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CHAPTER 1

First-Order Differential Equations and Their Applications

1.1 Introduction to Ordinary Differential Equations

Differential equations are found in many areas of mathematics, science, and engineering. Students taking a first course in differential equations have often already seen simple examples in their mathematics, physics, chemistry, or engineering courses. If you have not already seen differential equations, go to the library or Web and glance at some books or journals in your major field. You may be surprised to see the way in which differential equations dominate the study of many aspects of science and engineering.

Applied mathematics involves the relationships between mathematics and its applications. Often the type of mathematics that arises in applications is differential equations. Thus, the study of differential equations is an integral part of applied mathematics. Applied mathematics is said to have three fundamental aspects, and this course will involve a balance of the three:

1. The **modeling process** by which physical objects and processes are described by physical laws and mathematical formulations. Since so many physical laws involve rates of change (or the derivative), differential equations are often the natural language of science and engineering.
2. The **analysis** of the mathematical problems that are posed. This involves the complete investigation of the differential equation and its solutions, including detailed numerical studies. We will say more about this shortly.
3. However, the mathematical solution of the differential equation does not complete the overall process. The **interpretation** of the solution of the differential equation in the context of the original physical problem must be given, and the implications further analyzed.

Using the **qualitative** approach, we determine the behavior of the solutions without actually getting a formula for them. This approach is somewhat similar to the curve-sketching process in introductory calculus where we sketch curves by drawing

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maxima, minima, concavity changes, and so on. Qualitative ideas will be discussed in Section 1.4 and Chapters 4, 5, and 6.

Using the **numerical** approach, we compute estimates for the values of the unknown function at certain values of the independent variable. Numerical methods are extremely important and for many difficult problems are the only practical approach. We discuss numerical methods in Section 1.5. The safe and effective use of numerical methods requires an understanding of the basic properties of differential equations and their solutions.

Most of this book is devoted to developing **analytical** procedures, that is, obtaining explicit and implicit formulas for the solutions of various ordinary differential equations. We present a sufficient number of applications to enable the reader to understand how differential equations are used and to develop some feeling for the physical information they convey.

Asymptotic and **perturbation** methods are introduced in more advanced studies of differential equations. Asymptotic approximations are introduced directly from the exact analytic solutions in order to get a better understanding of the meaning of the exact analytic solutions. Unfortunately, many problems of physical interest do not have exact solutions. In this case, in addition to the previously mentioned numerical methods, approximation methods known as perturbation methods are often useful for understanding the behavior of differential equations.

In this book, we use the phrase "differential equation" to mean an ordinary differential equation (or a system of ordinary differential equations. An **ordinary differential equation** is an equation relating an unknown function of one variable to one or more functions of its derivatives. If the unknown x is a function of t, $x = x(t)$, then examples of ordinary differential equations are

$$
\begin{aligned}
\frac{dx}{dt} &= t^7 \cos x, \\
\frac{d^2x}{dt^2} &= x\frac{dx}{dt}, \\
\frac{d^4x}{dt^4} &= -5x^5.
\end{aligned}
\tag{1}
$$

The **order** of a differential equation is the order of the highest derivative of the unknown function (dependent variable) that appears in the equation. The differential equations in (1) are of first, second, and fourth order, respectively. Most of the equations we shall deal with will be of first or second order.

In applications, the dependent variables are frequently functions of time, which we denote by t. Some applications such as

1. population dynamics,
2. mixture and flow problems,
3. electronic circuits,
4. mechanical vibrations and systems

are discussed repeatedly throughout this text. Other applications, such as radioactive decay, thermal cooling, chemical reactions, and orthogonal trajectories, appear only as illustrations of more specific mathematical results. In all cases, modeling, analysis, and interpretation are important.

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Let us briefly consider the following motivating population dynamics problem.

Example 1.1.1 *Population Growth Problem*

Assume that the population of Washington, DC, grows due to births and deaths at the rate of 2% per year and there is a net migration into the city of 15,000 people per year. Write a mathematical equation that describes this situation.

● SOLUTION. We let

$$x(t) = \text{population as a function of time } t. \tag{2}$$

From calculus,

$$\frac{dx}{dt} = \text{rate of change of the population.} \tag{3}$$

In this example, 2% growth means 2% of the population $x(t)$. Thus, the population of Washington, DC satisfies

$$\frac{dx}{dt} = 0.02x + 15{,}000. \tag{4}$$

♦

Equation (4) is an example of a differential equation, and we develop methods to solve such equations in this text. We will discuss population growth models in more depth in Section 1.8 and Chapters 5 and 6.

In a typical application, physical laws often lead to a differential equation. As a simple example, we will consider later the vertical motion $x(t)$ of a constant mass. **Newton's law** says that the force F equals the mass m times the acceleration $\frac{d^2x}{dt^2}$:

$$m\frac{d^2x}{dt^2} = F. \tag{5}$$

If the forces are gravity $-mg$ and a force due to air resistance proportional to the velocity $\frac{dx}{dt}$, then the position satisfies the second-order differential equation

$$m\frac{d^2x}{dt^2} = -mg - c\frac{dx}{dt}, \tag{6}$$

where c is a proportionality constant determined by experiments. However, if in addition the mass is tied to a spring that exerts an additional force $-kx$ satisfying Hooke's law, then we will show that the mass satisfies the following second-order differential equation:

$$m\frac{d^2x}{dt^2} = -mg - kx - c\frac{dx}{dt}. \tag{7}$$

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In the main body of the book, we will devote some effort to the modeling process by which these differential equations arise, as well as learning how to solve these types of differential equations easily.

We will also see that a typical electronic circuit with a resistor, capacitor, and inductor can often be modeled by the following second-order differential equation:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = f(t). \tag{8}$$

Here the unknown variable is $i(t)$, the time-dependent current running through the circuit. We will present a derivation of this differential equation, and you will learn the meaning of the three positive constants R, L, and C (this is called an RLC circuit). For example, R represents the resistance of a resistor, and we will want to study how the current in the circuit depends on the resistance. The right-hand side $f(t)$ represents something that causes current in the circuit, such as a battery.

Physical problems frequently involve systems of differential equations. For example, we will consider the salt content in two interconnected well-mixed lakes, allowing for some inflow, outflow, and evaporation. If $x(t)$ represents the amount of salt in one of the lakes and $y(t)$ the amount in the other, then under a series of assumptions described in the book, the following coupled system of differential equations is an appropriate mathematical model:

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2} + 2\frac{y}{100} - 3\frac{x}{100}, \\ \frac{dy}{dt} &= 3\frac{y}{100} - \frac{5}{2}\frac{y}{100}. \end{aligned} \tag{9}$$

1.2 The Definite Integral and the Initial Value Problem

This chapter is concerned with first-order differential equations, in which the first derivative of a function $x(t)$ depends on the independent variable t and the unknown solution x. If $\frac{dx}{dt}$ is given directly in terms of t and x, the differential equation has the form

$$\frac{dx}{dt} = f(t, x). \tag{1}$$

In later sections, we will discuss a number of applications of first-order equations such as growth and decay problems for populations, radioactive decay, thermal cooling, mixture problems, evaporation and flow, electronic circuit theory, and several others. In these applications, some care is given to the development of mathematical models.

A **solution** of the differential equation (1) is a function that satisfies the differential equation for all values t of interest:

$$\frac{dx}{dt}(t) = f(t, x(t)) \qquad \text{for all } t.$$

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Example 1.2.1 *Showing That a Function Is a Solution*

Verify that $x = 3e^{t^2}$ is a solution of the first-order differential equation

$$\frac{dx}{dt} = 2tx. \tag{2}$$

● SOLUTION. We substitute $x = 3e^{t^2}$ in both the left- and right-hand sides of (2). On the left we get $\frac{d}{dt}(3e^{t^2}) = 2t(3e^{t^2})$, using the chain rule. Simplifying the right-hand side, we find that the differential equation (2) is satisfied

$$6te^{t^2} = 6te^{t^2},$$

which holds for all t. Thus $x = 3e^{t^2}$ is a solution of the differential equation (2). ♦

Before beginning our general development of first-order equations in Section 1.4, we will discuss some differential equations that can be solved with direct integration. These special cases will be used to motivate and illustrate some of the later development.

1.2.1 The Initial Value Problem and the Indefinite Integral

The simplest possible first-order differential equation arises if the function $f(t, x)$ in (1) does not depend on the unknown solution, so that the differential equation is

$$\frac{dx}{dt} = f(t). \tag{3}$$

Solving (3) for x is just the question of antidifferentiation in calculus. Although (3) can be solved by an integration, we can learn some important things concerning more general differential equations from it that will be useful later.

Example 1.2.2 *Indefinite Integration*

Consider the simple differential equation,

$$\frac{dx}{dt} = t^2. \tag{4}$$

By an integration, we obtain

$$x = \frac{1}{3}t^3 + c, \tag{5}$$

where c is an arbitrary constant. From this example, we see that differential equations usually have many solutions. We call (5) the **general solution** of (4), since it is a formula that gives all solutions. Because x depends on t, sometimes we use the notation $x(t)$. Often, especially in applications, we are interested in a specific solution of the differential equation to satisfy some additional condition. For example, suppose

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we are given that $x = 7$ at $t = 2$. This is written mathematically as $x(2) = 7$ and called an **initial condition**. By letting $t = 2$ and $x = 7$ in (5), we get

$$7 = \frac{8}{3} + c,$$

so that the constant c is determined to be $c = \frac{13}{3}$. Thus, the unique solution of the differential equation that satisfies the given initial condition is

$$x = \frac{1}{3}t^3 + \frac{13}{3}. \qquad \blacklozenge$$

More generally, we might wish to solve the differential equation

$$\frac{dx}{dt} = f(t), \tag{6}$$

subject to the initial condition

$$x(t_0) = x_0. \tag{7}$$

We introduce the symbol t_0 for the value of t at which the solution is given. Often $t_0 = 0$, but in the previous example, we had $t_0 = 2$ and $x_0 = 7$. We refer to (6) with the initial condition (7) as the **initial value problem** for the differential equation. We can always write a formula for the solution of (6) using an indefinite integral

$$x = \int f(t)dt + c. \tag{8}$$

Equation (5) is a specific example of (8). If we can obtain an explicit indefinite integral (antiderivative) of $f(t)$, then this initial value problem can be solved like Example 1.2.1. Explicit integrals of various functions $f(t)$ may be obtained by using any of the various techniques of integration from calculus. Tables of integrals or symbolic integration algorithms such as MAPLE or Mathematica that are available on more sophisticated calculators, personal computers, or larger computers may be used. However, if one cannot obtain an explicit integral, then it may be difficult to use (8) directly to satisfy the initial conditions.

1.2.2 The Initial Value Problem and the Definite Integral

A definite integral should usually be used to solve the differential equation $\frac{dx}{dt} = f(t)$ if an explicit integral is not used. The result can automatically incorporate the given initial condition $x(t_0) = x_0$. If both sides of the differential equation (6) are integrated with respect to t from t_0 to t, we get

$$\int_{t_0}^{t} \frac{dx}{d\bar{t}} d\bar{t} = \int_{t_0}^{t} f(\bar{t}) d\bar{t},$$

where we have introduced the dummy variable $\bar{t}$. The left-hand side equals $x(\bar{t}) \big|_{\bar{t}=t_0}^{\bar{t}=t} = x(t) - x(t_0)$, since the antiderivative of $\frac{dx}{dt}$ is x. We obtain the same result canceling

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$\bar{t}$, since $\frac{dx}{d\bar{t}}d\bar{t} = dx$. Thus, we have

$$x(t) - x(t_0) = \int_{t_0}^{t} f(\bar{t})d\bar{t}, \quad \text{or equivalently,}$$
$$x(t) = x(t_0) + \int_{t_0}^{t} f(\bar{t})d\bar{t}. \tag{9}$$

Note that $x(t_0) = x_0$ since $\int_{t_0}^{t_0} f(\bar{t})d\bar{t} = 0$. Any dummy variable of integration may be used. We have chosen $\bar{t}$.

Example 1.2.3 *Example with a Definite Integral*

Solve the differential equation

$$\frac{dx}{dt} = e^{-t^2}, \tag{10}$$

subject to the initial condition $x(3) = 7$.

● SOLUTION. The function e^{-t^2} does not have any explicit antiderivative. Thus, if we want to solve (10), we use definite integration from $t_0 = 3$, where $x_0 = 7$. Then (9) is

$$x - 7 = \int_{3}^{t} e^{-\bar{t}^2} d\bar{t},$$

and the solution of the initial value problem is

$$x = 7 + \int_{3}^{t} e^{-\bar{t}^2} d\bar{t}. \tag{11}$$

The function e^{-t^2} is important in probability, since $(1/\sqrt{2\pi})e^{-t^2/2}$ is the famous normal curve. ♦

There are many situations in which it is desirable to use a definite integral rather than an explicit antiderivative:

1. It is sometimes difficult to obtain an explicit antiderivative, and a formula like (11) suffices.
2. For some $f(t)$ (as in the previous example), it is impossible to obtain an explicit antiderivative in terms of elementary functions.
3. The function $f(t)$ might be expressed only by some data, in which case the definite integral (9) represents the area under the curve $f(t)$ and can be evaluated by an appropriate numerical integration method such as Simpson's or the trapezoid rule.

It is difficult to give general advice valid for all problems. If an integral is an elementary integral, then the explicit integral should be used. However, what is elementary to one person is not necessarily elementary to another. With the wide availability of computers, a definite integral can usually be evaluated by a numerical integration.

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General Solution Using a Definite Integral

Alternatively, in solving for the general solution of

$$\frac{dx}{dt} = f(t), \tag{12}$$

we may use the definite integral starting at any point a. In this case, we obtain the general solution of (12) as

$$x = \int_a^t f(\bar{t})d\bar{t} + c. \tag{13}$$

There appear to be two arbitrary constants, c and the lower limit of the integral a in (13). However, it can be shown that this is equivalent to one arbitrary constant. In practice, the lower limit a is often chosen to be the initial value of t so that $a = t_0$. To see that (13) actually solves the differential equation (12), it is helpful to recall the **fundamental theorem of calculus**:

$$\frac{d}{dt}\left(\int_a^t f(\bar{t})d\bar{t}\right) = f(t). \tag{14}$$

1.2.3 Mechanics I: Elementary Motion of a Particle with Gravity Only

Elementary motions of a particle are frequently described by differential equations. Simple integration can sometimes be used to analyze these elementary motions. For the one-dimensional vertical motion of a particle, we recall from calculus that

$$\begin{aligned} \text{Position} &= x(t), \\ \text{Velocity} &= v(t) = \frac{dx}{dt}, \\ \text{Acceleration} &= a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \end{aligned} \tag{15}$$

Newton's law of motion ($ma = F$) will yield a differential equation

$$m\frac{d^2x}{dt^2} = F\left(x, \frac{dx}{dt}, t\right), \tag{16}$$

where m is the mass and F is the sum of the applied forces, and we have allowed the forces to depend on position, velocity, and time. Equation (16) is a second-order differential equation, which we will study later in the book.

There are no techniques for solving (16) in all cases. However, (16) can be solved by simple integration if the force F does not depend on x and $\frac{dx}{dt}$. As an example, suppose that the only force on the mass is due to gravity. Then it is known that $F = -mg$, where g is the acceleration due to gravity. The minus sign is introduced because gravity acts downward, toward the surface of the earth. Here we are taking the coordinate system so that x increases toward the sky. The magnitude of the force

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due to gravity, mg, is called the weight of the body. Near the surface of the planet earth, g is approximately $g = 9.8$ m/s^2 in the mks system used by most of the world ($g = 32$ ft/s^2 when feet are used as the unit of length instead of meters). If we assume that we are interested in a mass that is located sufficiently near the surface of the earth, then g can be approximated by this constant. With the only force being gravity, (16) becomes

$$m\frac{d^2x}{dt^2} = -mg,$$

or equivalently,

$$\boxed{\frac{d^2x}{dt^2} = -g,} \tag{17}$$

since the mass m cancels. Integrating (17) yields

$$\frac{dx}{dt} = -gt + c_1, \tag{18}$$

where c_1 is an arbitrary constant of integration. We assume that the velocity at $t = 0$ is given and use the notation v_0 for this initial velocity. Since

$$v(t) = -gt + c_1$$

from (18), evaluating (18) at $t = 0$ gives $v_0 = c_1$. Thus, the velocity satisfies

$$\frac{dx}{dt} = -gt + v_0. \tag{19}$$

The position can be determined by integrating the velocity (19) to give

$$x = -\frac{1}{2}gt^2 + v_0t + c_2, \tag{20}$$

where c_2 is a second integration constant. We also assume that the position x_0 at $t = 0$ is given initially. Then, evaluating (20) at $t = 0$ gives $x_0 = c_2$, so that

$$x = -\frac{1}{2}gt^2 + v_0t + x_0. \tag{21}$$

Equations (19) and (21) are well-known formulas in physics and are used to solve for quantities such as the maximum height of a thrown object. We do not recommend memorizing (19) or (21). Instead, they should be derived in each case from the differential equation (17). If the applied force depends only on time and is not constant, then the formulas for velocity and position may be obtained by integration. If the

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applied force depends on other quantities, solving the differential equation is not so simple. We describe some more difficult problems in Section 1.11 and Chapter 2.

Example 1.2.4 *Motion with Gravity*

Suppose a ball is thrown upward from ground level with velocity v_0 and the only force is gravity. How high does the ball go before falling back toward the ground?

● SOLUTION. The differential equation (as before) is (17):

$$\frac{d^2x}{dt^2} = -g. \tag{22}$$

The initial conditions are that $x = 0$ and $\frac{dx}{dt} = v_0$ at $t = 0$. By successive integrations of (22) and by applying the initial conditions, we obtain

$$\frac{dx}{dt} = -gt + v_0 \tag{23}$$

and

$$x = -\frac{1}{2}gt^2 + v_0 t. \tag{24}$$

From (24), the height is known as a function of time. To determine the maximum height, we must first determine the time at which the ball reaches this height. From calculus, the maximum of a function $x = x(t)$ occurs at a critical point where $\frac{dx}{dt} = 0$. At the maximum height, the ball has stopped rising and has not started to fall, so the velocity is zero. Thus, the time of the maximum height is determined from (23):

$$0 = -gt + v_0, \quad \text{or equivalently,} \quad t = \frac{v_0}{g}. \tag{25}$$

When this time (25) is substituted into (24), a formula for the maximum height y is obtained:

$$y = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\frac{v_0}{g} = \frac{v_0^2}{g}\left(-\frac{1}{2} + 1\right) = \frac{v_0^2}{2g}. \tag{26}$$

♦

Example 1.2.5 *Car Braking*

Suppose that a car is going 76 m/s when brakes are applied at $t = 2$ s. Suppose that the nonconstant deceleration is known to be $a = -12t^2$. Determine the distance the car travels.

● SOLUTION. Let x measure the distance traveled once the brakes are applied. The differential equation is

$$\frac{d^2x}{dt^2} = -12t^2. \tag{27}$$

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By integrating (27) we obtain

$$\frac{dx}{dt} = -4t^3 + c_1. \tag{28}$$

By integrating (28) again, we obtain

$$x = -t^4 + c_1 t + c_2. \tag{29}$$

The initial conditions are $x = 0$ and $\frac{dx}{dt} = 76$ when $t = 2$. The initial velocity applied to (28) gives

$$76 = -4(2)^3 + c_1 = -32 + c_1,$$

so that $c_1 = 108$. The initial position applied to (29) gives

$$0 = -2^4 + c_1 t + c_2 = -16 + 216 + c_2,$$

and $c_2 = -200$. To determine the distance the car travels, we note that the car stops when the velocity is zero. That is,

$$\frac{dx}{dt} = -4t^3 + 108 = 0,$$

so that the stopping time is $t^3 = 27$ or $t = 3$. Then the distance traveled at time $t = 3$ is given by substituting $t = 3$ into (29):

$$x = -3^4 + 108(3) - 200 = -81 + 324 - 200 = 43\text{ m}. \qquad \blacklozenge$$

Exercises

In Exercises 1–8, verify that the given function x is a solution of the differential equation.

1. $x = 2e^{3t} + 1, \ \frac{dx}{dt} = 3x - 3.$
2. $x = t^2, \ \frac{dx}{dt} = 2\frac{x}{t}.$
3. $x = t - 1, \ \frac{dx}{dt} = \frac{x}{t-1}.$
4. $x = t^4, \ \frac{dx}{dt} = 4\frac{x^2}{t^5}.$
5. $x = e^{t^2}, \ \frac{dx}{dt} = 2tx.$
6. $x = 3e^{4t} + 2, \ \frac{dx}{dt} = 4x - 8.$
7. $x = e^{-2t}, \ \frac{dx}{dt} = -2e^{2t}x^2.$
8. $x = t + 3, \ \frac{dx}{dt} = \frac{x-3}{t}.$

--continued--