By the beginning of my senior year I had become a recognized regular and knew many of the musicians who played frequently. I would sit at the same spot, and if I was alone I'd carry along whatever I was reading. Sometimes Peter came with me, but he would always leave by midnight. "I need to get up early," he'd say without pride or regret. But one Thursday just before the first semester of our senior year, because classes were yet to start, he made an exception and stayed late. And that was the night I met Nico Aliprantis. It was Peter who pointed him out. "See that guy over there? He's the best math teacher I've ever had," he said.

At that time Nico was 62, which made him approximately three times as old as most people in the room. But he fit right in. His walk was tall and easy, his manner comfortable, and his mouth always on the verge of an amused smile. He chose a table near the stage, put his motorcycle helmet aside, and proceeded to roll his own cigarette.

"What class did you take from him?" I asked Peter. To date, I had been unaware that he cared about the quality of math teachers.

"Statistics," he said. "He was the only teacher that ever made math seem like a natural thing, not just a bunch of rules."

Nico listened to the music attentively. From time to time someone (probably a former student) would stop by his table and say hello. Most professors would have rated a brief nod and that too only if there happened to be some accidental eye contact, but in his case there seemed to be a reservoir of genuine goodwill. On two occasions a student pulled a chair up to his table and stayed for a chat.

Just before closing Nico went to the stage and asked for the saxophone. He played an old Charlie Parker tune whose name I could not place, but I had heard it before; it was one of Bauji's favorites. After establishing the refrain, he began to improvise, and I knew within a minute that he was good. He played effortlessly. He knew how to get from note to note seamlessly, with a light touch and his own unique style that he somehow intertwined with Parker's. You heard Charlie Parker but you also heard Nico Aliprantis, and the two coexisted with ease. Towards the end he got tangled up and lost his way. His eyebrows squeezed together and his forehead wrinkled, and for a second he looked angry with himself. But then he decided to finish and played a nice sequence to bring the tune to a logical conclusion. He bowed and everyone clapped—some, because he was different from the rest, and others, because he was good.

After everyone had played and the Coffee House was closing down, Peter and I caught up with Nico. "Dr. Aliprantis, you were fantastic!" said Peter.

He smiled and looked at us, recognizing Peter. "You've taken one of my classes," he said, peering at him from behind his glasses. And then after a second, "Peter Cage, right?"

I was surprised he remembered—he must have had hundreds if not thousands of students. But then he said, "You were great in that statistics class. I kept saying you should study mathematics instead of business," and I understood then that Peter had distinguished himself enough to be memorable.

"Dr Aliprantis, this is my friend Ravi Kapoor," Peter said, turning towards me. As I shook his hand I told Nico that I recognized the Charlie Parker tune, but couldn't recall the title.

"'Now's the Time'," he said, looking at me more closely. "You must know jazz because that's not one of Bird's most famous recordings."

Before I could reply Peter jumped in. "Ravi knows a lot about jazz."

Nico smiled. "Do you play?"

"Not well; otherwise I'd do it as a career," I said.

Nico nodded, earnest for the first time. "I'm the same way," he said. "This math gig was a fallback choice, though fortunately I love the subject and I'm good at it, much better than I am at jazz anyway."

"You were good," I told him. "That was a great sequence you created and it worked perfectly except for that little bit at the end."

He shook his head, "I may be good compared to some guy on the street, but I'm no Charlie Parker." He said it so matter-of-factly that there was nothing further for Peter and me to say. We stood there in silence for a few seconds and then I saw Nico notice the strain and make a conscious decision to steer the conversation away from himself. "So what else do you like besides jazz?" he asked me.

Nothing really, was the truth. "I used to love mathematics," was what I came up with instead.

"Used to?" Nico asked. He asked so softly and with such benevolent curiosity that I found myself telling him the truth.

"My grandfather made mathematics inspiring and fun. I've never had anyone else who could enthuse me the way he could."

Nico smiled at that. "There's a challenge!" he laughed. "Listen," he said arriving at a decision, "Why don't both of you sign up for the class I'm teaching this fall? It's called 'Thinking about Infinity'. It's Math 208, I think. You should check it out; it should be an interesting class. We start Monday."

Walking home that night Peter and I talked about whether we should accept Nico's invitation. Peter already had an offer from Morgan Stanley that

he was going to accept. He was one of the few people who had a job offer at the beginning of senior year—most people got offers later, typically in the fall semester. Without the pressure of the job hunt, he had the luxury of experimenting with "fun" classes and thought that Nico's class would fit the bill. I, on the other hand, had declared my major only a semester ago and needed to take five economics classes to graduate on schedule.

"It doesn't make sense for you, though," observed Peter making the same calculations I had just gone through. "You have to be Mr. Economics this semester." I knew he was right.

But that night, just before falling asleep, I decided to sign up for Nico's class after all. I knew this would mean having to take (a nearly impossible) *six* economics courses next semester or else taking a class in the summer, which would be a huge financial strain on my family (they were augmenting Bauji's bequest). But there was something about Nico.

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There were about 15 students who showed up for "Thinking about Infinity." Peter, as was his custom, had arrived early and found a spot in the front row. The other faces seemed unfamiliar save for a slender, curly-haired saxophone player I had seen a few times at Thursday Night Jazz. His music had not been memorable, but for some reason his name had stuck with me: Adin something. He sat in the front talking to Peter, who evidently knew him from somewhere.

Nico entered the room with the same languid ease that he had shown entering the Coffee House. "Good morning everybody. My name is Nico Aliprantis and we're going to spend this semester using our finite brains to think about infinity." He smiled at his own line. I wondered if it was improvised.

He quickly went over the logistics: we would meet once a week for the next 10 weeks, each class would be three hours long with a 10-minute break in the middle, office hours would be Wednesday afternoon, and the grades would be based on two take-home tests and the quality of class participation. No prior mathematics was required—This was a mathematics course for liberal arts majors. No textbook was required either; he would hand out notes when necessary. A student asked how people were supposed to study without a textbook.

"You'll see," Nico replied without disguising his sigh. He must have gotten this question all the time. "I think you'll find that attending class and thinking about the problems I present to you from time to time will provide all the structure you need."

He took off his glasses and faced the room. "There are two themes that are going to run throughout this course. I want to talk about them up front so that as we go into the subject matter you know what to look for. First, if you allow yourself to, you will find great beauty here. I think that mathematics is beautiful at its core; it is much more like a musical piece than an accounting formula." He looked up to see how the class was receiving this idea, and that's when he happened to catch my eye. "Much more like a jazz piece," he said with a half-wink. "G. H. Hardy, a famous English mathematician, said that good mathematics is about making good patterns. A painter makes patterns with shapes and colors, a poet with words. A mathematician makes patterns with ideas." He said "ideas" loudly and the word seem to reverberate in the silence that followed.

After a minute or so in which no one said anything, I could tell that Nico was scanning the room looking for someone to talk to. He settled on Adin. "You, sir," he said pointing, "what is your name and what do you study?"

Adin's deep voice did not match his slender frame. "Adin Kaminker. I'm majoring in philosophy," he said.

"Adin, do you have a favorite poem or song?"

"Sure," said Adin. "I quite like poetry actually."

"Excellent. Okay, may I ask you to recite a few lines from a poem that you find particularly beautiful?"

Adin did not hesitate. He picked an old favorite of Bauji's. "The woods are lovely, dark and deep / But I have promises to keep / And miles to go before I sleep / And miles to go before I sleep." His recital was practiced and smooth. "That's by Robert Frost," he concluded.

The class collectively turned towards Nico, their heads moving in unison right after Adin finished his recital, like a gallery watching a tennis match. "Thank you, Adin," Nico said, bowing his head in appreciation. "You recited the lines beautifully."

He looked up, addressing the whole room now, not just Adin. "Now let's imagine Robert Frost writing those lines. Maybe he played around with which words to use—perhaps at first he used the word 'forest', instead of 'woods'. Maybe he tried many different word-sequences in many different rhythms until he got this one. And when he did, you can bet he knew that he was onto something, that he had created something beautiful. As soon as he had those lines, I'm sure he knew that they were right. They appealed to his sense of aesthetics."

Nico was pacing. He was into it. "Mathematics is done in the same way," he continued "Most mathematicians have an aesthetic sense that

guides them toward the problems they try to solve and in the ways they approach them. They try many things and then, sometimes seemingly out of nowhere, an idea comes. The idea simplifies everything, puts everything in harmony. And when they have the idea they often know that they are right, even though they have not worked out all the details. With practice they get an aesthetic sense, not unlike a poet's I imagine."

A hand shot up in the back row. It was a goatee-wearing guy in beach flip-flops, shorts, and a longish—but surprisingly disciplined—pony tail.

"Your name please?" asked Nico. It turned out that unlike most teachers, Nico had a good memory for names.

"Percy Klug, but most people call me PK."

"Go ahead, PK," he said.

"If mathematics is so beautiful, why haven't I ever heard anyone talk about it that way before?"

He was right. Mathematics was seldom seen to be beautiful. Bauji saw it that way, but he was the only person I knew who held that opinion—until now.

"I'm not sure," said Nico. "Maybe it's because mathematics is not a spectator sport. You have to do it to appreciate it, and doing it requires patience and persistence. You can love a song without being able to sing, but that doesn't work in mathematics. Nevertheless, the beauty is there for you to find." He took a sip from his coffee mug, making a slurping noise. "So the first theme is beauty. Keep a look out for it. It's not really unique to this class; I find a lot of different branches of mathematics to be beautiful. But the second theme, I think, is especially true for us. This class is also about understanding how humans think and understanding the limits of what we can think." Nico paused and looked outside towards the courtyard. When he spoke again his voice was softer and more distant. "The story of infinity is a story of how far the human mind can take us. But it is also the story of boundaries that we may not cross, no matter what. We will see amazing facts that must be true but also raise tantalizing questions that seem to be unanswerable. Not because mathematicians just happened not to have found an answer so far, but rather because they couldn't possibly. Our current set of assumptions about infinity are not strong enough to lead to an answer to some questions. *Ever.*" I didn't understand all the things Nico said but was captivated by the way he said them—like a man of faith expressing reverence in a place of worship. There was motionless silence in the ensuing pause. Then I saw Adin fish out his notebook and write something down. His pencil sounded surprisingly loud.

"You'll see what I mean as the class progresses," said Nico, coming back to us. "But let's get started today by recalling our first memory of infinity. What made you think about infinity for the first time?"

PK the surfer guy raised his hand immediately. "Space," he said. "I grew up in the desert, and at night you could see the Milky Way and it was impossible not to think of infinity when you saw all those stars."

Nico nodded and wrote "Space" on the chalkboard. "Who else?" he asked.

A Chinese woman volunteered "time" because it kept on passing. "TIME" went on the list as well.

Peter said "God," feeling the need, in our secular times, to shrug his shoulders somewhat apologetically. In his later years Peter would become more certain about his faith.

"It is hard to imagine a finite God!" nodded Nico, adding the almighty to his list. "Counting," I volunteered. When I was five, I used to play a game with Bauji of naming larger and larger numbers. Invariably I'd find myself adding 1 to whatever strange number Bauji came up with.

"Yes, of course," said Nico. "Thank you Ravi." I was surprised he remembered my name from the other night.

After a pause Adin raised his head. "For me it was space—not in the unlimited sense, but in the sense of it being unendingly divisible. I first had that thought when my parents presented me with a microscope."

"That's right, Adin. Infinity has a dual aspect, the infinitely large and the infinitely small."

Nico's list read:

Space, without bound

Time

God

Number (counting)

Space, unendingly divisible

He looked at it for a few seconds. "It's a good list," he said, his back towards us. "In each of these examples we are observing a finite object or process and extrapolating it without limit. Where there are a billion stars there could be an infinite number; time keeps passing, so it may pass without end, forever; God almost by definition must be infinite—his powers are an unending extrapolation of our finite ones; numbers do go on and on

and on; and where we divide once, we could, at least in theory, divide again. By our ability to generalize and extrapolate we force infinity to exist, at least in our minds. Its existence is an affirmation of the human power of reasoning by recurrence."

"But does infinity really exist?" asked Adin. "I mean, do we know if anything on this list is actually infinite?"

Nico shrugged. "Some people say that space is unbounded but finite, that time has a beginning and an end, that God does not exist, and that numbers are only a product of the human mind. So according to this view there is nothing truly infinite in the physical universe."

"How can space be unbounded but finite?" PK wanted to know.

Nico laughed. "Good question. Perhaps space is like our planet. The earth is an unbounded surface. No matter how far you go, you'll never come to the edge. But the earth is also finite. So unbounded but finite things are certainly possible."

PK was not buying it. "That's because the earth has a flat, two-dimensional surface that curves upon itself in the third dimension to make a ball. But space is already three-dimensional; it has nothing to curve into!" PK was smarter than I had initially thought.

"Some people believe that there is a fourth dimension that we are unable to perceive. Perhaps the universe curves into the fourth dimension," said Nico.

Adin raised his hand. "There might be an infinity of dimensions then. Why stop at four?"

"It's possible, and then we could have another type of infinity, but we're only speculating here."

Peter, never one for science fiction–type theories, took us back to God. "Doesn't God have to be actually infinite in some sense?"

"If there is such a thing as God." It was Adin who replied, not Nico. Peter shrugged his shoulders without looking back. Peter seldom argued unless he thought he had a shot at changing the other person's opinion. Philosophical debates did not excite him.

Nico summarized where we were. "What we're seeing here is that there is no proof that infinity exists in nature. It may or it may not. But because numbers exist as an idea in the human mind, infinity must also exist in the human mind. If we acknowledge the existence of the number 1 and acknowledge that we can always add 1 to any number, we automatically acknowledge the concept of infinity. Any doubters?" He asked with curiosity, not with the intent to challenge. I thought that Adin was going to say

something, but on due consideration he apparently found Nico's statement to be airtight. "Very well. Since infinity exists, if not in nature, then at least as a valid idea in our minds, the first thing we ought to do is find a symbol for it. John Wallis, an English mathematician, did this in 1655. Most of you have probably seen it before. It's called the unending curve."

Nico drew the symbol "∞" on the chalkboard. "Now that we've got a symbol for it, we need to try to get a better handle on what it is." He looked up at us. "That, ladies and gentlemen, is much harder than you might suspect. In fact, it is much easier to say what infinity is not. For example, we can be sure that infinity is not a number, in the sense that 943 is a number."

"Why do you say that?" asked Peter.

Nico took a piece of chalk and wrote:

$$\infty - 1 = \infty$$

"If infinity was a number it would have to be its own predecessor. If you grant me that the only types of numbers are finite numbers and infinity, observe that 1 added to any finite number cannot give infinity, so infinity minus 1 must equal infinity. But if we were to treat infinity as we treat any other number, we could subtract ∞ from both sides and deduce that $-1 = 0$, which is absurd. So infinity is not a number and may not be treated as such."

"So then what is it?" asked PK.

"That's a tough question PK," said Nico. "The Greeks tried but couldn't answer it. And despite their discovery of zero the Hindu and Arabic mathematicians couldn't come to grips with infinity either. At one point, the Hindus defined infinity to be 1/0, but then wiser heads prevailed and they realized that it cannot make sense to divide anything by 0. Most of the medieval voices either repeated Greek ideas or made infinity into a theological issue and failed to make progress. It was not until very late in the 19th century that Georg Cantor came up with a framework that made sense of infinity."

Nico had a poster of Cantor, which he now unfurled. "This man," he said pointing at the photo, "was a genius in the true sense of the word. He is the hero of our story. He single-handedly created the mathematics of infinity. Cantor defined infinity. In fact, he defined many infinites, and we'll get to his precise definitions in due time. His thinking and methods are an important focus for us in class."

What grabbed me first about Cantor's face in Nico's poster were his eyes. They looked past the camera, focusing at nothing, but strained in

thought. I wondered if Cantor had been wrestling with some mathematical problem at the precise moment when the picture was taken. Only Cantor knew, and he was dead.

The bridge of Cantor's nose in the picture was straight and narrow—a Sherlock Holmes nose if there ever was one. His mouth was surrounded by a short beard that did not appear to have been trimmed carefully. It was dense in places and spotty in others. Despite the beard, I could make out the tension and the worry in his thin lips. His forehead was broad, and his scalp hairless. The photograph was grainy around the top of his head and gave the appearance of bubbling fizz on the surface of a freshly poured Coke.

"Cantor is most remembered for establishing the subject of set theory, the topic of this class. In doing this he single-handedly changed mathematics." Nico said this while looking at the photograph and slowly rubbing his chin. In the pause I felt that he would have loved to talk with Cantor in person, and frankly I would have loved to listen in on that conversation. Then, with a palpable gear shift, he turned and faced the classroom once more. He stood up straighter and his tone was firmer. It was time for mathematics.

"At an intuitive level a set is simply any collection of objects. Let me write out a few examples." He turned to the board and wrote:

A = {chair, elephant, tomato}
B = {16, watch, book, 23.75, saxophone}
C = {Godzilla, {A}}
N = {1, 2, 3, 4. . .}

"As you can see a set can have any object as a member. It is typical to collect the objects of a set within curly brackets." Nico pointed to the "{"and"}" which marked the opening and closing of his sets. "A tomato is an element (or a member) of set A, and the set A has three elements. The set **N** has an unlimited number of elements. It is not a finite set. *Infinity is simply defined as the order of a set that is not finite*."

It seemed a somewhat circular description to me, and I wasn't sure I saw the benefit. I looked up to protest but saw Nico looking at the class with an amused expression. He had anticipated our difficulties. "I can see from your faces that this definition is not the least bit satisfying. 'What is the point?' you all seem to be asking. You will see the point, I promise. More satisfying definitions of infinity require more mathematical machinery than we have at this stage. I ask you to keep this definition in the back

of your mind, for it will allow us to make progress and build an amazing structure to deeply understand the nature of infinity. It is a structure that still gives me goose bumps," said Nico without any air of pretense that I could detect. "We'll get to Cantor in due time. I put the definition up front because it seems odd to begin a class about infinity without defining it. But for now, let's stay with the Greeks."

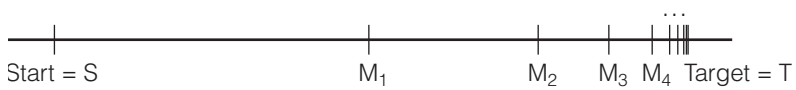
"The first Greek we'll meet is an odd bird by the name of Zeno, sometimes known as Zeno of Elea. He lived around the fifth century BC. He is said to have been a self-taught country boy. Zeno described several paradoxes built around the divisibility of space. The famous philosopher Plato dismissed these paradoxes as 'youthful efforts', yet he did nothing to resolve them. In fact, none of the best minds of the last two and a half thousand years could resolve the paradoxes raised by Zeno. Not bad for a country boy. The solutions came only about a hundred years ago. Today we're going to take a look at one of the most interesting of Zeno's paradoxes."

Nico went to the board and drew as he spoke. "Zeno asks us to consider a runner starting at a point S. He is going to his target T, which is 1 mile away. Now, to get to T he must first get to the midpoint between the starting point S and the target T. Call the midpoint M_1. It is a half-mile from T."

So far, so good. I was getting interested. There was always the faint hope that I would be able to crack a problem even though it had shown itself to be extraordinarily difficult. Nico had reintroduced me to the pleasures of the mathematical hunt; I would be chasing a puzzle!

Nico, meanwhile, was busy drawing another picture. "To get from M_1 to T the runner must once again get to the midpoint between the two. Call this midpoint M_2; it is a quarter of a mile away from T."

I began to see where this was going. To get from M_2 to T, the runner would have to get to M_3, then M_4, and so on forever. Nico's next drawing confirmed this.



"Because I'm constrained by a chalk of finite thickness I have not drawn M_5, M_6, and all the other M_n out there. But Zeno argued that the runner would indeed have to pass through an infinity of these points," said Nico. I could tell Nico was getting excited—his voice was louder and his pacing more intense. "You see," he said, walking over to the blackboard, "no matter how close the runner gets to T he still has to cover half the remaining distance, then half of what's left, and then half of what's left yet again. Essentially he has to keep making runs between successive M_i's. First he runs between S and M_1, then between M_1 and M_2, then between M_2 and M_3, and so on. Let's call each such run an 'M-run'."

Nico went to the blackboard again and wrote:

1. The runner would have to make an infinite number of M-runs.
2. It is impossible for the runner to make an infinite number of M-runs.
3. Therefore, the runner will never get to the target.

"Historians can't really be sure how Zeno himself saw his paradox. He may have seen it as a logical conundrum, or he may have used his argument to conclude that all motion is an illusion," said Nico.

"That's crazy!" exclaimed Peter. "Motion is not an illusion!"

"I agree, it sounds utterly crazy." said Nico. "Clearly, motion is possible. Clearly, people move and cover distances. Clearly, a runner can cover a mile without getting trapped within a sequence of M-runs." Nico was pacing again. "But just as clearly, logic works in our world. If an apparently logical argument leads to an absurd result, then either logic does not always work, or the argument is flawed in some subtle way. I firmly believe that logic works. So there must be a subtle flaw in Zeno's argument. And I want us to find it."

Peter nodded. Considering his impatience with philosophical arguments of this nature I knew that Nico had gotten through to him. I myself was beguiled by Zeno's argument. It seemed extraordinarily simple, trapping one in its iron-clad logic, and it was hard to avoid hurtling toward its inevitable but absurd conclusion.

"Let's examine the argument in pieces," resumed Nico. "First, does anyone doubt that the runner would have to make an infinite number of M-runs?"

"I do," said PK. "Toward the end the M-runs become so small that the runner's body itself would span over all of the last few M-intervals and would cover the target."