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Dean Corbae, Maxwell B. Stinchcombe & Juraj Zeman: An Introduction to Mathematical Analysis for Economic Theory and Econometrics

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Logic

The building blocks of modern economics are based on logical reasoning to prove the validity of a conclusion, \mathbb{B} , from well-defined premises, \mathbb{A} . In general, statements such as \mathbb{A} and/or \mathbb{B} can be represented using sets, and a “proof” is constructed by applying, sometimes ingeniously, a fixed set of rules to establish that the statement \mathbb{B} is true whenever \mathbb{A} is true. We begin with examples of how we represent statements as sets, then turn to the rules that allow us to form more and more complex statements, and then give a taxonomy of the major types of proofs that we use in this book.

1.1 ♦ Statements, Sets, Subsets, and Implication

The idea of a set (of things), or group, or collection is a “primitive,” one that we use without being able to clearly define it. The idea of belonging to a set (group, collection) is primitive in exactly the same sense. Our first step is to give the allowable rules by which we evaluate whether statements about sets are true.

We begin by fixing a set X of things that we might have an interest in. When talking about demand behavior, the set X has to include, at the very least, prices, incomes, affordable consumption sets, preference relations, and preference-optimal sets. The set X varies with the context, and is often not mentioned at all.

We express the primitive notion of membership by “ \in ,” so that “ $x \in A$ ” means that “ x is an element of the set A ” and “ $y \notin A$ ” means that “ y is not an element of A .”

Notation Alert 1.1.A *Capitalized letters are usually reserved for sets and smaller letters for points/things in the set that we are studying. Sometimes, several levels of analysis are present simultaneously, and we cannot do this. Consider the study of utility functions, u , on a set of options, X . A function u is a set of pairs of the form $(x, u(x))$, with x an option and $u(x)$ the number representing its utility. However, in our study of demand behavior, we want to see what happens as u varies. From this perspective, u is a point in the set of possible utility functions.*

Membership allows us to define subsets. We say “ A is a subset of B ,” written “ $A \subset B$,” if every $x \in A$ satisfies $x \in B$. Thus, subsets are defined in terms of the primitive relation “ \in .” We write $A = B$ if $A \subset B$ and $B \subset A$, and $A \neq B$ otherwise.

We usually specify the elements of a set explicitly by saying “The set A is the set of all elements x in X such that each x has the property \mathbb{A} , that is, that $\mathbb{A}(x)$ is true,” and write “ $A = \{x \in X : \mathbb{A}(x)\}$ ” as a shorter version of this. For example, with $X = \mathbb{R}^\ell$, the statement “ $\mathbf{x} \geq 0$ ” is identified with the set $\mathbb{R}_+^\ell = \{\mathbf{x} \in X : \mathbf{x} \geq 0\}$. In this way, we identify a statement with the set of elements of X for which the statement is true. There are deep issues in logic and the foundations of mathematics relating to the question of whether or not all sets can be identified by “properties.” Fortunately, these issues rarely impinge on the mathematics that economists need. Chapter 2 is more explicit about these issues.

We are very often interested in establishing the truth of statements of the form “If \mathbb{A} , then \mathbb{B} .” There are many equivalent ways of writing such a statement: “ $\mathbb{A} \Rightarrow \mathbb{B}$,” “ \mathbb{A} implies \mathbb{B} ,” “ \mathbb{A} only if \mathbb{B} ,” “ \mathbb{A} is sufficient for \mathbb{B} ,” or “ \mathbb{B} is necessary for \mathbb{A} .” To remember the sufficiency and necessity, it may help to subvocalize them as “ \mathbb{A} is sufficiently strong to guarantee \mathbb{B} ” and “ \mathbb{B} is necessarily true if \mathbb{A} is true.”

The logical relation of implication is a subset relation. If $A = \{x \in X : \mathbb{A}(x)\}$ and $B = \{x \in X : \mathbb{B}(x)\}$, then “ $\mathbb{A} \Rightarrow \mathbb{B}$ ” is the same as “ $A \subset B$.”

Example 1.1.1 *Let X be the set of numbers, $\mathbb{A}(x)$ the statement “ $x^2 < 1$,” and $\mathbb{B}(x)$ the statement “ $|x| \leq 1$.” Now, $\mathbb{A} \Rightarrow \mathbb{B}$. In terms of sets, $A = \{x \in X : \mathbb{A}(x)\}$ is the set of numbers strictly between -1 and $+1$, $B = \{x \in X : \mathbb{B}(x)\}$ is the set of numbers greater than or equal to -1 and less than or equal to $+1$, and $A \subset B$.*

The statements of interest can be quite complex to write out in their entirety. If X is the set of allocations in a model \mathcal{E} of an economy and $\mathbb{A}(x)$ is the statement “ x is a Walrasian equilibrium allocated for the economy \mathcal{E} ,” then a complete specification of the statement takes a great deal of work. Presuming some familiarity with general equilibrium models, we offer the following.

Example 1.1.2 *Let X be the set of allocations in a model \mathcal{E} of an economy; let $\mathbb{A}(x)$ be the statement “ x is a Walrasian equilibrium allocation”; and $\mathbb{B}(x)$ be the statement “ x is Pareto efficient for \mathcal{E} .” The first fundamental theorem of welfare economics is $\mathbb{A} \Rightarrow \mathbb{B}$. In terms of the definition of subsets, this is expressed as, “Every Walrasian equilibrium allocation is Pareto efficient.”*

In other cases, we are interested in the truth of statements of the form “ \mathbb{A} if and only if \mathbb{B} ,” often written “ \mathbb{A} iff \mathbb{B} .” Equivalently, such a statement can be written: “ $\mathbb{A} \Rightarrow \mathbb{B}$ and $\mathbb{B} \Rightarrow \mathbb{A}$,” which is often shortened to “ $\mathbb{A} \Leftrightarrow \mathbb{B}$.” Other frequently used formulations are: “ \mathbb{A} implies \mathbb{B} and \mathbb{B} implies \mathbb{A} ,” “ \mathbb{A} is necessary and sufficient for \mathbb{B} ,” or “ \mathbb{A} is equivalent to \mathbb{B} .” In terms of the corresponding sets A and B , these are all different ways of writing “ $A = B$.”

Example 1.1.3 *Let X be the set of numbers, $\mathbb{A}(x)$ the statement “ $0 \leq x \leq 1$,” and $\mathbb{B}(x)$ the statement “ $x^2 \leq x$.” From high school algebra, $\mathbb{A} \Leftrightarrow \mathbb{B}$. In terms of sets, $A = \{x \in X : \mathbb{A}(x)\}$ and $B = \{x \in X : \mathbb{B}(x)\}$ are both the sets of numbers greater than or equal to 0 and less than or equal to 1.*

1.2 ♦ Statements and Their Truth Values

Note that a statement of the form “ $\mathbb{A} \Rightarrow \mathbb{B}$ ” is simply a construct of two simple statements connected by “ \Rightarrow .” This is one of seven ways of constructing new statements that we use. In this section, we cover the first five of them: ands, ors, nots, implies, and equivalence. Repeated applications of these seven ways of constructing statements yield more and more elaboration and complication.

We begin with the simplest three methods, which construct new sets directly from a set or pair of sets that we start with. We then turn to the statements that are about relations between sets and introduce another formulation in terms of indicator functions. Later we give the other two methods, which involve the logical quantifiers “for all” and “there exists.” Throughout, interest focuses on methods of establishing the truth or falsity of statements, that is, on methods of proof.

1.2.a Ands/Ors/Not as Intersections/Unions/Complements

The simplest three ways of constructing new statements from other ones are using the connectives “and” or “or,” or by “not,” which is negation. Notationally: “ $\mathbb{A} \wedge \mathbb{B}$ ” means “ \mathbb{A} and \mathbb{B} ,” “ $\mathbb{A} \vee \mathbb{B}$ ” means “ \mathbb{A} or \mathbb{B} ,” and “ $\neg \mathbb{A}$ ” means “not \mathbb{A} .”

In terms of the corresponding sets: “ $\mathbb{A} \wedge \mathbb{B}$ ” is $A \cap B$, the intersection of A and B , that is, the set of all points that belong to both A and B ; “ $\mathbb{A} \vee \mathbb{B}$ ” is $A \cup B$, the union of A and B , that is, the set of all points that belong to A or belong to B ; and “ $\neg \mathbb{A}$ ” is $A^c = \{x \in X : x \notin A\}$, the complement of A , is the set of all elements of X that do *not* belong to A .

The meanings of these new statements, $\neg \mathbb{A}$, $\mathbb{A} \wedge \mathbb{B}$, and $\mathbb{A} \vee \mathbb{B}$, are given by a *truth table*, Table 1.a. The corresponding Table 1.b gives the corresponding set versions of the new statements.

\mathbb{A}	\mathbb{B}	$\neg \mathbb{A}$	$\mathbb{A} \wedge \mathbb{B}$	$\mathbb{A} \vee \mathbb{B}$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F
A	B	A^c	$A \cap B$	$A \cup B$
$x \in A$	$x \in B$	$x \notin A^c$	$x \in A \cap B$	$x \in A \cup B$
$x \in A$	$x \notin B$	$x \notin A^c$	$x \notin A \cap B$	$x \in A \cup B$
$x \notin A$	$x \in B$	$x \in A^c$	$x \notin A \cap B$	$x \in A \cup B$
$x \notin A$	$x \notin B$	$x \in A^c$	$x \notin A \cap B$	$x \notin A \cup B$

The first two columns of Table 1.a give possible truth values for the statements \mathbb{A} and \mathbb{B} . The last three columns give the truth values for $\neg \mathbb{A}$, $\mathbb{A} \wedge \mathbb{B}$, and $\mathbb{A} \vee \mathbb{B}$ as a function of the truth values of \mathbb{A} and \mathbb{B} . The first two columns of Table 1.b give the corresponding membership properties of an element x , and the last three columns give the corresponding membership properties of x in the sets A^c , $A \cap B$, and $A \cup B$.

Consider the second rows of both tables, the row where \mathbb{A} is true and \mathbb{B} is false. This corresponds to discussing an x with the properties that it belongs to A and does not belong to B . The statement “not \mathbb{A} ,” that is, $\neg \mathbb{A}$, is false, which corresponds to x not belonging to A^c , $x \notin A^c$. The statement “ \mathbb{A} and \mathbb{B} ,” that is, “ $\mathbb{A} \wedge \mathbb{B}$,” is also false. This is sensible: since \mathbb{B} is false, it is not the case that both \mathbb{A} and \mathbb{B} are true. This corresponds to x not being in the intersection of A and B , that is, $x \notin A \cap B$.

The statement “ \mathbb{A} or \mathbb{B} ,” that is, “ $\mathbb{A} \vee \mathbb{B}$,” is true. This is sensible: since \mathbb{A} is true, it is the case that at least one of \mathbb{A} and \mathbb{B} is true, corresponding to x being in the union of A and B .

It is important to note that we use the word “or” in its nonexclusive sense. When we describe someone as “tall or red-headed,” we mean to allow tall red-headed people. We do not mean “or” in the exclusive sense that the person is either tall or red-headed but not both. One sees this by considering the last columns in the two tables, the ones with the patterns $TTTF$ and $\epsilon\epsilon\epsilon\zeta$. “ \mathbb{A} or \mathbb{B} ” is true as long as at least one of \mathbb{A} and \mathbb{B} is true, and we do not exclude the possibility that both are true. The exclusive “or” is defined by $(\mathbb{A} \vee \mathbb{B}) \wedge (\neg(\mathbb{A} \wedge \mathbb{B}))$, which has the truth pattern $FTTF$. In terms of sets, the exclusive “or” is $(A \cup B) \cap (A \cap B)^c$, which has the corresponding membership pattern $\zeta\epsilon\epsilon\zeta$.

1.2.b Implies/Equivalence as Subset/Equality

Two of the remaining four ways of constructing new statements are: “ $\mathbb{A} \Rightarrow \mathbb{B}$,” which means “ \mathbb{A} implies \mathbb{B} ” and “ $\mathbb{A} \Leftrightarrow \mathbb{B}$,” which means “ \mathbb{A} is equivalent to \mathbb{B} .” In terms of sets, these are “ $A \subset B$ ” and “ $A = B$.” These are statements about relations between subsets of X .

Indicator functions are a very useful way to talk about the relations between subsets. For each $x \in X$ and $A \subset X$, define the **indicator of the set** A by

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \quad (1.1)$$

Remember, a proposition, \mathbb{A} , is a statement about elements $x \in X$ that can be either true or false. When it is true, we write $\mathbb{A}(x)$. The corresponding set A is $\{x \in X : \mathbb{A}(x)\}$. The indicator of A takes on the value 1 for exactly those x for which \mathbb{A} is true and takes on the value 0 for those x for which \mathbb{A} is false.

Indicator functions are ordered pointwise; that is, $1_A \leq 1_B$ when $1_A(x) \leq 1_B(x)$ for every point x in the set X . Saying “ $1_A \leq 1_B$ ” is the same as saying that “ $A \subset B$.” It is easy to give sets A and B that satisfy neither $A \subset B$ nor $B \subset A$. Therefore, unlike pairs of numbers r and s , for which it is always true that either $r \leq s$ or $s \leq r$, pairs of indicator functions may not be ranked by “ \leq .”

Example 1.2.1 *If X is the three-point set $\{a, b, c\}$, $A = \{a, b\}$, $B = \{b, c\}$, and $C = \{c\}$, then $1_A \leq 1_X$, $1_B \leq 1_X$, $1_C \leq 1_B$, $\neg(1_A \leq 1_B)$, and $\neg(1_B \leq 1_A)$.*

Proving statements of the form $\mathbb{A} \Rightarrow \mathbb{B}$ and $\mathbb{A} \Leftrightarrow \mathbb{B}$ is the essential part of mathematical reasoning. For the first, we take the truth of \mathbb{A} as given and then establish logically that the truth of \mathbb{B} follows. For the second, we take the additional step of taking the truth of \mathbb{B} as given and then establish logically that the truth of \mathbb{A} follows. In terms of sets, for proving the first, we take a point, x , assume only that $x \in A$, and establish that this implies that $x \in B$, thus proving that $A \subset B$. For proving the second, we take the additional step of taking a point, x , assume only that $x \in B$, and establish that this implies that $x \in A$. Here is the truth table for \Rightarrow and \Leftrightarrow , both for statements and for indicator functions.

Table 1.c				Table 1.d			
\mathbb{A}	\mathbb{B}	$\mathbb{A} \Rightarrow \mathbb{B}$	$\mathbb{A} \Leftrightarrow \mathbb{B}$	$x \in A$	$x \in B$	$1_A(x) \leq 1_B(x)$	$1_A(x) = 1_B(x)$
T	T	T	T	$x \in A$	$x \in B$	T	T
T	F	F	F	$x \in A$	$x \notin B$	F	F
F	T	T	F	$x \notin A$	$x \in B$	T	F
F	F	T	T	$x \notin A$	$x \notin B$	T	T

1.2.c The Empty Set and Vacuously True Statements

We now come to the idea of something that is vacuously true, and a substantial proportion of people find this idea tricky or annoying, or both. The idea that we are after is that starting from false premises, one can establish anything. In Table 1.c, if \mathbb{A} is false, then the statement $\mathbb{A} \Rightarrow \mathbb{B}$ is true, whether \mathbb{B} is true or false.

A statement that is false for all $x \in X$ corresponds to having an indicator function with the property that for all $x \in X$, $1_A(x) = 0$. In terms of sets, the notation for this is $A = \emptyset$, where we read “ \emptyset ” as the **empty set**, that is, the vacuous set, the one that contains no elements. No matter what the set B is, if $A = \emptyset$, then $1_A(x) \leq 1_B(x)$ for all $x \in X$.

Definition 1.2.2 *The statement $\mathbb{A} \Rightarrow \mathbb{B}$ is **vacuously true** if $A = \emptyset$.*

This definition follows the convention that we use throughout: we show the term or terms being defined in boldface type.

In terms of sets, this is the observation that for all B , $\emptyset \subset B$, that is, that every element of \emptyset belongs to B . What many people find distasteful is that “every element of \emptyset belongs to B ” suggests that there is an element of \emptyset , and since there is no such element, the statement feels wrong to them. There is nothing to be done except to get over the feeling.

1.2.d Indicators and Ands/Ors/Not

Indicator functions can also be used to capture ands, ors, and nots. Often this makes proofs simpler.

The pointwise minimum of a pair of indicator functions, 1_A and 1_B , is written as “ $1_A \wedge 1_B$,” and is defined by $(1_A \wedge 1_B)(x) = \min\{1_A(x), 1_B(x)\}$. Now, $1_A(x)$ and $1_B(x)$ are equal either to 0 or to 1. Since the minimum of 1 and 1 is 1, the minimum of 0 and 1 is 0, and the minimum of 0 and 0 is 0, $1_{A \cap B} = 1_A \wedge 1_B$. This means that the indicator associated with the statement “ $\mathbb{A} \wedge \mathbb{B}$ ” is $1_A \wedge 1_B$. By checking cases, we note that for all $x \in X$, $(1_A \wedge 1_B)(x) = 1_A(x) \cdot 1_B(x)$. As a result, $1_A \wedge 1_B$ is often written as $1_A \cdot 1_B$.

In a similar way, the pointwise maximum of a pair of indicator functions, 1_A and 1_B , is written as “ $1_A \vee 1_B$ ” and defined by $(1_A \vee 1_B)(x) = \max\{1_A(x), 1_B(x)\}$. Here, $1_{A \cup B} = 1_A \vee 1_B$, and the indicator associated with the statement “ $\mathbb{A} \vee \mathbb{B}$ ” is $1_A \vee 1_B$. Basic properties of numbers say that for all x , $(1_A \vee 1_B)(x) = 1_A(x) + 1_B(x) - 1_A(x) \cdot 1_B(x)$, so $1_A \vee 1_B$ could be defined as $1_A + 1_B - 1_A \cdot 1_B$.

The truth values in the fourth column, $\mathbb{B} \wedge \mathbb{C}$, are formed using the rules for \wedge and the truth values of \mathbb{B} and \mathbb{C} . The truth values in the next column, $\mathbb{A} \vee (\mathbb{B} \wedge \mathbb{C})$, are formed using the rules for \vee and the truth values of the \mathbb{A} column and the just-derived truth values of the $\mathbb{B} \wedge \mathbb{C}$ column. The truth values in the next three columns are derived analogously. Since the truth values in the column $\mathbb{A} \vee (\mathbb{B} \wedge \mathbb{C})$ match those in the column $(\mathbb{A} \vee \mathbb{B}) \wedge (\mathbb{A} \vee \mathbb{C})$, the two statements are equivalent. ■

Another Proof of Theorem 1.3.1. In terms of indicator functions,

$$\begin{aligned} 1_{A \cup (B \cap C)} &= 1_A \cdot (1_B + 1_C - 1_B \cdot 1_C) \\ &= 1_A \cdot 1_B + 1_A \cdot 1_C - 1_A \cdot 1_B \cdot 1_C, \end{aligned} \quad (1.2)$$

while

$$1_{(A \cap B) \cup (A \cap C)} = 1_A \cdot 1_B + 1_A \cdot 1_C + (1_A \cdot 1_B) \cdot (1_B \cdot 1_C). \quad (1.3)$$

Since indicators take only the values 0 and 1, for any set, for example, B , $1_B \cdot 1_B = 1_B$. Therefore $(1_A \cdot 1_B) \cdot (1_B \cdot 1_C) = 1_A \cdot 1_B \cdot 1_C$. ■

The following contains the commutative, associative, and distributive laws. To prove them, one can simply generate the appropriate truth table.

Theorem 1.3.2 *Let \mathbb{A} , \mathbb{B} , and \mathbb{C} be any statements. Then*

1. *commutativity holds, $(\mathbb{A} \vee \mathbb{B}) \Leftrightarrow (\mathbb{B} \vee \mathbb{A})$ and $(\mathbb{A} \wedge \mathbb{B}) \Leftrightarrow (\mathbb{B} \wedge \mathbb{A})$,*
2. *associativity holds, $((\mathbb{A} \wedge \mathbb{B}) \wedge \mathbb{C}) \Leftrightarrow (\mathbb{A} \wedge (\mathbb{B} \wedge \mathbb{C}))$, $((\mathbb{A} \vee \mathbb{B}) \vee \mathbb{C}) \Leftrightarrow (\mathbb{A} \vee (\mathbb{B} \vee \mathbb{C}))$, and*
3. *the distributive laws hold, $(\mathbb{A} \wedge (\mathbb{B} \vee \mathbb{C})) \Leftrightarrow ((\mathbb{A} \wedge \mathbb{B}) \vee (\mathbb{A} \wedge \mathbb{C}))$, $(\mathbb{A} \vee (\mathbb{B} \wedge \mathbb{C})) \Leftrightarrow ((\mathbb{A} \vee \mathbb{B}) \wedge (\mathbb{A} \vee \mathbb{C}))$.*

Exercise 1.3.3 Restate Theorem 1.3.2 in terms of sets and in terms of indicator functions. Then complete the proof of Theorem 1.3.2 both by generating the appropriate truth tables and by using indicator functions.

We now prove two results, Lemma 1.3.4 and Theorem 1.3.6, that form the basis for the methods of logical reasoning we pursue in this book. The following is used so many times that it is at least as important as a theorem.

Lemma 1.3.4 *\mathbb{A} implies \mathbb{B} iff \mathbb{A} is false or \mathbb{B} is true,*

$$(\mathbb{A} \Rightarrow \mathbb{B}) \Leftrightarrow ((\neg \mathbb{A}) \vee \mathbb{B}), \quad (1.4)$$

and a double negative makes a positive,

$$\neg(\neg \mathbb{A}) \Leftrightarrow \mathbb{A}. \quad (1.5)$$

Proof. In terms of indicator functions, (1.4) is $1_A(x) \leq 1_B(x)$ iff $1_A(x) = 0$ or $1_B(x) = 1$, which is true because $1_A(x)$ and $1_B(x)$ can only take on the values 0 and 1. (1.5) is simpler; it says that $1 - (1 - 1_A) = 1_A$. ■

An alternative method of proving (1.4) in Lemma 1.3.4 is to construct the truth table as follows.

Table 1.f

A	B	A \Rightarrow B	\neg A	(\neg A \vee B)
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the third and the fifth columns are identical, we have $(A \Rightarrow B) \Leftrightarrow ((\neg A) \vee B)$.

Exercise 1.3.5 Complete the proof of Lemma 1.3.4 by generating the appropriate truth tables and restate (1.5) in terms of sets.

The next result, Theorem 1.3.6, forms the basis for most of the logical reasoning in this book. The first (direct) approach (1.6) is the *sylogism*, which says that “if A is true and ‘A implies B’ is true, then B is true.” The second (indirect) approach (1.7) is the *contradiction*, which says in words that “if not A leads to a false statement of the form B and not B, then A is true. That is, one way to prove A is to hypothesize $\neg A$ and show that this leads to a contradiction. Another (indirect) approach (1.8) is the *contrapositive*, which says that “A implies B is the same as “whenever B is false, A is false.” In terms of sets, this last is “[$A \subset B$] \Leftrightarrow [$B^c \subset A^c$].”

Theorem 1.3.6 *If A is true and A implies B, then B is true,*

$$(A \wedge (A \Rightarrow B)) \Rightarrow B. \quad (1.6)$$

If A being false implies a contradiction, then A is true,

$$((\neg A) \Rightarrow (B \wedge (\neg B))) \Rightarrow A. \quad (1.7)$$

A implies B iff whenever B is false, A is false,

$$(A \Rightarrow B) \Leftrightarrow ((\neg B) \Rightarrow (\neg A)). \quad (1.8)$$

Proof. In the case of (1.6), $1_A(x) = 1$ and $1_A(x) \leq 1_B(x)$ imply that $1_B(x) = 1$.

In the case of (1.7), note that $(1_B(x) \wedge (1 - 1_B(x))) = 0$ for all x . Hence $(1 - 1_A(x)) \leq (1_B(x) \vee (1 - 1_B(x)))$ implies that $1 - 1_A(x) = 0$, that is, $1_A(x) = 1$.

In the case of (1.8), $1_A(x) \leq 1_B(x)$ iff $-1_B(x) \leq -1_A(x)$ iff $1 - 1_B(x) \leq 1 - 1_A(x)$. ■

Exercise 1.3.7 Give an alternative proof of Theorem 1.3.6 using truth tables and another using the distributive laws and Lemma 1.3.4.

As we are mostly interested in proving statements of the form “A implies B,” it is worth belaboring the notion of the contrapositive in (1.8). “A implies B” is the same as “whenever A is true, we know that B is true.” There is one and only one

way that this last statement could be false—if there is an $x \in X$ such that $\neg\mathbb{B}(x)$ while $\mathbb{A}(x)$. Therefore, “ \mathbb{A} implies \mathbb{B} ” is equivalent to $(\neg\mathbb{B}) \Rightarrow (\neg\mathbb{A})$.

In terms of sets, we are saying that $A \subset B$ is equivalent to $B^c \subset A^c$. Often it is easier to pick a point y , assume only that y does *not* belong to B , and establish that this implies that y does *not* belong to A .

Example 1.3.8 *Let X be the set of humans, let $\mathbb{A}(x)$ be the statement “ x has a Ph.D. in economics,” and let $\mathbb{B}(x)$ be the statement “ x is literate in at least one language.” Showing that $\mathbb{A} \Rightarrow \mathbb{B}$ is the same as showing that there are no completely illiterate economics Ph.D.s. Which method of proving the statement one would want to use depends on whether or not it is easier to check all the economics Ph.D.s in X for literacy or to check all the illiterates in X for Ph.D.s in economics.*

A final note: the contrapositive of “ $\mathbb{A} \Rightarrow \mathbb{B}$ ” is “ $(\neg\mathbb{B}) \Rightarrow (\neg\mathbb{A})$.” This is not the same as the *converse* of “ $\mathbb{A} \Rightarrow \mathbb{B}$,” which is “ $\mathbb{B} \Rightarrow \mathbb{A}$.”

1.4 ♦ Logical Quantifiers

The last two of our seven ways to construct statements use the two *quantifiers*, “ \exists ,” read as “there exists,” and “ \forall ,” read as “for all.” More specifically, “ $(\exists x \in A)[\mathbb{B}(x)]$ ” means “there exists an x in the set A such that $\mathbb{B}(x)$ ” and “ $(\forall x \in A)[\mathbb{B}(x)]$ ” means “for all x in the set A , $\mathbb{B}(x)$.” Our discussion of indicator functions has already used these quantifiers; for example, $1_A \leq 1_B$ was defined as $(\forall x \in X)[1_A(x) \leq 1_B(x)]$. We now formalize the ways in which we use the quantifiers.

Quantifiers should be understood as statements about the relations between sets, and here the empty set, \emptyset , is again useful. In terms of sets, “ $(\exists x \in A)[\mathbb{B}(x)]$ ” is the statement $(A \cap B) \neq \emptyset$, while “ $(\forall x \in A)[\mathbb{B}(x)]$ ” is the statement $A \subset B$.

Notation Alert 1.4.A *Following common usage, when the set A is supposed to be clear from context, we often write $(\exists x)[\mathbb{B}(x)]$ for $(\exists x \in A)[\mathbb{B}(x)]$. If A is not in fact clear from context, we run the risk of leaving the intended set A undefined.*

The two crucial properties of quantifiers are contained in the following, which gives the relationship among quantifiers, negations, and complements.

Theorem 1.4.1 *There is no x in A such that $\mathbb{B}(x)$ iff for all x in A , it is not the case that $\mathbb{B}(x)$,*

$$\neg(\exists x \in A)[\mathbb{B}(x)] \Leftrightarrow (\forall x \in A)[\neg\mathbb{B}(x)], \quad (1.9)$$

and it is not the case that for all x in A we have $\mathbb{B}(x)$ iff there is some x in A for which $\mathbb{B}(x)$ fails,

$$\neg(\forall x \in A)[\mathbb{B}(x)] \Leftrightarrow (\exists x \in A)[\neg\mathbb{B}(x)]. \quad (1.10)$$

Proof. In terms of sets, (1.9) is $[A \cap B = \emptyset] \Leftrightarrow [A \subset B^c]$. In terms of indicators, letting 0 be the function identically equal to 0, it is $1_A \cdot 1_B = 0$ iff $1_A \leq (1 - 1_B)$.

In terms of sets, (1.10) is $\neg[A \subset B] \Leftrightarrow [A \cap B^c \neq \emptyset]$. In terms of indicators, the left-hand side of (1.10) is $\neg[1_A \leq 1_B]$, which is true iff for some x in X , $1_A(x) > 1_B(x)$. This happens iff for some x , $1_A(x) = 1$ and $1_B(x) = 0$, that is, iff for some x , $1_A(x) \cdot (1 - 1_B(x)) = 1$, which is the right-hand side of (1.10). ■

The second tautology in Theorem 1.4.1 is important since it illustrates the concept of a *counterexample*. In particular, (1.10) states: “If it is not true that $\mathbb{B}(x)$ for all x in A , then there must exist a counterexample (i.e., an x satisfying $\neg\mathbb{B}(x)$), and vice versa.” Counterexamples are important tools, since knowing that $x \in A$ and $x \in B$ for hundreds and hundreds of x ’s does not prove that $A \subset B$, but a single counterexample shows that $\neg[A \subset B]$.

Often, one can profitably apply the rules in (1.9) and (1.10) time after time. The following anticipates material from the topics of convergence and continuity that we cover extensively later.

Example 1.4.2 A sequence of numbers is a list (x_1, x_2, x_3, \dots) , one x_n for each counting number $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, where the “ \dots ” indicates “keep going in this fashion.” Let \mathbb{R}_{++} denote the set of strictly positive numbers; for any $N \in \mathbb{N}$, let N_{\geq} be the set of integers greater than or equal to N ; and let $\mathbb{A}(\epsilon, x)$ be the statement that $|x| < \epsilon$. We say that a sequence converges to 0 if

$$(\forall \epsilon \in \mathbb{R}_{++})(\exists N \in \mathbb{N})(\forall n \in N_{\geq})[\mathbb{A}(\epsilon, x_n)],$$

which is more much conveniently, and just as precisely, written as

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|x_n| < \epsilon].$$

This captures the idea that the numbers in the sequence become and stay arbitrarily small as we move further and further out in the sequence. A verbal shorthand for this is that “for all positive ϵ (no matter how small), $|x_n|$ is smaller than ϵ for large n .”

Applying (1.9) and (1.10) repeatedly shows that the following are all equivalent to the sequence **not** converging to 0:

$$\begin{aligned} &\neg(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|x_n| < \epsilon], \\ &(\exists \epsilon > 0)\neg(\exists N \in \mathbb{N})(\forall n \geq N)[|x_n| < \epsilon], \\ &(\exists \epsilon > 0)(\forall N \in \mathbb{N})\neg(\forall n \geq N)[|x_n| < \epsilon], \\ &(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\exists n \geq N)\neg[|x_n| < \epsilon], \text{ and} \\ &(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\exists n \geq N)[|x_n| \geq \epsilon]. \end{aligned}$$

Thus, the statement “a sequence fails to converge to 0” is equivalent to “for some strictly positive ϵ , for all N (no matter how large), there is an even larger n such that $|x_n| \geq \epsilon$.”

One should also note that the commutative and distributive laws we found with “ \vee ” and “ \wedge ” in them may break down with quantifiers. While

$$(\exists x)[\mathbb{A}(x) \vee \mathbb{B}(x)] \Leftrightarrow (\exists x)[\mathbb{A}(x)] \vee (\exists x)[\mathbb{B}(x)], \quad (1.11)$$

$$(\exists x)[\mathbb{A}(x) \wedge \mathbb{B}(x)] \Rightarrow (\exists x)[\mathbb{A}(x)] \wedge (\exists x)[\mathbb{B}(x)]. \quad (1.12)$$

Example 1.4.3 To see why (1.12) cannot hold as an “if and only if” statement, suppose x is the set of countries in the world, $\mathbb{A}(x)$ is the property that x has a gross domestic product strictly above average, and $\mathbb{B}(x)$ is the property that x has a gross domestic product strictly below average. There will be at least one country above the mean and at least one country below the mean. That is, $(\exists x)[\mathbb{A}(x)] \wedge (\exists x)[\mathbb{B}(x)]$ is true, but clearly there cannot be a country that is both above and below the mean, $\neg(\exists x)[\mathbb{A}(x) \wedge \mathbb{B}(x)]$.

In terms of sets, (1.11) can be rewritten as $[(A \cup B) \neq \emptyset] \Leftrightarrow [(A \neq \emptyset) \vee (B \neq \emptyset)]$. The set form of (1.12) is $[A \cap B \neq \emptyset] \Rightarrow [(A \neq \emptyset) \wedge (B \neq \emptyset)]$. Hopefully this formulation makes the reason we do not have an “if and only if” relation in (1.12) even clearer.

We can also make increasingly complex statements by adding more variables. For example, statements of the form $\mathbb{A}(x, y)$ as x and y both vary across X . One can always view this as a statement about a pair (x, y) and change X to contain pairs, but this may not mitigate the additional complexity.

Example 1.4.4 When X is the set of numbers and $\mathbb{A}(x, y)$ states that “ y that is larger than x ,” where x and y are numbers, the statement $(\forall x)(\exists y)(x < y)$ says “for every x there is a y that is larger than x .” The statement $(\exists y)(\forall x)(x < y)$ says “there is a y that is larger than every x .” The former statement is true, but the latter is false.

1.5 ♦ Taxonomy of Proofs

We now discuss broadly the methodology of proofs you will frequently encounter in economics. The most intuitive is the *direct proof* in the form of “ $\mathbb{A} \Rightarrow \mathbb{B}$,” discussed in (1.6). The work is to fill in the intermediate steps so that $\mathbb{A} \Rightarrow \mathbb{A}_1$, $\mathbb{A}_1 \Rightarrow \mathbb{A}_2$, and $\dots \mathbb{A}_n \Rightarrow \mathbb{B}$ are all tautologies. In terms of sets, this involves constructing n sets A_1, \dots, A_n such that $A \subset A_1 \subset \dots \subset A_n \subset B$.

Notation Alert 1.5.A The “ \dots ” indicates A_2 through A_{n-1} in the first list. The “ \dots ” indicates the same sets in the second list, but we also mean to indicate that the subset relation holds for all the intermediate pairs.

In some cases, the sets A_1, \dots, A_n arise from splitting \mathbb{B} into cases. If we find $\mathbb{B}_1, \mathbb{B}_2$ such that $[\mathbb{B}_1 \vee \mathbb{B}_2] \Rightarrow \mathbb{B}$ and can show that $\mathbb{A} \Rightarrow [\mathbb{B}_1 \vee \mathbb{B}_2]$, then we are done.

In other cases it may be simpler to split \mathbb{A} into cases. That is, sometimes it is easier to find \mathbb{A}_1 and \mathbb{A}_2 for which $\mathbb{A} \Rightarrow [\mathbb{A}_1 \vee \mathbb{A}_2]$ and then to show that $[\mathbb{A}_1 \Rightarrow \mathbb{B}] \vee [\mathbb{A}_2 \Rightarrow \mathbb{B}]$.

Another direct method of proof, called *induction*, works only for the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$. Suppose we wish to show that $(\forall n \in \mathbb{N}) \mathbb{A}(n)$ is true. This is equivalent to proving $\mathbb{A}(1) \wedge (\forall n \in \mathbb{N}) (\mathbb{A}(n) \Rightarrow \mathbb{A}(n+1))$. This works since $\mathbb{A}(1)$ is true and $\mathbb{A}(1) \Rightarrow \mathbb{A}(2)$ and $\mathbb{A}(2) \Rightarrow \mathbb{A}(3)$ and so on. In Chapter 2 we show why induction works.

Proofs by contradiction are also known as indirect proofs. They may, initially, seem less natural than direct proofs. To help you on your way to becoming fluent in indirect proofs, we now give the exceedingly simple indirect proof of

the first fundamental theorem of welfare economics. This is perhaps one of the most important things you will learn in all of economics, and finding a direct proof seems rather difficult. Again, we presume some familiarity with general equilibrium models.

Definition 1.5.1 An *exchange economy model* is a triple, $\mathcal{E} = (I, \mathbf{y}_i, \succ_i)$, where I is a finite set (meant to represent the people in the economy), $\mathbf{y}_i \in \mathbb{R}_+^\ell$ is i *endowment* of the ℓ goods that are available in the model of the economy, and \succ_i is i 's *preference relation* over his or her own consumption.

There are two things to note here: first, we did not say what a preference relation is and we do it, in detail, in Chapter 2; and second, we assumed that preferences are defined only over own consumption, which is a very strong assumption, and we discuss it further.

Definition 1.5.2 An *allocation* is a list of vectors, written $(\mathbf{x}_i)_{i \in I}$, where $\mathbf{x}_i \in \mathbb{R}_+^\ell$ for each $i \in I$. An allocation $(\mathbf{x}_i)_{i \in I}$ is *feasible for* \mathcal{E} if for each good k , $k = 1, \dots, \ell$,

$$\sum_i x_{i,k} \leq \sum_i y_{i,k}, \quad (1.13)$$

where the summation is over all of the individuals in the economy.

Definition 1.5.3 A feasible allocation $(\mathbf{x}_i)_{i \in I}$ is *Pareto efficient* if there is no feasible allocation $(\mathbf{x}'_i)_{i \in I}$ such that all agents prefer \mathbf{x}'_i to \mathbf{x}_i .

Definition 1.5.4 A *price* \mathbf{p} is a nonzero vector in \mathbb{R}_+^ℓ . An allocation-price pair $((\mathbf{x}_i)_{i \in I}, \mathbf{p})$ is a *Walrasian equilibrium for* \mathcal{E} if it is feasible, and if \mathbf{x}'_i is preferred by i to \mathbf{x}_i , then i cannot afford \mathbf{x}'_i , that is,

$$\sum_k p_k x'_{i,k} > \sum_k p_k y_{i,k}. \quad (1.14)$$

Theorem 1.5.5 (First Fundamental Theorem of Welfare Economics) If $((\mathbf{x}_i)_{i \in I}, \mathbf{p})$ is a Walrasian equilibrium, then $(\mathbf{x}_i)_{i \in I}$ is Pareto efficient.

Let $A = \{(\mathbf{x}_i)_{i \in I} : (\exists \mathbf{p}) [((\mathbf{x}_i)_{i \in I}, \mathbf{p}) \text{ is a Walrasian equilibrium}]\}$, and $B = \{(\mathbf{x}_i)_{i \in I} : (\mathbf{x}_i)_{i \in I} \text{ is Pareto efficient}\}$. In terms of sets, Theorem 1.5.5 states that $A \subset B$.

Proof. \mathbb{A} is the statement “ $((\mathbf{x}_i)_{i \in I}, \mathbf{p})$ is a Walrasian equilibrium” and \mathbb{B} is the statement “ $(\mathbf{x}_i)_{i \in I}$ is Pareto efficient.” A proof by contradiction assumes $\mathbb{A} \wedge \neg \mathbb{B}$, and shows that this leads to a contradiction, $\mathbb{C} \wedge \neg \mathbb{C}$. In words, suppose that $((\mathbf{x}_i)_{i \in I}, \mathbf{p})$ is a Walrasian equilibrium but that $(\mathbf{x}_i)_{i \in I}$ is not Pareto efficient. We have to show that this leads to a contradiction.

By the definition of Pareto efficiency, failing to be Pareto efficient means that there exists a feasible allocation, $(\mathbf{x}'_i)_{i \in I}$, that has the property that all agents prefer

\mathbf{x}'_i to \mathbf{x}_i . By the definition of Walrasian equilibrium, we can sum (1.14) across all individuals to obtain

$$\sum_i \left(\sum_k p_k x'_{i,k} \right) > \sum_i \left(\sum_k p_k y_{i,k} \right). \quad (1.15)$$

Rearranging the summations in (1.15) gives

$$\begin{aligned} \sum_k \sum_i p_k x'_{i,k} &> \sum_k \sum_i p_k y_{i,k}, \quad \text{equivalently} \\ \sum_k p_k \left(\sum_i x'_{i,k} \right) &> \sum_k p_k \left(\sum_i y_{i,k} \right). \end{aligned} \quad (1.16)$$

Since $(\mathbf{x}'_i)_{i \in I}$ is a feasible allocation, multiplying each term in (1.13) by the nonnegative number p_k and then summing yields

$$\sum_k p_k \left(\sum_i x'_{i,k} \right) \leq \sum_k p_k \left(\sum_i p_k y_{i,k} \right). \quad (1.17)$$

Let r be the number $\sum_k p_k (\sum_i y_{i,k})$ and let s be the number $\sum_k p_k (\sum_i x'_{i,k})$. Equation (1.16) is the statement, \mathbb{C} , that $s > r$, whereas (1.17) is the statement $\neg \mathbb{C}$ that $s \leq r$. We have derived the contradiction $[\mathbb{C} \wedge \neg \mathbb{C}]$, which we know to be false, from the supposition $[\mathbb{A} \wedge \neg \mathbb{B}]$. From this, we conclude that $[\mathbb{A} \Rightarrow \mathbb{B}]$. ■

As one becomes more accustomed to the patterns of logical arguments, details of the arguments are suppressed. Here is a shorthand, three-sentence version of the foregoing proof.

Proof. If $(\mathbf{x}_i)_{i \in I}$ is not Pareto efficient, $\exists (\mathbf{x}'_i)_{i \in I}$ feasible and unanimously preferred to $(\mathbf{x}_i)_{i \in I}$. Summing (1.14) across individuals yields $\sum_k \sum_i p_k x'_{i,k} > \sum_k \sum_i p_k y_{i,k}$. Since $(\mathbf{x}'_i)_{i \in I}$ is feasible, summing (1.13) over goods, we have $\sum_k \sum_i p_k x'_{i,k} \leq \sum_k \sum_i p_k y_{i,k}$. ■

Just as “ $7x^2 + 9x < 3$ ” is a shorter and clearer version of “seven times the square of a number plus nine times that number adds to a number less than three,” the shortening of proofs is mostly meant to help. It can, however, feel like a diabolically designed code, one meant to obfuscate rather than elucidate.

Some decoding hints:

1. Looking at the statement of Theorem 1.5.5, we see that it ends in “then $(\mathbf{x}_i)_{i \in I}$ is Pareto efficient.” Since the shortened proof starts with the sentence “If $(\mathbf{x}_i)_{i \in I}$ is not Pareto efficient,” you should conclude that we are offering a proof by contradiction. This means that you should be looking for a conclusion that is always false. Reaching such a falsity completes the proof.
2. Despite what it says, the second sentence in the shortened proof does more than sum (1.14); it rearranges the summation. Your job as a reader is to

look at (1.14) and see that it leads to what is claimed. If the requisite rearrangement is tricky, then it should be given explicitly. Like beauty, trickiness is in the eye of the beholder.

3. The third sentence probably compresses too many steps. Sometimes, this will happen.

Throughout most of Chapter 2, we try to be explicit about the strategy of proof being used. As we get further and further into the book, we shorten proofs more and more. Hopefully, the early practice with proofs will help render our shortenings transparent.

Set Theory

In the foundations of economic theory, one worries about the existence of optima for single-person decision problems and about the existence of simultaneous optima for linked, multiple-person-decision problems. The simultaneous optima are called equilibria. Often more interesting than the study of existence questions is the study of the changes in these optima and equilibria as aspects of the economic environment change, which is called “comparative statics.” Since a change in one person’s behavior can result in a change in another’s optimal choices when the problems are linked, the comparative statics of equilibria will typically be a more complicated undertaking.

The early sections of this chapter cover notation, product spaces, relations, and functions. This is sufficient background for the foundational results in rational choice theory: conditions on preferences that guarantee the existence of optimal choices in finite contexts; representations of the optimal choices as solutions to utility maximization problems; and some elementary comparative statics results.

An introduction to weak orders, partial orders, and lattices provides sufficient background for the basics of monotone comparative statics based on supermodularity. It is also sufficient background for Tarski’s fixed-point theorem, the first of the fixed point theorems we cover. Fixed-point theorems are often the tool used to show the existence of equilibria. Tarski’s theorem also gives information useful for comparative statics, and we apply it to study the existence and properties of the set of stable matchings.

Whether or not the universe is infinite or finite but very large seems to be unanswerable. However, the mathematics of infinite sets often turns out to be much, much easier than finite mathematics. Imagine trying to study planar geometry under the simplifying assumption that the plane contains 293 million (or so) points. At the end of this chapter we deal with the basic results concerning infinite sets, results that we use extensively in our study of models of prices, quantities, and time, all of which begin in Chapter 3.

2.1 ♦ Some Simple Questions

Simple questions often have very complicated answers. In economics, a simple question with this property is, “What is money?” In mathematics, one can ask, “What is a set?” Intuitively, it seems that such simple questions ought to have answers of comparable simplicity. The existence of book-length treatments of both questions is an indication that these intuitions are wrong.

It would be wonderful if we could *always* associate with a property \mathbb{A} a set $A = \{x \in X : \mathbb{A}(x)\}$ consisting of all objects having property \mathbb{A} . Bertrand Russell taught us that we cannot.

Example 2.1.1 (Russell’s Paradox) *Let \mathbb{A} be the property “is a set and does not belong to itself.” Suppose there is a set A of all sets with property \mathbb{A} . If A belongs to itself, then it does not belong to itself—it is a set and it belongs to the set of sets that do not belong to themselves. But, if A does not belong to itself, then it does.*

Our way around this difficulty is to limit ourselves, ahead of time, to a smaller group of sets and objects, X , that we talk about. This gives a correspondingly smaller notion of membership in that group. To that end, in what follows, we *fix* a given universe (or space) X and consider only sets (or groups) whose elements (or members) are elements of X . This limits the properties that we can talk about. The limits are deep, complicated, and fascinating. They are also irrelevant to essentially everything we do with mathematics in the study of economics because the limits are loose enough to allow everything we use.

In the study of consumer demand behavior, X would have to contain, at a minimum, the positive orthant $(\mathbb{R}_+^\ell$, as a consumption set), the set of preference relations on the positive orthant, and the set of functions from price-income pairs to the positive orthant. Suppose we wish to discuss a result of the form, “The demand functions of all smooth preference relations with indifference curves lying inside the strictly positive orthant are themselves smooth.” This means that X has to include subsets of the positive orthant (e.g., the strictly positive orthant), subsets of the preference relations, and subsets of the possible demand functions.

The smaller group of objects that we talk about is called a **superstructure**. Superstructures are formally defined in §2.13 at the end of this chapter. The essential idea is that one starts with a set S . We have to start with some kind of primitive, we agree that S is a set, and we agree that none of the elements of S contains any elements. We then begin an inductive process, adding to S the class of all subsets of S , then the class of all subsets of everything we have so far, and so on and so on. As we will see, this allows us to construct and work with all of the spaces of functions, probabilities, preferences, stochastic process models, dynamic programming problems, equilibrium models, and so on, that we need to study economics. It also keeps us safely out of trouble by avoiding situations like Russell’s example and allows us to identify our restricted class of sets with the properties that they have.

2.2 ♦ Notation and Other Basics

As in Chapter 1, we express the notion of membership by “ \in ” so that “ $x \in A$ ” means “ x is an element of the set A ” and “ $x \notin A$ ” means “ x is not an element of A .” We usually specify the elements of a set explicitly by saying “ A is the set of all x in X having the property \mathbb{A} ,” and write $A = \{x \in X : \mathbb{A}(x)\}$. When the space X is understood, we may abbreviate this as $A = \{x : \mathbb{A}(x)\}$.

Example 2.2.1 If $\mathbb{A}(\mathbf{x})$ is the property “is affordable at prices \mathbf{p} and income w ” and $X = \mathbb{R}_+^\ell$, then the Walrasian budget set, denoted $B(\mathbf{p}, w)$, is defined by $B(\mathbf{p}, w) = \{\mathbf{x} \in X : \mathbb{A}(\mathbf{x})\}$. With more detail about the statement \mathbb{A} , this is $B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}_+^\ell : \mathbf{p} \cdot \mathbf{x} \leq w\}$.

Definition 2.2.2 For A and B subsets of X , we define:

1. $A \cap B$, the **intersection of A and B** , by $A \cap B = \{x \in X : [x \in A] \wedge [x \in B]\}$,
2. $A \cup B$, the **union of A and B** , by $A \cup B = \{x \in X : [x \in A] \vee [x \in B]\}$,
3. $A \subset B$, A is a **subset of B** , or B **contains A** , if $[x \in A] \Rightarrow [x \in B]$,
4. $A = B$, A is **equal to B** , if $[A \subset B] \wedge [B \subset A]$,
5. $A \neq B$, A is **not equal to B** , if $\neg[A = B]$,
6. $A \subsetneq B$, A is a **proper subset of B** , if $[A \subset B] \wedge [A \neq B]$,
7. $A \setminus B$, the **difference between A and B** , by $A \setminus B = \{x \in A : x \notin B\}$,
8. $A \Delta B$, the **symmetric difference between A and B** , by $A \Delta B = (A \setminus B) \cup (B \setminus A)$,
9. A^c , the **complement of A** , by $A^c = \{x \in X : x \notin A\}$,
10. \emptyset , the **empty set**, by $\emptyset = X^c$, and
11. A and B to be **disjoint** if $A \cap B = \emptyset$.

These definitions can be visualized using Venn diagrams as in Figure 2.2.2.

Example 2.2.3 If $X = \{1, 2, \dots, 10\}$, the counting numbers between 1 and 10, $A = \{\text{even numbers in } X\}$, $B = \{\text{odd numbers in } X\}$, $C = \{\text{powers of 2 in } X\}$, and $D = \{\text{primes in } X\}$, then $A \cap B = \emptyset$, $A \cap D = \{2\}$, $A \setminus C = \{6, 10\}$, $C \subsetneq A$, $B \neq C$, $C \cup D = \{2, 3, 4, 5, 7, 8\}$, and $B \Delta D = \{2, 9\}$.

There is a purpose to the notational choices made in defining “ \cap ” using “ \wedge ” and defining “ \cup ” using “ \vee .” Being in $A \cap B$ requires being in $A \wedge$ being in B , being in $A \cup B$ requires being in $A \vee$ being in B . The definitions of unions and intersections can easily be extended to arbitrary collections of sets. Let I be an index set, for example, $I = \mathbb{N} = \{1, 2, 3, \dots\}$ as in Example 1.4.2 (p. 10), and let A_i , $i \in I$ be subsets of X . Then $\cup_{i \in I} A_i = \{x \in X : (\exists i \in I)[x \in A_i]\}$ and $\cap_{i \in I} A_i = \{x \in X : (\forall i \in I)[x \in A_i]\}$.

We have seen the following commutative, associative, and distributive properties before in Theorem 1.3.2 (p. 7), and they are easily checked using Venn diagrams.

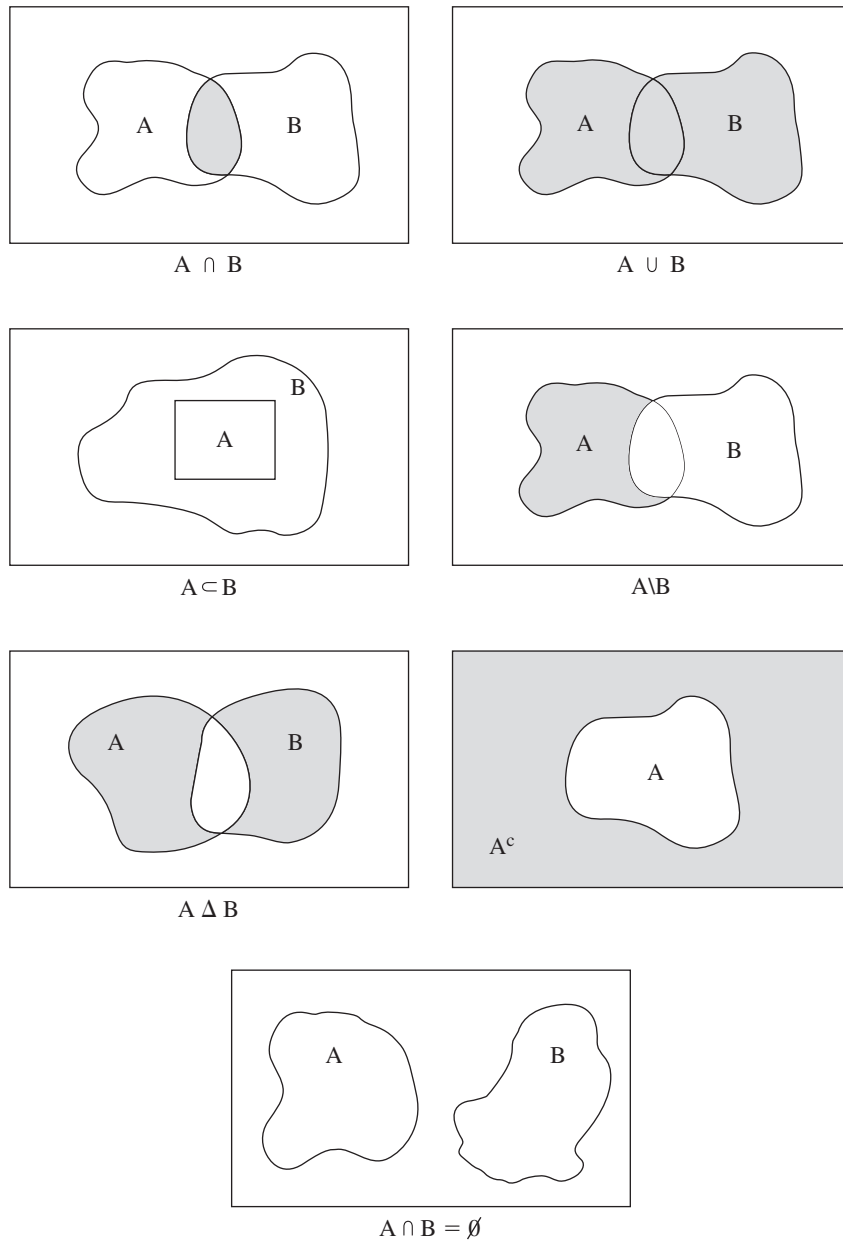


FIGURE 2.2.2

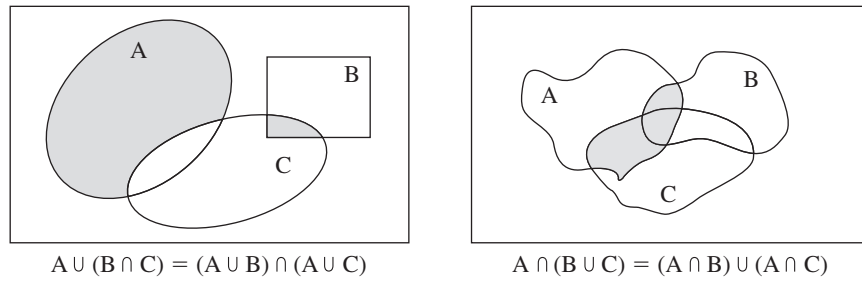


FIGURE 2.2.4

Theorem 2.2.4 For sets A , B , and C ,

1. $A \cap B = B \cap A$, $A \cup B = B \cup A$;
2. $(A \cap B) \cap C = A \cap (B \cap C)$, $(A \cup B) \cup C = A \cup (B \cup C)$; and
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Exercise 2.2.5 Prove Theorem 2.2.4 from Theorem 1.3.2. [See Figure 2.2.4. The proof amounts to applying the logical connectives and above definitions: to show $A \cap B = B \cap A$, it is sufficient to note that $x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B) \Leftrightarrow (x \in B) \wedge (x \in A) \Leftrightarrow x \in B \cap A$.]

The following properties are used extensively in probability theory and are easily checked in a Venn diagram. [See Figure 2.2.6.]

Theorem 2.2.6 (DeMorgan's Laws) If A , B , and C are any sets, then

1. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$, and
2. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

In particular, taking $A = X$, $(B \cup C)^c = B^c \cap C^c$ and $(B \cap C)^c = B^c \cup C^c$.

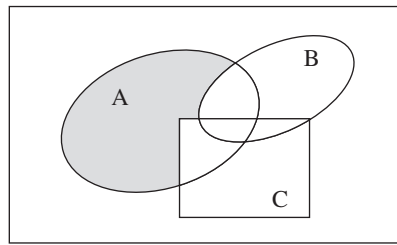
The last two equalities are “the complement of a union is the intersection of the complements” and “the complement of an intersection is the union of the complements.” When we think of B and C as statements, $(B \cup C)^c$ is “not B or C ,” which is equivalent to, “neither B nor C ,” which is equivalent to, “not B and not C ,” and this is $B^c \cap C^c$. In the same way, $(B \cap C)^c$ is “not both B and C ,” which is equivalent to “either not B or not C ,” and this is $B^c \cup C^c$.

Proof. For (1) we show that $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$, and $A \setminus (B \cup C) \supset (A \setminus B) \cap (A \setminus C)$.

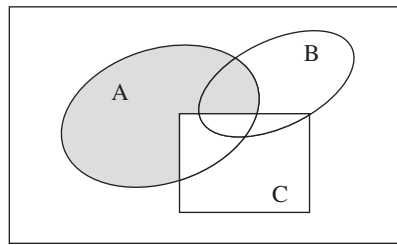
(\subset) Suppose $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin (B \cup C)$. Thus $x \in A$ and ($x \notin B$ and $x \notin C$). This implies $x \in A \setminus B$ and $x \in A \setminus C$. But this is just $x \in (A \setminus B) \cap (A \setminus C)$.

(\supset) Suppose $x \in (A \setminus B) \cap (A \setminus C)$. Then $x \in (A \setminus B)$ and $x \in (A \setminus C)$. Thus $x \in A$ and ($x \notin B$ and $x \notin C$). This implies $x \in A$ and $x \notin (B \cup C)$. But this is just $x \in A \setminus (B \cup C)$. ■

Exercise 2.2.7 Finish the proof of Theorem 2.2.6.



$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$



$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

FIGURE 2.2.6

Definition 2.2.8 For A a subset of X , the **power set of A** , denoted $\mathcal{P}(A)$, is the set of all subsets of A . A **collection** or **class of sets** is a subset of $\mathcal{P}(A)$, that is, a set of sets. A **family** is a set of collections.

Example 2.2.9 Let $X = \{a, b, c\}$. If $A = \{a, b\}$, $B = \{b\}$, $C = \{b, c\}$, then $\mathcal{P}(X)$, $\mathcal{C} = \{A\}$, $\mathcal{D} = \{A, B\}$, and $\mathcal{E} = \{A, C, \emptyset\}$ are collections, whereas $\mathbb{F} = \{\mathcal{D}, \mathcal{E}\}$ is a family.

To a great extent, the distinction among sets, collections, and families depends on where one starts the analysis. For example, we define functions as sets of pairs $(x, f(x))$. We are often interested in the properties of different sets of functions. If the possible pairs are the “points,” then a set of functions is a family. However, if X is the set of all functions, then a set of functions is just that, a set. We have the set/collection/family hierarchy in place for cases in which we have to distinguish among several levels in the same context. The following anticipates material on probability theory, where we assign probabilities to every set in a **field** of sets.

Example 2.2.10 A **field** is a collection, $\mathcal{F}^\circ \subset \mathcal{P}(X)$, such that $\emptyset \in \mathcal{F}^\circ$, $[A \in \mathcal{F}^\circ] \Rightarrow [A^c \in \mathcal{F}^\circ]$ and $[A, B \in \mathcal{F}^\circ] \Rightarrow [(A \cup B \in \mathcal{F}^\circ) \wedge (A \cap B \in \mathcal{F}^\circ)]$. For any collection $\mathcal{E} \subset \mathcal{P}(X)$, let $\mathbb{F}^\circ(\mathcal{E})$ denote the family of all fields containing \mathcal{E} , that is, $\mathbb{F}^\circ(\mathcal{E}) = \{\mathcal{F}^\circ : \mathcal{E} \subset \mathcal{F}^\circ, \mathcal{F}^\circ \text{ a field}\}$. The **field generated by \mathcal{E}** is defined as $\mathcal{F}^\circ(\mathcal{E}) = \bigcap \{\mathcal{F}^\circ : \mathcal{F}^\circ \in \mathbb{F}^\circ(\mathcal{E})\}$. This is a sensible definition because the intersection of any family of fields gives another field. In Example 2.2.9, $\mathcal{F}^\circ(\mathcal{C}) = \{\emptyset, X, A, \{c\}\}$ and $\mathcal{F}^\circ(\mathcal{D}) = \mathcal{F}^\circ(\mathcal{E}) = \mathcal{P}(X)$.

The following are some of the most important sets we encounter in this book:

- $\mathbb{N} = \{1, 2, 3, \dots\}$, the natural or “counting” numbers.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the integers.
- $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, the nonnegative integers.
- $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$, the quotients, or rational numbers.
- \mathbb{R} , the set of “real numbers,” that we construct in Chapter 3 by adding the so-called irrational numbers to \mathbb{Q} .

Note that \mathbb{Q} contains all of the finite-length decimals, for example, $7.96518 = \frac{m}{n}$ for $m = 796,518$ and $n = 100,000$. This means that \mathbb{Q} contains a representation for every physical measurement that we can make and every number we will ever see from a computer. The reason for introducing the extra numbers in \mathbb{R} is not one of realism. Rather, we shall see that \mathbb{Q} has “holes” in it, and even though the holes are infinitely small, they make analyzing some kinds of problems miserably difficult.

Even though we have not yet formally developed the set of numbers \mathbb{R} , the following example is worth seeing early and often.

Example 2.2.11 (The Field of Half-Closed Intervals) *Let $X = \mathbb{R}$ and for $a, b \in X$, $a < b$, define $(a, b] = \{x \in X : a < x \leq b\}$. Set $\mathcal{E} = \{(a, b] : a < b, a, b \in \mathbb{R}\}$ and let $\mathcal{X} = \mathcal{F}^\circ(\mathcal{E})$. A set E belongs to \mathcal{X} iff it can be expressed as a finite union of disjoint intervals of one of the following three forms: $(a, b]$; $(-\infty, b] = \{x \in X : x \leq b\}$; or $(a, +\infty) = \{x \in X : a < x\}$.*

It is worth noting the style we used in this last example. When we write “define $(a, b] = \{x \in X : a < x \leq b\}$,” we mean that whenever we use the symbols to the left of the equality, “ $(a, b]$ ” in this case, we intend that you will understand these symbols to mean the symbols to the right of the equality, “ $\{x \in X : a < x \leq b\}$ ” in this case. The word “let” is used in exactly the same way.

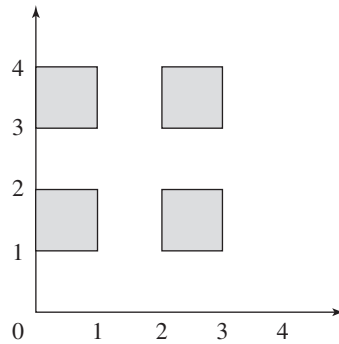
In a perfect world, we would take you through the construction of \mathbb{N} starting from the idea of the empty set. Had we done this construction properly, the following would be a result.

Axiom 1 *Every nonempty $S \subset \mathbb{N}$ contains a smallest element, that is, \leq is a well-ordering of \mathbb{N} .*

To be very explicit, we are assuming that if $S \in \mathcal{P}(\mathbb{N})$, $S \neq \emptyset$, then there exists $n \in S$ such that for all $m \in S$, $n \leq m$. There cannot be two such n , because $n \leq n'$ and $n' \leq n$ iff $n = n'$.

2.3 ♦ Products, Relations, Correspondences, and Functions

There is another way to construct new sets out of given ones, which involves the notion of an “ordered pair” of objects. In the set $\{a, b\}$, there is no preference

FIGURE 2.3.3 Cartesian product of $[0, 1] \cup [2, 3] \times [1, 2] \cup [3, 4]$.

given to a over b ; that is, $\{a, b\} = \{b, a\}$, so that it is an unordered pair.¹ We can also consider ordered pairs (a, b) , where we distinguish between the first and second elements.² Although what we mean when we use the phrase “we distinguish between the first and second elements” is very intuitive, it can be difficult to make it formal. One way is to say that “ (a, b) ” means the unordered pair of sets, “ $\{\{a\}, \{a, b\}\} = \{\{a, b\}, \{a\}\}$,” and we keep track of the order by noting that $\{a\} \subset \{a, b\}$, so that a comes before b . Throughout, A and B are any two sets, nonempty to avoid triviality.

Definition 2.3.1 The *product* or *Cartesian product* of A and B , denoted $A \times B$, is the set of all ordered pairs $\{(a, b) : a \in A \text{ and } b \in B\}$. The sets A and B are the *axes* of the product $A \times B$.

Example 2.3.2 $A = \{u, d\}$, $B = \{L, M, R\}$, $A \times B = \{(u, L), (u, M), (u, R), (d, L), (d, M), (d, R)\}$, and $A \times A = \{(u, u), (u, d), (d, u), (d, d)\}$.

Game theory is about the strategic interactions between people. If we analyze a two-person situation, we have to specify the options available to each. Suppose the first person’s options are the set $A = \{u, d\}$, mnemonically, *up* and *down*. Suppose the second person’s options are the set $B = \{L, M, R\}$, mnemonically *Left*, *Middle*, and *Right*. An equilibrium in a game is a vector of choices, that is, an ordered pair, some element of $A \times B$. A continuous example is as follows.

Example 2.3.3 $A = [0, 1] \cup [2, 3]$, $B = [1, 2] \cup [3, 4]$, $A \times B$ is the disjoint union of the four squares $[0, 1] \times [1, 2]$, $[0, 1] \times [3, 4]$, $[2, 3] \times [1, 2]$, and $[2, 3] \times [3, 4]$. [See Figure 2.3.3.]

In game theory with three or more players, a vector of choices belongs to a larger product space. In general equilibrium theory, an allocation is a list of the consumptions of the people in the economy. The set of all allocations is a product

1. The curly brackets, “{” and “}” will always enclose a set so that “ $\{a, b\}$ ” is the set containing elements a and b .

2. Hopefully, context will help you avoid confusing this notation with the interval consisting of all real numbers such that $a < x < b$.

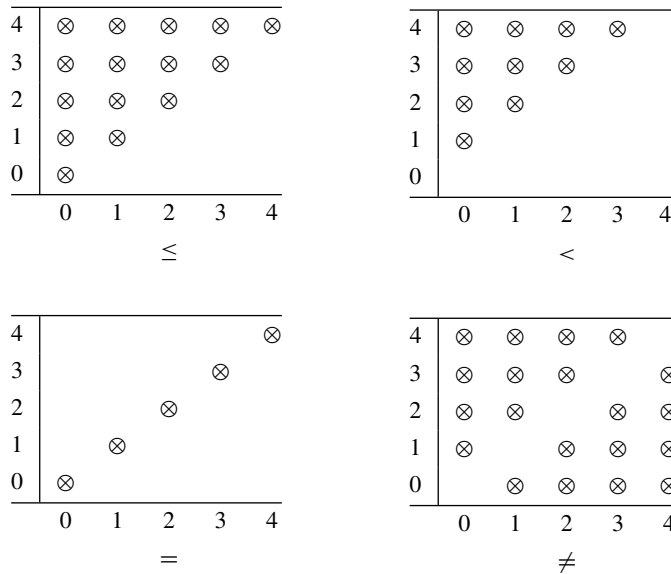
space, just a product of n spaces. The following is an example of an inductive definition.

Definition 2.3.4 Given a collection of sets, $\{A_m : m \in \mathbb{N}\}$, we define $\times_{m=1}^1 A_m = A_1$ and inductively define $\times_{m=1}^n A_m = \times_{m=1}^{n-1} A_m \times A_n$.

An ordered pair is called a **2-tuple** and an **n -tuple** is an element of $\times_{m=1}^n A_m$. Sets of 2-tuples are called binary relations and sets of n -tuples are called n -ary relations. Relations are the mathematical objects of interest.

Definition 2.3.5 Given two sets A and B , a **binary relation between A and B** , known simply as a **relation** if A and B can be inferred from context, is a subset $R \subset A \times B$. We use the notation $(a, b) \in R$ or aRb to denote the relation R holding for the ordered pair (a, b) and read it “ a is in the relation R to b .” If $R \subset A \times A$, we say that R is a **relation on A** . The **range of a relation R** is the set of $b \in B$ for which there exists $a \in A$ with $(a, b) \in R$.

Example 2.3.6 $A = \{0, 1, 2, 3, 4\}$, so that $A \times A$ has twenty-five elements. With the usual convention that x is on the horizontal axis and y on the vertical, the relations \leq , $<$, $=$, and \neq can be graphically represented by the \otimes 's in



Note that \leq , $<$, $=$, and \neq are sets. In terms of these sets, \leq is the union of the disjoint sets, $<$ and $=$, and the complement of $=$ is \neq .

Relations can also be used in all kinds of cute ways.

Example 2.3.7 Let $A = \{\text{Austin, Des Moines, Harrisburg}\}$ and $B = \{\text{Texas, Iowa, Pennsylvania}\}$. Then the relation $R = \{(\text{Austin, Texas}), (\text{Des Moines, Iowa}), (\text{Harrisburg, Pennsylvania})\}$ expresses “is the state capital of.”

A relation between A and B is a subset of $A \times B$. A function from A to B is a special kind of relation, and a correspondence from A to B is a way to view a relation as a function; that is, it is an alternate definition of a relation.

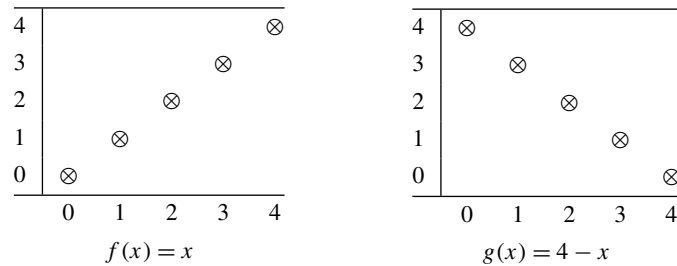
Definition 2.3.8 A **function (or mapping)** f , denoted $f : A \rightarrow B$, is a relation between A and B (i.e., $f \subset A \times B$) satisfying the following two properties:

1. for all $a \in A$, there exists $b \in B$ such that $(a, b) \in f$, and
2. if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.

For each $a \in A$, the unique b such that $(a, b) \in f$ is denoted $f(a)$. A function may be written as $a \mapsto f(a)$ (read “ a maps to $f(a)$ ”). The set A is called the **domain** of f , sometimes denoted $D(f)$. The **range** of f , denoted $\text{Range}(f)$ or $f(A)$, is $\{b \in B : \exists a \in A \text{ such that } (a, b) \in f\}$. The **graph** of f is $\text{Gr}(f) = \{(a, b) : (a, b) \in f\}$.

Verbally, for each a in A , f associates a unique b , denoted $b = f(a)$. A function f and its graph are one and the same. It is odd to distinguish verbally between a function and its graph, but we (daringly) do it anyway.³

Example 2.3.9 For $A = B = \{0, 1, 2, 3, 4\}$, the functions $f(x) = x$ and $g(x) = 4 - x$ can be represented by



Probabilities are an important example of functions.

Example 2.3.10 [↑Example 2.2.10 (p. 20)] A **probability** is a function, $P : \mathcal{F} \rightarrow [0, 1]$, from a field of sets, $\mathcal{F} \subset \mathcal{P}(X)$, to the interval $[0, 1]$ with the properties that $P(\emptyset) = 0$, $P(X) = 1$, and for disjoint $A, B \in \mathcal{F}$, $P(A \cup B) = P(A) + P(B)$. From these properties, we see that $P(A^c) = 1 - P(A)$ (since A^c and A are disjoint and their union is X), that $[A \subset B] \Rightarrow [P(B) = P(A) + P(B \setminus A)]$ (for the same sort of reason), and that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (since $A \cup B$ is the disjoint union of $A \setminus B$, $A \cap B$, and $B \setminus A$). Sometimes the number that a probability assigns to a set is called the **measure** of the set.

Example 2.3.11 [↑Example 2.2.11 (p. 21)] A **cumulative distribution function (cdf)** is a nondecreasing right-continuous function $F : \mathbb{R} \rightarrow [0, 1]$ that defines the probability, P_F , of an interval $(a, b]$ by $P_F((a, b]) = F(b) - F(a)$. We will see that P_F can be extended to the field of half-closed intervals and to the field of finite unions of disjoint intervals of all kinds: $(a, b]$, $(-\infty, b]$, $(a, +\infty)$, $[a, b) = \{x \in \mathbb{R} : a < x < b\}$, $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$; and (a, b) , $(-\infty, b)$, $[a, +\infty)$ defined analogously. We will also see that P_F can be extended to a much larger collection of sets.

3. Never let it be said that we lead dull lives of quiet desperation.

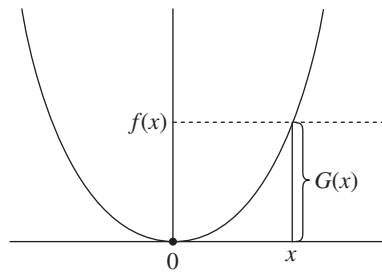


FIGURE 2.3.15

There are two equivalent ways to understand a correspondence from A to B : as a function from A to the subsets of B or as a subset of $A \times B$.

Definition 2.3.12 A **correspondence** G , denoted $G : A \twoheadrightarrow B$, is a relation between A and B . For each $a \in A$, the set of b such that $(a, b) \in G$ is denoted $G(a)$. Equivalently, G is a function from A to $\mathcal{P}(B)$ assigning a set, $G(a)$, to each element $a \in A$.

Exercise 2.3.13 For $A = B = [0, 1]$, draw three different correspondences from A to B that are not functions.

Exercise 2.3.14 Explicitly give the four relations in Example 2.3.6 (p. 23) as functions from A to $\mathcal{P}(A)$.

A correspondence G may have $G(a) = \emptyset$ or have $G(a)$ containing many elements. A function is a special kind of correspondence where for all a , $G(a)$ contains exactly one point.

Example 2.3.15 In Figure 2.3.15, you can see the graph of the function $f(x) = x^2$ and the correspondence $G(x) = [0, x^2]$, $f : \mathbb{R} \rightarrow \mathbb{R}$, and $G : \mathbb{R} \twoheadrightarrow \mathbb{R}$. $G(0)$ consists of one point; for $x \neq 0$, $G(x)$ is an interval.

Definition 2.3.16 Given a function $f : A \rightarrow B$ and $E \subset A$, $E \neq \emptyset$, the **image of E under f** is written $f(E)$ and is defined by $f(E) = \{b \in B : (\exists e \in E)[f(e) = b]\}$. The **restriction of f to E** is written $f|_E$ and is defined as the function $f : E \rightarrow B$ having as a graph the set $Gr(f|_E) = Gr(f) \cap (E \times B)$.

The image of a set E is the set of points to which it is mapped. The restriction of a function f to a set E ignores the behavior of the function outside of the set E .

Definition 2.3.17 A set X is **finite** if it is empty, in which case it has 0 elements, or if there exists $n \in \mathbb{N}$ and a function $f : \{1, \dots, n\} \rightarrow X$ such that $f(\{1, \dots, n\}) = X$. The smallest n with this property is called the **cardinality of X** and is denoted $\#X$.

This definition formalizes the idea that we would like to be able to count a finite set; that is, assign to each member of the set a number from $1, \dots, n$ for some n . By Axiom 1, the cardinality of a finite set is well defined.⁴

Example 2.3.18 *If $X = \{a, b\}$, $a \neq b$, then the function $f : \{1, \dots, 5\} \rightarrow X$ defined by $f(1) = f(2) = f(3) = a$ and $f(4) = f(5) = b$ shows that X is finite.*

The function f in this last example is a rather inefficient way to count the two-point set X . A function always takes a point to one single point. The given function f is **many-to-one**, that is, it takes many points to the same point. The following introduces the idea of a one-to-one function, and we return to it in more detail later.

Lemma 2.3.19 *If X is nonempty and finite and n is the cardinality of X , then there exists a function $f : \{1, \dots, n\} \rightarrow X$ such that $f(\{1, \dots, n\}) = X$, and for all $m \neq m'$, $m, m' \in \{1, \dots, n\}$, we have $f(m) \neq f(m')$.*

Proof. By the definition of cardinality, we know that n is the smallest natural number with the property that there exists an $f : \{1, \dots, n\} \rightarrow X$ with $f(\{1, \dots, n\}) = X$. Suppose that for some $m \neq m'$ in $\{1, \dots, n\}$, $f(m) = f(m')$. Consider the function $g : \{1, \dots, n-1\} \rightarrow X$ defined by $g(k) = f(k)$ for $k \in \{1, \dots, m-1\}$ and by $g(k) = f(k+1)$ for $k \in \{m, \dots, n-1\}$. Since $g(\{1, \dots, n-1\}) = X$, n was not the cardinality of X , a contradiction that completes the proof. ■

2.4 ♦ Equivalence Relations

As the name suggests, equivalence relations are relations of a special kind—a kind that appears frequently in the mathematics that economists use. The familiar equivalence classes from intermediate microeconomics are indifference curves, sets of consumption bundles that are all indifferent for the consumer, and isoprofit lines, sets of input-output vectors that yield the same profit for the producer. In game theory, one sees the strategic equivalence of strategies and the equivalence of games.

Definition 2.4.1 *An equivalence relation on a set A is a relation \sim that is:*

1. **reflexive**, $\forall a \in A, (a, a) \in \sim$,
2. **symmetric**, $\forall a, b \in A, [(a, b) \in \sim] \Leftrightarrow [(b, a) \in \sim]$, and
3. **transitive**, for all $a, b, c \in A, [(a, b) \in \sim] \wedge [(b, c) \in \sim] \Rightarrow [(a, c) \in \sim]$.

This is perhaps more intuitive with the aRb notation: $\sim \subset (A \times A)$ is an equivalence relation iff for all $a, b, c \in A$, $a \sim a$, $[a \sim b] \Leftrightarrow [b \sim a]$, and $[a \sim b \wedge b \sim c] \Rightarrow [a \sim c]$.

Example 2.4.2 *Equality is an equivalence relation on \mathbb{R} . If $u : X \rightarrow \mathbb{R}$ is a utility function representing preferences on a set X , then defining $x \sim y$ by $u(x) = u(y)$ gives the indifference equivalence relation.*

4. This is a fancy way of saying that our definition makes sense—that if a set is finite, then the cardinality of that set exists.

Example 2.4.3 Define the **congruence modulo 4** relation M_4 on \mathbb{Z} by $\forall x, y \in \mathbb{Z}, xM_4y$ if remainders obtained by dividing x and y by 4 are equal. For example, $13M_465$ because dividing 13 and 65 by 4 gives a remainder of 1.

Exercise 2.4.4 Show that congruence modulo 4 is an equivalence relation.

Definition 2.4.5 Given an equivalence relation \sim on a set A and an element $x \in A$, we define the **equivalence class determined by x** by $E_x = \{y \in A : y \sim x\}$. Note that $x \in E_x$ since $x \sim x$.

Example 2.4.6 The equivalence classes of \mathbb{Z} for the relation M_4 are determined by $x \in \{0, 1, 2, 3\}$, where $E_x = \{z \in \mathbb{Z} : \exists k \in \mathbb{Z}, z = 4k + x\}$, that is, x is the remainder when z is divided by 4.

Equivalence classes have the following property.

Theorem 2.4.7 Two equivalence classes E and E' are either disjoint or equal.

Proof. Let $E = \{y \in A : y \sim x\}$ and $E' = \{y \in A : y \sim x'\}$. If $E \cap E' = \emptyset$, then E and E' are disjoint. If $\exists z \in E \cap E'$, we show that $E = E'$. The first step is to demonstrate that $E \subset E'$, and the second step is to show that $E' \subset E$.

Let $w \in E$. We must show that $w \in E'$. Since $w \in E$, $w \sim x$. As $z \in E \cap E'$, we know that $z \sim x$ and $z \sim x'$. By transitivity $w \sim z$; hence $w \sim x'$, so that $w \in E'$.

Reversing the roles of E and E' in this argument demonstrates that $E' \subset E$. ■

Looking at Example 2.4.6 in light of Theorem 2.4.7, we see that if two elements are in relation, they have the same equivalence class. So $E_1 = E_5 = E_9 = \dots$ and $E_2 = E_6 = E_{10} = \dots$. More generally, for all $n, k \in \mathbb{Z}$, $E_n = E_{4k+n}$.

Notation 2.4.8 A/\sim denotes the collection of all \sim -equivalence classes.

Mnemonicly, \sim divides A into a collection of disjoint sets, so we write A/\sim . The union of all the sets in A/\sim equals all of A because every element a of A belongs to exactly one of the equivalence classes. Another way to understand A/\sim is as a partition of A .

Definition 2.4.9 A **partition** of a set A is a collection of nonempty disjoint subsets of A whose union is all of A .

We saw in Theorem 2.4.7 that equivalence relations give rise to partitions. The reverse is also true.

Exercise 2.4.10 For a partition, \mathcal{C} , of A , define $\sim_{\mathcal{C}}$ by $x \sim_{\mathcal{C}} y$ iff x and y belong to same element of \mathcal{C} . Show that $\sim_{\mathcal{C}}$ is an equivalence relation.

Example 2.4.11 The equivalence classes of \mathbb{Z} in Example 2.4.6 constitute a partition; $E_0 = \{\dots, -8, -4, 0, 4, 8, \dots\}$, $E_1 = \{\dots, -7, -3, 1, 5, \dots\}$, $E_2 = \{\dots, -6, -2, 2, 6, \dots\}$, and $E_3 = \{\dots, -5, -1, 3, 7, \dots\}$ are disjoint and their union is all of \mathbb{Z} . Generally, \mathbb{Z} can be partitioned to n subsets via the equivalence relation $x \sim y$ iff x, y have the same remainder after division by n . The partitioning sets contain those subsets having remainders $0, 1, \dots, n - 1$ to n . Another simple example is a coin toss experiment where the sample space $S = \{\text{Heads}, \text{Tails}\}$ has mutually exclusive events (i.e., $\text{Heads} \cap \text{Tails} = \emptyset$).

Example 2.4.12 Consider the relation \sim on \mathbb{R} given by $x \sim y$ iff $x - y \in \mathbb{Z}$. It can be easily checked that this is an equivalence relation. The equivalence of an arbitrary $x \in \mathbb{R}$ looks like $x + \mathbb{Z} = \{x + n : n \in \mathbb{Z}\}$. For all $n \in \mathbb{Z}$, x and $x + n$ are in the same equivalence class. Since for each x , there exists an $n \in \mathbb{Z}$ such that $n \leq x < n + 1$, x is in the same equivalence class as $x - n$, which we denote by (x) , where $x - n \in [0, 1)$. Thus for each x , (x) is a representation of the equivalence class of x . Note that if $x, y \in [0, 1)$, then $x - y \notin \mathbb{Z}$, so $x \not\sim y$.

What does the quotient space \mathbb{R}/\sim look like? This space consists of equivalence classes of \sim . By the above argument, we can make each member of $[0, 1)$ correspond to exactly one equivalence class of \sim . That is, we can think of $[0, 1)$ as \mathbb{R}/\sim .

Chapter 3 develops the real numbers, \mathbb{R} , as a collection of equivalence classes of sequences of elements of \mathbb{Q} .

2.5 ♦ Optimal Choice for Finite Sets

For all of this section, the set of options, X , is assumed to be finite.

Preference relations on a set of choices are at the core of economic theory. A decision maker's preferences are encoded in a preference relation, R , and “ a is in the relation R to b ” is interpreted as “ a is at least as good as b .” It is important to keep clear that the preference relation is assumed to be a property of the individual—your R is different than mine.

The two results in this section, Theorems 2.5.11 and 2.5.17, are the foundational results in the theory of *rational choice*:

- Theorem 2.5.11 shows that utility maximization is equivalent to preference maximization for complete and transitive preferences. This means that assuming that someone has a utility function and maximizes it is the same as assuming that the person can sensibly rank all of her options, perhaps allowing ties, and picks the option that she likes best or picks among the set of options that she likes best.
- Theorem 2.5.17 shows that preference maximizing behavior is equivalent to following a choice rule satisfying a minimal consistency condition called the weak axiom of revealed preference.

Theorem 2.5.14 shows that rational choice theory is not a mathematically empty one, and Theorem 2.5.15 gives the most basic comparative result for choice sets.

2.5.a The Basics

Let X be a finite set of options. We want to define the properties a relation \succsim on X should have in order to represent preferences that are rational. Remember that we write “ xRy ” for “ $(x, y) \in R$.”

Definition 2.5.1 A relation R on X is **complete** if for all $x, y \in X$, xRy or yRx ; it is **transitive** if for all $x, y, z \in X$, $[[xRy] \wedge [yRz]] \Rightarrow [xRz]$; and it is **rational** if it is both complete and transitive.

Example 2.5.2 *One of the crucial order properties of the set of numbers, \mathbb{R} , is that \leq and \geq are complete and transitive.*

Completeness neither implies nor is implied by transitivity. To see this, the following exercise gives an example of a relation that satisfies both completeness and transitivity, gives other relations that satisfy one of the conditions but not the other, and gives a relation that satisfies neither. When you see a new concept, you should develop the two habits that this exercise exemplifies: finding examples in which the new concept does and does not hold and finding examples that demonstrate how the new concept interacts with other, possibly related concepts.

Exercise 2.5.3 In Example 2.3.6 (p. 23), show that \leq is complete and transitive, that $<$ and $=$ are transitive but not complete, and that \neq is neither transitive nor complete. Check that the relation \succsim given later in Example 2.5.6 is complete but not transitive.

In thinking about preference relations, completeness is the requirement that any pair of choices can be compared for the purposes of making a choice. Given how much effort it is to make life decisions (jobs, marriage, kids), completeness is a strong requirement. When a relation is not complete, there are choices that cannot be compared and there may be two or more optimal choices in the set. For example, consider the relation \subset on the set of all subsets of $A = \{1, \dots, 10\}$ except A itself. Suppose we are looking for the largest subset. Then each of the subsets with nine elements is a largest element and they cannot be compared with each other. Transitivity is another rationality requirement. If violated, vicious cycles could arise among three or more options—any choice would have another that strictly beats it. To say “strictly beats” we need the following.

Definition 2.5.4 *Given a relation \succsim , define $x \succ y$ by $[x \succsim y] \wedge \neg[y \succsim x]$ and $x \sim y$ by $[x \succsim y] \wedge [y \succsim x]$.*

When talking about preference relations, “ $x \succ y$ ” is read as “ x is strictly preferred to y ” and “ $x \sim y$ ” is read as “ x is indifferent to y .” From the definitions, you can show that $[x \succsim y] \Leftrightarrow [[x \succ y] \vee [x \sim y]]$, and that the sets \succ and \sim are disjoint.

Exercise 2.5.5 Show that $x \sim y$ is an equivalence relation if \succsim is rational.

Example 2.5.6 *Suppose you are at a restaurant and you have a choice among four meals, pork, beef, chicken, or fish, all costing the same. Suppose that your preferences, \succsim , and strict preferences, \succ , are given by*

pork	⊗			⊗	pork	⊗			
beef			⊗	⊗	beef			⊗	
fish		⊗	⊗	⊗	fish		⊗	⊗	
chic	⊗	⊗	⊗		chic	⊗	⊗		
	chic	fish	beef	pork		chic	fish	beef	pork

The basic behavioral assumption in economics is that you choose the option that you like best. Here $p \succ b \succ f \succ c \succ p$. Suppose you try to find your favorite

meal. Start by thinking about (say) c , discover you like f better so you switch your decision to f , but you like b better, so you switch again, but you like p better so you switch again, but you like c better so you switch again, coming back to where you started. You become confused and starve to death before you make up your mind.

Exercise 2.5.7 Give the graphical representation \succsim , \succ , and \sim for the complete transitive preferences satisfying $c \succ f \sim b \succ p$.

Exercise 2.5.8 Give the relation \sim associated with the preferences given in Example 2.5.6. Is \sim an equivalence relation? Can it reasonably be interpreted as indifference?

2.5.b Representing Preferences

Definition 2.5.9 A utility function $u : X \rightarrow \mathbb{R}$ **represents** \succsim if $[x \succ y] \Leftrightarrow [u(x) > u(y)]$ and $[x \sim y] \Leftrightarrow [u(x) = u(y)]$.

Since u is a function, it assigns a numerical value to every point in X . Since we can compare any pair of numbers using \geq , any preference represented by a utility function is complete. As \geq is transitive on \mathbb{R} , any preference represented by a utility function is transitive.

Exercise 2.5.10 Show that u represents \succsim iff $[x \succsim y] \Leftrightarrow [u(x) \geq u(y)]$.

Theorem 2.5.11 The relation \succsim is rational iff there exists a utility function $u : X \rightarrow \mathbb{R}$ that represents \succsim .

Since X is finite, we can replace \mathbb{R} by \mathbb{N} or by some set $\{1, \dots, n\}$ in this result.

Proof. Suppose that \succsim is rational. We must show that there exists a utility function $u : X \rightarrow \mathbb{N}$ that represents \succsim . Let $W(x) = \{y \in X : x \succsim y\}$; this is the set of options that are weakly worse than x . A candidate utility function is $u(x) = \#W(x)$. By transitivity, $[x \succsim y] \Rightarrow [W(y) \subset W(x)]$. By completeness, either $W(x) \subset W(y)$ or $W(y) \subset W(x)$, and $W(x) = W(y)$ if $x \sim y$. Also, $[x \succ y]$ implies that $W(y)$ is a proper subset of $W(x)$. When we combine, if $x \succ y$, then $u(x) > u(y)$, and if $x \sim y$, then $W(x) = W(y)$, so that $u(x) = u(y)$.

Now suppose that $u : X \rightarrow \mathbb{R}$ represents \succsim . We must show that \succsim is complete and transitive. For $x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ (or both). By the definition of representing, $x \succsim y$ or $y \succsim x$. Suppose now that $x, y, z \in X$, $x \succsim y$, and $y \succsim z$. We must show that $x \succsim z$. We know that $u(x) \geq u(y)$ and $u(y) \geq u(z)$. This implies that $u(x) \geq u(z)$, so that $x \succsim z$. ■

The mapping $x \mapsto W(x)$ in the proof is yet another example of a correspondence, in this case from X to X . We now define the main correspondence used in rational choice theory.

Definition 2.5.12 A **choice rule** is a function $C : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, equivalently a correspondence from $\mathcal{P}(X)$ to X , such that $C(B) \subset B$ for all $B \in \mathcal{P}(X)$, and $C(B) \neq \emptyset$ if $B \neq \emptyset$.

The interpretation is that $C(B)$ is the set of options that might be chosen from the menu B of options. The best-known class of choice rules is made up of those

of the form $C^*(B) = C^*(B, \succsim) = \{x \in B : \forall y \in B, x \succsim y\}$. In light of Theorem 2.5.11, $C^*(B) = \{x \in B : \forall y \in B, u(x) \geq u(y)\}$, that is, $C^*(B)$ is the set of utility maximizing elements of B .

The set of maximizers, the **argmax**, is a sufficiently important construct in economics that it has its own notation.

Definition 2.5.13 For a nonempty set X and function $f : X \rightarrow \mathbb{R}$, $\arg \max_{x \in X} f(x)$ is the set $\{x^* \in X : (\forall x \in X)[f(x^*) \geq f(x)]\}$.

The basic existence result tells us that the preference-maximizing choice rule yields a nonempty set of choices.

Theorem 2.5.14 If B is a nonempty finite subset of X and \succsim is a rational preference relation on X , then $C^*(B) \neq \emptyset$.

Proof. Define $S^* = \bigcap_{x \in B} \{y \in B : y \succsim x\}$. It is clear that $S^* = C^*(B)$ (and you should check both directions of the inclusion if you are not used to writing proofs). All that is left is to show that $S^* \neq \emptyset$.

Let $n_B = \#B$ and pick a function $f : \{1, \dots, n_B\} \rightarrow B$ such that $B = f(\{1, \dots, n_B\})$. This means we order (or count) members of B as $f(1), \dots, f(n_B)$. For $m \in \{1, \dots, n_B\}$, let $S^*(m) = \{y \in B : \forall n \leq m, y \succsim f(n)\}$, so that $S^* = S^*(n_B)$. In other words, $S^*(m)$ contains the best elements between $f(1), \dots, f(m)$ with respect to \succsim . Now using a function f^* , we inductively pick up the largest element among $f(1), \dots, f(n)$ for all n . Define $f^*(1) = f(1)$. Given that $f^*(m-1)$ has been defined, define

$$f^*(m) = \begin{cases} f(m) & \text{if } f(m) \succsim f^*(m-1), \\ f^*(m-1) & \text{if } f^*(m-1) \succ f(m). \end{cases} \quad (2.1)$$

For each $m \in \{1, \dots, n_B\}$, $S^*(m) \neq \emptyset$ because it contains $f^*(m)$, and by transitivity, $f^*(n_B) \in S^*$. ■

The idea of the proof was simply to label the members of the finite set B and check its members step by step. We simply formalized this idea using logical tools and the definition of finiteness.

For $R, S \subset X$, we write $R \succsim S$ if $x \succsim y$ for all $x \in R$ and $y \in S$, and $R \succ S$ if $x \succ y$ for all $x \in R$ and $y \in S$. The basic comparison result for choice theory is that larger sets of options are at least weakly better.

Theorem 2.5.15 If $A \subset B$ are nonempty finite subsets of X and \succsim is a rational preference relation on X , then

1. $[x, y \in C^*(A)] \Rightarrow [x \sim y]$, optima are indifferent,
2. $C^*(B) \succsim C^*(A)$, larger sets are at least weakly better, and
3. $[C^*(B) \cap C^*(A) = \emptyset] \Rightarrow [C^*(B) \succ C^*(A)]$, a larger set is strictly better if it has a disjoint set of optima.

Proof. The proof of (1) combines two proof strategies: contradiction and splitting into cases. Suppose that $[x, y \in C^*(A)]$ but $\neg[x \sim y]$. We split the statement $\neg[x \sim y]$ into two cases, $[\neg[x \sim y]] \Leftrightarrow [[x \succ y] \vee [y \succ x]]$. If $x \succ y$, then $y \notin C^*(A)$, a contradiction. If $y \succ x$, then $x \notin C^*(A)$, a contradiction.

To prove (2), we must show that $[[x \in C^*(B)] \wedge [y \in C^*(A)]] \Rightarrow [x \succsim y]$. We again give a proof by contradiction. Suppose that $[x \in C^*(B)] \wedge [y \in C^*(A)]$ but $\neg[x \succsim y]$. Since \succsim is complete, $\neg[x \succsim y] \Rightarrow [y \succ x]$. As $y \in A$ and $A \subset B$, we know that $y \in B$. Therefore, $[y \succ x]$ contradicts $x \in C^*(A)$.

In what is becoming a pattern, we also prove (3) by contradiction. Suppose that $[C^*(B) \cap C^*(A) = \emptyset]$ but $\neg[C^*(B) \succ C^*(A)]$. By the definition of $R \succ S$ and the completeness of \succsim , $\neg[C^*(B) \succ C^*(A)]$ implies that there exists $y \in C^*(A)$ and $x \in C^*(B)$ such that $y \succsim x$. By (1), this implies that $y \in C^*(B, \succsim)$, which contradicts $[C^*(B) \cap C^*(A) = \emptyset]$. ■

2.5.c Revealed Preference

We now approach the choice problem starting with a choice rule rather than with a preference relation. The question is whether there is anything new or different when we proceed in this direction. The answer is “No, provided the choice rule satisfies a minimal consistency requirement, and satisfying this minimal consistency requirement reveals a preference relation.”

A choice rule C defines a relation, \succsim^* , “revealed preferred,” defined by $x \succsim^* y$ if $(\exists B \in \mathcal{P}(X))[[x, y \in B] \wedge [x \in C(B)]]$. Note that $\neg[x \succsim^* y]$ is $(\forall B \in \mathcal{P}(X))[\neg[x, y \in B] \vee \neg[x \in C(B)]]$, equivalently $(\forall B \in \mathcal{P}(X))[[x \in C(B)] \Rightarrow [y \notin B]]$. In words, x is revealed preferred to y if there is a choice situation, B , in which both x and y are available, and x belongs to the choice set.

From the relation \succsim^* we define “revealed strictly preferred,” \succ^* , as in Definition 2.5.4 (p. 29). It is both a useful exercise in manipulating logic and a good way to understand a piece of choice theory to explicitly write out two versions of the meaning of $x \succ^* y$:

$$\begin{aligned} & (\exists B_x \in \mathcal{P}(X))[[x, y \in B_x] \wedge [x \in C(B_x)]] \\ & \wedge (\forall B \in \mathcal{P}(X))[[y \in C(B)] \Rightarrow [x \notin B]], \end{aligned} \quad (2.2)$$

equivalently

$$\begin{aligned} & (\exists B_x \in \mathcal{P}(X))[[x, y \in B_x] \wedge [x \in C(B_x)] \wedge [y \notin C(B_x)]] \\ & \wedge (\forall B \in \mathcal{P}(X))[[y \in C(B)] \Rightarrow [x \notin B]]. \end{aligned}$$

In words, the latter of these says that there is a choice situation where x and y are both available, x is chosen but y is not, and if y is ever chosen, then we know that x was not available.

A set $B \in \mathcal{P}(X)$ reveals a strict preference of y over x , written $y \succ_B x$, if $x, y \in B$ and $y \in C(B)$ but $x \notin C(B)$.

Definition 2.5.16 *A choice rule satisfies the **weak axiom of revealed preference** if $[x \succsim^* y] \Rightarrow \neg(\exists B)[y \succ_B x]$.*

This is the minimal consistency requirement. Satisfying this requirement means that choosing x when y is available in one situation is not consistent with choosing y but not x in some other situation where they are both available.

Theorem 2.5.17 *If C is a choice rule satisfying the weak axiom, then \succsim^* is rational, and for all $B \in \mathcal{P}(X)$, $C(B) = C^*(B, \succsim^*)$. If \succsim is rational, then $B \mapsto C^*(B, \succsim)$ satisfies the weak axiom, and $\succsim = \succsim^*$.*

Proof. Suppose that C is a choice rule satisfying the weak axiom.

We must first show that \succsim^* is complete and transitive.

Completeness: For all $x, y \in X$, $\{x, y\} \in \mathcal{P}(X)$ is a nonempty set. Therefore $C(\{x, y\}) \neq \emptyset$, so that $x \succsim^* y$ or $y \succsim^* x$.

Transitivity: Suppose that $x \succsim^* y$ and $y \succsim^* z$. We must show that $x \succsim^* z$. To do this, it is sufficient to demonstrate that $x \in C(\{x, y, z\})$. Since $C(\{x, y, z\})$ is a nonempty subset of $\{x, y, z\}$, we know that there are three cases: $x \in C(\{x, y, z\})$, $y \in C(\{x, y, z\})$, and $z \in C(\{x, y, z\})$. We must show that each of these cases leads to the conclusion that $x \in C(\{x, y, z\})$.

Case 1: This one is clear.

Case 2: $y \in C(\{x, y, z\})$, the weak axiom, and $x \succsim^* y$ implies that $x \in C(\{x, y, z\})$.

Case 3: $z \in C(\{x, y, z\})$, the weak axiom, and $y \succsim^* z$ implies that $y \in C(\{x, y, z\})$. As we just saw in Case 2, this means that $x \in C(\{x, y, z\})$.

We now show that for all $B \in \mathcal{P}(X)$, $C(B) = C^*(B, \succsim^*)$. Pick an arbitrary $B \in \mathcal{P}(X)$. It is sufficient to establish that $C(B) \subset C^*(B, \succsim^*)$ and $C^*(B, \succsim^*) \subset C(B)$.

Pick an arbitrary $x \in C(B)$. By the definition of \succsim^* , for all $y \in B$, $x \succsim^* y$. By the definition of $C^*(\cdot, \cdot)$, this means that $x \in C^*(B, \succsim^*)$.

Now pick an arbitrary $x \in C^*(B, \succsim^*)$. By the definition of $C^*(\cdot, \cdot)$, this means that $x \succsim^* y$ for all $y \in B$. By the definition of \succsim^* , for each $y \in B$, there is a set B_y such that $x, y \in B_y$ and $x \in C(B_y)$. As C satisfies the weak axiom, for all $y \in B$, there is no set B_y with the property that $y \succ_{B_y} x$. Since $C(B) \neq \emptyset$, if $x \notin C(B)$, then we would have $y \succ_B x$ for some $y \in B$, a contradiction. ■

Exercise 2.5.18 What is left to be proved in Theorem 2.5.17? Provide the missing step(s).

It is important to note the reach and the limitation of Theorem 2.5.17.

Reach: First, we did not use X being finite at any point in the proof, so it applies to infinite sets. Second, the proof would go through so long as C is defined on all two- and three-point sets. This means that we can replace $\mathcal{P}(X)$ with a family of sets \mathcal{B} throughout, provided \mathcal{B} contains all two- and three-point sets.

Limitation: In many of the economic situations of interest, the two- and three-point sets are not the ones that people are choosing from. For example, the leading case has \mathcal{B} as the class of Walrasian budget sets.

2.6 ♦ Direct and Inverse Images, Compositions

Projections map products to their axes in a natural way: $\text{proj}_A : A \times B \rightarrow A$ is defined by $\text{proj}_A((a, b)) = a$; and $\text{proj}_B : A \times B \rightarrow B$ is defined by $\text{proj}_B((a, b)) = b$. The projections of a set $S \subset A \times B$ are defined by $\text{proj}_A(S) = \{a : \exists b \in B, (a, b) \in S\}$.

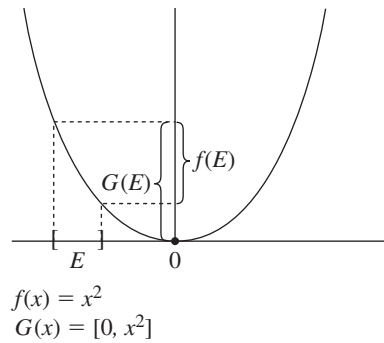


FIGURE 2.6.2

S) and $\text{proj}_B(S) = \{b : \exists a \in A, (a, b) \in S\}$. We use projections here to define direct and inverse images, we use them later in our study of sequence spaces and vector spaces.

2.6.a Direct Images

If $f : A \rightarrow B$ is a function and $E \subset A$, then $f(E) = \{f(a) \in B : a \in E\}$. We now extend this to correspondences/relations.

Definition 2.6.1 Let R be a relation from A to B . If $E \subset A$, then the (**direct image of E under the relation R**), denoted $R(E)$, is the set $\text{proj}_B(R \cap (E \times B))$.

That is, for a relation R that consists of some ordered pairs, $R(E)$ contains the second component of all ordered pairs whose first component comes from E .

Exercise 2.6.2 Show that if $E \subset A$ and f is a function mapping A to B , then $f(E) = \cup_{a \in E} \{f(a)\}$, and if G is a correspondence mapping A to B , then $G(E) = \cup_{a \in E} G(a)$. [See Figure 2.6.2.]

Exercise 2.6.3 Consider functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$ and $g(x) = x^3$. Clearly $f(\mathbb{R})$ is contained in the positive real numbers. For every $r \geq 0$, $f(\sqrt{r}) = f(-\sqrt{r}) = r$, which implies that $f(\mathbb{R}) = f(\mathbb{R}_+) = \mathbb{R}_+$. Also $g(\sqrt[3]{r}) = r$ shows that all real numbers appear in the range of g , that is, $g(\mathbb{R}) = \mathbb{R}$.

Theorem 2.6.4 Let f be a function from A to B and let $E, F \subset A$:

1. If $E \subset F$, then $f(E) \subset f(F)$,
2. $f(E \cap F) \subset f(E) \cap f(F)$,
3. $f(E \cup F) = f(E) \cup f(F)$,
4. $f(E \setminus F) \subset f(E)$, and
5. $f(E \Delta F) \subset f(E) \Delta f(F)$.

Exercise 2.6.5 Prove Theorem 2.6.4 and give examples in which subset relations are proper.

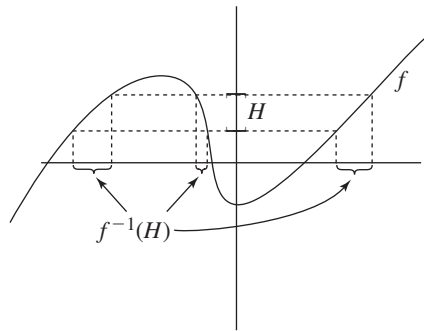


FIGURE 2.6.10

Exercise 2.6.6 Find and prove the analogue of Theorem 2.6.4 when the function f is replaced with a correspondence G , giving examples with the subset relations being proper.

2.6.b Inverse Relations

Inverse relations simply reverse the order in which we consider the axes.

Definition 2.6.7 Given a relation R between A and B , the **inverse of R** is the relation R^{-1} between B and A defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$. Images of sets under R^{-1} are called **inverse images**.

The inverse of a function need not be a function, though it will always be a correspondence.

Example 2.6.8 In general, functions are many-to-one. For example $f(x) = x^2$ from \mathbb{R} to \mathbb{R} maps both $+\sqrt{r}$ and $-\sqrt{r}$ to r when $r \geq 0$. In this case, the relation f^{-1} , viewed as a correspondence maps every nonnegative r to $\{-\sqrt{r}, +\sqrt{r}\}$, and maps every negative r to \emptyset .

Example 2.6.9 Let W be a finite set (of workers) and F a finite set (of firms). A function μ mapping W to $F \cup W$ is a **matching** if for all w , $[\mu(w) \in F] \vee [\mu(w) = w]$. We interpret $\mu(w) = w$ as the worker w being self-employed or unemployed. For $f \in F$, $\mu^{-1}(f) \subset W$ is the set of people who work at firm f .

It is well worth the effort to be even more specific for functions.

Definition 2.6.10 If f is a function from A to B and $H \subset B$, then the **inverse image of H under f** , denoted $f^{-1}(H)$, is the subset $\{a \in A \mid f(a) \in H\}$. [See Figure 2.6.10.]

When $H = \{b\}$ is a one-point set, we write $f^{-1}(b)$ instead of $f^{-1}(\{b\})$.

Exercise 2.6.11 Just to be sure that the notation is clear, prove the following and illustrate the results with pictures: $f^{-1}(H) = \cup_{b \in H} f^{-1}(b)$, $\text{proj}_B^{-1}(H) = A \times H$, and $\text{proj}_A^{-1}(E) = E \times B$.

Exercise 2.6.12 (Level Sets of Functions) Let $f : A \rightarrow B$ be a function. Define $a \sim_f a'$ if $\exists b \in B$ such that $a, a' \in f^{-1}(b)$. These equivalence classes are called **level sets** of the function f .

1. Show that \sim_f is an equivalence relation on A .
2. Show that $[a \sim_f a'] \Leftrightarrow [f(a) = f(a')]$.
3. Give an example with f, g being different functions from A to B but $\sim_f = \sim_g$.
4. Prove that the inverse images $f^{-1}(b)$ and $f^{-1}(b')$ are disjoint when $b \neq b'$. [This means that indifference curves never intersect.]

We return to inverse images under correspondences later. Since there are two ways to view them, as relations from A to B and as functions from A to $\mathcal{P}(B)$, there are two immediate possibilities for the definition of G inverse. It turns out that there is also a third possibility.

Inverse images under functions preserve the set operations, unions, intersections, and differences. As seen in Exercise 2.6.5, images need not have this property.

Theorem 2.6.13 Let f be a function mapping A to B , and let $G, H \subset B$:

1. if $G \subset H$, then $f^{-1}(G) \subset f^{-1}(H)$,
2. $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$,
3. $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$, and
4. $f^{-1}(G \setminus H) = f^{-1}(G) \setminus f^{-1}(H)$.

Proof. (1) If $a \in f^{-1}(G)$, then $f(a) \in G \subset H$ so $a \in f^{-1}(H)$. ■

Exercise 2.6.14 Finish the proof of Theorem 2.6.13.

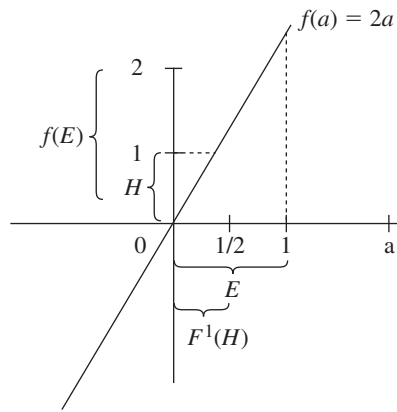
2.6.c Injections, Surjections, and Bijections

In general, functions are many-to-one. In Example 2.6.8 (p. 35), $f(x) = x^2$ maps both $+\sqrt{r}$ and $-\sqrt{r}$ to r . In Example 2.6.9 (p. 35), many workers may be matched to a single firm. When functions are not many-to-one but one-to-one, they have nice additional properties that flow from their inverses almost being functions.

Definition 2.6.15 $f : A \rightarrow B$ is **one-to-one** or an **injection** if $[f(a) = f(a')] \Rightarrow [a = a']$.

Since $[a = a'] \Rightarrow [f(a) = f(a')]$, being one-to-one is equivalent to $[f(a) = f(a')] \Leftrightarrow [a = a']$.

Recall the correspondence $G(b) = f^{-1}(b)$ from B to A introduced earlier. When f is many-to-one, then for some b , the correspondence $G(b)$ contains more than one point. When a correspondence always contains exactly one point, it is a function. Hence, the inverse of a one-to-one function is a function from the range of f to A . That is, the inverse of an injection $f : A \rightarrow B$ fails to be a function from B to A only in that it may not be defined for all of B .

FIGURE 2.6.18 $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(a) = 2a$.

Example 2.6.16 Let $E = \{2, 4, 6, \dots\}$ be the set of even natural numbers and define $f(n) = 2n$. Then f is one-to-one, $f(\mathbb{N}) = E$, and f^{-1} is a function from E to \mathbb{N} .

Definition 2.6.17 If $\text{Range}(f) = B$, f maps A onto B and we call f a **surjection**; f is a **bijection** if it is one-to-one and onto, that is, if it is both an injection and a surjection, in which case we write $f : A \leftrightarrow B$.

Note that surjectiveness of a map depends on the set into which the map is defined. For example, if we consider $f(x) = x^2$ as $f : \mathbb{R} \rightarrow \mathbb{R}_+$ then it is onto, whereas the same function viewed as $f : \mathbb{R} \rightarrow \mathbb{R}$ is not onto.

To summarize:

- injections are one-to-one and map A into B but may not cover all of B ;
- surjections put A all over B but may not be one-to-one; and
- bijections from A to B are one-to-one onto functions, which means that their inverse correspondences are functions from B to A .

Example 2.6.18 Let $E = [0, 1] \subset A = \mathbb{R}$, $H = [0, 1] \subset B = \mathbb{R}$, and $f(a) = 2a$. $\text{Range}(f) = \mathbb{R}$ so that f is a surjection; the image set is $f(E) = [0, 2]$; the inverse image set is $f^{-1}(H) = [0, \frac{1}{2}]$; and f is an injection, has inverse $f^{-1}(b) = \frac{1}{2}b$, and as a consequence of being one-to-one and onto is a bijection. [See Figure 2.6.18.]

Exercise 2.6.19 Show that if $f : A \leftrightarrow B$ is a bijection between A and B , then the subset relations in Theorem 2.6.4 (p. 34) hold with equality.

2.6.d Compositions of Functions

If we first apply f to an $a \in A$ to get $b = f(a)$, and then apply g to b , we have a new, composite function, $h(a) = g(f(a))$.

Definition 2.6.20 Let $f : A \rightarrow B$ and $g : B' \rightarrow C$, with $B' \subset B$ and $\text{Range}(f) \subset B'$. The **composition** $g \circ f$ is the function from A to C given by $g \circ f = \{(a, c) \in A \times C : (\exists b \in \text{Range}(f))[(a, b) \in f] \wedge [(b, c) \in g]\}$.

The order matters greatly here. If $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and $g : \mathbb{R}^3 \rightarrow \mathbb{R}$, then $h(x) = g(f(x))$ is perfectly well defined for $x \in \mathbb{R}^2$, but “ $f(g(y))$ ” is pure nonsense, since the domain of f is \mathbb{R}^2 , not \mathbb{R} . In matrix algebra, this corresponds to matrices having to be conformable in order for multiplication to be defined. However, even in conformable cases, order matters.

Example 2.6.21 $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

Here is a nonlinear example of the order mattering.

Example 2.6.22 Let $A \subset \mathbb{R}$, $f(a) = 2a$, and $g(a) = 3a^2 - 1$. Then $g \circ f = 3(2a)^2 - 1 = 12a^2 - 1$, whereas $f \circ g = 2(3a^2 - 1) = 6a^2 - 2$.

Compositions preserve surjectiveness and injectiveness.

Theorem 2.6.23 If $f : A \rightarrow B$ and $g : B \rightarrow C$ are surjections (injections), then their composition $g \circ f$ is a surjection (injection).

Exercise 2.6.24 Prove Theorem 2.6.23.⁵

Example 2.6.25 If $f(x) = x^2$ and $g(x) = x^3$, then g is a bijection between \mathbb{R} and itself, whereas $g \circ f$ is not even a surjection.

Theorem 2.6.26 Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$. Then

1. if $g \circ f$ is onto, then g is onto, and
2. if $g \circ f$ is one-to-one, then f is one-to-one.

Proof. For the first part, we show that if g is not onto, then $g \circ f$ cannot be onto. The function g is not onto iff $g(B) \subsetneq C$. Since $f(A) \subset B$, this implies that $g(f(A)) \subsetneq C$.

In a similar fashion, f is not one-to-one iff $(\exists a \neq a')[f(a) = f(a')]$, which implies that $(g \circ f)(a) = (g \circ f)(a')$, so that $g \circ f$ is not one-to-one. ■

To have $g \circ f$ be onto, one needs $g(\text{Range}(f)) = C$. To have $g \circ f$ be one-to-one, in addition to f being one-to-one, g should be injective on the $\text{Range}(f)$, but not necessarily on the whole B .

Exercise 2.6.27 Give an example of functions f, g where f is not onto but g and $g \circ f$ are onto. Also give an example where g is not injective, but $g \circ f$ is injective.

Definition 2.6.28 The **identity function on a set** A is the function $f : A \rightarrow A$ defined by $f(a) = a$ for all $a \in A$.

5. Hints: In the case of a surjection, we must show that for $(g \circ f)(a) := g(f(a))$, it is the case that $\forall c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = c$. To see this, let $c \in C$. Since g is a surjection, $\exists b \in B$ such that $g(b) = c$. Similarly, since f is a surjection, $\exists a \in A$ such that $f(a) = b$. Then $(g \circ f)(a) = g(f(a)) = g(b) = c$.

Example 2.6.29 Let M be a finite set (of men) and W a finite set (of women). A **matching** is a function μ from $M \cup W$ to itself such that for all $m \in M$, $[\mu(m) \in W] \vee [\mu(m) = m]$, for all $w \in W$, $[\mu(w) \in M] \vee [\mu(w) = w]$, and $\mu \circ \mu$ is the identity function.

2.7 ♦ Weak and Partial Orders, Lattices

Rational choice theory for an individual requires a complete and transitive preference relation. Equilibria involve simultaneous rational choices by many individuals. While we think it reasonable to require an individual to choose among different options, requiring a group to be able to choose is a less reasonable assumption. One of the prominent examples of a partial ordering, that is, one that fails to be complete, arises when one asks for unanimous agreement among the individuals in a group.

Let X be a nonempty set and \preceq a relation on X . We call (X, \preceq) an ordered set. Orders are relations, but we use a different name because our emphasis is on orders with interpretations reminiscent of the usual order, \leq , on \mathbb{R} . Table 2.7 gives names to properties that an order \preceq may or may not have.

Table 2.7

Property	Name
$(\forall x)[x \preceq x]$	Reflexivity
$(\forall x, y)[[x \preceq y] \Rightarrow [y \preceq x]]$	Symmetry
$(\forall x, y)[[[x \preceq y] \wedge [y \preceq x]] \Rightarrow [x = y]]$	Antisymmetry
$(\forall x, y, z)[[x \preceq y] \wedge [y \preceq z]] \Rightarrow [x \preceq z]$	Transitivity
$(\forall x, y)[[x \preceq y] \vee [y \preceq x]]$	Completeness
$(\forall x, y)(\exists u)[x, y \preceq u]$	Upper bound
$(\forall x, y)(\exists \ell)[\ell \preceq x, y]$	Lower bound
$(\forall x, y)(\exists u)[[x, y \preceq u] \wedge [[x, y \preceq u'] \Rightarrow [u \preceq u']]]$	Least upper bound
$(\forall x, y)(\exists \ell)[[\ell \preceq x, y] \wedge [[\ell' \preceq x, y] \Rightarrow [\ell' \preceq \ell]]]$	Greatest lower bound

To see that every complete relation is reflexive, take $x = y$ in the definition.

The three main kinds of ordered sets we study are in the following.

Definition 2.7.1 (X, \preceq) is a **weakly ordered set** if \preceq is complete and transitive.⁶ (X, \preceq) is a **partially ordered set (POSET)** if \preceq is reflexive, antisymmetric, and transitive. (X, \preceq) is a **lattice** if it is a POSET with the least upper bound and the greatest lower bound property.

The following are, for economists, the most frequently used examples of these kinds of ordered sets.

6. Earlier, in §2.5, we used “rational” instead of “weak” for complete and transitive relations. It would have been too hard for us, as economists, to accept the idea that the foundation of our theory of choice was “weak”; hence the choice of a grander name, “rational.”

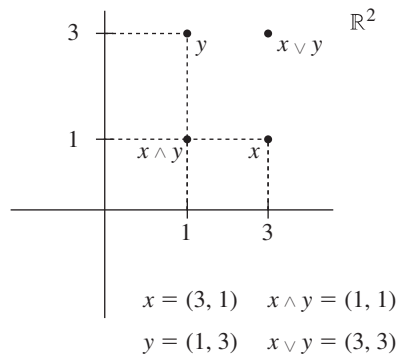


FIGURE 2.7.2

Example 2.7.2 Weak ordering: If $X = \mathbb{R}_+^2$ and \lesssim is defined by $x \lesssim y$ iff $u(x) \leq u(y)$ for a utility function u , then (X, \lesssim) is a weakly ordered set. We often represent this weak ordering by drawing representative indifference curves.

Partial ordering: If $X = \mathbb{R}_+^2$ and \lesssim is defined by the **usual vector order**, that is, by $\mathbf{x} \lesssim \mathbf{y}$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$, then (X, \lesssim) is a (POSET) that is not weakly ordered because it fails completeness, for example, the pair $(1, 3)$ and $(3, 1)$ are not comparable, as neither is greater in the vector order.

Lattices: \mathbb{R}^2 with the vector order is the canonical example of a lattice. For all $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$, the greatest lower bound, denoted $\mathbf{x} \wedge \mathbf{y}$, is the point $(\min\{x_1, y_1\}, \min\{x_2, y_2\})$, and the least upper bound, denote $\mathbf{x} \vee \mathbf{y}$, is the point $(\max\{x_1, y_1\}, \max\{x_2, y_2\})$. [See Figure 2.7.2.] By the same logic, \mathbb{R}^ℓ is a lattice with the vector order.

If $X = A \times B$, $A, B \subset \mathbb{R}$, is a rectangular set, then it is not only a POSET, but also it is a lattice. However, if X is the line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$ with the vector ordering, then it is a POSET that is not a lattice.

Notation 2.7.3 If the greatest lower bound of a pair x, y in a POSET X exists, it is called the **infimum** and denoted $\inf_X(x, y)$, $\inf(x, y)$, or $x \wedge y$. If the least upper bound of a pair $x, y \in X$ exists, it is called the **supremum** and denoted $\sup_X(x, y)$, $\sup(x, y)$, or $x \vee y$.

Example 2.7.4 The relation \leq on \mathbb{R} is reflexive, antisymmetric, transitive, complete; $\sup(x, y)$ is the maximum of the two numbers x and y and $\inf(x, y)$ is the minimum of the two. Thus, (\mathbb{R}, \leq) is a weakly ordered set, a POSET, and a lattice. On \mathbb{R} , the equality relation, $=$, is reflexive, symmetric, antisymmetric, and transitive, but fails the other properties.

There is a tight connection with the logical connectives \vee and \wedge introduced earlier. Recall that the indicator of a set $A \subset X$ is that function $1_A : X \rightarrow \{0, 1\}$ defined by

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \quad (2.3)$$

The set of indicator functions of subsets provides a classical example of lattices.

Example 2.7.5 The relation \leq on indicator functions is defined by $1_A \leq 1_B$ when $1_A(x) \leq 1_B(x)$ for every $x \in X$. Note that $1_A \leq 1_B$ iff $A \subset B$, so that \leq is not complete. On the class of indicator functions, \leq is reflexive, antisymmetric, transitive; $1_A \wedge 1_B = 1_{A \cap B}$ and $1_A \vee 1_B = 1_{A \cup B}$, so the class of indicators is a lattice with the relation \leq .

Exercise 2.7.6 Even though the relation \leq for vectors in \mathbb{R}^2 is not rational because it is not complete, defining $[(x_1, x_2) \sim (y_1, y_2)] \Leftrightarrow [(x_1, x_2) \leq (y_1, y_2)] \wedge [(y_1, y_2) \leq (x_1, x_2)]$ gives an equivalence relation. What is the equivalence relation?

You may be wondering how any reasonable relation with upper and lower bounds could fail to have the least upper or greatest lower bound property.

Example 2.7.7 The relation $<$ on \mathbb{R} (or on \mathbb{Q}) is not reflexive, is antisymmetric because $r < s$ and $s < r$ never happens, is transitive, and has the upper bound and the lower bound property but not the least upper bound nor the greatest lower bound property. This failure of $<$ to have the least upper and greatest lower bound has far-reaching consequences. To see why the failure occurs, let $r = s = 0$ and note that for any integer n , no matter how large, $1/10^n$ is an upper bound, but $1/10^{n+1}$ is a smaller upper bound.

The theory of rational choice by individuals studies weakly ordered sets.

Example 2.7.8 Suppose that \succsim is a rational preference ordering on X , that is, it is complete and transitive so that (X, \succsim) is a weakly ordered set. Completeness implies that \succsim is also reflexive because we can take $x = y$ in the definition of completeness. Completeness also implies the existence of at least one upper bound for any pair $x, y \in X$ because we can take $u = x$ if $x \succsim y$ and $u = y$ if $y \succsim x$. Parallel logic implies that there exists at least one lower bound. If there are two or more indifferent points in X , then antisymmetry fails, so that (X, \succsim) is not a lattice.

The subset relation is not the only interesting one for sets.

Exercise 2.7.9 Let X be the nonempty subsets of Y , that is, $X = \mathcal{P}(Y) \setminus \{\emptyset\}$. Define the relation t on X by $A t B$ if $A \cap B \neq \emptyset$. Mnemonically, “ $A t B$ ” if “ A touches B .” For $y \in Y$, let $X(y) = \{A \in X : y \in A\}$. Which, if any, of the properties given in Table 2.7 (p. 39) does the relation t have on X ? On $X(y)$?

From Definition 2.5.4 (p. 29), given a relation (e.g., a partial ordering) \succsim on a set X , we define $x \sim y$ if $x \succsim y$ and $y \succsim x$ and $x \succ y$ if $x \succsim y$ and $\neg(y \succsim x)$.

Exercise 2.7.10 Show that $\sim = (\succsim \cap \succsim^{-1})$ and that $\succ = (\succsim \setminus \sim)$.

Definition 2.7.11 A point $x \in X$ is **undominated in** \succsim if $\neg(\exists y \in X)[y \succ x]$.

Example 2.7.12 If $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$, then every point in X is undominated in \leq .

Definition 2.7.13 Let $\{\succsim_i : i \in I\}$ be a collection of rational preference orderings on a set X . Define the **unanimity order** on X by $x \succsim_U y$ if for all $i \in I$, $x \succsim_i y$. A point x is **Pareto optimal** or **Pareto efficient** if x is undominated in \succsim_U . If $x' \succ_U x$, then x' **Pareto improves on** x .

In all but the simplest cases, the unanimity order will fail completeness.

Example 2.7.14 Let $X = [0, 1]$ with $x \in X$ representing the proportion of the apple pie that person 1 will consume and $(1 - x)$ the proportion that person 2 will consume. Suppose that preferences are selfish, $[x > x'] \Leftrightarrow [x \succ_1 x']$ and $[x > x'] \Leftrightarrow [x \prec_2 x']$. No pair of points can be ranked by the unanimity order and every point is Pareto optimal.

The definition of Pareto optimality in Definition 2.7.13 is the usual one. Note that $y \succ_U x$ iff $(\forall i \in I)[y \succ_i x]$ and $(\exists j \in I)[y \succ_j x]$. Thus, x is Pareto efficient iff there is no y that everyone likes at least as well and someone strictly prefers. This differs from Definition 1.5.3 (p. 12), which is often known as *weak* Pareto efficiency: x is weakly Pareto efficient if there is no y such that $(\forall i \in I)[y \succ_i x]$.

Exercise 2.7.15 Show that if x is Pareto efficient, then it is weakly Pareto efficient. Modify the preferences of person 1 in Example 2.7.14 so that person 1 gains no extra utility from any pie beyond three-quarters of the pie; that is, three-quarters of the pie satiates person 1. With these preferences, give an example of an x that is weakly Pareto efficient but not Pareto efficient.

When preferences are nonsatiable, the allocation can be continuously adjusted, and preferences are continuous, then weak Pareto optimality and Pareto optimality are the same.

2.8 ♦ Monotonic Changes in Optima: Supermodularity and Lattices

Throughout economics, we are interested in how changes in one variable affect another variable. We answer such questions *assuming* that what we observe is the result of optimizing behavior. Part of learning to “think like an economist” involves internalizing this assumption. Given what we know about rational choice theory from §2.5, in many contexts optimizing behavior involves maximizing a utility function. In symbols, with $t \in T$ denoting a variable not determined by the individual, we let $x^*(t)$ denote the solution(s) to the problem $P(t)$,

$$\max_{x \in X} f(x, t), \quad (2.4)$$

and ask how $x^*(t)$ depends on t as t varies over T .

Since the problem $P(t)$ in (2.4) is meant as an approximation to, rather than a quantitative representation of, behavior, we are after “qualitative” results. These are results that are of the form “if t increases, then $x^*(t)$ will increase,” and they should be “fairly immune to” details of the approximation.⁷ If $x^*(\cdot)$ is differentiable, then we are after a statement of the form $dx^*/dt \geq 0$. If $x^*(\cdot)$ is not

7. Another part of learning to “think like an economist” involves developing an aesthetic sense of what “fairly immune” means. Aesthetics are complicated and subtle, best gained by immersion and indoctrination in and by the culture of economists, as typically happens during graduate school.

differentiable, then we are after a statement of the form that x^* is nondecreasing in t . We are going to go after such statements in three ways.

- 2.8.a. **The implicit function theorem**, using derivative assumptions on f when X and T are one-dimensional intervals.
- 2.8.b. **The simple univariate Topkis theorem**, using supermodularity assumptions on f when X and T are linearly ordered sets, for example, one-dimensional intervals.
- 2.8.c. **Monotone comparative statics**, using supermodularity when X is a lattice and T is partially ordered.

Unlike ranking by utility, supermodularity is not immune to monotonic rescaling, that is, supermodularity is a cardinal, not an ordinal concept. Quasi-supermodularity is the ordinal version of supermodularity, and we deal with it in the last part of this section.

Definition 2.8.1 *A partial order that also satisfies completeness is called a **total (or linear) ordering**, and (X, \approx) is called a **totally ordered set**. A **chain** in a partially ordered set is a subset, $X' \subset X$, such that (X', \approx) is totally ordered.*

The classical example of a linearly ordered set is the real line (\mathbb{R}, \leq) . In a total ordering, any two elements x and y in A can be compared, whereas in a partial ordering, there are noncomparable elements. For example, $(\mathcal{P}(\mathbb{N}), \subseteq)$ is a partially ordered set with many noncomparable elements. However, the set containing $\{1\}, \{1, 2\}, \dots, \{1, 2, \dots, n\}, \dots$ is a chain in $\mathcal{P}(\mathbb{N})$.

Exercise 2.8.2 Show that if $A \subset B$ and B is totally ordered, then A is totally ordered.

2.8.a The Implicit Function Approach

Assume that X and T are interval subsets of \mathbb{R} and that f is twice continuously differentiable.⁸ The terms f_x , f_t , f_{xx} , and f_{xt} denote the corresponding partial derivatives of f . To have $f_x(x, t) = 0$ characterize $x^*(t)$, we must have $f_{xx} < 0$ (which is a standard result about concavity in microeconomics). From the implicit function theorem, we know that $f_{xx} \neq 0$ is what is needed for there to exist a function $x^*(t)$ such that

$$f_x(x^*(t), t) \equiv 0. \quad (2.5)$$

To find dx^*/dt , take the derivative on both sides with respect to t and find

$$f_{xx} \frac{dx^*}{dt} + f_{xt} = 0, \quad (2.6)$$

so that $dx^*/dt = -f_{xt}/f_{xx}$. Since $f_{xx} < 0$, this means that dx^*/dt and f_{xt} have the same sign.

8. We study the continuity of functions in some detail later. For now, we are assuming that the reader has had a calculus-based microeconomics course.

This ought to be an intuitive result about the problem $P(t)$ in (2.4): if $f_{xt} > 0$, then increases in t increase f_x ; increases in f_x are increases in the marginal reward of x ; and as the marginal reward to x goes up, we expect the optimal level of x to go up. In a parallel fashion: if $f_{xt} < 0$, then increases in t decrease f_x ; decreases in f_x are decreases in the marginal reward of x ; and as the marginal reward to x goes down, we expect the optimal level of x to go down.

Exercise 2.8.3 Let $X = T = \mathbb{R}_+$ and $f(x, t) = x - \frac{1}{2}(x - t)^2$. Find $x^*(t)$ and verify directly that $dx^*/dt > 0$. Also find f_x , f_{xx} , and f_{xt} , and verify, using the sign test just given, that $dx^*/dt > 0$. If you can draw three-dimensional figures (and this is a skill worth developing), draw f and verify from your picture that $f_{xt} > 0$ and that it is this fact that makes $dx^*/dt > 0$. To practice with what goes wrong with derivative analysis when there are corner solutions, repeat this problem with $X = \mathbb{R}_+$, $T = \mathbb{R}$, and $g(x, t) = x - \frac{1}{2}(x + t)^2$.

Example 2.8.4 *The amount of a pollutant that can be emitted is regulated to be no more than $t \geq 0$. The cost function for a monopolist producing x is $c(x, t)$ with $c_t < 0$ and $c_{xt} < 0$. These derivative conditions mean that increases in the allowed emission level lower costs and lower marginal costs, so that the firm will always choose t . For a given t , the monopolist's maximization problem is therefore*

$$\max_{x \geq 0} f(x, t) = xp(x) - c(x, t), \quad (2.7)$$

where $p(x)$ is the (inverse) demand function. Since $f_{xt} = -c_{xt}$, we know that increases in t lead the monopolist to produce more, provided $f_{xx} < 0$.

The catch in the previous analysis is that $f_{xx} = xp_{xx} + p_x - c_{xx}$, so that it seems we have to know that $p_{xx} < 0$, or concavity of the inverse demand, and $c_{xx} > 0$, or convexity of the cost function, before we can reliably conclude that $f_{xx} < 0$. The global concavity of $f(\cdot, t)$ seems to have little to do with the intuition that it is the lowering of marginal costs that makes x^* depend positively on t . However, global concavity of $f(\cdot, t)$ is *not* what we need for the implicit function theorem, rather only the concavity of $f(\cdot, t)$ in the region of $x^*(t)$. With differentiability, this local concavity is an *implication* of $x^*(t)$ being a strict local maximum for $f(\cdot, t)$. Supermodularity makes it clear that the local maximum property is all that is being assumed and allows us to work with optima that are nondifferentiable.

2.8.b The Simple Supermodularity Approach

The simplest case has X and T being linearly ordered sets. The most common example has X and T being intervals in \mathbb{R} with the usual less-than-or-equal-to order. However, nothing rules out the sets X and T being discrete.

Definition 2.8.5 *For linearly ordered X and T , a function $f : X \times T \rightarrow \mathbb{R}$ is **supermodular** if for all $x' > x$ and all $t' > t$,*

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t), \quad (2.8)$$

equivalently

$$f(x', t') - f(x', t) \geq f(x, t') - f(x, t). \quad (2.9)$$

It is **strictly supermodular** if the inequalities are strict. For **submodularity** and **strictly submodularity**, reverse the inequalities.

At t , the benefit of increasing from x to x' is $f(x', t) - f(x, t)$, at t' , it is $f(x', t') - f(x, t')$. This assumption asks that the benefit of increasing x be increasing in t . A good verbal shorthand for this is that f has *increasing differences in x and t* . Three sufficient conditions in the differentiable case are: $\forall x, f_x(x, \cdot)$ is nondecreasing; $\forall t, f_t(\cdot, t)$ is nondecreasing; and $\forall x, t, f_{xt}(x, t) \geq 0$.

Theorem 2.8.6 (Topkis) *If X and T are linearly ordered, $f : X \times T \rightarrow \mathbb{R}$ is supermodular and $x^*(\tau)$ is the largest solution to $\max_{x \in X} f(x, \tau)$ for all τ , then $[t' > t] \Rightarrow [x^*(t') \succeq x^*(t)]$. Further, if there are unique, unequal maximizers at t' and t , then $x^*(t') > x^*(t)$.*

Proof. The idea of the proof is that having $x^*(t') < x^*(t)$ can only arise if f has strictly decreasing differences. Suppose that $t' > t$ but that $x' := x^*(t') < x := x^*(t)$. As $x^*(t)$ and $x^*(t')$ are maximizers, $f(x', t') \geq f(x, t')$ and $f(x, t) \geq f(x', t)$. Since x' is the largest of the maximizers at t' and $x > x'$, that is, x is larger than the largest maximizer at t' , we know a bit more—that $f(x', t') > f(x, t')$. Adding the inequalities, we get $f(x', t') + f(x, t) > f(x, t') + f(x', t)$, or

$$f(x, t) - f(x', t) > f(x, t') - f(x', t'),$$

that is, strictly decreasing differences in x and t . ■

Going back to the polluting monopolist of Example 2.8.4 (p. 44), we see that the supermodularity of f reduces to the supermodularity of $-c$. Thus assuming $-c$ (and hence f) is supermodular, we can use Theorem 2.8.6 to conclude that $x^*(t)$ is increasing. None of the second-derivative conditions except $c_{xt} < 0$ is necessary, and this can be replaced by the looser condition that $-c$ is supermodular.

Clever choices of T 's and f 's can make some analyses very easy.

Example 2.8.7 *Suppose that the one-to-one demand curve for a good produced by a monopolist is $x(p)$, so that $CS(p) = \int_p^\infty x(r) dr$ is the consumer surplus when the price p is charged. Let $p(\cdot)$ be $x^{-1}(\cdot)$, the inverse demand function. From intermediate microeconomics, you should know that the function $x \mapsto CS(p(x))$ is nondecreasing.*

The monopolist's profit when he produces x is $\pi(x) = x \cdot p(x) - c(x)$, where $c(x)$ is the cost of producing x . The maximization problems for the monopolist and for society are given by

$$\max_{x \geq 0} \pi(x) + 0 \cdot CS(p(x)), \quad \text{and} \quad (2.10)$$

$$\max_{x \geq 0} \pi(x) + 1 \cdot CS(p(x)). \quad (2.11)$$

Set $f(x, t) = \pi(x) + tCS(p(x))$, where $X = \mathbb{R}_+$ and $T = \{0, 1\}$. Since $CS(p(x))$ is nondecreasing, $f(x, t)$ is supermodular (and you should check this). Therefore $x^*(1) \geq x^*(0)$, so the monopolist always (weakly) restricts output relative to the social optimum.

Here is the externalities intuition: increases in x increase the welfare of people the monopolist does not care about, an effect external to the monopolist; the market gives the monopolist insufficient incentives to do the right thing. To fully appreciate how much simpler the supermodular analysis is, we have to see how complicated the differentiable analysis would be.

Example 2.8.8 [\uparrow Example 2.8.7 (p. 45)] Suppose that for every $t \in [0, 1]$, the problem

$$\max_{x \geq 0} \pi(x) + t \cdot CS(p(x))$$

has a unique solution, $x^*(t)$, and that the mapping $t \mapsto x^*(t)$ is continuously differentiable. (This can be guaranteed if we make the right kinds of assumptions on $\pi(\cdot)$ and $CS(p(\cdot))$.) To find the sign of $dx^*(t)/dt$, we assume that the first-order conditions,

$$\pi'(x^*(t)) + t dCS(p(x^*(t)))/dx \equiv 0,$$

characterize the optimum. In general, this means that we have to assume that $x \mapsto \pi(x) + tCS(p(x))$ is a smooth, concave function. We then take the derivative of both sides with respect to t . This involves evaluating $d(\int_{p(x^*(t))}^{\infty} x(r) dr)/dt$. In general, $d(\int_{f(t)}^{\infty} x(r) dr)/dt = -f'(t)x(f(t))$, so that when we take derivatives on both sides, we have

$$\pi''(x^*(t))(dx^*/dt) + dCS(p(x^*(t)))/dx - p'(x^*)(dx^*/dt)x(p(x^*)) = 0.$$

Gathering terms yields

$$[\pi''(x^*) - p'(x^*)x(p(x^*))](dx^*/dt) + dCS(p(x^*(t)))/dx = 0. \quad (2.12)$$

Since we are assuming that we are at an optimum, we know that $\pi''(x^*) \leq 0$. By assumption, $p'(x^*) < 0$ and $x > 0$, so the term in the square brackets is negative. As argued earlier, $dCS(p(x^*(t)))/dx > 0$. Therefore, the only way that (2.12) can be satisfied is if $dx^*/dt > 0$. Finally, by the fundamental theorem of calculus (which says that the integral of a derivative is the function itself), $x^*(1) - x^*(0) = \int_0^1 \frac{dx^*(r)}{dr} dr$. The integral of a positive function is positive, so this yields $x^*(1) - x^*(0) > 0$.

2.8.c Monotone Comparative Statics

Suppose that (X, \lesssim_X) and (T, \lesssim_T) are lattices. Define the order $\lesssim_{X \times T}$ on $X \times T$ by $(x', t') \lesssim_{X \times T} (x, t)$ iff $x' \lesssim_X x$ and $t' \lesssim_T t$. (This is the unanimity order again.)

Lemma 2.8.9 $(X \times T, \lesssim_{X \times T})$ is a lattice.

Proof. $(x', t') \vee (x, t) = (\max\{x', x\}, \max\{t', t\}) \in X \times T$, and $(x', t') \wedge (x, t) = (\min\{x', x\}, \min\{t', t\}) \in X \times T$. ■

Definition 2.8.10 For a lattice (L, \lesssim) , $f : L \rightarrow \mathbb{R}$ is **supermodular** if for all $\ell, \ell' \in L$,

$$f(\ell \wedge \ell') + f(\ell \vee \ell') \geq f(\ell) + f(\ell'), \quad (2.13)$$

equivalently

$$f(\ell \vee \ell') - f(\ell') \geq f(\ell) - f(\ell \wedge \ell'). \quad (2.14)$$

Taking $\ell' = (x', t)$ and $\ell = (x, t')$, we recover Definition 2.8.5 (and you should check this).

Exercise 2.8.11 Suppose that $(L, \lesssim) = (\mathbb{R}^n, \leq)$, $x \leq y$ iff $x_i \leq y_i, i = 1, \dots, n$. Show that L is a lattice. Show that $f : L \rightarrow \mathbb{R}$ is supermodular iff it has increasing differences in x_i and x_j for all $i \neq j$. Show that a twice continuously differentiable $f : L \rightarrow \mathbb{R}$ is supermodular iff $\partial^2 f / \partial x_i \partial x_j \geq 0$ for all $i \neq j$.

Example 2.8.12 On the lattices (\mathbb{R}^n, \leq) , the function $x \mapsto \min\{f_i(x_i) : i = 1, \dots, n\}$ is supermodular if each f_i is nondecreasing, as is the function $x \mapsto x_1 \cdot x_2 \cdot \dots \cdot x_{n-1} x_n$. For $x, t \in \mathbb{R}^n$, the function $(x, t) \mapsto x \cdot t$ is supermodular on $\mathbb{R}^n \times \mathbb{R}^n$.

Exercise 2.8.13 Show the following.

1. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice continuously differentiable, has a nonnegative gradient, and is supermodular, and $g : \mathbb{R} \rightarrow \mathbb{R}$ is twice continuously differentiable and convex, then $g(f(x))$ is increasing and supermodular.
2. If $f : [1, \infty) \times [1, \infty) \rightarrow \mathbb{R}$ is defined by $f(x_1, x_2) = x_1 + x_2$ and $g(r) = \log(r)$, then f is supermodular and strictly increasing, while g is strictly increasing and concave. However, $g(f(x))$ is strictly submodular.

Definition 2.8.14 For $A, B \subset L$, L a lattice, the **strong set order** is defined by $A \lesssim_{Strong} B$ iff $\forall (a, b) \in A \times B$, $a \wedge b \in A$ and $a \vee b \in B$.

Interval subsets of \mathbb{R} are sets of the form $(-\infty, r)$, $(-\infty, r]$, (r, s) , $(r, s]$, $[r, s)$, $[r, s]$, (r, ∞) , or $[r, \infty)$.

Exercise 2.8.15 Show that for intervals $A, B \subset \mathbb{R}$, $A \lesssim_{Strong} B$ iff every point in $A \setminus B$ is less than every point in $A \cap B$, and every point in $A \cap B$ is less than every point in $B \setminus A$. Also show that this is true when \mathbb{R} is replaced with any linearly ordered set.

The strong set order is not, in general, reflexive.

Example 2.8.16 If $A = \{x \in \mathbb{R}_+^2 : x_1 + x_2 \leq 1\}$, then $\neg[A \lesssim_{Strong} A]$. However, if $A \subset L$ is itself a lattice, then $[A \lesssim_{Strong} A]$. In particular, subsets of (\mathbb{R}, \leq) are linearly ordered, hence they are lattices.

Notation 2.8.17 For $S \subset L$ and $t \in T$, let $M(t, S) \subset S$ be the set of solutions to the problem $\max_{\ell \in S} f(\ell, t)$. For $t, t' \in T$, $S, S' \subset L$, define $(t', S') \gtrsim (t, S)$ if $t' \gtrsim_T t$ and $S' \gtrsim_{Strong} S$.

Theorem 2.8.18 *If (L, \lesssim_L) is a lattice, (T, \lesssim_T) is a partially ordered set, $f : L \times T \rightarrow \mathbb{R}$ is supermodular in ℓ for all t , and has increasing differences in ℓ and t , then $M(t, S)$ is nondecreasing in (t, S) for (t, S) with $M(t, S) \neq \emptyset$.*

Proof. Pick $(t', S') \succeq (t, S)$; we must show that $M(t', S') \succeq_{Strong} M(t, S)$.

Pick $\ell' \in M(t', S') \subset S'$ and $\ell \in M(t, S) \subset S$. By the definition of the strong set order, we have to show that $\ell \vee \ell' \in M(t', S')$ and $\ell \wedge \ell' \in M(t, S)$. We do the first one here; the second is an exercise.

Since $S' \succeq_{Strong} S$, $\ell' \wedge \ell \in S'$ and $\ell' \vee \ell \in S$. As ℓ is optimal in S and $\ell' \wedge \ell \in S$, we know that $f(\ell, t) - f(\ell' \wedge \ell) \geq 0$. Combining the supermodularity of $f(\cdot, t)$ and this last inequality, we have

$$f(\ell \vee \ell', t) - f(\ell', t) \geq f(\ell, t) - f(\ell \wedge \ell', t) \geq 0.$$

Increasing differences, $t' \succeq_T t$, $\ell \vee \ell' \succeq_L \ell'$, and this last inequality yield

$$f(\ell \vee \ell', t') - f(\ell', t') \geq f(\ell \vee \ell', t) - f(\ell', t) \geq 0.$$

Since ℓ' is optimal in S' and $\ell \vee \ell' \in S'$, we have just discovered that $\ell \vee \ell'$ is also optimal in S' , that is, $\ell \vee \ell' \in M(t', S')$. ■

Exercise 2.8.19 Complete the proof of Theorem 2.8.18.

The following is immediate from the last result, simply take $(t', S') = (t, S)$. However, a direct proof makes clearer how submodularity is working.

Corollary 2.8.20 *If (L, \lesssim) is a lattice, $f : L \rightarrow \mathbb{R}$ is supermodular, $S \subset L$ and (S, \lesssim) is a sublattice, that is, for $x, y \in S$, $x \vee y \in S$ and $x \wedge y \in S$, then $\arg \max_{x \in S} f(x)$ is a sublattice.*

2.8.d Quasi-Supermodularity

Sometimes people make the mistake of identifying supermodular utility functions as the ones for which there are complementarities. This is wrong.

Exercise 2.8.21 For $(x_1, x_2) \in \mathbb{R}_{++}^2$, define $u(x_1, x_2) = x_1 \cdot x_2$ and $v(x_1, x_2) = \log(u(x_1, x_2))$.

1. Show that $\partial^2 u / \partial x_1 \partial x_2 > 0$.
2. Show that $\partial^2 v / \partial x_1 \partial x_2 = 0$.
3. Find a monotonic transformation, f , of v such that $\partial^2 f(v) / \partial x_1 \partial x_2 < 0$ at some point $(x_1^\circ, x_2^\circ) \in \mathbb{R}_{++}^2$.

The problem is that supermodularity is not immune to monotonic transformations. That is, supermodularity, like expected utility theory, is a cardinal rather than an ordinal theory. Here is the ordinal version.

Definition 2.8.22 (Milgrom and Shannon) *A function $u : X \rightarrow \mathbb{R}$ is quasi-supermodular on the lattice X if, $\forall x, y \in X$,*

$$[u(x) \geq u(x \wedge y)] \Rightarrow [u(x \vee y) \geq u(y)], \quad \text{and}$$

$$[u(x) > u(x \wedge y)] \Rightarrow [u(x \vee y) > u(y)].$$

By way of contrast, $f : X \rightarrow \mathbb{R}$ is supermodular if $\forall x, y \in X$, and $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$, which directly implies that it is quasi-supermodular. The reason for the adjective “quasi” comes from intermediate economics, where you should have learned that a monotonically increasing transformation of a concave utility function is quasi-concave.

Lemma 2.8.23 *A monotonic increasing transformation of a supermodular function is quasi-supermodular.*

Exercise 2.8.24 Prove Lemma 2.8.23.

Recall that a binary relation, \lesssim , on a set X has a representation $u : X \rightarrow \mathbb{R}$ iff $[x \lesssim y] \Leftrightarrow [u(x) \leq u(y)]$. For choice theory on *finite* lattices with monotonic preferences, quasi-supermodularity and supermodularity of preferences are indistinguishable.

Theorem 2.8.25 (Chambers and Echenique) *A binary relation on a finite lattice X has a weakly increasing and quasi-supermodular representation iff it has a weakly increasing and supermodular representation.*

Proof. Since supermodularity implies q-supermodularity, we need only show that a weakly increasing q-supermodular representation can be monotonically transformed to be supermodular. Let u be quasi-supermodular, set $u(X) = \{u_1 < u_2 < \dots < u_N\}$, and define $g(u_n) = 2^{n-1}$, $n = 1, \dots, N$. You can show that the function $v(x) = g(u(x))$ is supermodular. ■

Exercise 2.8.26 There are three results that you should know about the relation between monotonicity and quasi-supermodularity:

1. Strong monotonicity implies quasi-supermodularity, hence supermodularity: Show that if a binary relation \lesssim on a finite lattice X has a strictly increasing representation, then that representation is quasi-supermodular. [By the Chambers and Echenique result above, this implies that the binary relation has a supermodular representation.]
2. Weak monotonicity does not imply quasi-supermodularity: Let $X = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ with $(x, y) \lesssim (x', y')$ iff $(x, y) \leq (x', y')$. Show that the utility function $u(x, y) = 0$ if $x = y = 0$ and $u(x, y) = 1$ otherwise is weakly monotonic, but no monotonic transformation of it is quasi-supermodular.
3. Supermodularity does not imply monotonicity: Let $X = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ with $(x, y) \lesssim (x', y')$ iff $(x, y) \leq (x', y')$. Show that the utility function $u(0, 0) = 0$, $u(0, 1) = -1$, $u(1, 0) = 2$ and that $u(1, 1) = 1.5$ is strictly supermodular but not monotonic.

2.9 ♦ Tarski's Lattice Fixed-Point Theorem and Stable Matchings

In this section we give a lattice formulation of Gale-Shapley matching problems. Our aim is to create pairs by choosing one person from each side of a market, in

such a fashion that everyone is happier with his or her part of the pairing than he or she would be alone, and there is no pair that could be rematched so as to make both better off.

2.9.a Matching Problems and Stability

A **matching problem** is a 4-tuple, $(M, W, (\succ_m)_{m \in M}, (\succ_w)_{w \in W})$. We assume that $M = \{m_1, \dots, m_n\}$ and $W = \{w_1, \dots, w_k\}$ are disjoint finite sets (mnemonically, *Men* and *Women*). We also assume that each $m \in M$ has strict rational preferences, \succ_m , over the set $W \cup \{m\}$. By strict, we mean that no distinct pair is indifferent. The expression $w \succ_m w'$ means that m prefers to be matched with w rather than w' , and $m \succ_m w$ means that m would prefer to be single, that is, matched with himself, rather than being matched to w . In exactly the same fashion, each $w \in W$ has strict rational preferences over $M \cup \{w\}$. Man m is *acceptable* to woman w if she likes him at least as much as staying single, and woman w is *acceptable* to man m if he likes her at least as much as staying single.

As in Example 2.6.29 (p. 39), a **matching** μ is a one-to-one onto function from $M \cup W$ to itself such that $\mu \circ \mu$ is the identity, $[\mu(m) \neq m] \Rightarrow [\mu(m) \in W]$, and $[\mu(w) \neq w] \Rightarrow [\mu(w) \in M]$. Interest focuses on properties of the set of matchings that are acceptable to everyone and that have the property that no one can find someone that he or she prefers to his or her own match and who prefers him or her to his or her match. Formally,

Definition 2.9.1 A matching μ is *stable* if it satisfies:

1. *individual rationality*: $\forall m, \mu(m) \succeq_m m$ and $\forall w, \mu(w) \succeq_w w$, and
2. *stability*: $\neg \exists (m, w) \in M \times W$ such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

For the purposes of a later proof, the following is worth having as a separate result, even though it is an immediate implication of the strict preferences we have assumed.

Lemma 2.9.2 If μ is individually rational, then it is stable iff $\neg \exists (m, w)$ such that $[w \succ_m \mu(m) \wedge m \succeq_w \mu(w)]$ or $[m \succ_w \mu(w) \wedge w \succeq_m \mu(m)]$.

Exercise 2.9.3 A matching μ' is indicated by **boldface** and the *'s in following table, which shows preferences (Ex. 2.4, Roth and Sotomayor 1990).

$$\begin{array}{lll} w_2 \succ_{m_1} \mathbf{w}_1^* \succ_{m_1} w_3 \succ_{m_1} m_1 & \mathbf{m}_1^* \succ_{w_1} m_3 \succ_{w_1} m_2 \succ_{w_1} w_1 \\ w_1 \succ_{m_2} \mathbf{w}_3^* \succ_{m_2} w_2 \succ_{m_2} m_2 & \mathbf{m}_3^* \succ_{w_2} m_1 \succ_{w_2} m_2 \succ_{w_2} w_2 \\ w_1 \succ_{m_3} \mathbf{w}_2^* \succ_{m_3} w_3 \succ_{m_3} m_3 & m_1 \succ_{w_3} m_3 \succ_{w_3} \mathbf{m}_2^* \succ_{w_3} w_3 \end{array}$$

1. Verify that μ' is stable.
2. Given a matching μ , for each m , define m 's **possibility set** as $P_\mu(m) = \{w \in W : m \succeq_w \mu(w)\} \cup \{m\}$. Show that the matching μ' maximizes each m 's preferences over $P_{\mu'}(m)$ and maximizes each w 's preferences over $P_{\mu'}(w)$. [The simultaneous optimization of preferences over the available set of options is a pattern we see repeatedly.]