Chapter One

Introduction

A *system* is a combination of components or parts that is perceived as a single entity. The parts making up the system may be clearly or vaguely defined. These parts are related to each other through a particular set of variables, called the *states* of the system, that completely determine the behavior of the system at any given time. A *dynamical system* is a system whose state changes with time. Specifically, the state of a dynamical system can be regarded as an information storage or memory of past system events. The set of (internal) states of a dynamical system must be sufficiently rich to completely determine the behavior of the system for any future time. Hence, the state of a dynamical system at a given time is uniquely determined by the state of the system at the initial time and the present input to the system. In other words, the state of a dynamical system in general depends on both the present input to the system and the past history of the system. Even though it is often assumed that the state of a dynamical system is the *least* set of state variables needed to completely predict the effect of the past upon the future of the system, this is often a convenient simplifying assumption.

We regard a dynamical system $\mathcal{G}$ as a mathematical model structure involving an input, state, and output that can capture the dynamical description of a given class of physical systems. Specifically, at each moment of time $t \in \mathbb{T}$, where $\mathbb{T}$ denotes a time-ordered subset of the reals, the dynamical system $\mathcal{G}$ receives an input $u(t)$ (e.g., matter, energy, information) and generates an output $y(t)$. The values of the input are taken from the fixed set U. Furthermore, over a time segment the input function $u : [t_1, t_2) \to U$ is not arbitrary but belongs to the admissible input class $\mathcal{U}$, that is, for every $u(\cdot) \in \mathcal{U}$ and $t \in \mathbb{T}$, $u(t) \in U$. The input class $\mathcal{U}$ depends on the physical description of the system. In addition, each system output $y(t)$ belongs to the fixed set Y with $y(\cdot) \in \mathcal{Y}$ over a given time segment, where $\mathcal{Y}$ denotes an output space. In general, the output of $\mathcal{G}$ depends on both the present input of $\mathcal{G}$ and the past history of $\mathcal{G}$. Thus, the state, and hence the output at some time $t \in \mathbb{T}$, depends on both the initial state $x(t_0) = x_0$ and the input segment $u : [t_0, t) \to U$. In other words, knowledge of both x_0 and $u \in \mathcal{U}$ is necessary and sufficient to determine the present and future state $x(t) = s(t, t_0, x_0, u)$ of $\mathcal{G}$.

In light of the above discussion, we view a dynamical system as a precise mathematical object defined on a time set as a mapping between vector spaces satisfying a set of axioms. A mathematical dynamical system thus consists of the space of states $\mathcal{D}$ of the system together with a rule or *dynamic* that determines the state of the system at a given future time from a given present state. This is formalized by the following definition. For this definition $\mathbb{T} = \mathbb{R}$ for continuous-time systems and $\mathbb{T} = \mathbb{Z}$ for discrete-time systems.

Definition 1.1. A *dynamical system* $\mathcal{G}$ on $\mathcal{D}$ is the octuple $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, \mathbb{T}, s, h)$, where $s : \mathbb{T} \times \mathbb{T} \times \mathcal{D} \times \mathcal{U} \to \mathcal{D}$ and $h : \mathbb{T} \times \mathcal{D} \times U \to Y$ are such that the following axioms hold:

i) (Continuity): For every $t_0 \in \mathbb{T}$, $x_0 \in \mathcal{D}$, and $u \in \mathcal{U}$, $s(\cdot, t_0, x_0, u)$ is continuous for all $t \in \mathbb{T}$.

ii) (Consistency): For every $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, and $t_0 \in \mathbb{T}$, $s(t_0, t_0, x_0, u) = x_0$.

iii) (Determinism): For every $t_0 \in \mathbb{T}$ and $x_0 \in \mathcal{D}$, $s(t, t_0, x_0, u_1) = s(t, t_0, x_0, u_2)$ for all $t \in \mathbb{T}$ and $u_1, u_2 \in \mathcal{U}$ satisfying $u_1(\tau) = u_2(\tau)$, $\tau \in [t_0, t]$.

iv) (Group property): $s(t_2, t_0, x_0, u) = s(t_2, t_1, s(t_1, t_0, x_0, u), u)$ for all $t_0, t_1, t_2 \in \mathbb{T}$, $t_0 \le t_1 \le t_2$, $x_0 \in \mathcal{D}$, and $u \in \mathcal{U}$.

v) (Read-out map): There exists $y \in \mathcal{Y}$ such that $y(t) = h(t, s(t, t_0, x_0, u), u(t))$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, $t_0 \in \mathbb{T}$, and $t \in \mathbb{T}$.

We denote the dynamical system $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, \mathbb{T}, s, h)$ by $\mathcal{G}$ and we refer to the map $s(\cdot, t_0, \cdot, u)$ as the *flow* or *trajectory* corresponding to $x_0 \in \mathcal{D}$, $t_0 \in \mathbb{T}$, and $u \in \mathcal{U}$; and for a given trajectory $s(t, t_0, x_0, u)$, $t \in \mathbb{T}$, we refer to $t_0 \in \mathbb{T}$ as an *initial time* of $\mathcal{G}$, $x_0 \in \mathcal{D}$ as an *initial condition* of $\mathcal{G}$, and $u \in \mathcal{U}$ as an *input* to $\mathcal{G}$. The dynamical system $\mathcal{G}$ is *isolated* if the input space consists of one element only, that is, $u(t) = u^*$, and the dynamical system is *undisturbed* if $u^* = 0$. If $\mathcal{G}$ is isolated, then $\mathcal{G}$ is isolated from any inputs and the environment is the only input acting on the system. This, for example, would correspond to a conservative mechanical system wherein the only external force acting on the system is gravity. In general, the output of $\mathcal{G}$ depends on both the present input of $\mathcal{G}$ and the past history of $\mathcal{G}$. Hence, the output of the dynamical system at some time $t \in \mathbb{T}$ depends on the state $s(t, t_0, x_0, u)$ of $\mathcal{G}$, which effectively serves as an information storage (memory) of past history. Furthermore, the determinism axiom ensures that the state, and hence the output, before some time $t \in \mathbb{T}$ is not influenced by the values of the output after time t. Thus, future inputs to $\mathcal{G}$ do not affect past and present outputs of $\mathcal{G}$. This is simply a statement of causality that holds for all physical systems. The notion of a dynamical system as defined in Definition 1.1 is far too general to develop useful practical deductions for dynamical systems. This notion of a dynamical system is introduced here to develop terminology and to introduce certain key concepts. As we will see in the next chapter, under additional regularity conditions the flow of a dynamical system describing the motion of the system as a function of time can generate a differential equation on the state space, allowing for the development of a large array of mathematical results leading to useful and practical analysis and control synthesis tools.

Determining the rule or dynamic that defines the state of physical and engineering systems at a given future time from a given present state is one of the central problems of science and engineering. Once the flow of a dynamical system describing the motion of the system starting from a given initial state is given, dynamical system theory can be used to describe the behavior of the system states over time for different initial conditions. Throughout the centuries—from the great cosmic theorists of ancient Greece to the present-day quest for a unified field theory—the most important dynamical system is our universe. By using abstract mathematical models and attaching them to the physical world, astronomers, mathematicians, and physicists have used abstract thought to deduce something that is true about the natural system of the cosmos.

The quest by scientists, such as Brahe, Kepler, Galileo, Newton, Huygens, Euler, Lagrange, Laplace, and Maxwell, to understand the regularities inherent in the distances of the planets from the sun and their periods and velocities of revolution around the sun led to the science of dynamical systems as a branch of mathematical physics. One of the most basic issues in dynamical system theory that was spawned from the study of mathematical models of our solar system is the stability of dynamical systems. System stability involves the investigation of small deviations from a system's steady state of motion. In particular, a dynamical system is *stable* if the system is allowed to perform persistent small oscillations about a system equilibrium, or about a state of motion. Among the first investigations of the stability of a given state of motion is by Isaac Newton. In particular, in his *Principia Mathematica* [335] Newton investigated whether a small perturbation would make a particle moving in a plane around a center of attraction continue to move near the circle, or diverge from it. Newton used his analysis to analyze the motion of the moon orbiting the Earth.

Numerous astronomers and mathematicians who followed made significant contributions to dynamical stability theory in an effort to show that the observed deviations of planets and satellites from fixed elliptical orbits were in agreement with Newton's principle of universal gravitation. Notable contributions include the work of Torricelli [431], Euler [116], Lagrange [252], Laplace [257], Dirichlet [106], Liouville [283], Maxwell [310], and Routh [369]. The most complete contribution to the stability analysis of dynamical systems was introduced in the late nineteenth century by the Russian mathematician Aleksandr Mikhailovich Lyapunov in his seminal work entitled *The General Problem of the Stability of Motion* [293–295]. Lyapunov's *direct method* states that if a positive-definite function (now called a Lyapunov function) of the state coordinates of a dynamical system can be constructed for which its time rate of change following small perturbations from the system equilibrium is always negative or zero, then the system equilibrium state is stable. In other words, Lyapunov's method is based on the construction of a Lyapunov function that serves as a generalized norm of the solution of a dynamical system. Its appeal comes from the fact that stability properties of the system solutions are derived directly from the governing dynamical system equations; hence the name, Lyapunov's direct method.

Dynamical system theory grew out of the desire to analyze the mechanics of heavenly bodies and has become one of the most fundamental fields of modern science as it provides the foundation for unlocking many of the mysteries in nature and the universe that involve the evolution of time. Dynamical system theory is used to study ecological systems, geological systems, biological systems, economic systems, neural systems, and physical systems (e.g., mechanics, thermodynamics, fluids, magnetic fields, galaxies, etc.), to cite but a few examples. Dynamical system theory has also played a crucial role in the analysis and control design of numerous complex engineering systems. In particular, advances in feedback control theory have been intricately coupled to progress in dynamical system theory, and conversely, dynamical system theory has been greatly advanced by the numerous challenges posed in the analysis and control design of increasingly complex feedback control systems.

Since most physical and engineering systems are inherently nonlinear, with system nonlinearities arising from numerous sources including, for example, friction (e.g., Coulomb, hysteresis), gyroscopic effects (e.g., rotational motion), kinematic effects (e.g., backlash), input constraints (e.g., saturation, deadband), and geometric constraints, system nonlinearities must be accounted for in system analysis and control design. Nonlinear systems, however, can exhibit a very rich dynamical behavior, such as multiple equilibria, limit cycles, bifurcations, jump resonance phenomena, and chaos, which can make general nonlinear system analysis and control notoriously difficult. Lyapunov's results provide a powerful framework for analyzing the stability of nonlinear dynamical systems. Lyapunov-based methods have also been used by control system designers to obtain stabilizing feedback controllers for nonlinear systems. In particular, for smooth feedback, Lyapunov-based methods were inspired by Jurdjevic and Quinn [224], who give sufficient conditions for smooth stabilization based on the ability to construct a Lyapunov function for the closed-loop system. More recently, Artstein [13] introduced the notion of a control Lyapunov function whose existence guarantees a feedback control law which globally stabilizes a nonlinear dynamical system. Even though for certain classes of nonlinear dynamical systems a universal construction of a feedback stabilizer can be obtained using control Lyapunov functions [13, 406], there does not exist a unified procedure for finding a Lyapunov function that will stabilize the closed-loop system for general nonlinear systems. In light of this, advances in Lyapunov-based methods have been developed for analysis and control design for numerous classes of nonlinear dynamical systems. As a consequence, Lyapunov's direct method has become one of the cornerstones of systems and control theory.

The main objective of this book is to present necessary mathematical tools for stability analysis and control design of nonlinear systems, with an emphasis on Lyapunov-based methods. The main contents of the book are as follows. In Chapter 2, we provide a systematic development of nonlinear ordinary differential equations, which is central to the study of nonlinear dynamical system theory. Specifically, we develop qualitative solutions properties, existence of solutions, uniqueness of solutions, continuity of solutions, and continuous dependence of solutions on system initial conditions for nonlinear dynamical systems.

In Chapter 3, we develop stability theory for nonlinear dynamical systems. Specifically, Lyapunov stability theorems are developed for time-invariant nonlinear dynamical systems. Furthermore, invariant set stability theorems, converse Lyapunov theorems, and Lyapunov instability theorems are also considered. Finally, we present several systematic approaches for constructing Lyapunov functions as well as stability of linear systems and Lyapunov's linearization method. Chapter 4 provides an advanced treatment of stability theory including partial stability, stability theory for time-varying systems, Lagrange stability, boundedness, ultimate boundedness, input-to-state stability, finite-time stability, semistability, and stability theorems via vector Lyapunov functions. In addition, Lyapunov and asymptotic stability of sets as well as stability of periodic orbits are also systematically addressed. In particular, local and global stability theorems are given using lower semicontinuous Lyapunov functions. Furthermore, generalized invariant set theorems are derived wherein system trajectories converge to a union of largest invariant sets contained on the boundary of the intersections over finite intervals of the closure of generalized Lyapunov level surfaces. These results provide transparent generalizations to standard Lyapunov and invariant set theorems.

In Chapter 5, using generalized notions of system energy storage and external energy supply, we present a systematic treatment of dissipativity theory [456]. Dissipativity theory provides a fundamental framework for the analysis and design of nonlinear dynamical systems using an input, state, and output system description based on system-energy-related considerations. As a direct application of dissipativity theory, absolute stability theory involving the stability of a feedback system whose forward path contains a dynamic linear time-invariant system and whose feedback path contains a memoryless (possibly time-varying) nonlinearity is also addressed. The Aizerman conjecture and the Luré problem, as well as the circle and Popov criteria, are extensively developed.

Using the concepts of dissipativity theory, in Chapter 6 we present feedback interconnection stability results for nonlinear dynamical systems. General stability criteria are given for Lyapunov, asymptotic, and exponential stability of feedback dynamical systems. Using quadratic supply rates corresponding to net system power and weighted input-output energy, we specialize these results to the classical positivity and small gain theorems. In addition, notions of a control Lyapunov function, feedback linearization, zero dynamics, minimum-phase systems, and stability margins for nonlinear feedback systems are also introduced. Finally, to address optimality issues within nonlinear control-system design we consider an optimal control problem in which a performance function is minimized over all possible closed-loop system trajectories. The value of the performance function is given by a solution to the Hamilton-Jacobi-Bellman equation. In Chapter 7, we provide a brief treatment of input-output stability and dissipativity theory. In particular, we introduce input-output system models as well as $\mathcal{L}_p$ stability. In addition, we develop connections between dissipativity theory of input, state, and output systems, and input-output dissipativity theory.

In Chapter 8, we develop a unified framework to address the problem of optimal nonlinear analysis and feedback control. Asymptotic stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function which can clearly be seen to be the solution to the steady-state form of the Hamilton-Jacobi-Bellman equation, and hence guarantees both stability and optimality. The overall framework provides the foundation for extending linear-quadratic controller synthesis to nonlinear-nonquadratic problems. Guaranteed stability margins for nonlinear optimal and inverse optimal regulators that minimize a nonlinear-nonquadratic performance criterion are also established. Using the optimal control framework of Chapter 8, in Chapter 9, we give a unification between nonlinear-nonquadratic optimal control and backstepping control. Backstepping control has received a great deal of attention in the nonlinear control literature [247,395]. The popularity of this control methodology is due in large part to the fact that it provides a systematic procedure for finding a Lyapunov function for nonlinear closed-loop cascade systems.

In Chapter 10, we develop an optimality-based framework to address the problem of nonlinear-nonquadratic control for disturbance rejection of nonlinear systems with bounded exogenous disturbances. Specifically, using dissipativity theory with appropriate storage functions and supply rates we transform the nonlinear disturbance rejection problem into an optimal control problem by modifying a nonlinear-nonquadratic cost functional to account for the exogenous disturbances. As a consequence, the resulting solution to the modified optimal control problem guarantees disturbance rejection for nonlinear systems with bounded input disturbances. Furthermore, it is shown that the Lyapunov function guaranteeing closed-loop stability is a solution to the steady-state Hamilton-Jacobi-Isaacs equation for the controlled system. The overall framework generalizes the Hamilton-Jacobi-Bellman conditions developed in Chapter 8 to address the design of optimal controllers for nonlinear systems with exogenous disturbances.

In Chapter 11, we concentrate on developing a unified framework to address the problem of optimal nonlinear robust control. As in the disturbance rejection problem, we transform the given robust control problem into an optimal control problem by properly modifying the cost functional to account for the system uncertainty. As a consequence, the resulting solution to the modified optimal control problem guarantees robust stability and performance for a class of nonlinear uncertain systems. In Chapter 12, we extend the framework developed in Chapter 11 to nonlinear systems with nonlinear time-invariant real parameter uncertainty. Robust stability of the closed-loop nonlinear system is guaranteed by means of a parameter-dependent Lyapunov function composed of a fixed (parameter-independent) and variable (parameter-dependent) part. The fixed part of the Lyapunov function can be seen to be the solution to the steady-state Hamilton-Jacobi-Bellman equation for the nominal system. Finally, in Chapters 13 and 14 we give a condensed presentation of the continuous-time analysis and control synthesis results developed in Chapters 2–12 for discrete-time systems.