Chapter One

Affine Coxeter Diagrams

By the results summarized in Chapter VI, Section 4.3, of [3], affine Coxeter groups can be characterized as groups generated by reflections of an affine space (by which is meant a Euclidean space without a fixed coordinate system or even a fixed origin). In an effort to get to affine buildings as quickly as possible, however, we will define them more succinctly, but with less motivation, simply as the Coxeter groups associated with the following Coxeter diagrams (and direct products of such groups):

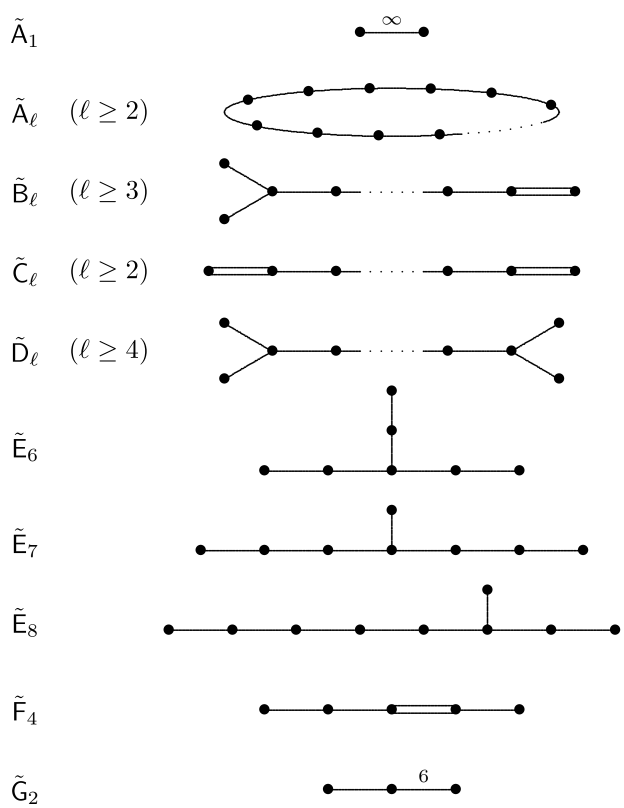

Figure 1.1 The Irreducible Affine Coxeter Diagrams

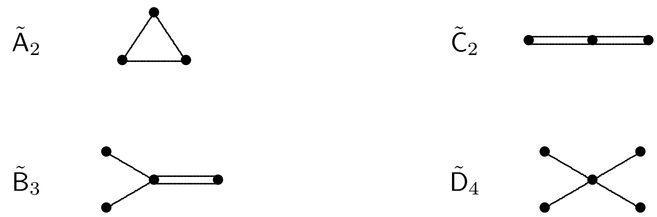

Figure 1.2 Some Small Affine Coxeter Diagrams

Each Coxeter diagram[1] in Figure 1.1 has a name of the form $\tilde{\mathsf{X}}_\ell$, where

$$\mathsf{X} = \mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F} \text{ or } \mathsf{G}$$

and ℓ is 1 less than the number of vertices. In each case, X_ℓ is the standard name of a spherical diagram with ℓ vertices that can be found in Figure 1.3. [2] Moreover, the spherical diagram X_ℓ can be obtained from $\tilde{\mathsf{X}}_\ell$ by deleting a single vertex (and the edge or double edge connected to it).

Following the usual convention, we assign the diagram $\tilde{\mathsf{C}}_2$ the alternative name $\tilde{\mathsf{B}}_2$. In Figure 1.2, we have reproduced a few of the smaller affine diagrams.

Definition 1.1. A vertex of $\tilde{\mathsf{X}}_\ell$ is called *special* if its deletion yields the diagram X_ℓ.

Thus, for example, every vertex of $\tilde{\mathsf{A}}_\ell$ is special, the special vertices of $\tilde{\mathsf{B}}_\ell$ are the two farthest to the left in Figure 1.1 and the special vertices of $\tilde{\mathsf{C}}_\ell$ are those at the two ends. Note that in each case, the special vertices of $\tilde{\mathsf{X}}_\ell$ form a single orbit under the action of $\mathrm{Aut}(\tilde{\mathsf{X}}_\ell)$ (the group of automorphisms of the underlying graph of $\tilde{X}_\ell$ that preserve the edge labels).

The names A_ℓ, B_ℓ, etc. are also the standard names of irreducible root systems. We explain this connection in Chapter 2.

Affine buildings are frequently called *Euclidean buildings* in the literature. These two terms are synonymous.

As in [37], we regard buildings exclusively as chamber systems throughout this book. We assume that the reader is familiar with the basic facts about

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[1] A Coxeter diagram is a graph whose edges are labeled by elements of the set

$$\{m \in \mathbb{N} \mid m \geq 3\} \cup \{\infty\}.$$

The Coxeter diagrams in Figures 1.1–1.3 are drawn using the usual conventions that an edge with label 3 is drawn simply as an edge with no label and an edge with label 4 is drawn as a double edge (also with no label).

[2] A Coxeter diagram is *spherical* if the associated Coxeter group W_Π (defined in 29.5) is finite. By [13], the only irreducible spherical Coxeter diagrams that do *not* appear in Figure 1.3 are the diagrams

(sometimes called $\mathsf{I}(m)$) for $m = 5$ and $m > 6$ and the diagrams H_3 and H_4.

Coxeter chamber systems and buildings—especially with properties of their roots, residues and projection maps—as covered in Chapters 1–5, 7–9 and 11 of [37]. We have summarized the most basic definitions and results from [37] in Appendix A (Chapter 29). We will also make frequent reference as we go along to the exact results in [37] and Appendix A that we require.

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An affine building is a building whose Coxeter diagram is affine. The apartments of a building with Coxeter diagram Π are all isomorphic to the Coxeter chamber system Σ_Π (as defined in 29.4). We begin our study of affine buildings, therefore, by studying properties of the Coxeter chamber systems Σ_Π for Π a connected affine diagram.[3]

In this chapter we describe three fundamental properties of these Coxeter chamber systems. These properties are stated in 1.3, 1.9 and 1.12 below. In Chapter 2 we will prove that these properties do, in fact, hold. To do this, we will rely on properties of root systems and thus on a representation of Σ_Π as a system of "alcoves" in a Euclidean space. In Chapters 3, we introduce the notion of a "root datum with valuation" which depends on this representation in an essential way. The notion of a root datum with valuation will not be required, however, until Chapter 13. Thus the reader who is impatient to see what affine buildings look like and who is willing to regard the three fundamental properties of affine Coxeter chamber systems as axioms can jump from the end of this first chapter straight to Chapter 4.[4]

For the rest of this chapter, we suppose that Π is one of the affine diagrams in Figure 1.1 and let $\Sigma = \Sigma_\Pi$ denote the Coxeter chamber system of type Π as defined in 29.4.

Definition 1.2. Two roots α and α' of Σ are *parallel* if $\alpha \subset \alpha'$ or $\alpha' \subset \alpha$.[5]

Here is the first of our fundamental properties:

Proposition 1.3. *For each root α there is a bijection $i \mapsto \alpha_i$ from $\mathbb{Z}$ to the set of roots of Σ parallel to α such that $\alpha_0 = \alpha$ and $\alpha_i \subset \alpha_{i+1}$ for each i.*

Proof. See 2.46. □

Corollary 1.4. *Parallelism is an equivalence relation on the set of roots of Σ.*

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[3]By 7.32 in [37], an arbitrary building Δ of finite rank is a direct product (in an appropriate sense) of buildings $\Delta_1, \dots, \Delta_k$ of type $\Pi_1, \dots, \Pi_k$, where $\Pi_1, \dots, \Pi_k$ are the connected components of the Coxeter diagram of Δ.

[4]Our intention is not, however, to elevate these three properties to the level of axioms, but rather to make clear and explicit the use we are making of the representation of Σ_Π in Euclidean space. Even in those chapters where we are not using it (i.e. Chapters 4–12), this representation remains, of course, an important source of intuition.

[5]The symbol $\subset$ is always to be interpreted in this book as allowing equality.

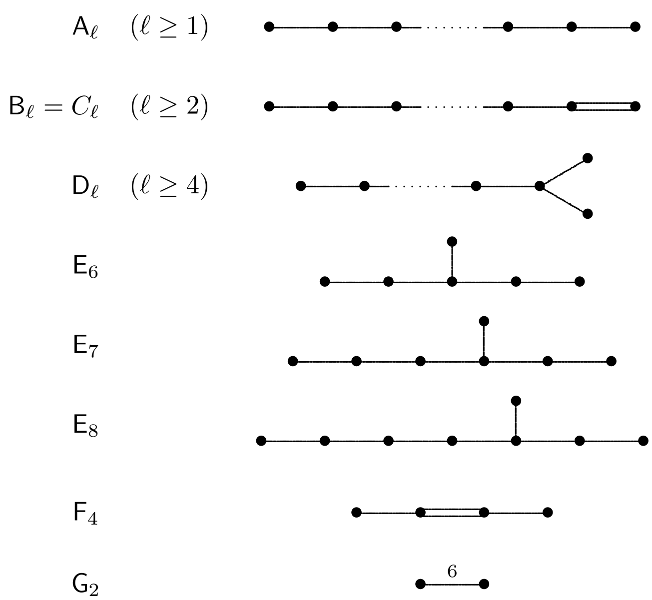

Figure 1.3 Some Spherical Coxeter Diagrams

Proof. This holds by 1.3. □

Notation 1.5. The parallel class of a root α will be denoted by $[\alpha]$.

Definition 1.6. A *gem* of Σ is a residue of type $I\backslash\{o\}$, where I denotes the vertex set of Π and o is a special vertex of Π as defined in 1.1. A gem of type $I\backslash\{o\}$ for some special vertex o is called *o-special*.

Proposition 1.7. *All proper residues of Σ are finite and all gems are, in fact, Coxeter chamber systems of type X_ℓ (up to isomorphism). In particular, all gems have the same number of chambers.*

Proof. This holds by 29.7.iii (and 5.17 of [37]). □

Definition 1.8. Let $\mathrm{Aut}(\Sigma)$ be as defined in 29.2 and let $T = T_\Pi$ denote the set of elements of $\mathrm{Aut}(\Sigma)$ that map each root of Σ to a parallel root. The elements of T are called *translations* and two objects in the same orbit of T are called *translates* of each other. By 1.4, T is a normal subgroup of $\mathrm{Aut}(\Sigma)$.

Here is our second fundamental property:

Proposition 1.9. *The group T of translations acts transitively on the set of gems of Σ.*

Proof. See 2.46. □

The set $[R,d]$ defined next will play a central role in this book.

Notation 1.10. Let R be a gem and let d be a chamber of R. We denote by $[R,d]$ the set of roots α of Σ that contain d but not some chamber of R adjacent to d.

Let R, d and $[R,d]$ be as in 1.10. As observed in 1.7, R is a Coxeter chamber system of type X_ℓ. This implies that there are exactly ℓ chambers in R adjacent to d. Let e be one of them. By 29.6.i, there is a unique root in $[R,d]$ that does not contain e. Thus $|[R,d]| = \ell$.

Definition 1.11. We will say that a root α *cuts* a residue E of Σ if both α and $-\alpha$ contain chambers of E. By 29.7.iv, this is equivalent to saying that $\alpha \cap E$ is a root of E.

Here is the last of our three fundamental properties:

Proposition 1.12. *Let R be a gem, let $d \in R$, let $[R,d]$ be the set defined in 1.10, let*

$$\alpha_1, \dots, \alpha_\ell$$

be the roots in $[R,d]$ and for each $i \in [1,\ell]$,[6] *let α_i' be a root parallel to α_i. Then there exists a unique gem that is cut by α_i' for all $i \in [1,\ell]$. In particular, R is the unique gem cut by all the roots in $[R,d]$.*

Proof. See 2.46. □

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We use the remainder of this chapter to record a number of simple consequences of 1.3, 1.7 and our three fundamental properties.

Proposition 1.13. *A proper residue (in particular, a gem) cannot be cut by two distinct parallel roots.*

Proof. Let E be a proper residue and suppose that α and β are parallel roots both cutting E. By 1.6, E is finite. By 29.7.iv, both $\alpha \cap E$ and $\beta \cap E$ are roots of E. By 29.6.i, therefore,

$$|\alpha \cap E| = |E|/2 = |\beta \cap E|.$$

Since α and β are parallel, it follows that $\alpha \cap E = \beta \cap E$. Hence $\alpha = \beta$ by 29.17. □

Proposition 1.14. *Let α and β be roots such that α is parallel to neither β nor $-\beta$. Then there exists a residue of rank 2 cut by both α and β.*

[6] We use interval notation throughout this book to denote intervals of integers. Thus, for example, by $[1,\ell]$, we mean the subset $\{1,2,\dots,\ell\}$ of $\mathbb{Z}$. This should not cause any confusion with the notation introduced in 1.5 or 1.10.

Proof. This holds by 29.24. □

Proposition 1.15. *Let α and β be roots such that α is parallel to neither β nor $-\beta$. Then there exist gems that are cut by β and contained in α.*

Proof. Let $i \mapsto \alpha_i$ be as in 1.3 with $\alpha_0 = \alpha$ (so $\alpha_i \subset \alpha_{i+1}$ for all $i \in \mathbb{Z}$) and choose $m \leq -2$. By 1.14, there exists a residue E of rank 2 cut by both α_m and β. Let P be the unique panel of E that is cut by β and contained in α_m. We choose a gem containing P if there is one and call it R.

Suppose that P is not contained in a gem and let o be its type. Then o is the only special vertex of Π and there exists (by inspection of Figure 1.1) a unique vertex i of Π adjacent to o and the label on the edge $\{o, i\}$ is 3. Let D be the residue of type $\{o, i\}$ containing P and let Q be the panel of D opposite P. Then Q is of type i and β cuts Q. Since $P \subset D \cap \alpha_m$, the root α_m either cuts D or contains D. By 1.13, it follows that $Q \subset D \subset \alpha_{m+1}$. Let R be the unique gem containing Q.

Thus whether or not P is contained in a gem, R is a gem cut by β that contains chambers in α_{m+1}. Thus either $R \subset \alpha_{m+1}$ or R is cut by α_{m+1}. By 1.13 again, it follows that $R \subset \alpha_{m+2} \subset \alpha_0 = \alpha$. □

Corollary 1.16. *Every root cuts gems.*

Proof. Let α be a root. Either there exists a root β parallel to neither α nor $-\alpha$, in which case 1.15 applies, or $\ell = 1$. If $\ell = 1$, then the unique panel cut by α is a gem. □

Corollary 1.17. *Let α be a root. Then there exists a gem R and a chamber $d \in R$ such that $\alpha \in [R, d]$ (where $[R, d]$ is as defined in 1.10).*

Proof. This holds by 1.16. □

We next observe that Σ satisfies a version of the parallel postulate:

Proposition 1.18. *For each root α and each gem R of Σ, there is a unique root parallel to α cutting R.*

Proof. Let R be a gem and let α be a root of Σ. By 1.16, there exists a gem R_1 cut by α. By 1.9, there exists a translation g mapping R_1 to R. By 1.8 and 1.11, α^g is a root parallel to α that cuts R. By 1.13, this root is unique. □

Proposition 1.19. *Let R be a gem of Σ. Then the map $\alpha \mapsto [\alpha]$ is a bijection from the set of roots cutting R to the set of parallel classes of roots of Σ.*

Proof. This map is injective by 1.13 and surjective by 1.18. □

Proposition 1.20. *The group* $\mathrm{Aut}(\Sigma)$ *acts faithfully on the set of roots of Σ.*

Proof. Let g be an element of $\mathrm{Aut}(\Sigma)$ that maps each root to itself and let d be a chamber of Σ. For each chamber e adjacent to d, there is (by 29.6.i) a unique root containing d but not e. The intersection of these roots is thus $\{d\}$. Since g maps this intersection to itself, it fixes d. □

Proposition 1.21. *Let R be a gem. The only element of T that maps R to itself is the identity.*

Proof. Let g be an element of T that maps R to itself and let α be a root. By 1.18, there is a unique root α_1 in the parallel class $[\alpha]$ that cuts R. By 1.13, g maps α_1 to itself. By 1.3, it follows that g acts trivially on $[\alpha]$. We conclude that g fixes every root of Σ. By 1.20, therefore, $g = 1$. □

Corollary 1.22. *The group T acts sharply transitively on the set of gems of Σ.*[7]

Proof. This holds by 1.9 and 1.21. □

Definition 1.23. Let R be a gem, let d be a chamber of R, let $[R, d]$ be as in 1.10 and let $\alpha \in [R, d]$. We denote by $T_{R,d,\alpha}$ the subgroup of T consisting of all translations that fix every root in $[R, d]$ *except* α.

Proposition 1.24. *Let R be a gem, let d be a chamber of R and let $T_{R,d,\alpha}$ for each $\alpha \in [R, d]$ be as in 1.23. Then $T_{R,d,\alpha}$ is isomorphic to $\mathbb{Z}$ and acts transitively and faithfully on the parallel class $[\alpha]$ for all $\alpha \in [R, d]$ and*

$$T = \bigoplus_{\alpha\in[R,d]} T_{R,d,\alpha}.$$

Thus, in particular, $T \cong \mathbb{Z}^\ell$.

Proof. Let

$$\alpha_1, \dots, \alpha_\ell$$

be the roots in $[R, d]$, let

$$X = [\alpha_1] \cup \cdots \cup [\alpha_\ell] \tag{1.25}$$

and let $\alpha'_i \in [\alpha_i]$ for all $i \in [1, \ell]$. Choose $i \in [1, \ell]$. By 1.12, there exists a unique residue R'_i cut by α'_i and by α_j for all $j \in [1, \ell]$ different from i. By 1.9, there exists a translation t mapping R to R'_i. Thus $t(\alpha_i)$ is a root in $[\alpha_i]$ cutting R'_i for each $i \in [1, \ell]$. By 1.13, it follows that $t(\alpha_i) = \alpha'_i$ and $t(\alpha_j) = \alpha_j$ for all $j \in [1, \ell]$ different from i. By 1.23, $t \in T_{R,d,\alpha_i}$. We conclude that T_{R,d,α_i} acts transitively on $[\alpha_i]$.

By 1.3, it follows that the group T_{R,d,α_i} induces a transitive group isomorphic to $\mathbb{Z}$ on $[\alpha_i]$ for each $i \in [1, \ell]$. In particular, the commutator group

$$[T_{R,d,\alpha_i}, T_{R,d,\alpha_j}]$$

acts trivially on the set X defined in 1.25 above, and if t is an arbitrary element of T, then there exist elements $t_i \in T_{R,d,\alpha_i}$ for $i \in [1, \ell]$ such that

[7] In other words, for each ordered pair (R_1, R_2) of gems, there is a *unique* element of T that maps R_1 to R_2.

the product $t^{-1}t_1 \cdots t_\ell$ also acts trivially on the set X. It thus suffices to show that T acts faithfully on X.

Let g be an element of T that acts trivially on X. Thus $g(R)$ is a gem cut by all the roots of $[R, d]$. By 1.12, R is the unique gem with this property. Hence $g(R) = R$. By 1.21, therefore, $g = 1$. □

Corollary 1.26. *Let α be a root. Then T acts transitively on the parallel class $[\alpha]$.*

Proof. This holds by 1.17 and 1.24. □

Definition 1.27. We will call a translation a *σ-translation* if it is a σ-automorphism of Σ for some $\sigma \in \mathrm{Aut}(\Pi)$ (as defined in 29.2). Let H be the image of T under the homomorphism from $\mathrm{Aut}(\Sigma)$ to $\mathrm{Aut}(\Pi)$ that sends a σ-automorphism to σ. Thus $H \cong T/T \cap W$ (since, by 29.4, W can be thought of as the group of special automorphisms of Σ). The elements of H will be called *translational automorphisms* of Π.

Proposition 1.28. *Let H be as defined in 1.27. Then H acts sharply transitively on the set of special vertices of* Π*. In particular, $|T/T \cap W|$ equals the number of special vertices.*

Proof. By 1.24, T is abelian. Therefore H is also abelian. It will suffice, therefore, to show that H acts transitively on the set of special vertices of Π. Let o, o_1 be two special vertices of Σ, let R be an o-special gem and let R_1 be an o_1-special gem. By 1.9, there exists a translation t mapping R to R_1. Then t is a σ-translation from some $\sigma \in H$ mapping o to o_1. □

Remark 1.29. By Section 4.7(XII) and Section 4.8(XII) in Chapter VI of [3], the group H defined in 1.27 is, in fact, cyclic except when $\Pi = \tilde{\mathsf{D}}_\ell$ for ℓ even, in which case $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Corollary 1.30. *The subgroup $T \cap W$ of* $\mathrm{Aut}^\circ(\Sigma)$ *(as defined in 29.2) acts sharply transitively on the set of gems of a given type.*

Proof. Let R and R_1 be two gems of the same type. By 1.9, there exists an element $t \in T$ mapping R to R_1. By 1.28, the element t acts trivially on Π. Thus $t \in W$. The claim holds, therefore, by 1.22. □

Definition 1.31. Let R be a gem and let x be a chamber of Σ. Then there is a unique gem R_1 of the same type as R containing x. By 1.30, there is a unique $g \in T \cap W$ mapping R_1 to R. We will call x^g the *special translate of x to R.*

Proposition 1.32. *Let R be an o-special gem for some special vertex o of* Π*, let $\{x, y\}$ be a panel of Σ of type $i \neq o$ and let u and v be the special translates of the chambers x and y to R (as defined in 1.31). Then $\{u, v\}$ is a panel of R of type i.*

Proof. This holds by 1.31 since the o-special gem containing x is the same as the o-special gem containing y. □

Proposition 1.33. *Let R be an o-special gem for some special vertex o of Π and let W_R be the stabilizer of R in $W = \mathrm{Aut}^\circ(\Sigma)$. For each chamber $x \in R$, let $\delta(x)$ be the special translate of y to R, where y is the unique chamber that is o-adjacent to x. Then $\delta(x^g) = \delta(x)^g$ for all $g \in W_R$.*

Proof. Let $g \in W_R$ and $x \in R$. Let y be the unique chamber that is o-adjacent to x and let t be the unique element of $T \cap W$ such that $y^t \in R$. Thus $g^{-1}tg$ is an element of $T \cap W$ (since $T \cap W$ is normal in W) that maps y^g to R. Therefore

$$\delta(x^g) = (y^g)^{g^{-1}tg} = y^{tg} = \delta(x)^g.$$

□

Proposition 1.34. *Let α and β be roots such that α is parallel to neither β nor $-\beta$. Then α contains some but not all chambers in $\partial\beta$ (where $\partial\beta$ is the border of β as in 29.40).*

Proof. By 1.14, there exists a residue E of rank 2 cut by both α and β. By 29.7.iv, both $\alpha \cap E$ and $\beta \cap E$ are roots of E, and by 29.17, $\alpha \cap E \neq \beta \cap E$. It follows that $\partial\beta \cap E$ contains two chambers and α contains exactly one of them. □

Proposition 1.35. *Let α be a root. Then every chamber is contained in some root parallel to α.*

Proof. Let d be a chamber and let R be a gem containing d. By 1.18, there is a root α' parallel to α that cuts R. By 1.3, there exists a root α'' that contains α' properly (and is, in particular, parallel to α'). By 1.13, α'' must contain R. □

Definition 1.36. Let α be a root. By 1.3, there exists a unique root $\beta \neq \alpha$ that contains α and is contained in every other root containing α. We will say that β *contains* α *minimally*, alternatively, that α is *contained in* β *maximally*.

Definition 1.37. A *strip* is a set of the form $-\alpha \cap \beta$, where α and β are roots such that β contains α minimally (as defined in 1.36).

Proposition 1.38. *Let α and β be roots such that β contains α minimally (as defined in 1.36) and let d be contained in the strip $-\alpha \cap \beta$. Then β is the convex hull of $\alpha \cup \{d\}$.*

Proof. By 1.36, every root containing α and d contains β. The claim holds, therefore, by 29.20. □

Proposition 1.39. *If β and β' are parallel roots and $\beta' \cap \partial\beta \neq \emptyset$, then $\beta \subset \beta'$ (where $\partial\beta$ is as in 29.40).*

Proof. Let $x \in \beta' \cap \partial\beta$ and let y be the chamber adjacent to x in $-\beta$. If $y \in \beta'$, then $\beta \subset \beta'$ simply because β and β' are parallel. If $y \notin \beta'$, then $\beta = \beta'$ by 29.6.i. □

Proposition 1.40. *Let α and β be roots such that β contains α minimally. Then the convex hull of $\partial\beta$ is the strip $-\alpha \cap \beta$.*

Proof. Let β' be a root containing $\partial\beta$. By 1.34, β' is a parallel either to β or to $-\beta$. By 1.39, $\beta \subset \beta'$ if β' is parallel to β. If β' is parallel to $-\beta$, then $-\alpha \subset \beta'$ simply by 1.36. The claim holds now by 29.20. □

Proposition 1.41. *Let α be a root, let $i \mapsto \alpha_i$ be the map from $\mathbb{Z}$ to $[\alpha]$ described in 1.3 and for each $i \in \mathbb{Z}$, let L_i denote the strip $\alpha_i \backslash \alpha_{i-1}$. Then Σ is the disjoint union of the strips L_i for all $i \in \mathbb{Z}$ and for each $k \in \mathbb{Z}$, the root α_k is the disjoint union of the strips L_i for all $i \leq k$.*

Proof. By 1.35, every chamber of Σ lies in a root parallel to α and in a root parallel to $-\alpha$. This means that for each chamber x in Σ the set of integers i such that $x \in \alpha_i$ is non-empty and bounded below. □

Proposition 1.42. *Let α and β be roots such that β contains α minimally and let P be a panel. Let $\mu(\alpha)$ and $\mu(\beta)$ be the walls of α and β as defined in 29.22. Then a panel P is contained in $\mu(\alpha) \cup \mu(\beta)$ if and only if P contains exactly one chamber in the strip $-\alpha \cap \beta$.*

Proof. Let $L = -\alpha \cap \beta$. Suppose that $P \in \mu(\alpha)$. Then P contains one chamber x in $-\alpha$ and one chamber y in α and (by 29.6.i) α is the unique root containing y but not x. Since $y \in \alpha \subset \beta$ but $\beta \neq \alpha$, it follows that β contains x. Thus $x \in L$ but $y \notin L$. Similarly, if $P \in \mu(\beta)$, then P contains exactly one chamber in L.

Suppose, conversely, that P is a panel that contains a unique chamber in L. Let x be this unique chamber and let y be the chamber in P distinct from x. Then $y \notin L$. Thus $y \in \alpha$ or $y \in -\beta$. In the first case, $P \in \mu(\alpha)$, and in the second case, $P \in \mu(\beta)$. □

Proposition 1.43. *For each $g \in T$, there exists $M \in \mathbb{N}$ such that*

$$\mathrm{dist}(x, x^g) \leq M$$

for all chambers x of Σ.

Proof. Let $g \in T$ and let R be a gem. Since R is finite, there exists a positive integer M such that

$$\mathrm{dist}(x, x^g) \leq M$$

for all $x \in R$. Now suppose that x is an arbitrary chamber of Σ. By 1.9, there exists an element $h \in T$ such that $x^h \in R$. By 1.24, T is abelian. Thus

$$\mathrm{dist}(x^g, x) = \mathrm{dist}(x^{gh}, x^h) = \mathrm{dist}(x^{hg}, x^h) = \mathrm{dist}((x^h)^g, x^h) \leq M.$$

□

The following notion plays a central role in the theory of affine buildings.

Definition 1.44. Let Γ be an arbitrary graph, let S be the vertex set of X and let

$$\mathrm{dist}(U, V) = \inf\{\mathrm{dist}(u, v) \mid u \in U,\ v \in V\}$$

for all subsets U and V of S. We will call two subsets X and Y of S *parallel* if the sets

$$\{\text{dist}(\{x\}, Y) \mid x \in X\} \text{ and } \{\text{dist}(X, \{y\}) \mid y \in Y\}$$

are both bounded.[8]

Proposition 1.45. *Let X be an arbitrary set of chambers of Σ and let $g \in T$. Then X and X^g are parallel as defined in 1.44.*

Proof. This holds by 1.43. □

Note that the parallel relation defined in 1.44 is an equivalence relation. We have already introduced the notion of parallel roots, so we need to observe that the two roots are parallel in the sense of 1.2 if and only if they are parallel in the sense of 1.44: By 1.26 and 1.43, roots that are parallel in the sense of 1.2 are parallel as defined in 1.44. Suppose, conversely, that α and β are roots that are not parallel in the sense of 1.2. We will say that a chamber x is *separated* from α by k roots if there are k roots of Σ containing α that do not contain x. If α is parallel to $-\beta$ in the sense of 1.2, then by 1.3, β contains chambers separated from α by arbitrarily many roots. If α is not parallel to $-\beta$ in the sense of 1.2, then by 1.14, $(-\alpha') \cap \beta \neq \emptyset$ for every root α' containing α, so again there are chambers in β separated from α by arbitrarily many roots. By 29.6.iii, we conclude that β contains chambers arbitrarily far from α (whether or not α is parallel to $-\beta$ in the sense of 1.2). Thus α and β are not parallel in the sense of 1.44.

We say that two walls of Σ (as defined in 29.22) are parallel if their chamber sets are parallel in the sense of 1.44.

Proposition 1.46. *Let α and α' be two roots. Then α is parallel to α' or $-\alpha'$ if and only if the walls $\mu(\alpha)$ and $\mu(\alpha')$ are parallel.*

Proof. Suppose that α is parallel to α'. By 1.26, α' is a translate of α. Hence $\mu(\alpha')$ is a translate of $\mu(\alpha)$. By 1.45, it follows that $\mu(\alpha)$ and $\mu(\alpha')$ are parallel. Since $\mu(\alpha') = \mu(-\alpha')$, the same conclusion holds if we assume that α is parallel to $-\alpha'$.

Suppose, conversely, that α is parallel to neither α' nor $-\alpha'$. Let $i \mapsto \alpha_i$ be the bijection from $\mathbb{Z}$ to the parallel class $[\alpha]$ as in 1.3. By 29.6.i, the chamber set of $\mu(\alpha)$ is contained in α_i for all $i > 0$ and α_i is parallel to neither α' nor $-\alpha'$ for all i. By 1.14, therefore, the chamber set of $\mu(\alpha')$ has a non-trivial intersection with $-\alpha_i$ for all i. Thus for each $N > 0$, there are chambers of $\mu(\alpha')$ not contained in any of the roots α_i for $i \in [1, N]$. By 29.6.iii, it follows that there exist chambers in $\mu(\alpha')$ that are arbitrarily far from $\mu(\alpha)$. Thus $\mu(\alpha)$ and $\mu(\alpha')$ are not parallel. □

Comment 1.47. If we identify Σ with its representation in a Euclidean space V as described in the next chapter, then parallel walls of Σ correspond

[8] "At bounded distance" might arguably be a more suitable name for this property than "parallel"; see, however, 1.47.

(by 2.34) to affine hyperplanes of V that are parallel in the usual sense of the word. This accounts for the use of the word "parallel" in 1.2 and 1.44.