" order-3 symmetry to $\mathbf{\Gamma}$. Let G_3 denote corresponding real hyperbolic group and $\rho : G_3 \to \mathbf{\Gamma_3}$ the corresponding representation. Here is a mild strengthening of our main result in [**S0**].

Theorem 1.4 *ρ is a horotube representation. Any parabolic element $g \in \mathbf{\Gamma_3}$ is conjugate to either $(I_0 I_2)^a$ for some $a \in \boldsymbol{Z}$ or $(I_0 I_1 I_2)^b$ for some $b \in 2\boldsymbol{Z}$. The quotient $\Omega/\mathbf{\Gamma_3}$ is homeomorphic to the Whitehead link complement.*

1.5 SPHERICAL CR STRUCTURES

A *spherical CR structure* on a 3-manifold is a system of coordinate charts into S^3, such that the overlap functions are restrictions of elements of $PU(2,1)$. The complete spherical CR structures mentioned in the Preface are special cases. One can ask the general question, Which 3-manifolds admit spherical CR structures? Here are some answers.

- The unit tangent bundle of a closed hyperbolic surface admits a "tautological" complete spherical CR structure. We will discuss the construction in Section 4.3, in the related context of triangle groups. Many other circle bundles over surfaces admit complete spherical CR structures (see [**GKL**] [**AGG**]) as do many Seifert fiber spaces (see [**KT**]).

- In very recent work Anan'in and Gusevskii [**AG**] have announced that there is a complete spherical CR structure on the trivial circle bundle over a closed hyperbolic surface. This has been a long-standing open problem.

- In recent work, Falbel [**F**] has shown that there is no complete spherical CR structure on the figure-eight knot complement whose cusps have purely parabolic holonomy. In the language above, this means that there is no horotube group Γ such that Ω/Γ is the figure-8 knot complement.

- Here is an example of the kind of result that can be proved using contact-/symplectic-based manifold invariants: Let P be the Poincaré homology 3-sphere. Then P is the quotient of S^3 by a finite subgroup of $PU(2,1)$, and hence, it admits a complete spherical CR structure. In [**L**] it is proved that the oppositely oriented version of P is not *symplectically semifillable*. Hence, the oppositely oriented version of P does not admit a complete spherical CR structure compatible with that orientation.

- In [**S2**], we gave a computer-aided construction of a closed hyperbolic 3-manifold with a complete spherical CR structure. The associated group is the $(4,4,4)$-complex reflection triangle group in which $I_0I_1I_0I_2$ is elliptic of order $n = 7$. Experimental evidence suggests that the same result holds when $n > 7$, and this evidence led us to the HST. (We originally wanted to use the HST to deal with the $(4,4,4)$ triangle groups, but we were unable to prove the analogue of Theorem 1.4 in the $(4,4,4)$ case.)

If M is a 3-manifold with n torus ends, then we can identify the set of closed fillings of M (all cusps filled) with the subset $\mathcal{D}(M) \subset \boldsymbol{Z}^{(2n)}$ consisting of the $2n$-tuples $(p_1, q_1, ..., p_n, q_n)$, where p_j and q_j are relatively prime for all j. We say that a cusped manifold is *CR positive* if there is an open cone in $\boldsymbol{R}^{2n}$ such that all but finitely many points in $\mathcal{D}(M) \cap C$ result in fillings that have complete spherical CR structures. If M is both hyperbolic and CR positive, then a positive density subset of fillings of M admit both hyperbolic and spherical CR structures. Combining the HST with an analysis of the flexibility of the group $\boldsymbol{\Gamma_3}$ from Theorem 1.4, we prove the following theorem.

Theorem 1.5 *This Whitehead link complement is CR positive.*

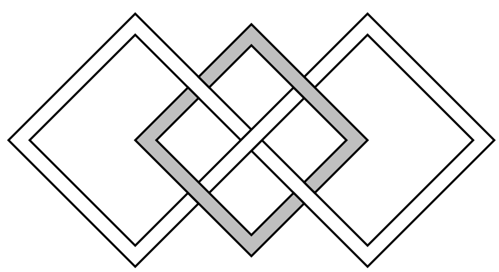

Figure 1.1: The Whitehead link

In [**A**], it is shown that any infinite list of closed hyperbolic 3-manifolds, with uniformly bounded volume, contains an infinite number of commensurability classes. Thus, Theorem 1.5 and Theorem 1.1 combine to prove the following corollary.

Corollary 1.6 *There is an infinite list of pairwise incommensurable closed hyperbolic 3-manifolds which admit complete spherical CR structures.*

Generalizing Theorem 1.5, we will prove the following.

Theorem 1.7 *Let T be a finite tree, with an odd number N of vertices, such that every vertex has valence at most 3. Suppose also that at least one vertex of valence 1 is incident to a vertex of valence 2. Then there is an $(N+3)$ cusped finite cover of the Whitehead link complement, canonically associated to T, which is CR positive.*

The side condition about the valence-2 vertex seems not to be necessary; it is an artifact of our proof.

One of the great consequences of Theorem 1.1 is that the set of volumes of hyperbolic 3-manifolds is well ordered and has the ordinal structure of ω^ω. See [**T0**]. Using Theorem 1.7, we easily get a related statement.

Corollary 1.8 *Let S be the set of volumes of closed hyperbolic 3-manifolds, which admit complete spherical CR structures. Let $S^{(0)} = S$, and let $S^{(n+1)}$ be the accumulation set of $S^{(n)}$. Then $S^{(n)} \neq \emptyset$ for all n.*

The basic idea in proving Corollary 1.8 is to note that $\mathrm{vol}(M_n) \to \mathrm{vol}(M)$ if $\{M_n\}$ is a nonrepeating sequence of Dehn fillings of M. If we have a CR positive manifold with many cusps, then we can produce a lot of limit points to the set S simply by controlling the rates at which we do the fillings on different cusps. This is the essentially the same trick that Thurston performs in [**T0**].

1.6 THE GOLDMAN-PARKER CONJECTURE

We continue the notation from Section 1.4. Say that a word in a reflection triangle group G has *genuine length* k if the word has length k in the generators and is not conjugate to a shorter word. Referring to the representation space $\mathrm{Rep}(\zeta)$, the endpoint ρ_0 maps the words of genuine length 3 and 4 to loxodromic elements. The following is a central conjecture about the complex hyperbolic reflection triangle groups.

Conjecture 1.9 *Suppose that $\rho \in \mathrm{Rep}(\zeta)$ maps all words of genuine length 3 and 4 to loxodromic elements. Then ρ is discrete.*

The (∞, ∞, ∞) case of Conjecture 1.9, the first case studied, was introduced by Goldman and Parker in [**GP**]. Goldman and Parker made substantial progress on this case, and we gave a computer-aided proof of the conjecture in [**S1**]. Then we gave a better and entirely traditional proof in [**S5**]. Computer experiments done by Wyss-Gallifent [**W-G**] and me suggested the general version of the conjecture.

In Part 3 we will combine the HST with Theorems 1.3 and 1.4 to prove the following result.

Theorem 1.10 *Suppose that $\rho \in \mathrm{Rep}(\zeta)$ maps all words of genuine length 3 to loxodromic elements. If $|\zeta|$ is sufficiently large, then ρ is a horotube representation and, hence, discrete.*

Here $|\zeta| = \min(\zeta_0, \zeta_1, \zeta_2)$. Unfortunately, we don't have an effective bound on $|\zeta|$. For $|\zeta| > 14$, it turns out that the words of genuine length 4 are automatically loxodromic if the words of genuine length 3 are loxodromic. Compare [**P**] or [**S3**].

To help understand the triangle groups, we will prove the following *addendum* to the HST.

Theorem 1.11 *Let ρ, $\widehat{\rho}$, and G be as in the HST. Additionally suppose that $\widehat{\rho}$ is injective and that $\widehat{\rho}(g)$ is parabolic iff $\rho(g)$ is parabolic for all $g \in G$. If $\widehat{\rho}$ is sufficiently far along in a sequence that converges nicely to ρ, then there is a homeomorphism from Ω to $\widehat{\Omega}$ that conjugates Γ to $\widehat{\Gamma}$.*

With a bit more work, we deduce the following corollary.

Corollary 1.12 *Suppose that $|\zeta|$ is sufficiently large and Γ_1 and Γ_2 are even subgroups of two ζ-complex reflection triangle groups, whose words of genuine length 3 are loxodromic. Then Γ_1 and Γ_2 have topologically conjugate actions on S^3. In particular, the limit sets of these groups are topological circles.*

1.7 ORGANIZATIONAL NOTES

The monograph has four parts. Parts 2, 3, and 4 all depend on Part 1 but are essentially independent from each other. Parts 2, 3, and 4 can be read in any order after part 1 has been finished.

Part 1 (Chapters 1–5) is the introductory part. Chapter 2 presents some background material on complex hyperbolic geometry. Chapter 3 presents some background material on discrete groups and topology. Chapter 4 introduces the reflection triangle groups, relating them to the HST. In Chapter 5 we give a heuristic explanation of the HST, comparing it with Thurston's theorem. Chapter 5 is the conceptual heart of the HST. Most of our proof of the HST amounts to making the discussion in Chapter 5 rigorous.

In Part 2 (Chapters 6–13), we prove the HST and Theorem 1.11. We give a more extensive overview of Part 2 just before starting Chapter 6.

In Part 3 (Chapters 14–18), we derive all our applications, using Theorems 1.3 and 1.4 as black boxes.

In Part 4 (Chapters 19–23), we prove Theorems 1.3 and 1.4. Our proof of Theorem 1.4, which is fairly similar to what we did in [**S0**], is almost self-contained. We omit a few minor and tedious calculations, and in those places we refer the reader to [**S0**] for details. Our proof of Theorem 1.3 is sketchier but tries to hit the main ideas in [**S5**]. Most of Part 4 appears in our other published work, but here we take the opportunity to improve on the exposition given in our earlier accounts and also to correct a few glitches. (See Sections 21.1.2 and 22.4.)

We have written an extensive Java applet that illustrates the constructions in Part 4 graphically and in great detail. We encourage the reader to use this applet as a guide to the mathematics in Part 4. As of this writing, in 2006, the applet is called *Applet 45*. It currently resides on my Web site, at http://www.math.brown.edu/∼res/applets.html.