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**Gaurav Suri & Hartosh Singh Bal: A Certain Ambiguity**

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CHAPTER

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YESTERDAY I FOUND THE CALCULATOR my grandfather gave me on my 12th birthday. It had fallen behind the bookcase and I saw it when I was rearranging the study. I had not thought about it for years, yet when I held it, it seemed as familiar as ever. The “I” in Texas Instruments was missing, as it had been for all but two days of its life; the buttons still made a confirming clicking sound when pressed; and when I put in some new batteries, the numbers on the LCD shone through with a blazing greenness, more extravagant than the dull grey of the modern calculator. My grandfather had intended this calculator to mark a change in my life—a new direction. As it turned out, it did mark a change, though not the one he had in mind.

I punched in the number 342 without thinking about it. It was the same number I had entered 25 years ago when the calculator was brand new.

“Want to see some number magic?” my grandfather had asked as he watched me push the buttons more or less randomly. I was sitting in his room completely taken by his birthday present, if not quite sure what to do with it. He put his notebook down, temporarily giving up on the math problem that had resisted solution since morning.

“Yes, Bauji!” I had rushed over to him.

“Enter any three-digit number in your calculator and do not let me see it.” That is when I had first entered 342, the same three digits I entered now. “OK. Now enter the same number again, so you have a six-digit number,” he had said. I punched in 342 again, so now I had 342342 entered in

my calculator. “Now, I do not know the number you have in there, Ravi, but I do know that it is evenly divisible by 13.”

By “evenly divisible” he meant that there would be no remainder. For example, 9 is evenly divisible by 3 but not by 4.

Bauji’s claim seemed fantastic to me. How could he know that my number, randomly chosen and completely unknown to him, would be evenly divisible by 13? But it was! I divided 342342 by 13 and I got 26334 exactly, with no remainder.

“You’re right,” I said, amazed.

He wasn’t finished, though. “Now, Ravi, I also know that whatever number you got after you divided by 13 is further divisible by 11.” He was right once again. 26334 divided by 11 was 2394. Why was this working? “Take the number you got and divide by 7. Not only will it divide evenly, but you will be surprised at what you get.” He had begun his pacing and I knew that he was as excited as I was.

I divided 2394 by 7 and I got 342! “Oh! Oh! It’s the number I started with! Bauji, how did this happen?”

My grandfather just sat there, grinning at the completeness of my astonishment. “You will just have to figure that one out Ravi,” he said, walking out to check the state of his tomato plants, the newest additions to his vegetable garden in the backyard. He seemed to be the only person who could grow tomatoes in New Delhi’s dry summer heat.

The first thing I did was to check the divisions by hand. My hypothesis was that Bauji had rigged the new calculator somehow. But no, the numbers worked out exactly the same way when I did the long divisions by hand. Next, I decided to try this with some other three-digit numbers. The same thing worked every time. Whatever the repeated number, I could divide it evenly by 13, 11, and 7, and each time I got back to the number I started with. A few minutes of checking and rechecking convinced me that this property was true of any three-digit number. I tried doing the same thing with four-digit numbers, and it did not work any more. Neither did it work for two-digit numbers. What was going on?

I tried reversing the order. Instead of dividing first by 13, then 11, and then 7, I divided the six-digit number first by 7, then 11, and then 13. It made no difference at all. After dividing by each of those three numbers I would get back my original three-digit number. Why was this happening?

I wasn’t getting anywhere and it was getting to be dinnertime. Ma had already called me twice. I knew that risking a third “I’ll be right there” would be unwise, and so I put my notebook away and headed to the

kitchen. I stopped thinking about the problem. And then mysteriously, out of nowhere, just when I was wondering if Ma would let me have ice cream, a new idea occurred to me. Even now, with more experience in such things, I cannot quite explain the inception of the moment of insight—the Aha moment—when out of nowhere a new idea comes, and chaos is replaced by understanding.

My first real Aha moment was at the dinner table, two days after my 12th birthday. The idea that loosened the knot was the realization that division was the reverse of multiplication, a fact I had long known, but never applied in quite the fashion this problem demanded. Instead of dividing by 13, 11, and 7 one at a time, why not multiply them together and then divide by the product all at once? Would this approach even lead to the same answer? I thought it would, and a confirming example showed this to be the case. In my head I divided the number 24 first by 2 and then by 3. I got 4 as the answer. Next I divided 24 by 6 (which is  $2 \times 3$ ) and got 4 as well. So, it should work to divide the six-digit number by the product  $13 \times 11 \times 7$ . I did some more examples on a paper towel just to be sure. It appeared that I might be onto something.

Ma noticed that I was completely distracted. “Ravi, what’s going on? Why aren’t you eating?” But I hardly heard her. I had to find out what  $13 \times 11 \times 7$  was; perhaps that would lead me to understand why Bauji’s magic worked.

“I’ll be right back, Ma,” I said, getting up quickly before she could react.

“No, sir, you won’t. Sit here and finish your dinner.” She looked like she meant it.

But my grandfather must have known what I was going through.

“Its okay Anita. Let him go.” He must have been convincing enough, for I saw unwilling permission in my mother’s eyes.

I ran up the stairs two at a time and fired up the calculator.  $13 \times 11 \times 7$  was . . . 1001.

I knew this was terribly significant, though as yet I was not quite sure why. I tried dividing 342342 by 1001. As I expected, I got 342. But wait a minute. That must mean that the reverse is true as well. So if I multiply 342 by 1001 I should get 342342. Of course!  $342 \times (1001) = 342 \times (1000 + 1) = 342,000 + 342 = 342342$ . So, taking a three-digit number and repeating it was just like multiplying it by 1001. And if you multiplied it by 1001, you could divide the six-digit number by 1001 to get the original three-digit number. What had confused me was dividing by 13, 11, and 7—but by dividing

by those numbers I was in effect dividing by 1001. How simple! How could I have not seen it?

“Bauji! Bauji! I’ve got it!”

And he too bounded up the stairs two at a time, just as I had a few minutes before. “Tell me,” he gasped. He was out of breath, but not overly so—not bad for 85.

“When you asked me to repeat the three-digit number, you were actually having me multiply it by 1001. And then you made me divide it by 1001, except you did it in three stages. So of course I ended up with the same number I started with!”

He looked at me and smiled. “Good work,” he said ruffling my hair, his most characteristic gesture of affection. “I’ll give you another one to think about tomorrow.” When I told him I wanted another one right then, he laughed. “Looks like you’ll be the next mathematician in the family. We might have to send you to The Institute of Advanced Studies! Maybe we could collaborate on some research.”

Sitting in his lap, surrounded by his books and papers, I could not imagine a better fate.

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The next evening Bauji died. I remember going to his room, calculator in hand, ready for my next puzzle, but as I neared the door I heard my mother’s voice—really a whisper—coming from inside the room. The door was ajar, about half open. I could hear an urgent pleading in her tone even though her sounds did not seem to have the rhythm of words. From the hallway I saw that she sat on the floor, cross-legged, near my grandfather’s desk. She had Bauji’s—her father’s—head in her lap and she was massaging his forehead, beseeching him. Even though I had never seen a dead person before, and even though I was not standing near him, I could tell that Bauji was gone. His posture had an unalterable finality that sleep lacks. For the longest time I couldn’t move. I stood in the doorway of his room which suddenly seemed extraordinary in its sameness: his desk was piled high with its usual mountain of mathematics books; three of them were open. On the far wall his books were spilling out of the two large bookcases; many were on the floor. On the wall near me were his music records and cassettes, mostly instrumental jazz. The tape-recorder was set on auto-replay and was then softly playing Louis Armstrong’s trumpet rendition of “Summertime.”

I walked across the room to Ma unsure what to say or do. After many minutes she seemed to understand that she was no longer alone. For a second

I saw her face collapse with grief. Her eyes closed tightly on themselves and tears flowed from the corners. Then she quickly pulled me towards her so I wouldn't see her cry. From over her shoulder I could see Bauji's last expression. He wore the face of happy surprise, as if he had at last glimpsed the solution to a difficult mathematical problem, and the answer was not at all the one he had expected.

As my mother hugged me more tightly the calculator slipped from my hands and fell to the floor near Bauji's hands. The "I" of Texas Instruments fell out and could not be reattached despite my, and then my father's, best attempts.

Many years later my mother told me that Bauji had decided that I had a mathematician's mind and he had wanted to push me to excel in the subject. The calculator was to have been a catalyst in my development as a mathematician. "I'm going to use it to get Ravi passionate about mathematics," Bauji had said.

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Although Bauji was secular, he participated in religious functions with some regularity. "Religion is about community," he would announce after each such function, "and everyone needs community." I once heard an uncle refer to him as an "atheist with goodwill towards God." I didn't know quite what he meant, but it somehow seemed to fit Bauji.

So when he died, no one was quite sure what type of ceremony was required. He would be cremated, that much was clear. All Hindus are cremated, and while Bauji never referred to himself as a Hindu, he never repudiated the affiliation either. But the family divided on the extent of the rites that should accompany the sacrament. Bauji's sister insisted upon a full recitation from the scripture, sprinkling of holy Ganga-jal, spreading of gold dust, application of sandalwood paste, the presence of six Brahmans, and the lighting of the pyre by the eldest male descendant.

My mother disagreed. "Bauji liked simplicity, he would not have wanted all this," she said. After much wrangling (that got her crying) Ma prevailed on every point except the lighting of the pyre by the eldest male descendant.

"There is no other option. It has to be a male descendent or his soul will not be properly liberated. It says so in the Vedas," insisted my great aunt, and on this one issue she refused to give in. And since Bauji had no sons and I was the only grandchild, the eldest male descendant they were talking about was me.

When I reached out with the kindling to light the funeral pyre, my hand started to shake. The shaking was strong, and it grew stronger when

I unexpectedly started to shiver. I was embarrassed: I had wanted to present an image of dignified grief to my relatives—so becoming in a 12-year-old faced with the passing of his grandfather, whom he so adored—but it wasn't working. I had the odd sensation of watching myself from outside my own body, as if I were floating around the entire ceremony, watching it from above. I could see the boy in the middle who now (as a final embarrassment) appeared to be crying, who was losing the battle to still his hands, which were stubbornly refusing to obey his commands. Then I saw my father hold the boy from behind and steady his wrists. At his touch, I came back into myself. The flame caught. The pyre hissed.

. . .

A few days after all the relatives left I got in the habit of going to Bauji's room every afternoon after school. I would lie down on the floor in the center of the room and imagine he was still sitting on his desk by the window. When he was wrestling with a problem he would sit there with his eyes shut and his body perfectly still. He would stay like that for a long time and then, every once in a while, he would sit up very straight and furiously start writing in his notebook. If he liked what he wrote, he would jump up, as if released by a spring, and pace with great energy and intensity, muttering to himself, or sometimes to me, "Could this be it? Could this be it?" Sometimes he would end these walkabouts with a loud "Ha!" and take me in his arms and throw me high up, nearly to the ceiling, and then catch me under my armpits as I came down. "I see it now, Ravi! I see it!" Now and then he would challenge me to have a go at a mathematical question, and, if I succeeded, he and I would do a joyful postmortem on the insight that cracked the case.

Bauji saw grace in mathematics, and sometimes I could see glimpses of what he saw. Now, without him, there was only the monotonous drone of doing what needed to be done.

I did try to read his mathematics books, but their pages seemed cold and lifeless. The symbols spread themselves on page after page without reason or beauty. I looked at his handwritten notebooks and they too had an alien feel, except for one page whose margin contained the notation "Show this to Ravi" next to some ominous-looking calculations. For two days I tried and failed to decipher what he might have wanted to show me. I could only tell that it seemed to have something to do with prime numbers and infinity.

After I was beaten by his math, I took to going to his room and listening to his jazz records. At first they, too, seemed without order, like the mathematical symbols in his books. There was none of the predictable, repeating

structure of the music I was accustomed to; instead there were notes that seemed floated on the spur of the moment, without a planned arrangement to guide them.

Then suddenly one day, I got it. I was listening to Charlie Parker (*Crazedology*) and I had a musical Aha moment. I realized that in most of these records there was, in fact, an underlying structure which allowed for inspired improvisation: the ensemble would first play the tune from beginning to end, with the melody played by the horns, and the harmony played by the rhythm section—the piano, bass, and drums. Then, as the tune went on, the rhythm section would continue to play the harmony while each horn improvised a solo. The soloist would select notes available within the harmonic structure while incorporating the soul of the original melody, but with his notes he would create something new each time he played. And it was all done with a casual, understated coolness. Within two weeks I was hooked.

So I spent the afternoons and evenings listening to Parker, Armstrong, Ellington, and then Goodman and Getz. For two months my parents let me be. They must have heard the music coming from Bauji's room—the music that everybody except Bauji thought was strange—yet they did not ask me about it.

It was not until a week before my final examinations that my mother came to institute some course correction. I was sprawled on the floor, my eyes shut and my feet keeping pace with the changing moods of “West End Blues.” She tapped me on the shoulder and asked if she could turn off the record player. “We need to talk,” she said. I could tell she was picking her words carefully. She told me she knew how much I missed Bauji and she knew I was listening to his music to “stay connected with him.” I wasn't sure of this—I thought I was listening to his music because I liked it—but I did not think it best to volunteer this information. “Ravi, it's time to move on. Bauji wanted you to follow a path in life and you can't get on that path if all you do is listen to this . . . music.”

Then she gave me the big news. “Bauji has left you a lot of money. Most of his life savings, actually. His will says that you must use this money to go to college in America. And you can't do that unless you keep doing well in school. You have one week to rescue your grades. If you want to go, you have to bear down—not just this week, but all the way through high school.”

America. Land of freedom. Land of Louis Armstrong, The Institute of Advanced Studies, and wide open roads that could take you anywhere. Of course I wanted to go. But more importantly, Bauji wanted me to go.

So I studied hard. Over the next six years I became a repository of facts about accounting rules, thermodynamics laws, Sanskrit verb types, inorganic compounds, and different rock structures. My grades were excellent, my capacity to store information phenomenal. But it was a joyless endeavor. I excelled in all subjects but saw beauty in none. Even mathematics lost its luster; it became like a game with well-defined rules to be followed for the sole purpose of acing examinations.

But get the grades I did, and one Friday evening just before my eighteenth birthday, I got a letter from Stanford University inviting me to come and study there.

. . .

The first person I met in California was Peter Cage. Peter had volunteered to pick up international students from the airport and drive them to campus. He greeted me with a wide smile and, unexpectedly, a hug. “Welcome to America,” he said, appearing to mean it. When we were in his car (a newish Toyota), I asked him what motivated him to volunteer to help foreign students. “Looks good on the resume,” was his answer. “A multinational company might look upon this experience very favorably.” He said this without guilt or apology, and I instantly liked him for it. I knew others who signed up for causes to get credit of one kind or another, but invariably they would ascribe their volunteerism to a higher calling. Not Peter. He and I became friends, then roommates.

Peter was stubbornly wholesome. He ran three miles every morning, ate right steadfastly, kept his room orderly (all his books were always stacked; mine never were), did his homework on time, and was never noticeably down. Even more incredibly, he seemed to be singularly free of any doubt. He was majoring in business because it was the best way to get into investment banking; he would practice banking because it was the best way to get rich; he would get rich because it would lead to freedom. And this wasn’t all talk either. Now, 19 years after our first freshman semester, Peter is one of the leading investment bankers in the technology sector in Silicon Valley. He decided what he was going to do, and then he did it.

I, on the other hand, was filled with doubt. I had difficulty getting interested in any subject. Getting good grades was not really an end in itself anymore because, in my mind, my contract with Bauji was fulfilled the day I was admitted to Stanford. And without the grade imperative, I drifted. I had brief flashes of interest in astronomy, Roman history, and game theory, but nothing really took. I had great difficulty choosing a major. It was not

until the second semester of my junior year that I finally did so—and that, only at my father’s urging. He thought that economics would make me attractive to a wide variety of corporate recruiters. Having no vision of my own, I went along with his.

Peter, too, was enthusiastic about my choice of major. “You can’t go wrong with economics. You can do anything with it, even investment banking.” But the thought of dedicating my life to any one thing seemed too heavy to me. Whatever I picked, how could I be sure that I had picked well? How could I be sure that what I was going to dedicate my life to was worthy? This lack of certainty became a theme for me. The big choices of career path and specialization had laid the seed, but gradually even lesser things, such as which class to take or which book to read next, caused internal debate. I wanted some way to know that my choice was right.

The one event free of any such internal strife was “Thursday Night Jazz,” a student jam session where anyone could come and perform. It typically started at 11:00 p.m. and went on ‘til 2:00 a.m., or until everyone who wanted to play had gotten a turn on stage. The best musicians were allowed to play early—before midnight—while there was still an audience to be had. After midnight most everybody left, and the only people remaining in the audience were other musicians who were yet to play. Except me. I would come early and, more often than not, stay till closing, even though I came to listen, not to play. I did play once—it was during my sophomore year—when I allowed beer and the enthusiastic goading of Peter Cage to overcome my self-consciousness and banged out an insipid “The Way You Look Tonight” on the piano. But truth be told (as Peter frequently reminded me), I was no worse than 90% of the people who got up there; I just had a harsher internal critic.

But my critic was more sympathetic when it came to other people. I easily tolerated their mistakes. I told myself that I didn’t come to listen to jazz perfection—a Miles Davis CD was all anyone needed for that. I came to listen to live performances that, though flawed, were more immediate and powerful than any recording (both for the player and the listener). With few exceptions, every person who came up on stage had heard the beauty of a perfectly executed improvisation. The fact that the tones playing in their heads were somewhat different from the ones coming out of their instruments was unfortunate, but in my view did not negate the nobility of their attempt. Despite all the botched harmonics and the poor timing, there were still moments of beauty. You just had to wait for them.

By the beginning of my senior year I had become a recognized regular and knew many of the musicians who played frequently. I would sit at the same spot, and if I was alone I'd carry along whatever I was reading. Sometimes Peter came with me, but he would always leave by midnight. "I need to get up early," he'd say without pride or regret. But one Thursday just before the first semester of our senior year, because classes were yet to start, he made an exception and stayed late. And that was the night I met Nico Aliprantis. It was Peter who pointed him out. "See that guy over there? He's the best math teacher I've ever had," he said.

At that time Nico was 62, which made him approximately three times as old as most people in the room. But he fit right in. His walk was tall and easy, his manner comfortable, and his mouth always on the verge of an amused smile. He chose a table near the stage, put his motorcycle helmet aside, and proceeded to roll his own cigarette.

"What class did you take from him?" I asked Peter. To date, I had been unaware that he cared about the quality of math teachers.

"Statistics," he said. "He was the only teacher that ever made math seem like a natural thing, not just a bunch of rules."

Nico listened to the music attentively. From time to time someone (probably a former student) would stop by his table and say hello. Most professors would have rated a brief nod and that too only if there happened to be some accidental eye contact, but in his case there seemed to be a reservoir of genuine goodwill. On two occasions a student pulled a chair up to his table and stayed for a chat.

Just before closing Nico went to the stage and asked for the saxophone. He played an old Charlie Parker tune whose name I could not place, but I had heard it before; it was one of Bauji's favorites. After establishing the refrain, he began to improvise, and I knew within a minute that he was good. He played effortlessly. He knew how to get from note to note seamlessly, with a light touch and his own unique style that he somehow intertwined with Parker's. You heard Charlie Parker but you also heard Nico Aliprantis, and the two coexisted with ease. Towards the end he got tangled up and lost his way. His eyebrows squeezed together and his forehead wrinkled, and for a second he looked angry with himself. But then he decided to finish and played a nice sequence to bring the tune to a logical conclusion. He bowed and everyone clapped—some, because he was different from the rest, and others, because he was good.

After everyone had played and the Coffee House was closing down, Peter and I caught up with Nico. "Dr. Aliprantis, you were fantastic!" said Peter.

He smiled and looked at us, recognizing Peter. “You’ve taken one of my classes,” he said, peering at him from behind his glasses. And then after a second, “Peter Cage, right?”

I was surprised he remembered—he must have had hundreds if not thousands of students. But then he said, “You were great in that statistics class. I kept saying you should study mathematics instead of business,” and I understood then that Peter had distinguished himself enough to be memorable.

“Dr Aliprantis, this is my friend Ravi Kapoor,” Peter said, turning towards me. As I shook his hand I told Nico that I recognized the Charlie Parker tune, but couldn’t recall the title.

“Now’s the Time’,” he said, looking at me more closely. “You must know jazz because that’s not one of Bird’s most famous recordings.”

Before I could reply Peter jumped in. “Ravi knows a lot about jazz.”

Nico smiled. “Do you play?”

“Not well; otherwise I’d do it as a career,” I said.

Nico nodded, earnest for the first time. “I’m the same way,” he said. “This math gig was a fallback choice, though fortunately I love the subject and I’m good at it, much better than I am at jazz anyway.”

“You were good,” I told him. “That was a great sequence you created and it worked perfectly except for that little bit at the end.”

He shook his head, “I may be good compared to some guy on the street, but I’m no Charlie Parker.” He said it so matter-of-factly that there was nothing further for Peter and me to say. We stood there in silence for a few seconds and then I saw Nico notice the strain and make a conscious decision to steer the conversation away from himself. “So what else do you like besides jazz?” he asked me.

Nothing really, was the truth. “I used to love mathematics,” was what I came up with instead.

“Used to?” Nico asked. He asked so softly and with such benevolent curiosity that I found myself telling him the truth.

“My grandfather made mathematics inspiring and fun. I’ve never had anyone else who could enthuse me the way he could.”

Nico smiled at that. “There’s a challenge!” he laughed. “Listen,” he said arriving at a decision, “Why don’t both of you sign up for the class I’m teaching this fall? It’s called ‘Thinking about Infinity’. It’s Math 208, I think. You should check it out; it should be an interesting class. We start Monday.”

Walking home that night Peter and I talked about whether we should accept Nico’s invitation. Peter already had an offer from Morgan Stanley that

he was going to accept. He was one of the few people who had a job offer at the beginning of senior year—most people got offers later, typically in the fall semester. Without the pressure of the job hunt, he had the luxury of experimenting with “fun” classes and thought that Nico’s class would fit the bill. I, on the other hand, had declared my major only a semester ago and needed to take five economics classes to graduate on schedule.

“It doesn’t make sense for you, though,” observed Peter making the same calculations I had just gone through. “You have to be Mr. Economics this semester.” I knew he was right.

But that night, just before falling asleep, I decided to sign up for Nico’s class after all. I knew this would mean having to take (a nearly impossible) *six* economics courses next semester or else taking a class in the summer, which would be a huge financial strain on my family (they were augmenting Bauji’s bequest). But there was something about Nico.

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There were about 15 students who showed up for “Thinking about Infinity.” Peter, as was his custom, had arrived early and found a spot in the front row. The other faces seemed unfamiliar save for a slender, curly-haired saxophone player I had seen a few times at Thursday Night Jazz. His music had not been memorable, but for some reason his name had stuck with me: Adin something. He sat in the front talking to Peter, who evidently knew him from somewhere.

Nico entered the room with the same languid ease that he had shown entering the Coffee House. “Good morning everybody. My name is Nico Aliprantis and we’re going to spend this semester using our finite brains to think about infinity.” He smiled at his own line. I wondered if it was improvised.

He quickly went over the logistics: we would meet once a week for the next 10 weeks, each class would be three hours long with a 10-minute break in the middle, office hours would be Wednesday afternoon, and the grades would be based on two take-home tests and the quality of class participation. No prior mathematics was required—This was a mathematics course for liberal arts majors. No textbook was required either; he would hand out notes when necessary. A student asked how people were supposed to study without a textbook.

“You’ll see,” Nico replied without disguising his sigh. He must have gotten this question all the time. “I think you’ll find that attending class and thinking about the problems I present to you from time to time will provide all the structure you need.”

He took off his glasses and faced the room. “There are two themes that are going to run throughout this course. I want to talk about them up front so that as we go into the subject matter you know what to look for. First, if you allow yourself to, you will find great beauty here. I think that mathematics is beautiful at its core; it is much more like a musical piece than an accounting formula.” He looked up to see how the class was receiving this idea, and that’s when he happened to catch my eye. “Much more like a jazz piece,” he said with a half-wink. “G. H. Hardy, a famous English mathematician, said that good mathematics is about making good patterns. A painter makes patterns with shapes and colors, a poet with words. A mathematician makes patterns with ideas.” He said “ideas” loudly and the word seem to reverberate in the silence that followed.

After a minute or so in which no one said anything, I could tell that Nico was scanning the room looking for someone to talk to. He settled on Adin. “You, sir,” he said pointing, “what is your name and what do you study?”

Adin’s deep voice did not match his slender frame. “Adin Kaminker. I’m majoring in philosophy,” he said.

“Adin, do you have a favorite poem or song?”

“Sure,” said Adin. “I quite like poetry actually.”

“Excellent. Okay, may I ask you to recite a few lines from a poem that you find particularly beautiful?”

Adin did not hesitate. He picked an old favorite of Bauji’s. “The woods are lovely, dark and deep / But I have promises to keep / And miles to go before I sleep / And miles to go before I sleep.” His recital was practiced and smooth. “That’s by Robert Frost,” he concluded.

The class collectively turned towards Nico, their heads moving in unison right after Adin finished his recital, like a gallery watching a tennis match. “Thank you, Adin,” Nico said, bowing his head in appreciation. “You recited the lines beautifully.”

He looked up, addressing the whole room now, not just Adin. “Now let’s imagine Robert Frost writing those lines. Maybe he played around with which words to use—perhaps at first he used the word ‘forest’, instead of ‘woods’. Maybe he tried many different word-sequences in many different rhythms until he got this one. And when he did, you can bet he knew that he was onto something, that he had created something beautiful. As soon as he had those lines, I’m sure he knew that they were right. They appealed to his sense of aesthetics.”

Nico was pacing. He was into it. “Mathematics is done in the same way,” he continued “Most mathematicians have an aesthetic sense that

guides them toward the problems they try to solve and in the ways they approach them. They try many things and then, sometimes seemingly out of nowhere, an idea comes. The idea simplifies everything, puts everything in harmony. And when they have the idea they often know that they are right, even though they have not worked out all the details. With practice they get an aesthetic sense, not unlike a poet's I imagine."

A hand shot up in the back row. It was a goatee-wearing guy in beach flip-flops, shorts, and a longish—but surprisingly disciplined—pony tail.

"Your name please?" asked Nico. It turned out that unlike most teachers, Nico had a good memory for names.

"Percy Klug, but most people call me PK."

"Go ahead, PK," he said.

"If mathematics is so beautiful, why haven't I ever heard anyone talk about it that way before?"

He was right. Mathematics was seldom seen to be beautiful. Bauji saw it that way, but he was the only person I knew who held that opinion—until now.

"I'm not sure," said Nico. "Maybe it's because mathematics is not a spectator sport. You have to do it to appreciate it, and doing it requires patience and persistence. You can love a song without being able to sing, but that doesn't work in mathematics. Nevertheless, the beauty is there for you to find." He took a sip from his coffee mug, making a slurping noise. "So the first theme is beauty. Keep a look out for it. It's not really unique to this class; I find a lot of different branches of mathematics to be beautiful. But the second theme, I think, is especially true for us. This class is also about understanding how humans think and understanding the limits of what we can think." Nico paused and looked outside towards the courtyard. When he spoke again his voice was softer and more distant. "The story of infinity is a story of how far the human mind can take us. But it is also the story of boundaries that we may not cross, no matter what. We will see amazing facts that must be true but also raise tantalizing questions that seem to be unanswerable. Not because mathematicians just happened not to have found an answer so far, but rather because they couldn't possibly. Our current set of assumptions about infinity are not strong enough to lead to an answer to some questions. *Ever.*" I didn't understand all the things Nico said but was captivated by the way he said them—like a man of faith expressing reverence in a place of worship. There was motionless silence in the ensuing pause. Then I saw Adin fish out his notebook and write something down. His pencil sounded surprisingly loud.

“You’ll see what I mean as the class progresses,” said Nico, coming back to us. “But let’s get started today by recalling our first memory of infinity. What made you think about infinity for the first time?”

PK the surfer guy raised his hand immediately. “Space,” he said. “I grew up in the desert, and at night you could see the Milky Way and it was impossible not to think of infinity when you saw all those stars.”

Nico nodded and wrote “Space” on the chalkboard. “Who else?” he asked.

A Chinese woman volunteered “time” because it kept on passing. “TIME” went on the list as well.

Peter said “God,” feeling the need, in our secular times, to shrug his shoulders somewhat apologetically. In his later years Peter would become more certain about his faith.

“It is hard to imagine a finite God!” nodded Nico, adding the almighty to his list. “Counting,” I volunteered. When I was five, I used to play a game with Bauji of naming larger and larger numbers. Invariably I’d find myself adding 1 to whatever strange number Bauji came up with.

“Yes, of course,” said Nico. “Thank you Ravi.” I was surprised he remembered my name from the other night.

After a pause Adin raised his head. “For me it was space—not in the unlimited sense, but in the sense of it being unendingly divisible. I first had that thought when my parents presented me with a microscope.”

“That’s right, Adin. Infinity has a dual aspect, the infinitely large and the infinitely small.”

Nico’s list read:

Space, without bound

Time

God

Number (counting)

Space, unendingly divisible

He looked at it for a few seconds. “It’s a good list,” he said, his back towards us. “In each of these examples we are observing a finite object or process and extrapolating it without limit. Where there are a billion stars there could be an infinite number; time keeps passing, so it may pass without end, forever; God almost by definition must be infinite—his powers are an unending extrapolation of our finite ones; numbers do go on and on

and on; and where we divide once, we could, at least in theory, divide again. By our ability to generalize and extrapolate we force infinity to exist, at least in our minds. Its existence is an affirmation of the human power of reasoning by recurrence.”

“But does infinity really exist?” asked Adin. “I mean, do we know if anything on this list is actually infinite?”

Nico shrugged. “Some people say that space is unbounded but finite, that time has a beginning and an end, that God does not exist, and that numbers are only a product of the human mind. So according to this view there is nothing truly infinite in the physical universe.”

“How can space be unbounded but finite?” PK wanted to know.

Nico laughed. “Good question. Perhaps space is like our planet. The earth is an unbounded surface. No matter how far you go, you’ll never come to the edge. But the earth is also finite. So unbounded but finite things are certainly possible.”

PK was not buying it. “That’s because the earth has a flat, two-dimensional surface that curves upon itself in the third dimension to make a ball. But space is already three-dimensional; it has nothing to curve into!” PK was smarter than I had initially thought.

“Some people believe that there is a fourth dimension that we are unable to perceive. Perhaps the universe curves into the fourth dimension,” said Nico.

Adin raised his hand. “There might be an infinity of dimensions then. Why stop at four?”

“It’s possible, and then we could have another type of infinity, but we’re only speculating here.”

Peter, never one for science fiction–type theories, took us back to God. “Doesn’t God have to be actually infinite in some sense?”

“If there is such a thing as God.” It was Adin who replied, not Nico. Peter shrugged his shoulders without looking back. Peter seldom argued unless he thought he had a shot at changing the other person’s opinion. Philosophical debates did not excite him.

Nico summarized where we were. “What we’re seeing here is that there is no proof that infinity exists in nature. It may or it may not. But because numbers exist as an idea in the human mind, infinity must also exist in the human mind. If we acknowledge the existence of the number 1 and acknowledge that we can always add 1 to any number, we automatically acknowledge the concept of infinity. Any doubters?” He asked with curiosity, not with the intent to challenge. I thought that Adin was going to say

something, but on due consideration he apparently found Nico's statement to be airtight. "Very well. Since infinity exists, if not in nature, then at least as a valid idea in our minds, the first thing we ought to do is find a symbol for it. John Wallis, an English mathematician, did this in 1655. Most of you have probably seen it before. It's called the unending curve."

Nico drew the symbol " $\infty$ " on the chalkboard. "Now that we've got a symbol for it, we need to try to get a better handle on what it is." He looked up at us. "That, ladies and gentlemen, is much harder than you might suspect. In fact, it is much easier to say what infinity is not. For example, we can be sure that infinity is not a number, in the sense that 943 is a number."

"Why do you say that?" asked Peter.

Nico took a piece of chalk and wrote:

$$\infty - 1 = \infty$$

"If infinity was a number it would have to be its own predecessor. If you grant me that the only types of numbers are finite numbers and infinity, observe that 1 added to any finite number cannot give infinity, so infinity minus 1 must equal infinity. But if we were to treat infinity as we treat any other number, we could subtract  $\infty$  from both sides and deduce that  $-1 = 0$ , which is absurd. So infinity is not a number and may not be treated as such."

"So then what is it?" asked PK.

"That's a tough question PK," said Nico. "The Greeks tried but couldn't answer it. And despite their discovery of zero the Hindu and Arabic mathematicians couldn't come to grips with infinity either. At one point, the Hindus defined infinity to be  $1/0$ , but then wiser heads prevailed and they realized that it cannot make sense to divide anything by 0. Most of the medieval voices either repeated Greek ideas or made infinity into a theological issue and failed to make progress. It was not until very late in the 19th century that Georg Cantor came up with a framework that made sense of infinity."

Nico had a poster of Cantor, which he now unfurled. "This man," he said pointing at the photo, "was a genius in the true sense of the word. He is the hero of our story. He single-handedly created the mathematics of infinity. Cantor defined infinity. In fact, he defined many infinities, and we'll get to his precise definitions in due time. His thinking and methods are an important focus for us in class."

What grabbed me first about Cantor's face in Nico's poster were his eyes. They looked past the camera, focusing at nothing, but strained in

thought. I wondered if Cantor had been wrestling with some mathematical problem at the precise moment when the picture was taken. Only Cantor knew, and he was dead.

The bridge of Cantor's nose in the picture was straight and narrow—a Sherlock Holmes nose if there ever was one. His mouth was surrounded by a short beard that did not appear to have been trimmed carefully. It was dense in places and spotty in others. Despite the beard, I could make out the tension and the worry in his thin lips. His forehead was broad, and his scalp hairless. The photograph was grainy around the top of his head and gave the appearance of bubbling fizz on the surface of a freshly poured Coke.

“Cantor is most remembered for establishing the subject of set theory, the topic of this class. In doing this he single-handedly changed mathematics.” Nico said this while looking at the photograph and slowly rubbing his chin. In the pause I felt that he would have loved to talk with Cantor in person, and frankly I would have loved to listen in on that conversation. Then, with a palpable gear shift, he turned and faced the classroom once more. He stood up straighter and his tone was firmer. It was time for mathematics.

“At an intuitive level a set is simply any collection of objects. Let me write out a few examples.” He turned to the board and wrote:

$$\begin{aligned} A &= \{\text{chair, elephant, tomato}\} \\ B &= \{16, \text{watch, book, } 23.75, \text{ saxophone}\} \\ C &= \{\text{Godzilla, } \{A\}\} \\ \mathbf{N} &= \{1, 2, 3, 4, \dots\} \end{aligned}$$

“As you can see a set can have any object as a member. It is typical to collect the objects of a set within curly brackets.” Nico pointed to the “{” and “}” which marked the opening and closing of his sets. “A tomato is an element (or a member) of set A, and the set A has three elements. The set  $\mathbf{N}$  has an unlimited number of elements. It is not a finite set. *Infinity is simply defined as the order of a set that is not finite.*”

It seemed a somewhat circular description to me, and I wasn't sure I saw the benefit. I looked up to protest but saw Nico looking at the class with an amused expression. He had anticipated our difficulties. “I can see from your faces that this definition is not the least bit satisfying. ‘What is the point?’ you all seem to be asking. You will see the point, I promise. More satisfying definitions of infinity require more mathematical machinery than we have at this stage. I ask you to keep this definition in the back

of your mind, for it will allow us to make progress and build an amazing structure to deeply understand the nature of infinity. It is a structure that still gives me goose bumps,” said Nico without any air of pretense that I could detect. “We’ll get to Cantor in due time. I put the definition up front because it seems odd to begin a class about infinity without defining it. But for now, let’s stay with the Greeks.”

“The first Greek we’ll meet is an odd bird by the name of Zeno, sometimes known as Zeno of Elea. He lived around the fifth century BC. He is said to have been a self-taught country boy. Zeno described several paradoxes built around the divisibility of space. The famous philosopher Plato dismissed these paradoxes as ‘youthful efforts’, yet he did nothing to resolve them. In fact, none of the best minds of the last two and a half thousand years could resolve the paradoxes raised by Zeno. Not bad for a country boy. The solutions came only about a hundred years ago. Today we’re going to take a look at one of the most interesting of Zeno’s paradoxes.”

Nico went to the board and drew as he spoke. “Zeno asks us to consider a runner starting at a point S. He is going to his target T, which is 1 mile away. Now, to get to T he must first get to the midpoint between the starting point S and the target T. Call the midpoint  $M_1$ . It is a half-mile from T.”

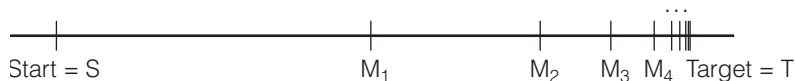


So far, so good. I was getting interested. There was always the faint hope that I would be able to crack a problem even though it had shown itself to be extraordinarily difficult. Nico had reintroduced me to the pleasures of the mathematical hunt; I would be chasing a puzzle!

Nico, meanwhile, was busy drawing another picture. “To get from  $M_1$  to T the runner must once again get to the midpoint between the two. Call this midpoint  $M_2$ ; it is a quarter of a mile away from T.”



I began to see where this was going. To get from  $M_2$  to T, the runner would have to get to  $M_3$ , then  $M_4$ , and so on forever. Nico’s next drawing confirmed this.



“Because I’m constrained by a chalk of finite thickness I have not drawn  $M_5$ ,  $M_6$ , and all the other  $M_n$  out there. But Zeno argued that the runner would indeed have to pass through an infinity of these points,” said Nico. I could tell Nico was getting excited—his voice was louder and his pacing more intense. “You see,” he said, walking over to the blackboard, “no matter how close the runner gets to T he still has to cover half the remaining distance, then half of what’s left, and then half of what’s left yet again. Essentially he has to keep making runs between successive  $M_i$ ’s. First he runs between S and  $M_1$ , then between  $M_1$  and  $M_2$ , then between  $M_2$  and  $M_3$ , and so on. Let’s call each such run an ‘M-run.’”

Nico went to the blackboard again and wrote:

1. The runner would have to make an infinite number of M-runs.
2. It is impossible for the runner to make an infinite number of M-runs.
3. Therefore, the runner will never get to the target.

“Historians can’t really be sure how Zeno himself saw his paradox. He may have seen it as a logical conundrum, or he may have used his argument to conclude that all motion is an illusion,” said Nico.

“That’s crazy!” exclaimed Peter. “Motion is not an illusion!”

“I agree, it sounds utterly crazy.” said Nico. “Clearly, motion is possible. Clearly, people move and cover distances. Clearly, a runner can cover a mile without getting trapped within a sequence of M-runs.” Nico was pacing again. “But just as clearly, logic works in our world. If an apparently logical argument leads to an absurd result, then either logic does not always work, or the argument is flawed in some subtle way. I firmly believe that logic works. So there must be a subtle flaw in Zeno’s argument. And I want us to find it.”

Peter nodded. Considering his impatience with philosophical arguments of this nature I knew that Nico had gotten through to him. I myself was beguiled by Zeno’s argument. It seemed extraordinarily simple, trapping one in its iron-clad logic, and it was hard to avoid hurtling toward its inevitable but absurd conclusion.

“Let’s examine the argument in pieces,” resumed Nico. “First, does anyone doubt that the runner would have to make an infinite number of M-runs?”

“I do,” said PK. “Toward the end the M-runs become so small that the runner’s body itself would span over all of the last few M-intervals and would cover the target.”

“Valid point. Anyone care to shoot that down?” asked Nico.

Adin spoke up. “Nothing stops us from assuming that the runner is a dimensionless point that will have to travel a finite distance no matter how small the M-run.”

“Exactly correct,” said Nico. “I think Zeno was indeed correct in saying that an infinite number of M-runs would have to be completed for the runner to reach his target. But the next leg of his argument is even more interesting. He claims that it is impossible for the runner to complete an infinity of M-runs. What do we think of that?”

PK thought that there was no question Zeno was right. “If the runner has to cover a finite distance an infinite number of times he would never reach his target.”

“That certainly seems correct. Anyone see a way out?”

The class was silent. If you kept adding a diminishing but finite quantity to itself forever, wouldn’t you get a sum that grew forever?

Nico wanted us to think with specific numbers. “Let us say that the runner keeps a constant pace of running one mile in four minutes. How long will it take him to run a half-mile?” asked Nico.

This was simple. “Two minutes,” someone said.

“And a 1/4 mile?”

“One minute.”

“Right. 1/8 mile takes 1/2 a minute, and I’m sure you see the trend. Now if we sum up how long it will take the runner to complete each M-run we get an infinite series because there are an infinite number of M-runs.” Nico wrote out the series on the blackboard.

$$\text{Time taken by runner} = 2 + 1 + 1/2 + 1/4 + 1/8 + 1/16 + \dots$$

“As I’m sure you’ve realized the 2 is for the first half-mile, the 1 is for the next 1/4 mile, and so on. The three dots denote the unending nature of this series. Now one objection Zeno might have had is that this series grows unboundedly large. Would he have been correct?”

Nico’s question hung in the classroom. Calculators were pulled out, pens uncapped, and notebooks opened. The class was experimenting. I just stared at the unending sum. I recalled from the Bauji days that an infinite number of additions could yield a finite sum, but I couldn’t immediately see how that was possible. After all, wouldn’t one always have more terms to add?

Meanwhile, Peter had some interesting statistics to report. “If you add the first five terms, the sum is 3.875,” he said. “If you add the first ten terms you get 3.996. Adding more terms gets you closer to 4, but the terms get smaller

so quickly that they contribute almost nothing to the sum. It seems very probable to me you can never pass 4 no matter how many terms you add.”

“Brilliant!” said Nico. He actually pumped his fist. “What do other people think of Peter’s bold guess?” There was agreement in the room. Others had come up with similar calculations. “I see several nods,” said Nico. “Peter, you are correct that the sum of the terms never exceeds 4. But in mathematics, unlike any other branch of human learning, you have to prove your case. It is not enough to simply state your result as a guess; an airtight justification is necessary, and without an airtight justification, nothing is resolved. This is not meant to discourage you,” he said to the class at large. “In fact, I am telling you that Peter’s guess is correct and that a series of infinite terms can indeed yield a finite sum. I will even tell you that the sum of the series is 4. So not only does the series not exceed 4, as Peter suggests, but I claim that it exactly adds up to 4. Once we understand *why* this is true we will have removed what may have been one of Zeno’s concerns.”

“I understand why,” said a woman’s voice from behind me. She appeared unaware that her declaration had come across as boastful for the emphasis had been on the “I.” The entire class (even Adin) turned around to look at her. Her strong jawline was the first thing I noticed about her. Today, as she has entered her thirties, the rest of her face has softened a little, but the jawline remains as well defined as ever.

“You understand why? Have you studied infinite series in a Calculus class?” asked Nico.

“No, I just figured it out,” she said seemingly unaffected by the scrutiny.

“What’s your name?” asked Nico.

“Claire Stern.”

“Well, Claire, come on down then, and show us,” said Nico handing her a piece of chalk.

Without saying a word she wrote:

$$\begin{array}{r} \text{Sum} = 2 + 1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \dots \\ - 1/2 * \text{Sum} = \quad 1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \dots \end{array}$$

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$$= 1/2 * \text{Sum} = 2 + 0 + 0 + 0 + 0 + 0 + 0 + \dots$$

$$\Rightarrow 1/2 * \text{Sum} = 2$$

$$\Rightarrow \text{Sum} = 4$$

She nonchalantly tossed the chalk back towards Nico, who was by then smiling broadly at her. “Claire, that is a beautiful argument! For the class, please explain what you’ve done here.”

“We’re trying to determine the sum of the series, so I put it on the left-hand side,” she said. “I wrote out the infinite sum on the right, the way you did earlier. Next I multiplied both sides of the equation by  $1/2$  and shifted all the terms over by 1. I subtracted the second equation from the first, and all the terms got cancelled. The tails in the two equations are exactly the same and may be zeroed out upon subtraction. It left me with  $1/2 \times \text{Sum}$  on the left-hand side, equaling 2 on the right-hand side, which implies that the Sum = 4.”

“Bravo! Bravo! That is such an elegant argument, Claire. I love the way all the terms seem to cancel out. I’m proud of you,” Nico said.

I could tell he was immensely pleased with what Claire had just demonstrated, and so it was a true surprise when I heard him announce, “Claire’s proof is clever, but it is not correct,” as she headed back to her seat. Nico waited until she was settled in and then spoke directly to her. “Claire, you applied the rule of finite mathematics to an infinite sum. In a finite equality you can multiply both sides of the equation by  $1/2$ . But what does it mean to multiply all terms of an infinite sum by half? How do you do that?”

“I don’t know what you mean,” Claire told him.

“You cannot treat infinite series as finite ones. Strange things can happen. Let me give you an example,” he said, going to the blackboard. “It involves rearranging the terms of an infinite series.”

$$\begin{aligned}0 &= (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \dots \\ &= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots \\ &= 1 + 0 + 0 + 0 + 0 + \dots = 1.\end{aligned}$$

“I’ve done nothing but rearrange the terms, which is perfectly legal in a finite sum, but I get an absurd result of 0 being equal to 1. Claire’s argument leads to the right answer—unlike what I just showed you—but it yields the right answer without the right method. To fully understand and appreciate what it means for an infinite sum to converge to a point, we have to tackle the fascinating idea of limits. We’ll get into that next week.”

Nico’s example was exactly on point. I saw that infinite sums are strange animals and may not always behave the way one might expect. For the first time in a long time, I felt the stirrings of interest; I wanted to figure out what was going on with these series.

Meanwhile, Nico was summarizing the discussion so far. “Zeno’s argument said that it is impossible to make an infinite number of M-runs. One reason for his claim may have been the assumption that an infinite number of M-runs must add to an infinite distance. We’ve begun to see that this need not be the case, and we will complete our understanding in the next class.”

Nico’s passion for accuracy impelled him to leave us with a warning. “Zeno’s paradoxes are one of the most discussed pieces in the history of philosophy. It is entirely possible that Zeno himself had some other reason for thinking that it is impossible to have an infinite number of M-runs. But all I expect you to take away from our discussion is that an infinite number of terms can and do converge to a finite quantity, which by itself is one of the great ideas of human history. The Greeks never fully understood this; actually, no one did until a hundred or so years ago, when the concept of limits began to be developed. Many mathematicians used heuristic methods to develop the mathematics of infinite series, but their methods—like Claire’s fantastic argument—lacked rigor. As we saw, Claire’s argument was almost correct, but we’ll make it air-tight by taking a closer look at limits and infinite sums next week. Right now let’s take a quick break. When we get back I’d like to talk about the treatment of infinity through medieval times. There are stories of great stupidity and great genius. Ten minutes.”

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Peter introduced me to Adin during the break. “This is my roommate Ravi,” he told him, “and Ravi, this is Adin. Adin and I used to work out together. We used to race each other in the swimming pool every morning.” Both Peter and Adin were tall, but whereas Peter was muscular, Adin was slender.

“Who won?” I asked.

“It was always pretty close,” said Peter. “That’s what made it fun.”

“Peter would win in the shorter lengths, but I could usually take him if we swam more than five laps,” Adin said.

I told Adin that I remembered seeing him play at the Coffee House. “You play the sax, don’t you?”

He was obviously pleased that I had remembered. “Yeah, a little bit.”

“So why is a musician-philosopher taking this math class?” I asked him.

Adin laughed. “Musician-philosopher! I’m afraid I’m neither. But I am interested in both—for very different reasons. This class actually connects up pretty well to several key ideas I’m interested in. Math has a great deal to say about philosophy.”

“Really?” I was surprised. “Like what?”

“Well, Nico touched upon it in class. Like he said, mathematics requires proof, and proof confirms truth. I’ve always been interested in how one can be sure of something, and mathematics seems to provide the way to certain truth. Certainty is very important to me.”

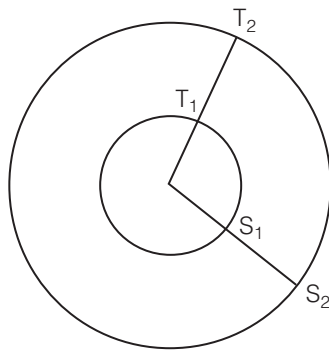
“What do you want to be certain about?” Peter asked.

“The purpose of life, for instance,” he said, without missing a beat.

Peter laughed. He thought Adin was joking. But Adin’s faced remained earnest and steady.

• • •

After the break, Nico drew a curious-looking drawing of two concentric circles on the blackboard.



“Zeno’s paradox was not the only problem that worried mathematicians through history,” he said. “There was also this curious example of two concentric circles. Each circle has an infinity of points on its circumference, but since the inner circle is smaller, one would think that it contains fewer points than the larger outer circle. But if we draw a radius to the outer circle, we can see that each time the radius touches a point on the outer circle, it also touches a point on the inner circle. So the sweeping radius sets up a correspondence between points on the circumference of circles of different sizes. In our picture  $S_1$  corresponds to  $S_2$  and  $T_1$  to  $T_2$ . This correspondence seems to suggest that the two circles have the same number of points, even though one is bigger. How can that be?”

We sat there looking at the circles. There was no doubt in my mind that for every point on the larger circle there was a point on the smaller circle. But this defied common sense! Surely the larger circle had more points. Meanwhile, Nico amused himself by making slurping sounds in his coffee cup and said nothing further.

Finally, it was Adin who thought he had found a way out: “I agree this looks very strange, but unlike Zeno’s problem there is no logical paradox

here. Zeno's suggested that there could be no motion, which we know to be false. All this problem is saying is that the two circles have the same number of points, which is strange, but in my opinion there is no contradiction here."

"That's pretty good, Adin," said Nico. "You're right, it's strange, but the strangeness might be just because many of us don't have strong intuitions about infinite sets." I saw Adin's point, yet was not fully satisfied. He was right—there was no apparent logical contradiction—but I was not comfortable with the result, even though the evidence was in front of me.

Apparently, my uneasiness had been shared by other mathematicians through history, who, according to Nico, concluded from such results that infinity was a slippery, even dangerous concept. "Until modern times, most of what was written on infinity after the Greeks was more theological than mathematical," Nico said. "Medieval mathematicians saw infinity as an awe-inspiring and sometimes a fear-inspiring idea. 'Only God is infinite' was their conclusion; everything else is limited. An Italian thinker, Giordano Bruno, was tortured for nine years in part because he refused to retract his idea that the universe was infinite and extended forever. Bruno believed that reason and philosophy are superior to faith, and to knowledge founded on faith. He refused to accept the finiteness of the universe merely because the Church decreed that only God could be truly infinite. At his trial, which ended in 1600, he was as defiant as ever. Upon hearing his death sentence, he responded, 'Perhaps your fear in passing judgment on me is greater than mine in receiving it'. He was then gagged and burned alive."

A shocked "Oh God!" escaped out of Claire. It came out louder than she had intended.

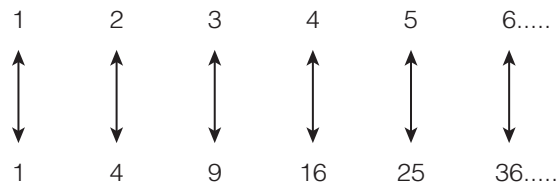
"I know; sometimes life is terrible," said Nico nodding in her direction. "Bruno was killed by people who valued power over truth. Unforgivable." It was the only time in the entire semester that I saw Nico look angry. His usual amused half-smile was replaced by a downward scowl that deepened the lines running down along his nose. His eyebrows crept together, and his large, usually clear forehead became riven with wavy furrows. It seemed to me that ideas were as important to Nico Aliprantis as they had been to Giordano Bruno.

After a while he continued: "The one shining exception to mediocre thinking about infinity before Cantor was Galileo."

"The telescope guy?" asked PK.

"The very same. Galileo, who invented the telescope and first saw the moons of Jupiter and the rings of Saturn, also had an insightful idea about infinity."

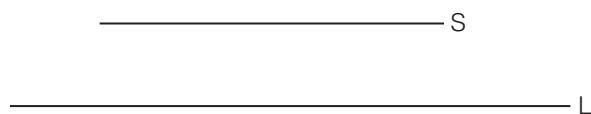
He turned to the blackboard and wrote:



“Galileo observed that you could put a number and its square in a one-to-one correspondence. There is an infinity of numbers and an infinity of square numbers. This seems to show that there are as many numbers as there are squares, but at the same time there are a lot of numbers that are not squares.” Nico was pacing now, his coffee mug forgotten. “Many people before Galileo had observed this correspondence and they had deemed it a paradox. ‘How can a part be equal to a whole?’ they asked. Galileo’s insight was this: *he realized that you cannot apply the laws of finite mathematics to infinite sets*. Claire tried to do that earlier, and even though her method was elegant, I had to stop her. Galileo said that for infinite sets it is possible for a part to equal the whole; that this was no paradox, only a property of infinity. With this simple conclusion, Galileo introduced the modern age of infinity. And that is as far as we are going to go today.”

It had been a perfect class. Nico had been clear, engaging, and stimulating. Half of me felt like standing up and giving him a hand. Of course I did no such thing. But now, many years later, from my perch in adulthood, I wish I had.

“Before you go,” said Nico, “I’d like you to spend 10 minutes thinking about a simple question related to the concentric circle problem we discussed earlier. He drew two straight lines, one longer than the other.



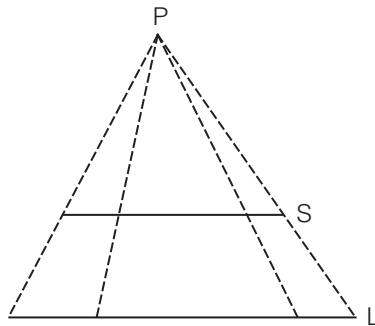
“Let’s label the shorter line S and the longer line L. Here’s my question: Does L have more points than S, or do they have the same number of points? Please prove your answer, write it down, and drop it off before you go.” With that Nico stopped talking. He found a seat, something to read, and an old cigar to chew on.

My first instinct was that the number of points had to be the same. If two circles of different lengths could have the same number of points, then

surely these two lines would as well. But try as I might I couldn't find a mapping that led to a one-to-one correspondence. I was missing the equivalent of the center of the circle.

And then I saw it. Aha!

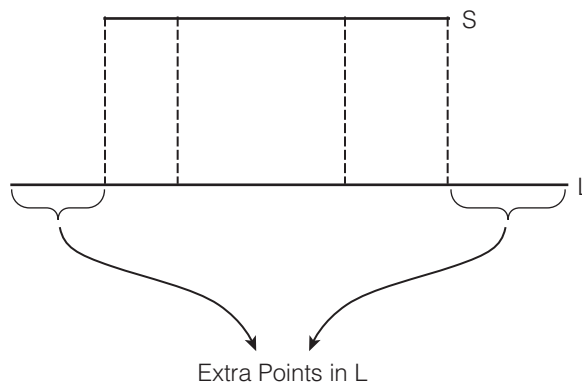
I quickly wrote my answer: Draw 2 lines connecting the pair of endpoints. Extend these lines to intersect at P.



Now the lines that pass through P and connect S and L establish a one-to-one correspondence. Therefore, S and L have the same number of points. I handed my answer to Nico on the way out. He looked at it, nodded, and then grinned. But I hadn't been the first to finish. Claire Stern was finished almost as soon as Nico had presented the problem. By the time I got out, she was gone.

When Adin and Peter didn't come out of the classroom ten minutes after everyone else had left, I went in to see what was going on. I found them embroiled in a heated discussion.

"Ravi, take a look at this," said Peter. "Adin says this is not correct." Peter showed me his drawing:



“This shows that there are more points in L. I’ve mapped everything in S onto L, and L still has points left over,” said Peter pointing to the parts of the longer line L that extended out from the section directly under S.

“Peter, I’m not saying that your mapping is incorrect. I’m saying that it doesn’t prove anything,” said Adin. “We are here required to show one of two things: either that there exists some one-to-one correspondence between the two lines, in which case they are of the same size, or that no such correspondence can exist, in which case one is bigger than the other. Finding a correspondence that is not one-to-one doesn’t really show anything. There could still be another correspondence that is, in fact, one-to-one.”

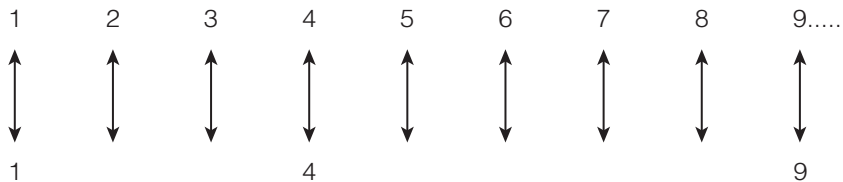
“I don’t get it,” said Peter, looking towards me.

I saw what Adin was getting at. He was right, but was not explaining himself too well. Peter needed an example.

“Peter, you remember the correspondence where Galileo matched each number to its square?” I asked.

“Yeah,” he said, unsure of where I was taking him.

“Suppose I match each square with itself,” I said, asking for his pen. I wrote out an alternative correspondence:



“See, Peter, here all the squares are matched up, and there are all these numbers left over which makes it seem like there are fewer numbers in the bottom row. But this correspondence does not show anything, because we know another correspondence exists that matches the collections in a one-to-one manner.”

Peter got it. He hit himself on the back of his head. “Duh! I should have seen that,” he said.

• • •

Later that day I saw Claire. She was sitting in the main quad, her back very straight, with a notebook in her lap and a tapping pencil in her left hand. As I approached her and could make out the drawings in her notebook, I saw that she was still trying to map Nico’s shorter segment into the longer one.

“You were the first one to be done with that; how come you’re still working on it?” I asked her, quashing the impulse to play it safe and walk on without a word. She looked up reluctantly, a little annoyed, I felt, to have had her thought stream intruded upon. The sun reflected off the lenses of her glasses and I couldn’t see her eyes.

“I wasn’t too happy with my proof,” she said. “I’m looking for a more direct approach.”

She didn’t invite me to sit; she didn’t ask my name and she didn’t inquire if I had solved the problem. Instead, she went back to her notebook, leaving me standing there, feeling like a schmuck.

I stubbornly refused to leave. Pulling out a sheet of paper of my own (not daring to ask her for one) I drew out my solution that joined the two endpoints and extended the two lines until they met. Brashly, and without a word, I placed my drawing on top of her notebook.

She looked at my drawing for about a minute and then nodded. “That’s good,” she said. And then she shifted to the left, making room for me. She pointed to her notebook. “My idea was that any line of a given length can be transformed into a circle with a circumference of that exact length. So with the two lines you can draw two circles with unequal circumferences, and we’ve already shown that all circles of any circumference have the same number of points.”

It was an indirect proof, and in its way it was quite efficient. It used the result that we’d already proven in class.

“That’s clever,” I told her.

She shook her head. “No, it’s not as direct as what you did. I think yours is what Nico was looking for.” She shut her notebook, preparing to leave. Impetuously and impulsively and risking rejection I asked her if she wanted to grab lunch at Tressider. Much to my surprise she said yes.

On our way to lunch we passed by Nico’s office. The door was open and he saw us walk by. “Hey, you guys, can you come in here for a minute?” Nico was sitting with his legs on his desk looking at everyone’s responses to his question. “You two were the only ones who got the answer right, although your approaches were quite different.”

He asked us how we liked the class.

“It was stimulating,” said Claire.

I was more effusive in my praise. “That was one of the best classes I’ve ever had. I don’t think they can get any better, only different.”

Nico smiled. “So it meets the standards set by your grandfather?”

Once again I was surprised at his memory. “Absolutely,” I told him.

“I meant to ask you—your grandfather was a professional mathematician?”

“Yeah. He never formally got his Ph.D., but he published a lot of papers.”

“What was his field?” asked Nico.

“He published in many different areas, but his concentration was in algebraic number theory.”

“You’re kidding!” said Nico. “That was the subject of my dissertation. What was his name?”

“Vijay Sahni.”

“Vijay Sahni . . . I know that name. Wait a minute, did he do work on elliptic fields?”

I didn’t know. “He might have,” I said.

“I do know that name. In fact I think your grandfather may have written a paper that was a particular favorite of mine. Hold on a minute, let me find it.” Nico’s bookcase must have been more organized than it looked, for a few minutes later he had found what he was looking for. He fished out an old book titled *Classic Papers in Algebraic Number Theory*. Most of the book’s pages were heavily underlined.

“This book used to be a great favorite of mine. It was published in 1961 and not many people noticed it when it came out, but I must have spent endless hours with it,” he said quickly turning the pages. Then he found it. “There it is! ‘Elliptic Curves Over Function Fields’ by Vijay Sahni. Is that how he spelled his name?” Nico asked, pointing to the open page.

It was his name.

“Look at the footnote,” said Claire, looking at the page from behind me.

I did. There it was, in smudgy black and white. The handful of words after which things would never be the same: *Note from the Editors: Mr. Vijay Sahni informs us that the key ideas contained in this paper were formulated while he was serving a prison sentence in Morisette, New Jersey, in 1919.*

I did know that Bauji had come to America towards the end of the Great War. But Bauji in prison? There must be some mistake. Yet, the paper was in his field. And the date mentioned in the footnote made sense—surely there could not have been more than one Vijay Sahni in the United States in 1919. Could it be that he had been imprisoned? But for what crime?

And what in the world was he doing in Morisette, New Jersey?