Chapter One

Introduction and Outline of Contents

1.1 INTRODUCTION

The use of various types of wave energy as a probe is an increasingly promising nondestructive means of detecting objects and of diagnosing the properties of quite complicated materials.

An analysis of this technique requires a detailed understanding of, first, how waves evolve in the medium of interest in the absence of any inhomogeneities and, second, the nature of the scattered or echo waves generated when the original wave is perturbed by inhomogeneities that might exist in the medium. The overall aim of the analysis is to calculate the relationships between the unperturbed waveform and the echo waveform and to indicate how these relationships can be used to characterise inhomogeneities in the medium.

The central problem with which we shall be concerned in this monograph can be simply stated as follows.

A system consists of a medium containing a transmitter and a receiver. The transmitter emits a signal that is eventually detected at the receiver, possibly after it has been perturbed, that is, scattered, by some inhomogeneity in the medium. We are interested in the manner in which the emitted signal evolves through the medium and the form that it assumes at the receiver. Properties of the scattered or echo signal are then used to estimate the properties of any inhomogeneity in the medium.

Classifying inhomogeneities in the medium into identifiable classes by means of their echoes is known as the *inverse scattering problem*. An associated problem is that of *waveform design*, which is concerned with the choice of the signal waveform that optimises the echo signal from classes of prescribed inhomogeneities. These problems are of considerable interest and importance in engineering and the applied sciences. However, in order to be able to investigate them, the problem of knowing how to predict the echo signal when the emitted signal and the inhomogeneities are known must be well understood. This is called the *direct scattering problem*.

When the media involved are either stationary or possess time-independent characteristics—these are called *autonomous problems* (APs)—the mathematical analysis of the associated scattering effects is now quite well developed and a number of efficient techniques are available for constructing solutions to both the direct and the inverse problems. However, when the media are either moving or have time-dependent characteristics—these are known as *nonautonomous problems* (NAPs)—the investigations of corresponding scattering phenomena have not reached such a well-developed stage. Nevertheless, there are many significant problems of interest in the applied sciences that are NAPs. For instance, this type of

problem can often arise when investigating sonar, radar, nondestructive testing and ultrasonic medical diagnosis methods. Indeed, they occur in any system that is either in motion or has components that either can be switched on or off or can be altered periodically. We shall study some of these systems in later chapters. These NAPs are intriguing both from a theoretical standpoint and from the point of view of developing constructive methods of solution; they certainly present a nontrivial challenge.

In our study here of NAPs we take as a starting point the assumption that all media involved consist of a continuum of interacting infinitesimal elements. Consequently, a disturbance in some small region of a medium induces an associated disturbance in neighbouring regions with the result that some sort of disturbance eventually spreads throughout the medium. We call the progress or evolution of such disturbances *propagation*. Typical examples of this phenomenon include, for instance, waves on water, where the medium is the layer of water close to the surface, the interaction forces are fluid pressure and gravity and the resulting waveform is periodic. Again, acoustic waves in gases, liquids and solids are supported by an elastic interaction and exhibit a variety of waveforms which can be, for example, sinusoidal, periodic, transient pulse or arbitrary. However, in principle any waveform can be set in motion in a given system provided suitable initial or source conditions are imposed.

The above discussion can be conveniently expressed in symbolic form as follows.

Consider first a system that has no inhomogeneities. Let $f_0(x, s)$ be a quantity that characterises the state of the system at some initial time $t = s$ and let $u_0(x, t)$ be a quantity that characterises the state of the system at some later time $t > s$. We shall be concerned with systems for which states can be related by means of an "evolution rule," denoted by $U_0(t - s)$, that determines the *evolution* in time of the system from its initial state $f_0(x, s)$ to a state $u_0(x, t)$ at a later time $t > s$. This being the case, we write

$$u_0(x, t) = U_0(t - s) f_0(x, s),$$

where it is understood that $U_0(0) = I =$ the identity.

In a similar manner, when inhomogeneities are present in the system, we will assume that we can express the evolution of the system from an initial state $f_1(x, s)$ to a state $u_1(x, t)$ at a later time $t > s$ in the form

$$u_1(x, t) = U_1(t - s) f_1(x, s), \qquad U_1(0) = I,$$

where $U_1(t - s)$ denotes an appropriate evolution rule. Thus we see that we are concerned with two classes of problems. When there are no inhomogeneities present in the system, we shall say that we have a *free problem* (FP). When inhomogeneities are present in a system, we shall say that we have a *perturbed problem* (PP). We shall express this situation symbolically in the form

$$u_j(x, t) = U_j(t - s) f_j(x, s), \qquad U_j(0) = I, \qquad j = 0, 1,$$

where when $j = 0$ we will assume that we have a FP whilst when $j = 1$ we have a PP.

The principal aim of this monograph is to make the preceding discussions more precise and, in so doing, indicate means of developing sound, constructive methods of solution from what might be originally thought to be a purely abstract mathematical framework. In this connection we are immediately faced with a number of fundamental questions.

- What are the mathematical equations that define (model) the systems of interest?
- What is meant by a solution of the defining equations?
- Under what conditions do the defining equations have unique solutions?
- When solutions of the defining equations exist, can they be expressed in the form

$$u_j(x,t) = U_j(t-s)\, f_j(x,s), \qquad U_j(0) = I, \qquad j = 0, 1,$$

  where $U_j(t-s)$, $j = 0, 1$, is an evolution rule?
- How can $U_j(t-s)$ be determined?
- What are the basic properties of $U_j(t-s)$, $j = 0, 1$?
- If a given problem is regarded as a PP, then an associated FP can be taken to be a problem that is more easily solved than the PP. We then ask, Is it possible to determine an initial state of the system defined by the FP so that the state of this system at some later time t, denoted $u_0(x,t)$, is equal in some sense to $u_1(x,t)$, the state at time t of the system defined by the PP? If we can do this, then it is readily seen that we will have taken a large step towards creating a firm basis from which to develop robust, constructive methods for determining the required quantity $u_1(x,t)$ in terms of a more readily obtainable quantity $u_0(x,t)$.

In connection with the possible equality of $u_j(x,t)$, $j = 0, 1$, we first recognise and make use of the fact that in most experimental procedures, measurements in a system are made far away from any inhomogeneities that might exist in the system. Consequently, we will be mainly concerned here with the nature of the solutions $u_j(x,t)$, $j = 0, 1$, and of their differences in the *far field* of any nonhomogeneity. With this in mind we shall see that it will be sufficient for our purposes to ask, What is the behaviour of $u_j(x,t)$, $j = 0, 1$, as $t \to \infty$? Once this asymptotic behaviour is known, we can clarify what we mean by the equality of $u_j(x,t)$, $j = 0, 1$, and turn to determining the conditions that will actually ensure when, in the far field at least, $u_j(x,t)$, $j = 0, 1$, can be considered equal.

Although we are particularly interested in addressing these various questions when dealing with NAPs, we would point out that even for APs a detailed mathematical analysis of such questions can be technically very demanding. However, we would emphasise that this monograph is not a book on such topics as functional analysis, mathematical scattering theory, linear operator theory and semigroup theory but rather is meant as a guide through these various areas with the intention of highlighting their uses in practical problems. Examples will be given whenever it is practical to do so, as it is felt that abstract theories are often best appreciated, at first, by means of examples. The presentation in this monograph will frequently be quite

formal, and we rely very much on the often held view expressed by Goldberger and Watson that "any formal manipulations which are not obviously wrong are assumed to be correct" [44]. Nevertheless, references will always be given in the text to where more precise and often quite general details can be found. Furthermore, we emphasise that in this monograph we are not interested in investigating the evolutionary processes mentioned above in full generality but rather shall confine our attention to those systems involving waves.

1.2 SOME ILLUSTRATIONS

As we have already mentioned, our main interest in this monograph will centre on those physical phenomena whose evolution can be described in terms of propagating waves. In the case of APs a simple example of a propagating wave is provided by a physical quantity y, which is defined by

$$y(x,t) = f(x-ct), \qquad (x,t) \in \mathbf{R} \times \mathbf{R}, \tag{1.1}$$

where c is a real constant. We notice that y has the same value at all those x and t for which $(x-ct)$ has the same values. Thus (1.1) represents a wave that moves with constant velocity c along the x-axis without changing shape. If f is assumed to be sufficiently differentiable, then on differentiating (1.1) twice with respect to x and t, we obtain

$$\left\{ \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right\} y(x,t) = 0, \qquad (x,t) \in \mathbf{R} \times \mathbf{R}. \tag{1.2}$$

Similarly, we notice that a physical quantity w defined by

$$w(x,t) = g(x+ct), \qquad (x,t) \in \mathbf{R} \times \mathbf{R}, \tag{1.3}$$

where c is a real constant, represents a wave moving with constant velocity c along the x-axis without changing shape but moving in the opposite direction to the wave $y(x,t)$ defined in (1.1). Furthermore, we see that $w(x,t)$ also satisfies an equation of the form (1.2). Equation (1.2) is referred to as the *classical wave equation.* The term "classical" will only be used in order to avoid possible confusion with other wave equations that we might consider.

Because the wave equation is linear, the compound wave

$$u(x,t) = y(x,t) + w(x,t) = f(x-ct) + g(x+ct), \tag{1.4}$$

where f, g are arbitrary functions, is also a solution of the wave equation (1.2). This is the celebrated d'Alembert solution of the wave equation. In specific problems the functions f, g are determined in terms of the imposed initial conditions that solutions of (1.2) are required to satisfy [71, 105]. Indeed, if $u(x,t)$ is required to satisfy the initial conditions

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x),$$

then (1.4) can be expressed in the form

$$u(x,t) = \frac{1}{2}\{\varphi(x-ct)+\varphi(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds \tag{1.5}$$

(see chapter 2 and [71, 105]). Consequently, if a system is defined by the wave equation (1.2), then (1.5) indicates how the initial state of the system, characterised by the functions φ and ψ, evolves into a state defined by $u(x,t)$, $t>0$.

As a simple illustration of such an evolutionary process, consider a system consisting of an infinite string in the particular case when $\psi \equiv 0$. The initial state of this system is then completely characterised by the function φ. For simplicity, assume that the initial displacement of the string is of finite amplitude and is only nonzero over a finite length of the string. The function φ defines this profile. The result (1.5) indicates that this initial state of the system evolves into a state consisting of two propagating waves travelling in opposite directions along the string, each having the same profile as the initial state but with only half the initial amplitude.

As an illustration of a scattering process, again consider waves on a string. We assume, as before, that $\psi \equiv 0$, but now we impose the extra requirement that the displacement of the string should be zero at $x=0$ for all time. Essentially, this means that we are considering waves on a semi-infinite string. Arguing as before, we see that the initial state of the system, which again is completely characterised by the function φ, evolves into a state consisting of two propagating waves travelling in opposite directions and having the same profile as the initial state but with only half the initial amplitude. However, this situation does not persist. After a certain time, say T, which is dependent on the velocity of the waves, one of the propagating waves will strike the "barrier" at $x=0$. This wave will bounce off the barrier; that is, it will be reflected or scattered by the barrier. Consequently, from time T onwards the initial state of the system evolves into a state consisting of three components. Specifically, it will consist of the two propagating waves mentioned earlier and a reflected or scattered wave. Thus the evolved state can become quite difficult to describe analytically. Quite how difficult matters can become will be demonstrated in chapter 2. Further complications arise when we have to work either in more than one dimension or with defining equations that are more difficult to analyse than the wave equation (1.2). It was largely with these prospects in mind that scattering theory was developed.

We would emphasise at this stage that not all solutions of the wave equation yield propagating waves. For example, if the wave equation is solved using a separation of variables technique, then *stationary wave* solutions can be obtained. They are called stationary waves because they have certain features, such as nodes and antinodes, that retain their positions permanently with respect to time. Such solutions can be related to the *bound states* appearing in quantum mechanics and to the *trapped wave* phenomenon of classical wave theory.

Since the classical wave equation occurs in so many areas of mathematical physics, we shall adopt it as a prototype for much of our present study. We shall, of course, discuss other types of wave equations after we have settled the main features of wave scattering associated with this particular equation.

1.3 TOWARDS GENERALISATIONS

Scattering means different things to different people. Broadly speaking, it can be thought of as the interaction of an evolutionary process with a nonhomogeneous and possibly nonlinear medium. Certainly, the study of scattering phenomena has played a central role in mathematical physics over the years, with perhaps the earliest investigation of them being attributed to Leonardo da Vinci who studied the scattering of light into the shadow of an opaque body. Subsequently, other scattering effects have been discovered and investigated in such diverse fields as acoustics, quantum mechanics, medical diagnosis and many other nondestructive testing procedures.

Scattering phenomena arise as a consequence of some perturbation of a given known system, and they are analysed by developing an associated scattering theory. These scattering theories are concerned, broadly speaking, with the comparison of two systems, one being regarded as a perturbation of the other, and with the study of the manner in which the two systems might eventually become equal in some sense. Since a NAP can often be regarded as a perturbation of an AP, the development of scattering theories involving NAPs and APs seems to offer good prospects for providing a sound basis from which to develop robust approximation methods for obtaining solutions to a NAP in terms of the more readily obtainable solutions to an associated AP.

An initial aim in this monograph is to present an introductory study of acoustic wave propagation and scattering in the presence of *time-dependent perturbations*. Later chapters of the book will indicate how the analysis of the acoustic case can be extended to similar problems in electromagnetism and elasticity. The study will be made from the standpoint of modern spectral analysis and scattering theory, and the emphasis throughout will be on the development of constructive methods.

Scattering processes can be conveniently characterised in terms of either one or the other of two basic problems. In one the governing differential equation is perturbed but the spatial domain of interest is unaffected. This gives rise to *potential scattering*. In the other the governing differential equation is unaltered but the spatial domain of interest is perturbed. This gives rise to *target scattering*. In developing the material that follows we shall often begin with a discussion of APs, in particular with potential scattering problems, where the notion of a perturbation is perhaps rather more clear, as much because the technical requirements are less than those for target scattering as because the rich theory and analytical techniques of quantum scattering can quite readily be used to indicate the way we might proceed in our analysis. Afterwards we turn our attention to target scattering.

For APs there are traditionally two main approaches to the analysis of scattering phenomena. The first is by means of a *time-independent* or *stationary scattering* theory. This consists of separating out the time dependence in the problems and then studying solutions of the resulting spatial equations. This is a *frequency domain* analysis in which a central interest is the asymptotic behaviour of solutions at large distances. The second approach is by means of a *time-dependent scattering* theory in which the time evolution of the states of the systems and their asymptotic behaviour for large time is a dominant interest. This is a *time domain analysis.*

We remark that not all states of a system necessarily lead to scattering events; for example, as mentioned earlier, a system can have bound states. However, those states of a system that do lead to scattering events and which, as $t \to \pm\infty$, are *asymptotically equal* (AE) in some sense to scattering events of some simpler system are said to satisfy an *asymptotic condition*. An investigation of this condition is a principal study of time-dependent scattering theory. The importance of the time-independent theory for APs lies in the fact that the actual calculation of expressions can often be more readily carried out than for similar expressions occurring in the time-dependent theory. However, we shall see that for NAPs *the separation of variables technique is no longer available*. Consequently, an associated time-independent theory is *not* immediately available.

In the quantum mechanics of the scattering of elementary particles by a potential, the wave packets that characterise scattered particles can be shown to be AE, for large time, to the corresponding wave packets for free particles. The correspondence between these two systems, one describing the scattered particles and the other describing the free particles, is effected by means of Møller operators [71]. Wilcox [154, 153] has developed analogous concepts for wave propagation problems associated with the equations of classical physics. Specifically, Wilcox established conditions that ensure that waves propagating in a inhomogeneous medium are AE, for large time, to corresponding waves propagating in a homogeneous medium. The correspondence between the two wave systems is given by an analogue of the Møller operators of quantum scattering, which we shall simply refer to as *wave operators* (WOs). Since wave propagation problems in homogeneous media can often be solved explicitly, a knowledge of an appropriate WO then provides information concerning the asymptotic behaviour of solutions to wave propagation problems in inhomogeneous media. The intention here is to indicate how these ideas of Wilcox might extend to an investigation of NAPs.

As we remarked above, the study of NAPs has been motivated by problems arising in such areas as, for example, radar, sonar, nondestructive testing and ultrasonic medical diagnosis. In all these areas a powerful diagnostic is the dynamical response of the media to the emitted signal. Mathematically, many of these problems can be conveniently modelled in terms of an *initial boundary value problem* (IBVP). To fix ideas, we shall confine our attention initially to acoustic problems and to IBVPs for the classical wave equation. Results for electromagnetic problems and for elastic problems will be discussed in later chapters.

Specifically, for NAPs we shall be interested in IBVPs that have the following typical form.

Determine a quantity $w(x,t)$ that satisfies

$$\left\{\partial_t^2 + L(x,t)\right\} w(x,t) = f(x,t), \qquad (x,t) \in Q, \tag{1.6}$$

$$w(x,s) = w_{0s}(x), \quad w_t(x,s) = w_{1s}(x), \quad x \in \Omega(s), \tag{1.7}$$

$$w(x,t) \in (bc)(t), \quad (x,t) \in \partial\Omega(s) \times \mathbf{R}, \tag{1.8}$$

where

$$L(x,t) = -c^2\Delta + q(x,t), \tag{1.9}$$

with Δ denoting the usual Laplacian in $\mathbf{R}^n$ and

$$Q := \{(x,t) \in \mathbf{R}^n \times \mathbf{R}\},$$
$$\Omega(t) := \{x \in \mathbf{R}^n : (x,t) \in Q\},$$
$$B(t) := \{x \in \mathbf{R}^n : (x,t) \notin Q\}.$$

The region Q is an open set in $(\mathbf{R}^n \times \mathbf{R})$, and $\Omega(t)$ denotes the exterior, at time t, of a scattering target $B(t)$. For each value of t the domain $\Omega(t)$ is open in $\mathbf{R}^n$ and is assumed to have a smooth boundary $\partial\Omega(t)$. The lateral surface of Q, denoted ∂Q, is defined by

$$\partial Q := \bigcup_{t \in I} \partial\Omega(t), \tag{1.10}$$

where, for some fixed $T > 0$, $I := \{t \in \mathbf{R} : 0 \le t < T\}$.

The quantities f, c, w_{0s}, w_{1s} and q are given *data functions*, and $s \in \mathbf{R}$ denotes a *fixed initial time*. The notation in (1.8) indicates that the solution $w(x,t)$ is required to satisfy certain conditions, denoted (bc), imposed on the boundary that might also depend on t.

For purposes of illustration in this introductory chapter, we will discuss here, in an entirely formal manner, scattering phenomena associated with *initial value problems* (IVPs) of the form

$$\{\partial_t^2 + L(x,t)\}u(x,t) = 0, \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}, \tag{1.11}$$

$$u(x,s) = f_s(x), \quad u_t(x,s) = g_s, \quad x \in \mathbf{R}^n, \tag{1.12}$$

where $s \in \mathbf{R}$ is a fixed initial time, L is defined via (1.9), and f_s and g_s are given data functions. We remark that when investigating (1.6), an appropriate version of Duhamel's principle (see chapter 5) indicates that in the first instance it is sufficient to study an equation of the form (1.11).

Problems of the form (1.6)–(1.8) divide quite naturally into the following two broad classes.

Target scattering problems: (i) $q(x,t) \equiv 0$; (ii) the (bc) depend on t.
Potential scattering problems: (i) $q(x,t) \not\equiv 0$; (ii) the (bc) are independent of t.

Target scattering problems divide quite readily into two important types.

- Exterior problems in which waves are scattered by obstacles of either finite or infinite extent. These arise, for example, in studies of antennae, propellers and layered media.
- Interior problems in which waves are scattered within some cavity.

We shall be particularly interested in this monograph in the NAP versions of these two types of problems, for example, when an obstacle might be moving or a cavity pulsating.

The NAP versions of potential scattering problems are typical of those that can occur as a consequence of linearising about a time-periodic solution of some nonlinear problem. These can arise, for example, when investigating nonlinear materials and also in certain approaches to inverse scattering problems.

Work on these two types of problems has presented a number of interesting features. For instance, for problems posed in a bounded region, that is, for an interior problem, it has been found that solutions can grow exponentially in energy yet remain finite in amplitude. Whether or not this is also true for exterior problems is not yet fully settled. Numerical evidence indicates that growth of solution occurs by virtue of a parametric resonance similar to that found in studies of the Hill and the Mathieu equations. Again, for such NAPs the familiar separation of variables technique and associated Fourier transform methods are not immediately available. Consequently, a thorough time-dependent theory is required for such problems. Towards this end we shall often give relatively simple worked examples of wave motions on strings to illustrate the salient features of the various analytical techniques that will be required when developing more general time-dependent scattering theories for both APs and NAPs.

To illustrate some of the strategies that will be adopted in this monograph when developing scattering theories, let us begin by considering two systems that are governed, respectively, by IVPs of the form

$$\{\partial_t^2 + L_0(x)\}u_0(x,t) = 0, \quad u_0(x,0) = \varphi_0(x), \quad u_{0t}(x,0) = \psi_0(x), \tag{1.13}$$

$$\{\partial_t^2 + L_1(x)\}u_1(x,t) = 0, \quad u_1(x,0) = \varphi_1(x), \quad u_{1t}(x,0) = \psi_1(x), \tag{1.14}$$

where $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ and, recalling (1.11), (1.12), we have set, for ease of presentation and without any loss of generality, $s = 0$.

In (1.14) the differential expression $L_1(x)$ is assumed to be some perturbation of the differential expression $L_0(x) = -\Delta$, the n-dimensional Laplacian. For this reason we shall refer to (1.13) as a free problem and (1.14) as a perturbed problem.

We remark that, as they are written, (1.13) and (1.14) are APs. However, when q and/or (bc) is also time-dependent, (1.13) and (1.14) are NAPs. We shall restrict our attention, for the moment, to the AP case. We emphasise that the approach we adopt in this introductory chapter will be almost entirely formal. The necessary mathematical structure required to support much of the presentation will be provided in chapters 3–5.

An analysis of such problems must begin with a statement that indicates what is meant by solutions of (1.13) and (1.14); that is, a *solution concept* must be introduced. For example, we might want to confine our attention to a *local classical solution* of (1.13) (or (1.14)). This would be an element v with the property that

$$v \in C(\mathbf{R}^n \times [0,T), \mathbf{R}) \cap C^2(\mathbf{R}^n \times [0,T), \mathbf{R}), \tag{1.15}$$

which, for some $T > 0$, satisfies (1.13) (or (1.14)). The notation here means that v is a continuous function, defined on $\mathbf{R}^n \times [0,T)$ with values in $\mathbf{R}$, that is also twice continuously differentiable (see chapter 3 for more details). A local classical

solution is called a *global solution* if we can take $T = \infty$. A *p-periodic solution* is a global solution that is p-periodic in $t \in \mathbf{R}$.

Since we shall be concerned with problems that model real-life situations, we shall always require that mathematically the problem be *well posed*; that is, it is expected that

- a solution exists and is unique for the class of data of interest,
- the solution depends continuously on the data.

Even when it is possible in principle, the actual determination of a classical solution is often a very difficult task in practice. The situation can be eased considerably by realising, that is, by interpreting, IVPs (1.13) and (1.14) in some more convenient collection of functions than that indicated in (1.15). For example, we might choose to realise IVPs (1.13) and (1.14) in the collection of functions denoted by

$$H := L_2(\mathbf{R}^n \times \mathbf{R}) \equiv L_2(\mathbf{R}^n \times \mathbf{R}, \mathbf{C}),$$

which consists of all those functions that are square-integrable functions on their domain of definition $(\mathbf{R}^n \times \mathbf{R})$ and which take values in $\mathbf{C}$, the collection of complex numbers. This choice is not as contrived as it might appear at first sight. We shall see that such a selection will enable us to deal quite automatically with important physical quantities such as, for example, energy. We can obtain a realisation of IVPs (1.13) and (1.14) in the collection $H := L_2(\mathbf{R}^n \times \mathbf{R}) \equiv L_2(\mathbf{R}^n \times \mathbf{R}, \mathbf{C})$ by introducing the spatial *mappings* or *operations* A_j, $j = 0, 1$, defined by (see chapter 3 for more details)

$$\begin{aligned} &A_j : H \to H, \qquad j = 0, 1 \\ &A_j u_j = L_j u_j, \qquad u_j \in D(A_j), \qquad j = 0, 1, \\ &D(A_j) = \{u_j \in H : L_j u_j \in H,\ u_j \in (bc)\}. \end{aligned} \tag{1.16}$$

Here $D(A_j)$ denotes the *domain of definition* of the mapping A_j. In the event that we are only dealing with potential scattering problems, the qualifier $u_j \in (bc)$ is omitted from $D(A_j)$.

Notice the following.

- We have assumed, for ease of presentation, that (1.13) and (1.14) can be posed in the same collection H. This will not always be the case.
- The definition of $D(A_j)$ ensures that throughout any mathematical manipulations involving A_j, we always "stay in H."
- When we have more mathematical structure available, we will be able to replace the word "collection" by "space" and the word "operations" by "operators" (see chapter 3).

The *classical IVPs* (1.13) and (1.14) can now be replaced by the *abstract IVPs*

$$\begin{gathered} \{\partial_t^2 + A_j\} u_j(x, t) = 0, \\ u_j(x, 0) = \varphi_j(x), \qquad u_{jt}(x, 0) = \psi_j(x), \qquad j = 0, 1. \end{gathered} \tag{1.17}$$

If we now understand u_j to be a "rule" defined by

$$u_j = u_j(.,.) : t \to u_j(.,t) =: u_j(t) \in H \qquad \text{for all } t \in \mathbf{R},$$

then IVPs (1.17) can be written equivalently in the form

$$\{d_t^2 + A_j\}u_j(t) = 0, \qquad u_j(0) = \varphi_j, \qquad u_{jt}(0) = \psi_j, \qquad j = 0, 1, \tag{1.18}$$

which is an *ordinary differential equation* in H. Consequently, with this understanding, problems (1.18) have solutions (in H) that can be written in the form

$$u_j(t) = (\cos t A_j^{1/2})\varphi_j + A_j^{-1/2}(\sin t A_j^{1/2})\psi_j, \qquad j = 0, 1. \tag{1.19}$$

Hence the solutions of (1.17) can be written in the form

$$\begin{aligned} u_j(x,t) &= (\cos t A_j^{1/2})\varphi_j(x) + A_j^{-1/2}(\sin t A_j^{1/2})\psi_j(x), \qquad j = 0, 1 \\ &= \mathrm{Re}\{v_j(x,t)\}, \qquad j = 0, 1, \end{aligned} \tag{1.20}$$

where

$$\begin{aligned} v_j(x,t) &= \exp\{-it A_j^{1/2}\}h_j(x) =: U_j(t)h_j(x), \\ h_j(x) &= \varphi_j(x) + i A_j^{-1/2}\psi_j(x). \end{aligned} \tag{1.21}$$

The quantities $v_j(x,t)$, $j = 0, 1$, in (1.21) define the complex-valued solutions of IVPs (1.17). The quantity $U_j(t)$ is a rule that governs the evolution (in time) of $h_j(x)$, the initial state of the jth system, into $v_j(x,t)$, the state of the jth system at some other time t. For this reason it will be called, eventually, an *evolution operator*.

In order to develop a scattering theory, we must first show that the FP and the PP actually have solutions in some convenient collections of functions. Once this has been done, we will need a means of comparing these solutions; that is, we will need some suitable formula for measuring their distance apart as either $t \to \pm\infty$ or $x \to \infty$. In general, such a formula is called a *norm*. A familiar example of this is the formula for measuring the distance, in $\mathbf{R}^2$, between the origin O and a point P with coordinates (x, y). The length of the line OP, denoted $\|OP\|$, is well known to be

$$\|OP\| := \sqrt{x^2 + y^2}. \tag{1.22}$$

In (1.22) we are working with *numbers*. We shall see (chapter 3) that the notion of a norm can be generalised so that we will be able to measure the separation of two *functions* rather than having to deal with the separation of the *numerical values* of these functions. With all this in mind we shall write

$$v_j(.,t) := v_j(t), \qquad j = 0, 1, \tag{1.23}$$

and compare the solutions of the FP and the PP by considering an expression of the form

$$\|v_1(t) - v_0(t)\|, \tag{1.24}$$

where $\|.\|$ denotes some suitably chosen norm defined in terms of spatial coordinates. We then find that

$$\begin{aligned}\|v_1(t) - v_0(t)\| &= \|U_1(t)h_1 - U_0(t)h_0\| \\ &= \left\|U_0^*(t)U_1(t)h_1 - h_0\right\| \\ &= \|W(t)h_1 - h_0\|\,, \end{aligned} \tag{1.25}$$

where the symbol $*$ denotes that $U_0(t)$ has a modified form (something like an inverse) as a result of moving it from one quantity to another.

The time dependence in the above comparison can be removed by taking the limit as $t \to \pm\infty$. We then obtain

$$\lim_{t\to\pm\infty} \|v_1(t) - v_0(t)\| = \|W_\pm h_1 - h_0\|, \tag{1.26}$$

where

$$W_\pm := \lim_{t\to\pm\infty} W(t) = \lim_{t\to\pm\infty} U_0^*(t)U_1(t). \tag{1.27}$$

If it can be shown that the two solutions $v_1(t)$ and $v_0(t)$ exist and that the various steps leading to (1.27) can be justified, then it will still remain to be shown that the limits in (1.27), which define the *wave operators* $W_\pm$, actually exist.

When all the above has been achieved, we see that if the initial data for the FP and PP are related according to

$$h_0 = W_\pm h_1, \tag{1.28}$$

then the limit in (1.26) is zero, thus indicating that the PP is *asymptotically free* as $t \to \pm\infty$. That is, solutions of the PP with initial data (state) h_1 are *asymptotically equal* in time to solutions of a FP with initial data (state) h_0, which is given by (1.28).

Consequently, if solutions of the two systems are known to exist, then, keeping (1.28) in mind, we would expect there to exist elements $h_\pm$ such that

$$v_1(t) \sim U_0(t)h_\pm \quad \text{as}\, t \to \pm\infty, \tag{1.29}$$

where $\sim$ denotes *asymptotic equality* and the $\pm$ are used to indicate the possibly different limits as $t \to \pm\infty$. We would emphasise that it is not automatic that both the limits implied by (1.29) should exist. Indeed, a solution such as v_1 could be asymptotically free as $t \to +\infty$ but not as $t \to -\infty$.

If we use (1.29) in conjunction with the definition $v_j = U_j h_j,\ j = 0, 1$, then we have

$$U_0^*(t)U_1(t)h_1 =: W(t)h_1 \sim h_\pm \tag{1.30}$$

and we can conclude that

$$W_\pm : h_1 \to h_\pm. \tag{1.31}$$

The two initial conditions $h_\pm$ for the FP are related. This is illustrated by noticing that the above discussion implies

$$h_+ = W_+ h_1 = W_+ W_-^* h_- =: S h_-, \tag{1.32}$$

where S is called the *scattering operator* (SO) for the problem.

Thus we see that the existence of the wave operators $W_\pm$ is equivalent to the *asymptotic condition*

$$\lim_{t\to\pm\infty} \|v_1(t) - v_0(t)\| = \|W_\pm h_1 - h_0\| = 0. \tag{1.33}$$

Consequently, the construction of $W_\pm$ will be one of our primary goals.

For the approach outlined above to be of any practical use, we have to deal with two basic tasks.

Task 1: Provide a means of interpreting, in a constructive manner, such terms as $(\cos t A_j^{1/2})$ and $\exp\{-it A_j^{1/2}\}$, $j = 0, 1$.

Task 2: Determine the far-field forms of the solutions $v_j(x, t)$, $j = 0, 1$, as these are the quantities to be compared.

In addressing these two tasks we shall look first at the FP ($j = 0$). Our intention will be to use as much as possible of our treatment of the FP as a pattern for analysis of the associated PP ($j = 1$).

With regard to task 1 recall that when working in a finite-dimensional space setting, that is, when we are concerned with matrix equations rather than with differential equations, the corresponding situation is resolved in terms of eigenvector expansions (also called spectral decompositions). We shall see (chapter 4) that in an infinite-dimensional setting the matter can be resolved by means of the celebrated spectral theorem. However, there is an inherent difficulty associated with this particular approach. It centres on the practical, constructive determination of an appropriate collection of *projection operators*, the *spectral family*, associated with A_0. The situation can be eased considerably by noticing that the FP we are considering at the moment is posed in all of $\mathbf{R}^n$. This means that, for the wave equation at least, we can make use of the Plancherel theory of Fourier transforms [144]. This theory indicates that for any $f \in H \equiv H(\mathbf{R}^n)$, the following integrals exist and are convergent.

$$(Ff)(p) = \hat{f}(p) = \int_{\mathbf{R}^n} \overline{w_0(x, p)}\, f(x)\, dx, \tag{1.34}$$

$$f(x) = (F^* \hat{f})(x) = \int_{\mathbf{R}^n} w_0(x, p)\, \hat{f}(p)\, dp, \tag{1.35}$$

$$((\Phi A_0) f)(x) = \int_{\mathbf{R}^n} w_0(x, p) \Phi(|p|^2)\, \hat{f}(p)\, dp, \tag{1.36}$$

where Φ is a function that is "sufficiently nice" for our purposes and $w_0(x, p)$ is the usual Fourier kernel

$$w_0(x, p) = \frac{1}{(2\pi)^{n/2}} \exp(ix.p), \qquad x, p \in \mathbf{R}^n. \tag{1.37}$$

We notice that $w_0(x, p)$ satisfies the Helmholtz equation

$$(A_0 + |p|^2)w_0(x, p) = 0, \qquad x, p \in \mathbf{R}^n. \tag{1.38}$$

Recalling the results obtained when dealing with matrix equations, it might be thought that w_0 is an eigenfunction of $A_0 = -\Delta$ with associated eigenvalue $|p|^2$ and as such could provide the basis for a spectral decomposition of A_0. However, we shall find that for most cases of interest here, w_0 does *not* belong to H, the underlying collection of functions. Consequently, w_0 will have to be regarded as being in some sense a generalised eigenfunction of A_0. (See chapter 4 for details and also the remarks in Chapter 11.)

We emphasise that the integrals in (1.34)–(1.36) are improper integrals and as such must be interpreted by means of a limiting process intimately connected with the specific problem being considered. This aspect will be discussed in detail in chapter 6.

With these several remarks in mind we see that the Plancherel theory, which has been developed *quite independently* of any scattering phenomena, indicates that (1.34)–(1.36) provide the required spectral decomposition of A_0 and as such will be referred to as a *generalised eigenfunction expansion theorem* for A_0.

The above generalised eigenfunction expansion theorem provides the means for interpreting the various terms in (1.20) and (1.21). Indeed, a straightforward application of (1.36) yields the required solutions in the following forms.

$$u_0(x, t) = \int_{\mathbf{R}^n} w_0(x, p) \left\{ \hat{\varphi}_0(p) \cos t\,|p| + \hat{\psi}_0(p) \frac{\sin t\,|p|}{|p|} \right\} dp \tag{1.39}$$

and

$$\begin{aligned} v_0(x, t) &= \int_{\mathbf{R}^n} w_0(x, p) \exp(-it\,|p|)\, \hat{h}_0(p)\, dp \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} \exp\{i(x.p - t\,|p|)\}\, \hat{h}_0(p)\, dp. \end{aligned} \tag{1.40}$$

The results (1.39) and (1.40) complete task 1 for the case $j = 0$.

As far as task 2 is concerned, we shall use (1.40) to calculate the asymptotic behaviour of the complex waveform v_0. To this end and for convenience at this stage, we shall assume

$$\operatorname{supp} \hat{h}_0(p) = \{p : 0 < a \le |p| \le b\}.$$

Here "supp" denotes "support." The support of a function is defined as the closure of the set where it is nonvanishing (see chapter 3). In this case (1.40) will read

$$\begin{aligned} v_0(x, t) &= \frac{1}{(2\pi)^{n/2}} \int_{a \le |p| \le b} \exp\{i(x.p - t\,|p|)\}\, \hat{h}_0(p)\, dp \\ &= \int_{a \le |p| \le b} w_0(x, p) \exp(-it\,|p|)\, \hat{h}_0(p)\, dp. \end{aligned} \tag{1.41}$$

If we now introduce spherical polar coordinates for p in (1.41) according to

$$p = \rho\omega, \qquad \rho \geqslant 0, \qquad dp = \rho^{n-1}\, d\rho\, d\omega, \qquad \omega \in S^{n-1} := \{x \in \mathbf{R}^n : |x| = 1\}, \tag{1.42}$$

then (1.41) becomes

$$v_0(x,t) = \frac{1}{(2\pi)^{n/2}} \int_a^b \int_{S^{n-1}} \exp\{i\rho(x.\omega - t)\}\, \hat{h}_0(\rho\omega)\rho^{n-1}\, d\omega\, d\rho \tag{1.43}$$

$$= \frac{1}{(2\pi)^{n/2}} \int_a^b \exp(-i\rho t) V_0(x,\rho)\rho^{n-1}\, d\rho, \tag{1.44}$$

where

$$V_0(x,\rho) := \int_{S^{n-1}} \exp\{i\rho(x.\omega)\}\, \hat{h}_0(\rho\omega)\, d\omega \tag{1.45}$$

and $x = |x|\,\eta$ with $\eta \in S^{n-1}$. Furthermore, it is easily verified that $V_0(x,\rho)$ is a solution of the Helmholtz equation $(\Delta + \rho^2)V_0(x,\rho) = 0$ for all $x \in \mathbf{R}^n$. Our aim now is to display the asymptotic behaviour of v_0 as $t \to \infty$. This we do by first calculating the asymptotic behaviour of $V_0(x,\rho)$ as $|x| \to \infty$ and then using this knowledge to determine the required behaviour of $v_0(x,t)$ as $t \to \infty$. To give some indication of how this can be done, we begin by noticing that the integral in (1.45) suggests use of the method of stationary phase. A precise version of this method has been developed for integrals of the form (1.45) by Littman [76] and by Matsumura [78]. Their results for our present case lead to the following representations.

$$v_0(x,t) = |x|^{(1-n)/2}\, G\left(\frac{x}{|x|}, |x| - t\right) + |x|^{(1-n)/2}\, G_1\left(\frac{x}{|x|}, |x| + t\right) + q_1(x,t), \tag{1.46}$$

where $G(\eta, r)$ and $G_1(\eta, r)$ are functions of $\eta \in S^{n-1}$ and $r \in \mathbf{R}$. Here

$$G(\eta, r) = \frac{1}{(2\pi)^{1/2}} \int_a^b \exp(i\rho r)\, \hat{h}_0(\rho\eta)(-i\rho)^{(n-1)/2}\, d\rho. \tag{1.47}$$

The precise forms of G_1 and q_1 are not really of immediate interest to us in this introductory chapter since in many specific cases of interest it can be proved that they tend to zero as $t \to \infty$; consequently, we shall set them to zero here.

The relation (1.47) has the flavour of a Fourier transform. Indeed, in terms of the usual one-dimensional Fourier transform, denoted here by F_1, we have the following defining relations.

$$\hat{G}_{(1)}(\eta,\rho) = (F_1 G)(\eta,\rho) = \lim_{M\to\infty} \frac{1}{(2\pi)^{1/2}} \int_{|r|\leq M} \exp(-i\rho r) G(\eta, r)\, dr, \tag{1.48}$$

$$G(\eta, r) = (F_1^* \hat{G}_{(1)})(\eta, r) = \lim_{M\to\infty} \frac{1}{(2\pi)^{1/2}} \int_{|\rho|\leq M} \exp(i\rho r)\, \hat{G}_{(1)}(\eta,\rho)\, d\rho, \tag{1.49}$$

where the limits are taken in the $L_2(S^{n-1} \times \mathbf{R})$ sense. Furthermore, the Fourier-Plancherel theory [144, 154] indicates that

$$\|G\| \quad = \left\| \hat{G}_{(1)} \right\| \quad \text{for all } G \in L_2(S^{n-1} \times \mathbf{R}),$$

where $\|.\|$ is the "'size" rule, or norm, in $L_2(S^{n-1} \times \mathbf{R})$. We also notice that for any $h \in L_2(\mathbf{R}^n)$, and in particular for h_0 above, the Fourier transforms (1.34)–(1.36) in conjunction with the Parseval equality, using $\|.\|$ to denote the size rule in $L_2(\mathbf{R}^n)$, indicate

$$\begin{aligned}
\|h_0\|^2 = \left\| \hat{h}_0 \right\|^2 &= \int_{\mathbf{R}^n} \left| \hat{h}_0(p) \right|^2 dp = \int_0^\infty \int_{S^{n-1}} \left| \hat{h}_0(\rho\eta) \right|^2 \rho^{n-1}\, d\rho\, d\eta \\
&= \int_0^\infty \int_{S^{n-1}} \left| (-i\rho)^{(n-1)/2} \hat{h}_0(\rho\eta) \right|^2 d\rho\, d\eta \\
&= \int_0^\infty \int_{S^{n-1}} \left| \hat{G}_{(1)}(\eta, \rho) \right|^2, \qquad (1.50)
\end{aligned}$$

where

$$\hat{G}_{(1)}(\eta, \rho) := \begin{cases} (-i\rho)^{(n-1)/2} \hat{h}_0(\rho\eta), & \rho \geqslant 0, \\ 0, & \rho < 0. \end{cases} \qquad (1.51)$$

To see that this notation makes sense, we rewrite the definition (1.47) in a slightly different form as follows.

$$\begin{aligned}
G(\eta, r) &= \frac{1}{(2\pi)^{1/2}} \int_a^b \exp(i\rho r)\, \hat{h}_0(\rho\eta)(-i\rho)^{(n-1)/2}\, d\rho \\
&= \frac{1}{(2\pi)^{1/2}} \int_{\mathbf{R}} \chi_{ab}(\rho) \exp(i\rho r)\, \hat{h}_0(\rho\eta)(-i\rho)^{(n-1)/2}\, d\rho, \qquad (1.52)
\end{aligned}$$

where $\chi_{ab}(\rho)$ is the characteristic function of the interval $[a, b]$.

We now use (1.49) to obtain

$$G(\eta, r) = F_1^*\{\chi_{ab}(.)\, \hat{h}_0((.)\eta)(-i)(.))^{(n-1)/2}\}(r, \eta). \qquad (1.53)$$

Hence

$$(F_1 G)(\eta, \rho) \quad = \quad \hat{G}_{(1)}(\eta, \rho) \quad = \quad \chi_{ab}(\rho)\, \hat{h}_0(\rho\eta)(-i\rho)^{(n-1)/2}, \qquad (1.54)$$

and we conclude that the Fourier transform of $G(\eta, r)$ is

$$\chi_{ab}(\rho)\, \hat{h}_0(\rho\eta)(-i\rho)^{(n-1)/2}.$$

Bearing in mind (1.54), we see that the right-hand side of (1.51) is a Fourier transform in the sense of (1.48) and as such defines a unique function $G \in L_2(S^{n-1} \times \mathbf{R})$, hence the notation. These several remarks suggest the following.

DEFINITION 1.1 *(i) The function G defined by (1.51) and (1.49) is the wave profile associated with each $h_0 \in L_2(\mathbf{R}^n)$.*
(ii) Corresponding to the wave function v_0 defined by

$$v_0(x,t) = \exp\{-it A_0^{1/2}\} h_0(x), \qquad h_0 \in L_2(\mathbf{R}^n),$$

there is an asymptotic wave function v_0^∞ defined by

$$v_0^\infty(x,t) := |x|^{(1-n)/2}\, G\left(\frac{x}{|x|}, |x| - t\right), \tag{1.55}$$

where $x \in \mathbf{R}^n \setminus \{0\}$, $t \in \mathbf{R}$, and G is the wave profile for h_0.

The manner in which we arrived at definition 1.1 indicates that it will only have practical significance if we can actually *prove* a result of the following form.

THEOREM 1.2 *For every $h_0 \in L_2(\mathbf{R}^n)$*

$$\lim_{t\to\infty} \left\| v_0(,.t) - v_0^\infty(.,t) \right\| = 0.$$

A proof of such a theorem requires that the solutions involved have certain decay properties. We develop this aspect in later chapters.

Once we reach this stage, we have completed tasks 1 and 2 for the FP. We would now like to have a similar pattern of results for the PP.

When the free system is perturbed either by the presence of an obstacle (target scattering) or by additional terms being attached to L_0 (potential scattering), the FP solution $v_0(x,t)$ will be perturbed to yield $v_1(x,t)$, the solution of the PP. Furthermore, we can expect that the generalised eigenfunction expansion given by (1.34)–(1.36) will also become perturbed. Essentially, this means that in order that we might have a generalised eigenfunction expansion theorem appropriate for the PP and which has the same form as (1.34)–(1.36), we can expect to have to use a kernel $w(x,p)$, which is some perturbation of the kernel $w_0(x,p)$ used for the FP.

Now, $w_0(x,p)$ satisfies the Helmholtz equation (1.38). Furthermore, a separation of variables technique indicates that it is a *steady-state* solution of the wave equation and represents a plane wave propagating in the direction p with a time dependence of the form $\exp\{-it|p|\}$. Consequently, in order to obtain a generalised eigenfunction expansion theorem for the mapping A_1 that characterises the PP as a perturbation of that obtained for the mapping A_0 that characterises the FP, we can expect to have to use a perturbed (distorted) kernel $w(x,p)$ satisfying

$$\{A_1 + |p|^2\} w(x,p) = 0, \tag{1.56}$$

where we assume

$$w(x,p) = w_0(x,p) + w'(x,p). \tag{1.57}$$

This will then emphasise that the total wave field associated with the PP will be characterised by the plane wave $w(x,p)$, which consists of a *free* or *incident wave*,

characterised by $w_0(x, p)$, and a *scattered wave*, characterised by $w'(x, p)$. For this reason we say that $w(x, p)$ is a *distorted plane wave* [54].

We will find that there are two families of generalised eigenfunction expansions arising from the use of the distorted plane wave $w(x, p)$. For one family the distorted plane wave consists of an incident free wave and a scattered wave that is *outgoing*. For the other family the scattered wave is *incoming*.

The outgoing and incoming distorted plane waves will be denoted by $w_+(x, p)$ and $w_-(x, p)$, respectively. Hence we write

$$w(x, p) \equiv w_\pm(x, p) = w_0(x, p) + w'_\pm(x, p). \tag{1.58}$$

The precise meaning of "outgoing" and "incoming" will be given in chapter 6 and in later chapters as required. For the time being just think of them as waves that either approach or recede from a barrier.

The actual construction of $w'_\pm(x, p)$ is achieved via a study of the Helmholtz equation [62]. We remark that this will rely on a detailed investigation of $\sigma(A_1)$, the *spectrum* of A_1, and on $R_\lambda(A_1) = (A_1 - \lambda I)^{-1}$, the *resolvent* of A_1. This lies in the province of *stationary* (*steady-state*) scattering theory and will be dealt with later.

Once more is known about the distorted plane wave kernels $w_\pm(x, p)$, in many cases of practical interest it is possible to *prove* generalised eigenfunction expansion theorems of the following form.

$$(F_\pm f)(p) := \hat{f}_\pm(p) = \int_{\mathbf{R}^n} \overline{w_\pm(x, p)}\, f(x)\, dx, \tag{1.59}$$

$$f(x) = (F_\pm^* \hat{f}_\pm)(x) = \int_{\mathbf{R}^n} w_\pm(x, p)\, \hat{f}_\pm(p)\, dp, \tag{1.60}$$

$$(\Phi(A) f)(x) = \int_{\mathbf{R}^n} w_\pm(x, p) \Phi(|p|^2)\, \hat{f}_\pm(p)\, dp, \tag{1.61}$$

where, as before, we emphasise that the integrals in (1.59)–(1.61) are improper integrals and must be interpreted by means of a limiting process intimately connected with the particular problem being studied.

The above theorems, when proved, will then provide outgoing and incoming spectral representations of the complex-valued solutions

$$v_1(x, t) = \exp\{-it A_1^{1/2}\} h_1(x). \tag{1.62}$$

Specifically, we will have the following.

<u>Outgoing representation</u>:

$$v_1(x, t) = \int_{\mathbf{R}^n} w_+(x, p) \exp\{-it\,|p|\}\, \hat{h}_+(p)\, dp. \tag{1.63}$$

Incoming representation:

$$v_1(x,t) = \int_{\mathbf{R}^n} w_-(x,p)\exp\{-it\,|p|\}\,\hat{h}_-(p)\,dp, \tag{1.64}$$

where

$$\hat{h}_\pm(p) = \int_{\mathbf{R}^n} \overline{w_\pm(x,p)}h_1(x)\,dx = (F_\pm h_1)(p).$$

If we use the incoming representation (1.64) and substitute $w_-(x,p)$ given by (1.58) into (1.64), then we obtain

$$v_1(x,t) = v_0^+(x,t) + v_1^+(x,t), \tag{1.65}$$

where

$$v_0^+(x,t) = \int_{\mathbf{R}^n} w_0(x,p)\exp\{-it\,|p|\}\,\hat{h}_-(p)\,dp, \tag{1.66}$$

$$v_1^+(x,t) = \int_{\mathbf{R}^n} w'_-(x,p)\exp\{-it\,|p|\}\,\hat{h}_-(p)\,dp. \tag{1.67}$$

Recalling our discussion of the FP, we see that the first two terms under the integral sign in (1.66) indicate that $v_0^+(x,t)$ is a free wave of the form

$$v_0^+(x,t) = \exp\{-it\,A_0^{1/2}\}h_0^+(x), \tag{1.68}$$

where

$$h_0^+(x) = v_0^+(x,0). \tag{1.69}$$

Expanding (1.68) by means of (1.34)–(1.36) yields

$$v_0^+(x,t) = \exp\{-it\,A_0^{1/2}\}h_0^+(x) = \int_{\mathbf{R}^n} w_0(x,p)\exp\{-it\,|p|\}\,\hat{h}_0^+(p)\,dp. \tag{1.70}$$

For (1.66) and (1.70) to be compatible we must have $\hat{h}_-(p) = \hat{h}_0^+(p)$, that is,

$$h_0^+(x) = (F^*\hat{h}_-)(x) = (F^*F_-h_1)(x). \tag{1.71}$$

Recalling the definitions of AE and the wave operators (see (1.28)), we see that (1.71) suggests that the wave operator that relates $h_0^+(x)$, the initial state of the free system, to $h_1(x)$, the initial state of the perturbed system, is

$$W_+ = F^*F_-. \tag{1.72}$$

Similarly, if we start with the outgoing representation and substitute for w_+ from (1.58), we can obtain

$$W_- = F^*F_+$$

and so conclude that

$$W_{\pm} = F^* F_{\mp}. \tag{1.73}$$

The free wave $v_0^+(x,t)$ defined in (1.68) is asymptotically equal to the asymptotic wave function (see (1.55) and theorem 1.2)

$$v_0^{+,\infty}(x,t) = |x|^{(1-n)/2}\, G\left(\frac{x}{|x|}, |x| - t\right), \tag{1.74}$$

where (see (1.54))

$$\hat{G}(\eta,\rho) = (-i\rho)^{(n-1)/2} H(\rho)\, \hat{h}_0^+(\rho\eta), \qquad \rho \in \mathbf{R}, \qquad \eta \in S^{n-1}, \tag{1.75}$$

with $H(\rho)$ denoting the Heaviside unit function.

Since by (1.71) we have

$$\hat{h}_0^+(p) = (Fh_0^+)(p) = (F_- h_1)(p) = \hat{h}_-(p), \tag{1.76}$$

(1.75) can be written in the form

$$\hat{G}(\eta,\rho) = (-i\rho)^{(n-1)/2} H(\rho)\, \hat{h}_-(\rho\eta), \qquad \rho \in \mathbf{R}, \qquad \eta \in S^{n-1}. \tag{1.77}$$

In most cases of physical interest, and certainly for those investigated in this monograph, it will be possible to obtain results of the following form.

$$\lim_{t\to\infty} \left\| v_1(.,t) - v_0^+(.,t) \right\| = 0, \tag{1.78}$$

$$\lim_{t\to\infty} \left\| v_0^+(.,t) - v_0^{+,\infty}(.,t) \right\| = 0, \tag{1.79}$$

$$\lim_{t\to\infty} \left\| v_1(.,t) - v_0^{+,\infty}(.,t) \right\| = 0. \tag{1.80}$$

These several observations and remarks suggest the following definition.

Definition 1.3 *The asymptotic wave function associated with the wave function v defined by*

$$v(x,t) = \exp\{-itA^{1/2}\}h(x)$$

is

$$v^{\infty}(x,t) = |x|^{(1-n)/2}\, G\left(\frac{x}{|x|}, |x| - t\right), \qquad x \in \mathbf{R}^n, \qquad t \in \mathbf{R},$$

where G is defined via (1.77).

We notice that G defined through (1.77) has been obtained with AE very much in mind, from (1.75), which is a result for FPs, by using (1.71) and (1.76). Nevertheless, in order for definition 1.3 to have practical significance, we must be able to establish a result of the following form.

THEOREM 1.4

$$\lim_{t\to\infty} \left\| v(.,t) - v^{\infty}(.,t) \right\| = 0.$$

This result follows from (1.78)–(1.80) and the fact that when the initial data are related according to (1.71) and (1.76), we have v^{∞} and $v_0^{+,\infty}$ coincident.

When we have *proved* that we can reach this stage, we will have completed tasks 1 and 2 for the PP. In addition we will have shown, by theorem 1.4 and the remarks following it, that the asymptotic wave function for the FP and for the PP can be made to coincide.

An alternative method frequently used when discussing wave motions in the autonomous case is to reduce the given IVP to an equivalent first-order system. We illustrate the method here, briefly and entirely formally, for an AP.

The FP (1.13) and the PP (1.14) can be written in the form

$$\begin{bmatrix} u_j \\ u_{jt} \end{bmatrix}_t (x,t) + \begin{bmatrix} 0 & -I \\ A_j & 0 \end{bmatrix} \begin{bmatrix} u_j \\ u_{jt} \end{bmatrix} (x,t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad j = 0, 1, \tag{1.81}$$

$$\begin{bmatrix} u_j \\ u_{jt} \end{bmatrix} (x,0) = \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} (x), \qquad j = 0, 1. \tag{1.82}$$

The problem (1.81), (1.82) can be written compactly as

$$\{\partial_t - i\mathbf{M}_j\}\mathbf{\Phi}_j(x,t) = 0, \qquad \mathbf{\Phi}_j(x,0) = \mathbf{\Phi}_{j0}(x), \qquad j = 0, 1, \tag{1.83}$$

where

$$\mathbf{\Phi}_j(x,t) = \begin{bmatrix} u_j \\ u_{jt} \end{bmatrix} (x,t) =: \langle u_j, u_{jt} \rangle (x,t),$$

$$\mathbf{\Phi}_{j0}(x) = \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} (x) =: \langle \varphi_j, \psi_j \rangle (x),$$

$$-i\mathbf{M}_j = \begin{bmatrix} 0 & -I \\ A_j & 0 \end{bmatrix}.$$

If we now decide to analyse the problem (1.83) in some certain collection of functions, denoted by H_E, then we can interpret $\mathbf{\Phi}_j$, $j = 0, 1$, as H_E-valued functions of t in the sense

$$\mathbf{\Phi}_j \equiv \mathbf{\Phi}_j(.,.) : t \to \mathbf{\Phi}_j(.,t) =: \mathbf{\Phi}_j(t) \in H_E.$$

Implicit in this notation is the requirement that the collection H_E can be decomposed as a "sum" of two collections, H_1 and H_2. We denote this symbolically by setting $H_E = H_1 \oplus H_2$, with the understanding that we have the interpretations

$$u_j = u_j(.,.) : t \to u_j(.,t) =: u_j(t) \in H_1, \qquad j = 0, 1,$$

$$u_{jt} = u_{jt}(.,.) : t \to u_{jt}(.,t) =: uj(t) \in H_2, \qquad j = 0, 1.$$

We can now reformulate (1.83) as an IVP for an *ordinary differential equation* in H_E of the form

$$\{d_t - i\mathbf{M}_j\}\mathbf{\Phi}_j(t) = 0, \qquad \mathbf{\Phi}_j(0) = \mathbf{\Phi}_{j0}. \tag{1.84}$$

Thus we see that we have a means of replacing the original partial differential equation that involves numerically valued functions with an ordinary differential equation that involves functions with values in some collection of functions (H_E in the present case). We shall see that a bonus in adopting this approach will turn out to be that we can use analogues of familiar results for scalar ordinary differential equations to solve abstract ordinary differential equations such as (1.84).

If we now make what appears to be the outrageous assumption that the $\mathbf{M}_j$, $j = 0, 1$, in (1.84) are constants, then, using an integrating factor technique, a solution of (1.84) can be obtained in the form

$$\mathbf{\Phi}_j(t) = e^{it\mathbf{M}_j}\mathbf{\Phi}_{j0} =: \mathbf{U}_j(t)\mathbf{\Phi}_{j0}, \qquad j = 0, 1. \tag{1.85}$$

Hence

$$\mathbf{\Phi}_j(x, t) = e^{it\mathbf{M}_j}\mathbf{\Phi}_{j0}(x) =: \mathbf{U}_j(t)\mathbf{\Phi}_{j0}(x), \tag{1.86}$$

and we notice the following features of this solution of the system governed by IVP (1.13).

- The vector function $\mathbf{\Phi}_{j0}$ can be considered as defining the *state* of the system of interest at time $t = 0$, that is, the *initial state* of the system.
- The function $\mathbf{\Phi}_j$ can be considered as defining the state of the system at some other time t.
- The quantity $\mathbf{U}_j$ can be regarded as an "operator" that controls the manner in which the initial state of the system $\mathbf{\Phi}_{j0}(x)$ evolves to the state $\mathbf{\Phi}_j(x, t)$, and for this reason it will eventually be called an *evolution operator*.

We remark that the collection H_E will later be identified as an energy space, hence the subscript.

We now notice that if the $\mathbf{U}_j$, $j = 0, 1$, can be determined and interpreted in a meaningful and practical manner, then the required solutions of (1.13) and (1.14) will be given by the first component of $\mathbf{\Phi}_j$ in (1.86).

Once the formal nature of the above illustrations has been removed and questions of existence and uniqueness of solution have been properly settled, we can turn our attention to constructing an appropriate scattering theory and to developing methods for actually determining such entities as the wave operators and ultimately the required solutions, albeit often only asymptotically, as $t \to \pm\infty$.

It turns out that a profitable way of investigating the IVPs for NAPs is again to reduce them to first-order systems. This we shall do in essentially the same way as we did for APs.

With (1.11), (1.12) and (1.81), (1.82) in mind and proceeding as for APs, we find that we are led to a consideration of the following NAPs.

$$\{d_t - i\mathbf{N}_j(t)\}\mathbf{\Psi}_j(t) = 0, \qquad \mathbf{\Psi}_j(s) = \mathbf{\Psi}_{js}, \qquad j = 0, 1, \tag{1.87}$$

where for $j = 0, 1$,

$$\mathbf{\Psi}_j(t) = \langle u_j(t), u_{jt}(t)\rangle, \qquad -i\mathbf{N}_j(t) = \begin{bmatrix} 0 & -I \\ A_j(t) & 0 \end{bmatrix}, \tag{1.88}$$

$$\mathbf{N}_j(t) : H_E(\mathbf{R}^n) \supseteq D(\mathbf{N}_j(t)) \to H_E(\mathbf{R}^n), \tag{1.89}$$

$$D(\mathbf{N}_j(t)) := \{\boldsymbol{\xi} = \langle \xi_1, \xi_2\rangle \in H_E(R^n) : A_j(t)\xi_1 \in H_2(R^n), \xi_2 \in H_1(R^n)\}.$$

$$A_j(t) = L_j(x, t),$$

$$\mathbf{\Psi}_j(s) = \langle u_j, u_{jt}\rangle(s) = \langle \varphi_j, \psi_j\rangle = \mathbf{\Psi}_{js},$$

where $0 < s \in \mathbf{R}$ is a fixed initial time and, as before, $H_E \equiv H_E(\mathbf{R}^n) = H_1(\mathbf{R}^n) \oplus H_2(\mathbf{R}^n)$.

In contrast to the AP case the matrix operator in (1.88) is now a function of t. Nevertheless, (1.87) can still be solved by an integrating factor technique. In this case we obtain the required solution in the following form.

$$\mathbf{\Psi}_j(t) = \exp\{i \int_s^t \mathbf{N}_j(\eta)\, d\eta\}\mathbf{\Psi}_{js} =: \mathbf{U}_j(t, s)\mathbf{\Psi}_{js}, \tag{1.90}$$

where the entities $\mathbf{U}(t, s)$, $j = 0, 1$, are known as the *propagators* for the NAP (1.87).

A wide range of physically significant problems in such fields as acoustics, electromagnetics and elasticity are in fact NAPs. When discussing these problems in this monograph, we shall have in mind the following three main aims.

- To provide results concerning the existence and uniqueness of solutions to problems of the type (1.87).
- To develop constructive methods for determining the propagators $\mathbf{U}_j(t, s)$ and hence the required solutions.
- To develop an appropriate and constructive scattering theory.

We shall first discuss NAPs in an abstract setting and then illustrate the practical relevance of the results obtained by indicating their use when dealing with some particular problems.

We remark that in a series of papers Cooper and Strauss have provided an elegant extension to NAPs of the Lax-Phillips theory, which was developed originally for scattering phenomena associated with APs (see chapter 11 and the cited references). Here we adopt a different approach that is more in keeping with the strategies and theory developed for APs by Wilcox [154] and his colleagues. This, we shall see, leads in quite a natural manner to constructive methods for solving (1.11), (1.12) and also to the development of an associated scattering theory.

1.4 CHAPTER SUMMARIES

Since this book is meant to be an introductory text for some if not all readers, it is felt that it should be as self-contained as possible. The intention is first to give,

at a leisurely pace, a reasonably comprehensive overview of the various concepts, techniques and strategies associated with the development of scattering theories for APs and then to show how these various ideas have to be adjusted and added to in order to deal with corresponding NAPs. With this in mind the book has the following structure.

In chapter 2 we give examples, each in one space variable, that illustrate relatively simply some of the notations, methods and techniques that will be used later when studying more complicated systems than acoustic waves on a string. Free problems are considered first, and a number of solution methods are indicated. This is followed by a discussion of associated perturbed problems of both the AP and NAP types. The APs are used to illustrate such features as reflected and transmitted waves, a scattering matrix and resonances. The NAPs are introduced in turn to illustrate such aspects as lack of energy conservation, the nonavailability of the separation of variables technique and scattering frequencies. The final section deals with a scattering problem generated by a moving bead on a semi-infinite string. This gives us an opportunity to introduce, amongst other things, the quasi-stationary approximation method.

We introduce in chapter 3 a number of concepts and results from analysis that are used regularly in subsequent chapters. The presentation is mainly restricted to giving basic definitions and formulating theorems. Few, if any, proofs will be given, but all the results will be referenced and, in chapter 11, a commentary chapter, indications for further reading will be provided. More advanced topics in analysis than those appearing here will be introduced as required. This will have the virtue of emphasising their particular roles in the development of scattering theories.

In finite-dimensional spaces a linear operator can be represented in terms of a matrix with respect to a basis of associated eigenvectors. A generalisation of this notion to an infinite-dimensional space setting is complicated by the fact that the spectrum of an operator on such a space can now consist of more than just eigenvalues. To show how this situation can be eased, we introduce, in chapter 4, the spectral theorem and indicate how it can provide the required generalisations of the notions of spectral representation of an operator and the spectral decomposition of the underlying function space, with respect to an operator, that were available in a finite-dimensional space setting.

In chapter 5 we give a number of results from the theory of semigroups and also provide an overview of results and techniques from the abstract theory of Volterra integral equations. We indicate how these various results can be used to settle questions of existence and uniqueness of solutions to equations governing scattering processes. The final section deals with determination of the propagator (evolution operator) for NAPs and related approximation methods.

In chapter 6 we recall salient features of scattering theories that have been developed for APs. Some of these have already been alluded to in chapter 1. However, here we provide a rather more precise account. Topics to be covered include propagation aspects, solution decay, scattering states, solutions with finite energy, representations of solutions, expansion theorems and construction of solutions. The comparison of solutions for large time is discussed, as are the evolution operator for a wave equation and the asymptotic equality of solutions. Results are

recalled concerning the existence, uniqueness and completeness of wave and scattering operators. Mention is also made of the principles of limiting absorption and limiting amplitude. Furthermore, a method is outlined for the construction of wave and scattering operators. In the final section of chapter 6 indications are given of how the philosophy used and the results obtained when dealing with APs can be adapted when dealing with NAPs.

The material presented in chapters 7–9 deals almost entirely with NAPs and relies very much on the preparation offered in chapters 2–6.

Chapter 7 is concerned with determination of the echo wave field. In chapter 8 the influence of time-periodic perturbations on the scattered field is examined. In chapter 9 mention is made of inverse scattering problems in a nonautonomous setting. A method that offers good prospects for dealing with this class of NAPs is introduced and discussed.

In chapter 10 indications are given of the manner in which methods for dealing with nonautonomous acoustic problems can be extended to similar problems in electromagnetism and elasticity.

All references are collected together in a much extended bibliography in chapter 11. Furthermore, a commentary on the material presented in this monograph can be found in this final chapter. This will provide more historical background, additional references and a guide to further reading and more specialised texts.