CHAPTER 1

Logic

The building blocks of modern economics are based on logical reasoning to prove the validity of a conclusion, $\mathbb{B}$, from well-defined premises, $\mathbb{A}$. In general, statements such as $\mathbb{A}$ and/or $\mathbb{B}$ can be represented using sets, and a "proof" is constructed by applying, sometimes ingeniously, a fixed set of rules to establish that the statement $\mathbb{B}$ is true whenever $\mathbb{A}$ is true. We begin with examples of how we represent statements as sets, then turn to the rules that allow us to form more and more complex statements, and then give a taxonomy of the major types of proofs that we use in this book.

1.1 ◆ Statements, Sets, Subsets, and Implication

The idea of a set (of things), or group, or collection is a "primitive," one that we use without being able to clearly define it. The idea of belonging to a set (group, collection) is primitive in exactly the same sense. Our first step is to give the allowable rules by which we evaluate whether statements about sets are true.

We begin by fixing a set X of things that we might have an interest in. When talking about demand behavior, the set X has to include, at the very least, prices, incomes, affordable consumption sets, preference relations, and preference-optimal sets. The set X varies with the context, and is often not mentioned at all.

We express the primitive notion of membership by "$\in$," so that "$x \in A$" means that "x is an element of the set A" and "$y \notin A$" means that "y is not an element of A."

Notation Alert 1.1.A *Capitalized letters are usually reserved for sets and smaller letters for points/things in the set that we are studying. Sometimes, several levels of analysis are present simultaneously, and we cannot do this. Consider the study of utility functions, u, on a set of options, X. A function u is a set of pairs of the form $(x, u(x))$, with x an option and $u(x)$ the number representing its utility. However, in our study of demand behavior, we want to see what happens as u varies. From this perspective, u is a point in the set of possible utility functions.*

1.2 ◆ Statements and Their Truth Values

Note that a statement of the form "$\mathbb{A} \Rightarrow \mathbb{B}$" is simply a construct of two simple statements connected by "$\Rightarrow$." This is one of seven ways of constructing new statements that we use. In this section, we cover the first five of them: ands, ors, nots, implies, and equivalence. Repeated applications of these seven ways of constructing statements yield more and more elaboration and complication.

We begin with the simplest three methods, which construct new sets directly from a set or pair of sets that we start with. We then turn to the statements that are about relations between sets and introduce another formulation in terms of indicator functions. Later we give the other two methods, which involve the logical quantifiers "for all" and "there exists." Throughout, interest focuses on methods of establishing the truth or falsity of statements, that is, on methods of proof.

1.2.a Ands/Ors/Nots as Intersections/Unions/Complements

The simplest three ways of constructing new statements from other ones are using the connectives "and" or "or," or by "not," which is negation. Notationally: "$\mathbb{A} \wedge \mathbb{B}$" means "$\mathbb{A}$ and $\mathbb{B}$," "$\mathbb{A} \vee \mathbb{B}$" means "$\mathbb{A}$ or $\mathbb{B}$," and "$\neg\mathbb{A}$" means "not $\mathbb{A}$."

In terms of the corresponding sets: "$\mathbb{A} \wedge \mathbb{B}$" is $A \cap B$, the intersection of A and B, that is, the set of all points that belong to both A *and* B; "$\mathbb{A} \vee \mathbb{B}$" is $A \cup B$, the union of A and B, that is, the set of all points that belong to A *or* belong to B; and "$\neg\mathbb{A}$" is $A^c = \{x \in X : x \notin A\}$, the complement of A, is the set of all elements of X that do *not* belong to A.

The meanings of these new statements, $\neg\mathbb{A}$, $\mathbb{A} \wedge \mathbb{B}$, and $\mathbb{A} \vee \mathbb{B}$, are given by a *truth table*, Table 1.a. The corresponding Table 1.b gives the corresponding set versions of the new statements.

Table 1.a

$\mathbb{A}$	$\mathbb{B}$	$\neg\mathbb{A}$	$\mathbb{A} \wedge \mathbb{B}$	$\mathbb{A} \vee \mathbb{B}$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

Table 1.b

A	B	A^c	$A \cap B$	$A \cup B$
$x \in A$	$x \in B$	$x \notin A^c$	$x \in A \cap B$	$x \in A \cup B$
$x \in A$	$x \notin B$	$x \notin A^c$	$x \notin A \cap B$	$x \in A \cup B$
$x \notin A$	$x \in B$	$x \in A^c$	$x \notin A \cap B$	$x \in A \cup B$
$x \notin A$	$x \notin B$	$x \in A^c$	$x \notin A \cap B$	$x \notin A \cup B$

The first two columns of Table 1.a give possible truth values for the statements $\mathbb{A}$ and $\mathbb{B}$. The last three columns give the truth values for $\neg\mathbb{A}$, $\mathbb{A} \wedge \mathbb{B}$, and $\mathbb{A} \vee \mathbb{B}$ as a function of the truth values of $\mathbb{A}$ and $\mathbb{B}$. The first two columns of Table 1.b give the corresponding membership properties of an element x, and the last three columns give the corresponding membership properties of x in the sets A^c, $A \cap B$, and $A \cup B$.

Consider the second rows of both tables, the row where $\mathbb{A}$ is true and $\mathbb{B}$ is false. This corresponds to discussing an x with the properties that it belongs to A and does not belong to B. The statement "not $\mathbb{A}$," that is, $\neg\mathbb{A}$, is false, which corresponds to x not belonging to A^c, $x \notin A^c$. The statement "$\mathbb{A}$ and $\mathbb{B}$," that is, "$\mathbb{A} \wedge \mathbb{B}$," is also false. This is sensible: since $\mathbb{B}$ is false, it is not the case that both $\mathbb{A}$ and $\mathbb{B}$ are true. This corresponds to x not being in the intersection of A and B, that is, $x \notin A \cap B$.

Table 1.c

$\mathbb{A}$	$\mathbb{B}$	$\mathbb{A} \Rightarrow \mathbb{B}$	$\mathbb{A} \Leftrightarrow \mathbb{B}$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

Table 1.d

$x \in A$	$x \in B$	$1_A(x) \leq 1_B(x)$	$1_A(x) = 1_B(x)$
$x \in A$	$x \in B$	T	T
$x \in A$	$x \notin B$	F	F
$x \notin A$	$x \in B$	T	F
$x \notin A$	$x \notin B$	T	T

1.2.c The Empty Set and Vacuously True Statements

We now come to the idea of something that is vacuously true, and a substantial proportion of people find this idea tricky or annoying, or both. The idea that we are after is that starting from false premises, one can establish anything. In Table 1.c, if $\mathbb{A}$ is false, then the statement $\mathbb{A} \Rightarrow \mathbb{B}$ is true, whether $\mathbb{B}$ is true or false.

A statement that is false for all $x \in X$ corresponds to having an indicator function with the property that for all $x \in X$, $1_A(x) = 0$. In terms of sets, the notation for this is $A = \emptyset$, where we read "$\emptyset$" as the **empty set**, that is, the vacuous set, the one that contains no elements. No matter what the set B is, if $A = \emptyset$, then $1_A(x) \leq 1_B(x)$ for all $x \in X$.

Definition 1.2.2 *The statement $\mathbb{A} \Rightarrow \mathbb{B}$ is **vacuously true** if $A = \emptyset$.*

This definition follows the convention that we use throughout: we show the term or terms being defined in boldface type.

In terms of sets, this is the observation that for all B, $\emptyset \subset B$, that is, that every element of $\emptyset$ belongs to B. What many people find distasteful is that "every element of $\emptyset$ belongs to B" suggests that there is an element of $\emptyset$, and since there is no such element, the statement feels wrong to them. There is nothing to be done except to get over the feeling.

1.2.d Indicators and Ands/Ors/Nots

Indicator functions can also be used to capture ands, ors, and nots. Often this makes proofs simpler.

The pointwise minimum of a pair of indicator functions, 1_A and 1_B, is written as "$1_A \wedge 1_B$," and is defined by $(1_A \wedge 1_B)(x) = \min\{1_A(x), 1_B(x)\}$. Now, $1_A(x)$ and $1_B(x)$ are equal either to 0 or to 1. Since the minimum of 1 and 1 is 1, the minimum of 0 and 1 is 0, and the minimum of 0 and 0 is 0, $1_{A \cap B} = 1_A \wedge 1_B$. This means that the indicator associated with the statement "$\mathbb{A} \wedge \mathbb{B}$" is $1_A \wedge 1_B$. By checking cases, we note that for all $x \in X$, $(1_A \wedge 1_B)(x) = 1_A(x) \cdot 1_B(x)$. As a result, $1_A \wedge 1_B$ is often written as $1_A \cdot 1_B$.

In a similar way, the pointwise maximum of a pair of indicator functions, 1_A and 1_B, is written as "$1_A \vee 1_B$" and defined by $(1_A \vee 1_B)(x) = \max\{1_A(x), 1_B(x)\}$. Here, $1_{A \cup B} = 1_A \vee 1_B$, and the indicator associated with the statement "$\mathbb{A} \vee \mathbb{B}$" is $1_A \vee 1_B$. Basic properties of numbers say that for all x, $(1_A \vee 1_B)(x) = 1_A(x) + 1_B(x) - 1_A(x) \cdot 1_B(x)$, so $1_A \vee 1_B$ could be defined as $1_A + 1_B - 1_A \cdot 1_B$.

The truth values in the fourth column, $\mathbb{B} \wedge \mathbb{C}$, are formed using the rules for $\wedge$ and the truth values of $\mathbb{B}$ and $\mathbb{C}$. The truth values in the next column, $\mathbb{A} \vee (\mathbb{B} \wedge \mathbb{C})$, are formed using the rules for $\vee$ and the truth values of the $\mathbb{A}$ column and the just-derived truth values of the $\mathbb{B} \wedge \mathbb{C}$ column. The truth values in the next three columns are derived analogously. Since the truth values in the column $\mathbb{A} \vee (\mathbb{B} \wedge \mathbb{C})$ match those in the column $(\mathbb{A} \vee \mathbb{B}) \wedge (\mathbb{A} \vee \mathbb{C})$, the two statements are equivalent. ■

Another Proof of Theorem 1.3.1. In terms of indicator functions,

$$\begin{aligned} 1_{A\cup(B\cap C)} &= 1_A \cdot (1_B + 1_C - 1_B \cdot 1_C) \\ &= 1_A \cdot 1_B + 1_A \cdot 1_C - 1_A \cdot 1_B \cdot 1_C, \end{aligned} \tag{1.2}$$

while

$$1_{(A\cap B)\cup(A\cap C)} = 1_A \cdot 1_B + 1_A \cdot 1_C + (1_A \cdot 1_B) \cdot (1_B \cdot 1_C). \tag{1.3}$$

Since indicators take only the values 0 and 1, for any set, for example, B, $1_B \cdot 1_B = 1_B$. Therefore $(1_A \cdot 1_B) \cdot (1_B \cdot 1_C) = 1_A \cdot 1_B \cdot 1_C$. ■

The following contains the commutative, associative, and distributive laws. To prove them, one can simply generate the appropriate truth table.

Theorem 1.3.2 *Let $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ be any statements. Then*

1. *commutativity holds,* $(\mathbb{A} \vee \mathbb{B}) \Leftrightarrow (\mathbb{B} \vee \mathbb{A})$ *and* $(\mathbb{A} \wedge \mathbb{B}) \Leftrightarrow (\mathbb{B} \wedge \mathbb{A})$,
2. *associativity holds,* $((\mathbb{A} \wedge \mathbb{B}) \wedge \mathbb{C}) \Leftrightarrow (\mathbb{A} \wedge (\mathbb{B} \wedge \mathbb{C}))$, $((\mathbb{A} \vee \mathbb{B}) \vee \mathbb{C}) \Leftrightarrow (\mathbb{A} \vee (\mathbb{B} \vee \mathbb{C}))$, *and*
3. *the distributive laws hold,* $(\mathbb{A} \wedge (\mathbb{B} \vee \mathbb{C})) \Leftrightarrow ((\mathbb{A} \wedge \mathbb{B}) \vee (\mathbb{A} \wedge \mathbb{C}))$, $(\mathbb{A} \vee (\mathbb{B} \wedge \mathbb{C})) \Leftrightarrow ((\mathbb{A} \vee \mathbb{B}) \wedge (\mathbb{A} \vee \mathbb{C}))$.

Exercise 1.3.3 Restate Theorem 1.3.2 in terms of sets and in terms of indicator functions. Then complete the proof of Theorem 1.3.2 both by generating the appropriate truth tables and by using indicator functions.

We now prove two results, Lemma 1.3.4 and Theorem 1.3.6, that form the basis for the methods of logical reasoning we pursue in this book. The following is used so many times that it is at least as important as a theorem.

Lemma 1.3.4 *$\mathbb{A}$ implies $\mathbb{B}$ iff $\mathbb{A}$ is false or $\mathbb{B}$ is true,*

$$(\mathbb{A} \Rightarrow \mathbb{B}) \Leftrightarrow ((\neg\mathbb{A}) \vee \mathbb{B}), \tag{1.4}$$

and a double negative makes a positive,

$$\neg(\neg\mathbb{A}) \Leftrightarrow \mathbb{A}. \tag{1.5}$$

Proof. In terms of indicator functions, (1.4) is $1_A(x) \leq 1_B(x)$ iff $1_A(x) = 0$ or $1_B(x) = 1$, which is true because $1_A(x)$ and $1_B(x)$ can only take on the values 0 and 1. (1.5) is simpler; it says that $1 - (1 - 1_A) = 1_A$. ■

way that this last statement could be false—if there is an $x \in X$ such that $\neg\mathbb{B}(x)$ while $\mathbb{A}(x)$. Therefore, "$\mathbb{A}$ implies $\mathbb{B}$" is equivalent to $(\neg\mathbb{B}) \Rightarrow (\neg\mathbb{A})$.

In terms of sets, we are saying that $A \subset B$ is equivalent to $B^c \subset A^c$. Often it is easier to pick a point y, assume only that y does *not* belong to B, and establish that this implies that y does *not* belong to A.

Example 1.3.8 *Let X be the set of humans, let $\mathbb{A}(x)$ be the statement "x has a Ph.D. in economics," and let $\mathbb{B}(x)$ be the statement "x is literate in at least one language." Showing that $\mathbb{A} \Rightarrow \mathbb{B}$ is the same as showing that there are no completely illiterate economics Ph.D.s. Which method of proving the statement one would want to use depends on whether or not it is easier to check all the economics Ph.D.s in X for literacy or to check all the illiterates in X for Ph.D.s in economics.*

A final note: the contrapositive of "$\mathbb{A} \Rightarrow \mathbb{B}$" is "$(\neg\mathbb{B}) \Rightarrow (\neg\mathbb{A})$." This is not the same as the *converse* of "$\mathbb{A} \Rightarrow \mathbb{B}$," which is "$\mathbb{B} \Rightarrow \mathbb{A}$."

1.4 ◆ Logical Quantifiers

The last two of our seven ways to construct statements use the two *quantifiers*, "$\exists$," read as "there exists," and "$\forall$," read as "for all." More specifically, "$(\exists x \in A)[\mathbb{B}(x)]$" means "there exists an x in the set A such that $\mathbb{B}(x)$" and "$(\forall x \in A)[\mathbb{B}(x)]$" means "for all x in the set A, $\mathbb{B}(x)$." Our discussion of indicator functions has already used these quantifiers; for example, $1_A \leq 1_B$ was defined as $(\forall x \in X)[1_A(x) \leq 1_B(x)]$. We now formalize the ways in which we use the quantifiers.

Quantifiers should be understood as statements about the relations between sets, and here the empty set, $\emptyset$, is again useful. In terms of sets, "$(\exists x \in A)[\mathbb{B}(x)]$" is the statement $(A \cap B) \neq \emptyset$, while "$(\forall x \in A)[\mathbb{B}(x)]$" is the statement $A \subset B$.

Notation Alert 1.4.A *Following common usage, when the set A is supposed to be clear from context, we often write $(\exists x)[\mathbb{B}(x)]$ for $(\exists x \in A)[\mathbb{B}(x)]$. If A is not in fact clear from context, we run the risk of leaving the intended set A undefined.*

The two crucial properties of quantifiers are contained in the following, which gives the relationship among quantifiers, negations, and complements.

Theorem 1.4.1 *There is no x in A such that $\mathbb{B}(x)$ iff for all x in A, it is not the case that $\mathbb{B}(x)$,*

$$\neg(\exists x \in A)[\mathbb{B}(x)] \Leftrightarrow (\forall x \in A)[\neg\mathbb{B}(x)], \tag{1.9}$$

and it is not the case that for all x in A we have $\mathbb{B}(x)$ iff there is some x in A for which $\mathbb{B}(x)$ fails,

$$\neg(\forall x \in A)[\mathbb{B}(x)] \Leftrightarrow (\exists x \in A)[\neg\mathbb{B}(x)]. \tag{1.10}$$

Proof. In terms of sets, (1.9) is $[A \cap B = \emptyset] \Leftrightarrow [A \subset B^c]$. In terms of indicators, letting 0 be the function identically equal to 0, it is $1_A \cdot 1_B = 0$ iff $1_A \leq (1 - 1_B)$.

Example 1.4.3 *To see why* (1.12) *cannot hold as an "if and only if" statement, suppose* x *is the set of countries in the world,* $\mathbb{A}(x)$ *is the property that* x *has a gross domestic product strictly above average, and* $\mathbb{B}(x)$ *is the property that* x *has a gross domestic product strictly below average. There will be at least one country above the mean and at least one country below the mean. That is,* $(\exists x)[\mathbb{A}(x)] \wedge (\exists x)[\mathbb{B}(x)]$ *is true, but clearly there cannot be a country that is both above and below the mean,* $\neg(\exists x)[\mathbb{A}(x) \wedge \mathbb{B}(x)]$.

In terms of sets, (1.11) can be rewritten as $[(A \cup B) \neq \emptyset] \Leftrightarrow [(A \neq \emptyset) \vee (B \neq \emptyset)]$. The set form of (1.12) is $[A \cap B \neq \emptyset] \Rightarrow [(A \neq \emptyset) \wedge (B \neq \emptyset)]$. Hopefully this formulation makes the reason we do not have an "if and only if" relation in (1.12) even clearer.

We can also make increasingly complex statements by adding more variables. For example, statements of the form $\mathbb{A}(x, y)$ as x and y both vary across X. One can always view this as a statement about a pair (x, y) and change X to contain pairs, but this may not mitigate the additional complexity.

Example 1.4.4 *When* X *is the set of numbers and* $\mathbb{A}(x, y)$ *states that "*y *that is larger than* x*," where* x *and* y *are numbers, the statement* $(\forall x)(\exists y)(x < y)$ *says "for every* x *there is a* y *that is larger than* x*." The statement* $(\exists y)(\forall x)(x < y)$ *says "there is a* y *that is larger than every* x*." The former statement is true, but the latter is false.*

1.5 ◆ Taxonomy of Proofs

We now discuss broadly the methodology of proofs you will frequently encounter in economics. The most intuitive is the *direct proof* in the form of "$\mathbb{A} \Rightarrow \mathbb{B}$," discussed in (1.6). The work is to fill in the intermediate steps so that $\mathbb{A} \Rightarrow \mathbb{A}_1$, $\mathbb{A}_1 \Rightarrow \mathbb{A}_2$, and . . . $\mathbb{A}_n \Rightarrow \mathbb{B}$ are all tautologies. In terms of sets, this involves constructing n sets $A_1, \ldots, A_n$ such that $A \subset A_1 \subset \cdots \subset A_n \subset B$.

Notation Alert 1.5.A *The "*. . .*" indicates* A_2 *through* A_{n-1} *in the first list. The "*$\cdots$*" indicates the same sets in the second list, but we also mean to indicate that the subset relation holds for all the intermediate pairs.*

In some cases, the sets $A_1, \ldots, A_n$ arise from splitting $\mathbb{B}$ into cases. If we find $\mathbb{B}_1$, $\mathbb{B}_2$ such that $[\mathbb{B}_1 \vee \mathbb{B}_2] \Rightarrow \mathbb{B}$ and can show that $\mathbb{A} \Rightarrow [\mathbb{B}_1 \vee \mathbb{B}_2]$, then we are done.

In other cases it may be simpler to split $\mathbb{A}$ into cases. That is, sometimes it is easier to find $\mathbb{A}_1$ and $\mathbb{A}_2$ for which $\mathbb{A} \Rightarrow [\mathbb{A}^1$ and $\mathbb{A}^2]$ and then to show that $[\mathbb{A}_1 \Rightarrow \mathbb{B}] \vee [\mathbb{A}_2 \Rightarrow \mathbb{B}]$.

Another direct method of proof, called *induction*, works only for the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. Suppose we wish to show that $(\forall n \in \mathbb{N})\ \mathbb{A}(n)$ is true. This is equivalent to proving $\mathbb{A}(1) \wedge (\forall n \in \mathbb{N})\ (\mathbb{A}(n) \Rightarrow \mathbb{A}(n+1))$. This works since $\mathbb{A}(1)$ is true and $\mathbb{A}(1) \Rightarrow \mathbb{A}(2)$ and $\mathbb{A}(2) \Rightarrow \mathbb{A}(3)$ and so on. In Chapter 2 we show why induction works.

Proofs by contradiction are also known as indirect proofs. They may, initially, seem less natural than direct proofs. To help you on your way to becoming fluent in indirect proofs, we now give the exceedingly simple indirect proof of

$\mathbf{x}'_i$ to $\mathbf{x}_i$. By the definition of Walrasian equlibrium, we can sum (1.14) across all individuals to obtain

$$\sum_i \left(\sum_k p_k x'_{i,k} \right) > \sum_i \left(\sum_k p_k y_{i,k} \right). \tag{1.15}$$

Rearranging the summations in (1.15) gives

$$\sum_k \sum_i p_k x'_{i,k} > \sum_k \sum_i p_k y_{i,k}, \quad \text{equivalently}$$

$$\sum_k p_k \left(\sum_i x'_{i,k} \right) > \sum_k p_k \left(\sum_i y_{i,k} \right). \tag{1.16}$$

Since $(\mathbf{x}'_i)_{i \in I}$ is a feasible allocation, multiplying each term in (1.13) by the nonnegative number p_k and then summing yields

$$\sum_k p_k \left(\sum_i x'_{i,k} \right) \leq \sum_k p_k \left(\sum_i p_k y_{i,k} \right). \tag{1.17}$$

Let r be the number $\sum_k p_k \left(\sum_i y_{i,k} \right)$ and let s be the number $\sum_k p_k \left(\sum_i x'_{i,k} \right)$. Equation (1.16) is the statement, $\mathbb{C}$, that $s > r$, whereas (1.17) is the statement $\neg\mathbb{C}$ that $s \leq r$. We have derived the contradiction $[\mathbb{C} \wedge \neg\mathbb{C}]$, which we know to be false, from the supposition $[\mathbb{A} \wedge \neg\mathbb{B}]$. From this, we conclude that $[\mathbb{A} \Rightarrow \mathbb{B}]$. ■

As one becomes more accustomed to the patterns of logical arguments, details of the arguments are suppressed. Here is a shorthand, three-sentence version of the foregoing proof.

Proof. If $(\mathbf{x}_i)_{i \in I}$ is not Pareto efficient, $\exists (\mathbf{x}'_i)_{i \in I}$ feasible and unanimously preferred to $(\mathbf{x}_i)_{i \in I}$. Summing (1.14) across individuals yields $\sum_k \sum_i p_k x'_{i,k} > \sum_k \sum_i p_k y_{i,k}$. Since $(\mathbf{x}'_i)_{i \in I}$ is feasible, summing (1.13) over goods, we have $\sum_k \sum_i p_k x'_{i,k} \leq \sum_k \sum_i p_k y_{i,k}$. ■

Just as "$7x^2 + 9x < 3$" is a shorter and clearer version of "seven times the square of a number plus nine times that number adds to a number less than three," the shortening of proofs is mostly meant to help. It can, however, feel like a diabolically designed code, one meant to obfuscate rather than elucidate.

Some decoding hints:

1. Looking at the statement of Theorem 1.5.5, we see that it ends in "then $(\mathbf{x}_i)_{i \in I}$ is Pareto efficient." Since the shortened proof starts with the sentence "If $(\mathbf{x}_i)_{i \in I}$ is not Pareto efficient," you should conclude that we are offering a proof by contradiction. This means that you should be looking for a conclusion that is always false. Reaching such a falsity completes the proof.
2. Despite what it says, the second sentence in the shortened proof does more than sum (1.14); it rearranges the summation. Your job as a reader is to

CHAPTER 2

Set Theory

In the foundations of economic theory, one worries about the existence of optima for single-person decision problems and about the existence of simultaneous optima for linked, multiple-person-decision problems. The simultaneous optima are called equilibria. Often more interesting than the study of existence questions is the study of the changes in these optima and equilibria as aspects of the economic environment change, which is called "comparative statics." Since a change in one person's behavior can result in a change in another's optimal choices when the problems are linked, the comparative statics of equilibria will typically be a more complicated undertaking.

The early sections of this chapter cover notation, product spaces, relations, and functions. This is sufficient background for the foundational results in rational choice theory: conditions on preferences that guarantee the existence of optimal choices in finite contexts; representations of the optimal choices as solutions to utility maximization problems; and some elementary comparative statics results.

An introduction to weak orders, partial orders, and lattices provides sufficient background for the basics of monotone comparative statics based on supermodularity. It is also sufficient background for Tarski's fixed-point theorem, the first of the fixed point theorems we cover. Fixed-point theorems are often the tool used to show the existence of equilibria. Tarski's theorem also gives information useful for comparative statics, and we apply it to study the existence and properties of the set of stable matchings.

Whether or not the universe is infinite or finite but very large seems to be unanswerable. However, the mathematics of infinite sets often turns out to be much, much easier than finite mathematics. Imagine trying to study planar geometry under the simplifying assumption that the plane contains 293 million (or so) points. At the end of this chapter we deal with the basic results concerning infinite sets, results that we use extensively in our study of models of prices, quantities, and time, all of which begin in Chapter 3.

2.2 ◆ Notation and Other Basics

As in Chapter 1, we express the notion of membership by "$\in$" so that "$x \in A$" means "x is an element of the set A" and "$x \notin A$" means "x is not an element of A." We usually specify the elements of a set explicitly by saying "A is the set of all x in X having the property $\mathbb{A}$," and write $A = \{x \in X : \mathbb{A}(x)\}$. When the space X is understood, we may abbreviate this as $A = \{x : \mathbb{A}(x)\}$.

Example 2.2.1 *If $\mathbb{A}(\mathbf{x})$ is the property "is affordable at prices $\mathbf{p}$ and income w" and $X = \mathbb{R}^\ell_+$, then the Walrasian budget set, denoted $B(\mathbf{p}, w)$, is defined by $B(\mathbf{p}, w) = \{\mathbf{x} \in X : \mathbb{A}(\mathbf{x})\}$. With more detail about the statement $\mathbb{A}$, this is $B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^\ell_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}$.*

Definition 2.2.2 *For A and B subsets of X, we define:*

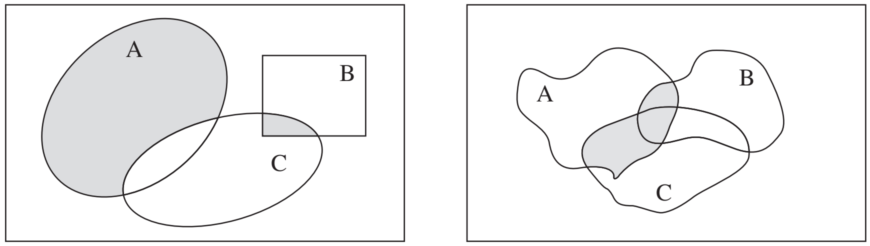
1. *$A \cap B$, the **intersection of** A **and** B, by $A \cap B = \{x \in X : [x \in A] \wedge [x \in B]\}$,*
2. *$A \cup B$, the **union of** A **and** B, by $A \cup B = \{x \in X : [x \in A] \vee [x \in B]\}$,*
3. *$A \subset B$, A **is a subset of** B, or B **contains** A, if $[x \in A] \Rightarrow [x \in B]$,*
4. *$A = B$, A **is equal to** B, if $[A \subset B] \wedge [B \subset A]$,*
5. *$A \neq B$, A **is not equal to** B, if $\neg[A = B]$,*
6. *$A \subsetneq B$, A **is a proper subset of** B, if $[A \subset B] \wedge [A \neq B]$,*
7. *$A \setminus B$, **the difference between** A **and** B, by $A \setminus B = \{x \in A : x \notin B\}$,*
8. *$A \Delta B$, **the symmetric difference between** A **and** B, by $A \Delta B = (A \setminus B) \cup (B \setminus A)$,*
9. *A^c, the **complement of** A, by $A^c = \{x \in X : x \notin A\}$,*
10. *$\emptyset$, **the empty set**, by $\emptyset = X^c$, and*
11. *A and B to be **disjoint** if $A \cap B = \emptyset$.*

These definitions can be visualized using Venn diagrams as in Figure 2.2.2.

Example 2.2.3 *If $X = \{1, 2, \ldots, 10\}$, the counting numbers between 1 and 10, $A = \{$even numbers in $X\}$, $B = \{$odd numbers in $X\}$, $C = \{$powers of 2 in $X\}$, and $D = \{$primes in $X\}$, then $A \cap B = \emptyset$, $A \cap D = \{2\}$, $A \setminus C = \{6, 10\}$, $C \subsetneq A$, $B \neq C$, $C \cup D = \{2, 3, 4, 5, 7, 8\}$, and $B \Delta D = \{2, 9\}$.*

There is a purpose to the notational choices made in defining "$\cap$" using "$\wedge$" and defining "$\cup$" using "$\vee$." Being in $A \cap B$ requires being in A $\wedge$ being in B, being in $A \cup B$ requires being in A $\vee$ being in B. The definitions of unions and intersections can easily be extended to arbitrary collections of sets. Let I be an index set, for example, $I = \mathbb{N} = \{1, 2, 3, \ldots\}$ as in Example 1.4.2 (p. 10), and let A_i, $i \in I$ be subsets of X. Then $\cup_{i \in I} A_i = \{x \in X : (\exists i \in I)[x \in A_i]\}$ and $\cap_{i \in I} A_i = \{x \in X : (\forall i \in I)[x \in A_i]\}$.

We have seen the following commutative, associative, and distributive properties before in Theorem 1.3.2 (p. 7), and they are easily checked using Venn diagrams.



$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

FIGURE 2.2.4

Theorem 2.2.4 *For sets A, B, and C,*

1. $A \cap B = B \cap A$, $A \cup B = B \cup A$;
2. $(A \cap B) \cap C = A \cap (B \cap C)$, $(A \cup B) \cup C = A \cup (B \cup C)$; *and*
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Exercise 2.2.5 Prove Theorem 2.2.4 from Theorem 1.3.2. [See Figure 2.2.4. The proof amounts to applying the logical connectives and above definitions: to show $A \cap B = B \cap A$, it is sufficient to note that $x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B) \Leftrightarrow (x \in B) \wedge (x \in A) \Leftrightarrow x \in B \cap A$.]

The following properties are used extensively in probability theory and are easily checked in a Venn diagram. [See Figure 2.2.6.]

Theorem 2.2.6 (DeMorgan's Laws) *If A, B, and C are any sets, then*

1. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$, *and*
2. $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

In particular, taking $A = X$, $(B \cup C)^c = B^c \cap C^c$ *and* $(B \cap C)^c = B^c \cup C^c$.

The last two equalities are "the complement of a union is the intersection of the complements" and "the complement of an intersection is the union of the complements." When we think of B and C as statements, $(B \cup C)^c$ is "not B or C," which is equivalent to, "neither B nor C," which is equivalent to, "not B and not C," and this is $B^c \cap C^c$. In the same way, $(B \cap C)^c$ is "not both B and C," which is equivalent to "either not B or not C," and this is $B^c \cup C^c$.

Proof. For (1) we show that $A \backslash (B \cup C) \subset (A \backslash B) \cap (A \backslash C)$, and $A \backslash (B \cup C) \supset (A \backslash B) \cap (A \backslash C)$.

($\subset$) Suppose $x \in A \backslash (B \cup C)$. Then $x \in A$ and $x \notin (B \cup C)$. Thus $x \in A$ and ($x \notin B$ and $x \notin C$). This implies $x \in A \backslash B$ and $x \in A \backslash C$. But this is just $x \in (A \backslash B) \cap (A \backslash C)$.

($\supset$) Suppose $x \in (A \backslash B) \cap (A \backslash C)$. Then $x \in (A \backslash B)$ and $x \in (A \backslash C)$. Thus $x \in A$ and ($x \notin B$ and $x \notin C$). This implies $x \in A$ and $x \notin (B \cup C)$. But this is just $x \in A \backslash (B \cup C)$. ■

Exercise 2.2.7 Finish the proof of Theorem 2.2.6.

The following are some of the most important sets we encounter in this book:

- $\mathbb{N} = \{1, 2, 3, \ldots\}$, the natural or "counting" numbers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the integers.
- $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, the nonnegative integers.
- $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$, the quotients, or rational numbers.
- $\mathbb{R}$, the set of "real numbers," that we construct in Chapter 3 by adding the so-called irrational numbers to $\mathbb{Q}$.

Note that $\mathbb{Q}$ contains all of the finite-length decimals, for example, $7.96518 = \frac{m}{n}$ for $m = 796{,}518$ and $n = 100{,}000$. This means that $\mathbb{Q}$ contains a representation for every physical measurement that we can make and every number we will ever see from a computer. The reason for introducing the extra numbers in $\mathbb{R}$ is not one of realism. Rather, we shall see that $\mathbb{Q}$ has "holes" in it, and even though the holes are infinitely small, they make analyzing some kinds of problems miserably difficult.

Even though we have not yet formally developed the set of numbers $\mathbb{R}$, the following example is worth seeing early and often.

Example 2.2.11 (The Field of Half-Closed Intervals) *Let $X = \mathbb{R}$ and for $a, b \in X$, $a < b$, define $(a, b] = \{x \in X : a < x \leq b\}$. Set $\mathcal{E} = \{(a, b] : a < b,\ a, b \in \mathbb{R}\}$ and let $\mathcal{X} = \mathcal{F}^\circ(\mathcal{E})$. A set E belongs to $\mathcal{X}$ iff it can be expressed as a finite union of disjoint intervals of one of the following three forms: $(a, b]$; $(-\infty, b] = \{x \in X : x \leq b\}$; or $(a, +\infty) = \{x \in X : a < x\}$.*

It is worth noting the style we used in this last example. When we write "define $(a, b] = \{x \in X : a < x \leq b\}$," we mean that whenever we use the symbols to the left of the equality, "$(a, b]$" in this case, we intend that you will understand these symbols to mean the symbols to the right of the equality, "$\{x \in X : a < x \leq b\}$" in this case. The word "let" is used in exactly the same way.

In a perfect world, we would take you through the construction of $\mathbb{N}$ starting from the idea of the empty set. Had we done this construction properly, the following would be a result.

Axiom 1 *Every nonempty $S \subset \mathbb{N}$ contains a smallest element, that is, $\leq$ is a well-ordering of $\mathbb{N}$.*

To be very explicit, we are assuming that if $S \in \mathcal{P}(\mathbb{N})$, $S \neq \emptyset$, then there exists $n \in S$ such that for all $m \in S$, $n \leq m$. There cannot be two such n, because $n \leq n'$ and $n' \leq n$ iff $n = n'$.

2.3 ◆ Products, Relations, Correspondences, and Functions

There is another way to construct new sets out of given ones, which involves the notion of an "ordered pair" of objects. In the set $\{a, b\}$, there is no preference

space, just a product of n spaces. The following is an example of an inductive definition.

Definition 2.3.4 *Given a collection of sets, $\{A_m : m \in \mathbb{N}\}$, we define $\times_{m=1}^{1} A_m = A_1$ and inductively define $\times_{m=1}^{n} A_m = \times_{m=1}^{n-1} A_m \times A_n$.*

An ordered pair is called a 2**-tuple** and an n**-tuple** is an element of $\times_{m=1}^{n} A_m$. Sets of 2-tuples are called binary relations and sets of n-tuples are called n-ary relations. Relations are the mathematical objects of interest.

Definition 2.3.5 *Given two sets A and B, a* ***binary relation between*** *A* ***and*** *B, known simply as a* ***relation*** *if A and B can be inferred from context, is a subset $R \subset A \times B$. We use the notation $(a, b) \in R$ or aRb to denote the relation R holding for the ordered pair (a, b) and read it "a is in the relation R to b." If $R \subset A \times A$, we say that R is a* ***relation on*** *A. The* ***range of a relation*** *R is the set of $b \in B$ for which there exists $a \in A$ with $(a, b) \in R$.*

Example 2.3.6 *$A = \{0, 1, 2, 3, 4\}$, so that $A \times A$ has twenty-five elements. With the usual convention that x is on the horizontal axis and y on the vertical, the relations $\leq$, $<$, $=$, and $\neq$ can be graphically represented by the $\otimes$'s in*

4	⊗	⊗	⊗	⊗	⊗
3	⊗	⊗	⊗	⊗	
2	⊗	⊗	⊗		
1	⊗	⊗			
0	⊗				
	0	1	2	3	4

$\leq$

4	⊗	⊗	⊗	⊗	
3	⊗	⊗	⊗		
2	⊗	⊗			
1	⊗				
0					
	0	1	2	3	4

$<$

4					⊗
3				⊗	
2			⊗		
1		⊗			
0	⊗				
	0	1	2	3	4

$=$

4	⊗	⊗	⊗	⊗	
3	⊗	⊗	⊗		⊗
2	⊗	⊗		⊗	⊗
1	⊗		⊗	⊗	⊗
0		⊗	⊗	⊗	⊗
	0	1	2	3	4

$\neq$

Note that $\leq$, $<$, $=$, and $\neq$ are sets. In terms of these sets, $\leq$ is the union of the disjoint sets, $<$ and $=$, and the complement of $=$ is $\neq$.

Relations can also be used in all kinds of cute ways.

Example 2.3.7 *Let $A = \{$Austin, Des Moines, Harrisburg$\}$ and $B = \{$Texas, Iowa, Pennsylvania$\}$. Then the relation $R = \{$(Austin,Texas),(Des Moines,Iowa), (Harrisburg,Pennsylvania)$\}$ expresses "is the state capital of."*

A relation between A and B is a subset of $A \times B$. A function from A to B is a special kind of relation, and a correspondence from A to B is a way to view a relation as a function; that is, it is an alternate definition of a relation.

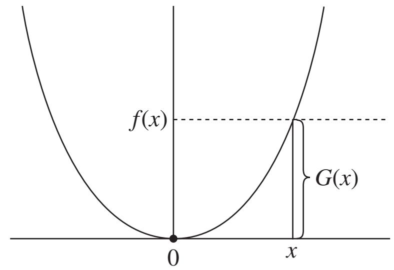


FIGURE 2.3.15

There are two equivalent ways to understand a correspondence from A to B: as a function from A to the subsets of B or as a subset of $A \times B$.

Definition 2.3.12 *A* ***correspondence*** *G, denoted $G : A \twoheadrightarrow B$, is a relation between A and B. For each $a \in A$, the set of b such that $(a, b) \in G$ is denoted $G(a)$. Equivalently, G is a function from A to $\mathcal{P}(B)$ assigning a set, $G(a)$, to each element $a \in A$.*

Exercise 2.3.13 For $A = B = [0, 1]$, draw three different correspondences from A to B that are not functions.

Exercise 2.3.14 Explicitly give the four relations in Example 2.3.6 (p. 23) as functions from A to $\mathcal{P}(A)$.

A correspondence G may have $G(a) = \emptyset$ or have $G(a)$ containing many elements. A function is a special kind of correspondence where for all a, $G(a)$ contains exactly one point.

Example 2.3.15 *In Figure 2.3.15, you can see the graph of the function $f(x) = x^2$ and the correspondence $G(x) = [0, x^2]$, $f : \mathbb{R} \to \mathbb{R}$, and $G : \mathbb{R} \twoheadrightarrow \mathbb{R}$. $G(0)$ consists of one point; for $x \neq 0$, $G(x)$ is an interval.*

Definition 2.3.16 *Given a function $f : A \to B$ and $E \subset A$, $E \neq \emptyset$,* ***the image of*** *E* ***under*** *f is written $f(E)$ and is defined by $f(E) = \{b \in B : (\exists e \in E)[f(e) = b]\}$. The* ***restriction of*** *f* ***to*** *E is written $f_{|E}$ and is defined as the function $f : E \to B$ having as a graph the set $Gr(f_{|E}) = Gr(f) \cap (E \times B)$.*

The image of a set E is the set of points to which it is mapped. The restriction of a function f to a set E ignores the behavior of the function outside of the set E.

Definition 2.3.17 *A set X is* ***finite*** *if it is empty, in which case it has 0 elements, or if there exists $n \in \mathbb{N}$ and a function $f : \{1, \ldots, n\} \to X$ such that $f(\{1, \ldots, n\}) = X$. The smallest n with this property is called the* ***cardinality of*** *X and is denoted $\#X$.*

Example 2.4.3 *Define the* ***congruence modulo 4*** *relation* M_4 *on* $\mathbb{Z}$ *by* $\forall x, y \in \mathbb{Z}$, xM_4y *if remainders obtained by dividing* x *and* y *by* 4 *are equal. For example,* $13M_465$ *because dividing* 13 *and* 65 *by* 4 *gives a remainder of* 1.

Exercise 2.4.4 Show that congruence modulo 4 is an equivalence relation.

Definition 2.4.5 *Given an equivalence relation* $\sim$ *on a set* A *and an element* $x \in A$, *we define the* ***equivalence class determined by*** x *by* $E_x = \{y \in A : y \sim x\}$. *Note that* $x \in E_x$ *since* $x \sim x$.

Example 2.4.6 *The equivalence classes of* $\mathbb{Z}$ *for the relation* M_4 *are determined by* $x \in \{0, 1, 2, 3\}$, *where* $E_x = \{z \in \mathbb{Z} : \exists k \in \mathbb{Z}, z = 4k + x\}$, *that is,* x *is the remainder when* z *is divided by* 4.

Equivalence classes have the following property.

Theorem 2.4.7 *Two equivalence classes* E *and* E' *are either disjoint or equal.*

Proof. Let $E = \{y \in A : y \sim x\}$ and $E' = \{y \in A : y \sim x'\}$. If $E \cap E' = \emptyset$, then E and E' are disjoint. If $\exists z \in E \cap E'$, we show that $E = E'$. The first step is to demonstrate that $E \subset E'$, and the second step is to show that $E' \subset E$.

Let $w \in E$. We must show that $w \in E'$. Since $w \in E$, $w \sim x$. As $z \in E \cap E'$, we know that $z \sim x$ and $z \sim x'$. By transitivity $w \sim z$; hence $w \sim x'$, so that $w \in E'$.

Reversing the roles of E and E' in this argument demonstrates that $E' \subset E$. ■

Looking at Example 2.4.6 in light of Theorem 2.4.7, we see that if two elements are in relation, they have the same equivalence class. So $E_1 = E_5 = E_9 = \ldots$ and $E_2 = E_6 = E_{10} = \ldots$. More generally, for all $n, k \in \mathbb{Z}$, $E_n = E_{4k+n}$.

Notation 2.4.8 $A/\sim$ *denotes the collection of all* $\sim$*-equivalence classes.*

Mnemonically, $\sim$ divides A into a collection of disjoint sets, so we write $A/\sim$. The union of all the sets in $A/\sim$ equals all of A because every element a of A belongs to exactly one of the equivalence classes. Another way to understand $A/\sim$ is as a partition of A.

Definition 2.4.9 *A* ***partition*** *of a set* A *is a collection of nonempty disjoint subsets of* A *whose union is all of* A.

We saw in Theorem 2.4.7 that equivalence relations give rise to partitions. The reverse is also true.

Exercise 2.4.10 For a partition, $\mathcal{C}$, of A, define $\sim_{\mathcal{C}}$ by $x \sim_{\mathcal{C}} y$ iff x and y belong to same element of $\mathcal{C}$. Show that $\sim_{\mathcal{C}}$ is an equivalence relation.

Example 2.4.11 *The equivalence classes of* $\mathbb{Z}$ *in Example 2.4.6 constitute a partition;* $E_0 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$, $E_1 = \{\ldots, -7, -3, 1, 5, \ldots\}$, $E_2 = \{\ldots, -6, -2, 2, 6, \ldots\}$, *and* $E_3 = \{\ldots, -5, -1, 3, 7, \ldots\}$ *are disjoint and their union is all of* $\mathbb{Z}$. *Generally,* $\mathbb{Z}$ *can be partitioned to* n *subsets via the equivalence relation* $x \sim y$ *iff* x, y *have the same remainder after division by* n. *The partitioning sets contain those subsets having remainders* $0, 1, \ldots, n - 1$ *to* n. *Another simple example is a coin toss experiment where the sample space* $S = \{Heads, Tails\}$ *has mutually exclusive events (i.e.,* $Heads \cap Tails = \emptyset$*).*

Example 2.5.2 *One of the crucial order properties of the set of numbers, $\mathbb{R}$, is that $\leq$ and $\geq$ are complete and transitive.*

Completeness neither implies nor is implied by transitivity. To see this, the following exercise gives an example of a relation that satisfies both completeness and transitivity, gives other relations that satisfy one of the conditions but not the other, and gives a relation that satisfies neither. When you see a new concept, you should develop the two habits that this exercise exemplifies: finding examples in which the new concept does and does not hold and finding examples that demonstrate how the new concept interacts with other, possibly related concepts.

Exercise 2.5.3 In Example 2.3.6 (p. 23), show that $\leq$ is complete and transitive, that $<$ and $=$ are transitive but not complete, and that $\neq$ is neither transitive nor complete. Check that the relation $\succsim$ given later in Example 2.5.6 is complete but not transitive.

In thinking about preference relations, completeness is the requirement that any pair of choices can be compared for the purposes of making a choice. Given how much effort it is to make life decisions (jobs, marriage, kids), completeness is a strong requirement. When a relation is not complete, there are choices that cannot be compared and there may be two or more optimal choices in the set. For example, consider the relation $\subset$ on the set of all subsets of $A = \{1, \ldots, 10\}$ except A itself. Suppose we are looking for the largest subset. Then each of the subsets with nine elements is a largest element and they cannot be compared with each other. Transitivity is another rationality requirement. If violated, vicious cycles could arise among three or more options—any choice would have another that strictly beats it. To say "strictly beats" we need the following.

Definition 2.5.4 *Given a relation $\succsim$, define $x \succ y$ by $[x \succsim y] \wedge \neg[y \succsim x]$ and $x \sim y$ by $[x \succsim y] \wedge [y \succsim x]$.*

When talking about preference relations, "$x \succ y$" is read as "x is strictly preferred to y" and "$x \sim y$" is read as "x is indifferent to y." From the definitions, you can show that $[x \succsim y] \Leftrightarrow [[x \succ y] \vee [x \sim y]]$, and that the sets $\succ$ and $\sim$ are disjoint.

Exercise 2.5.5 Show that $x \sim y$ is an equivalence relation if $\succsim$ is rational.

Example 2.5.6 *Suppose you are at a restaurant and you have a choice among four meals, pork, beef, chicken, or fish, all costing the same. Suppose that your preferences, $\succsim$, and strict preferences, $\succ$, are given by*

pork	⊗			⊗
beef			⊗	⊗
fish		⊗	⊗	⊗
chic	⊗	⊗	⊗	
	chic	fish	beef	pork

$\succsim$

pork	⊗			
beef				⊗
fish			⊗	⊗
chic		⊗	⊗	
	chic	fish	beef	pork

$\succ$

The basic behavioral assumption in economics is that you choose the option that you like best. Here $p \succ b \succ f \succ c \succ p$. Suppose you try to find your favorite

of the form $C^*(B) = C^*(B, \succsim) = \{x \in B : \forall y \in B, \ \ x \succsim y\}$. In light of Theorem 2.5.11, $C^*(B) = \{x \in B : \forall y \in B, \ \ u(x) \geq u(y)\}$, that is, $C^*(B)$ is the set of utility maximizing elements of B.

The set of maximizers, the **argmax**, is a sufficiently important construct in economics that it has its own notation.

Definition 2.5.13 *For a nonempty set X and function $f : X \to \mathbb{R}$, $\arg\max_{x \in X} f(x)$ is the set $\{x^* \in X : (\forall x \in X)[f(x^*) \geq f(x)]\}$.*

The basic existence result tells us that the preference-maximizing choice rule yields a nonempty set of choices.

Theorem 2.5.14 *If B is a nonempty finite subset of X and $\succsim$ is a rational preference relation on X, then $C^*(B) \neq \emptyset$.*

Proof. Define $S^* = \cap_{x \in B}\{y \in B : y \succsim x\}$. It is clear that $S^* = C^*(B)$ (and you should check both directions of the inclusion if you are not used to writing proofs). All that is left is to show that $S^* \neq \emptyset$.

Let $n_B = \#B$ and pick a function $f : \{1, \ldots, n_B\} \to B$ such that $B = f(\{1, \ldots, n_B\})$. This means we order (or count) members of B as $f(1), \ldots, f(n_B)$. For $m \in \{1, \ldots, n_B\}$, let $S^*(m) = \{y \in B : \forall n \leq m, \ \ y \succsim f(n)\}$, so that $S^* = S^*(n_B)$. In other words, $S^*(m)$ contains the best elements between $f(1), \ldots, f(m)$ with respect to $\succsim$. Now using a function f^*, we inductively pick up the largest element among $f(1), \ldots, f(n)$ for all n. Define $f^*(1) = f(1)$. Given that $f^*(m-1)$ has been defined, define

$$f^*(m) = \begin{cases} f(m) & \text{if } f(m) \succsim f^*(m-1), \\ f^*(m-1) & \text{if } f^*(m-1) \succ f(m). \end{cases} \tag{2.1}$$

For each $m \in \{1, \ldots, n_B\}$, $S^*(m) \neq \emptyset$ because it contains $f^*(m)$, and by transitivity, $f^*(n_b) \in S^*$. ■

The idea of the proof was simply to label the members of the finite set B and check its members step by step. We simply formalized this idea using logical tools and the definition of finiteness.

For $R, S \subset X$, we write $R \succsim S$ if $x \succsim y$ for all $x \in R$ and $y \in S$, and $R \succ S$ if $x \succ y$ for all $x \in R$ and $y \in S$. The basic comparison result for choice theory is that larger sets of options are at least weakly better.

Theorem 2.5.15 *If $A \subset B$ are nonempty finite subsets of X and $\succsim$ is a rational preference relation on X, then*

1. $[x, y \in C^*(A)] \Rightarrow [x \sim y]$, *optima are indifferent,*
2. $C^*(B) \succsim C^*(A)$, *larger sets are at least weakly better, and*
3. $[C^*(B) \cap C^*(A) = \emptyset] \Rightarrow [C^*(B) \succ C^*(A)]$, *a larger set is strictly better if it has a disjoint set of optima.*

Proof. The proof of (1) combines two proof strategies: contradiction and splitting into cases. Suppose that $[x, y \in C^*(A)]$ but $\neg[x \sim y]$. We split the statement $\neg[x \sim y]$ into two cases, $[\neg[x \sim y]] \Leftrightarrow [[x \succ y] \vee [y \succ x]]$. If $x \succ y$, then $y \notin C^*(A)$, a contradiction. If $y \succ x$, then $x \notin C^*(A)$, a contradiction.

To prove (2), we must show that $[[x \in C^*(B)] \wedge [y \in C^*(A)]] \Rightarrow [x \succsim y]$. We again give a proof by contradiction. Suppose that $[x \in C^*(B)] \wedge [y \in C^*(A)]$ but $\neg[x \succsim y]$. Since $\succsim$ is complete, $\neg[x \succsim y] \Rightarrow [y \succ x]$. As $y \in A$ and $A \subset B$, we know that $y \in B$. Therefore, $[y \succ x]$ contradicts $x \in C^*(A)$.

In what is becoming a pattern, we also prove (3) by contradiction. Suppose that $[C^*(B) \cap C^*(A) = \emptyset]$ but $\neg[C^*(B) \succ C^*(A)]$. By the definition of $R \succ S$ and the completeness of $\succsim$, $\neg[C^*(B) \succ C^*(A)]$ implies that there exists $y \in C^*(A)$ and $x \in C^*(B)$ such that $y \succsim x$. By (1), this implies that $y \in C^*(B, \succsim)$, which contradicts $[C^*(B) \cap C^*(A) = \emptyset]$. ■

2.5.c Revealed Preference

We now approach the choice problem starting with a choice rule rather than with a preference relation. The question is whether there is anything new or different when we proceed in this direction. The answer is "No, provided the choice rule satisfies a minimal consistency requirement, and satisfying this minimal consistency requirement reveals a preference relation."

A choice rule C defines a relation, $\succsim^*$, "revealed preferred," defined by $x \succsim^* y$ if $(\exists B \in \mathcal{P}(X))[[x, y \in B] \wedge [x \in C(B)]]$. Note that $\neg[x \succsim^* y]$ is $(\forall B \in \mathcal{P}(X))[\neg[x, y \in B] \vee \neg[x \in C(B)]]$, equivalently $(\forall B \in \mathcal{P}(X))[[x \in C(B)] \Rightarrow [y \notin B]]$. In words, x is revealed preferred to y if there is a choice situation, B, in which both x and y are available, and x belongs to the choice set.

From the relation $\succsim^*$ we define "revealed strictly preferred," $\succ^*$, as in Definition 2.5.4 (p. 29). It is both a useful exercise in manipulating logic and a good way to understand a piece of choice theory to explicitly write out two versions of the meaning of $x \succ^* y$:

$$(\exists B_x \in \mathcal{P}(X))[[x, y \in B_x] \wedge [x \in C(B_x)]] \\ \wedge (\forall B \in \mathcal{P}(X))[[y \in C(B)] \Rightarrow [x \notin B]], \tag{2.2}$$

equivalently

$$(\exists B_x \in \mathcal{P}(X))[[x, y \in B_x] \wedge [x \in C(B_x)] \wedge [y \notin C(B_x)]] \\ \wedge (\forall B \in \mathcal{P}(X))[[y \in C(B)] \Rightarrow [x \notin B]].$$

In words, the latter of these says that there is a choice situation where x and y are both available, x is chosen but y is not, and if y is ever chosen, then we know that x was not available.

A set $B \in \mathcal{P}(X)$ reveals a strict preference of y over x, written $y \succ_B x$, if $x, y \in B$ and $y \in C(B)$ but $x \notin C(B)$.

Definition 2.5.16 *A choice rule satisfies the* ***weak axiom of revealed preference*** *if* $[x \succsim^* y] \Rightarrow \neg(\exists B)[y \succ_B x]$.

This is the minimal consistency requirement. Satisfying this requirement means that choosing x when y is available in one situation is not consistent with choosing y but not x in some other situation where they are both available.

Theorem 2.5.17 *If C is a choice rule satisfying the weak axiom, then $\succsim^*$ is rational, and for all $B \in \mathcal{P}(X)$, $C(B) = C^*(B, \succsim^*)$. If $\succsim$ is rational, then $B \mapsto C^*(B, \succsim)$ satisfies the weak axiom, and $\succsim = \succsim^*$.*

Proof. Suppose that C is a choice rule satisfying the weak axiom.

We must first show that $\succsim^*$ is complete and transitive.

Completeness: For all $x, y \in X$, $\{x, y\} \in \mathcal{P}(X)$ is a nonempty set. Therefore $C(\{x, y\}) \neq \emptyset$, so that $x \succsim^* y$ or $y \succsim^* x$.

Transitivity: Suppose that $x \succsim^* y$ and $y \succsim^* z$. We must show that $x \succsim^* z$. To do this, it is sufficient to demonstrate that $x \in C(\{x, y, z\})$. Since $C(\{x, y, z\})$ is a nonempty subset of $\{x, y, z\}$, we know that there are three cases: $x \in C(\{x, y, z\})$, $y \in C(\{x, y, z\})$, and $z \in C(\{x, y, z\})$. We must show that each of these cases leads to the conclusion that $x \in C(\{x, y, z\})$.

Case 1: This one is clear.

Case 2: $y \in C(\{x, y, z\})$, the weak axiom, and $x \succsim^* y$ implies that $x \in C(\{x, y, z\})$.

Case 3: $z \in C(\{x, y, z\})$, the weak axiom, and $y \succsim^* z$ implies that $y \in C(\{x, y, z\})$. As we just saw in Case 2, this means that $x \in C(\{x, y, z\})$.

We now show that for all $B \in \mathcal{P}(X)$, $C(B) = C^*(B, \succsim^*)$. Pick an arbitrary $B \in \mathcal{P}(X)$. It is sufficient to establish that $C(B) \subset C^*(B, \succsim^*)$ and $C^*(B, \succsim^*) \subset C(B)$.

Pick an arbitrary $x \in C(B)$. By the definition of $\succsim^*$, for all $y \in B$, $x \succsim^* y$. By the definition of $C^*(\cdot, \cdot)$, this means that $x \in C^*(B, \succsim^*)$.

Now pick an arbitrary $x \in C^*(B, \succsim^*)$. By the definition of $C^*(\cdot, \cdot)$, this means that $x \succsim^* y$ for all $y \in B$. By the definition of $\succsim^*$, for each $y \in B$, there is a set B_y such that $x, y \in B_y$ and $x \in C(B_y)$. As C satisfies the weak axiom, for all $y \in B$, there is no set B_y with the property that $y \succ_{B_y} x$. Since $C(B) \neq \emptyset$, if $x \notin C(B)$, then we would have $y \succ_B x$ for some $y \in B$, a contradiction. ■

Exercise 2.5.18 What is left to be proved in Theorem 2.5.17? Provide the missing step(s).

It is important to note the reach and the limitation of Theorem 2.5.17.

Reach: First, we did not use X being finite at any point in the proof, so it applies to infinite sets. Second, the proof would go through so long as C is defined on all two- and three-point sets. This means that we can replace $\mathcal{P}(X)$ with a family of sets $\mathcal{B}$ throughout, provided $\mathcal{B}$ contains all two- and three-point sets.

Limitation: In many of the economic situations of interest, the two- and three-point sets are not the ones that people are choosing from. For example, the leading case has $\mathcal{B}$ as the class of Walrasian budget sets.

2.6 ◆ Direct and Inverse Images, Compositions

Projections map products to their axes in a natural way: $\text{proj}_A : A \times B \to A$ is defined by $\text{proj}_A((a, b)) = a$; and $\text{proj}_B : A \times B \to B$ is defined by $\text{proj}_A((a, b)) = b$. The projections of a set $S \subset A \times B$ are defined by $\text{proj}_A(S) = \{a : \exists b \in B,\ \ (a, b) \in$

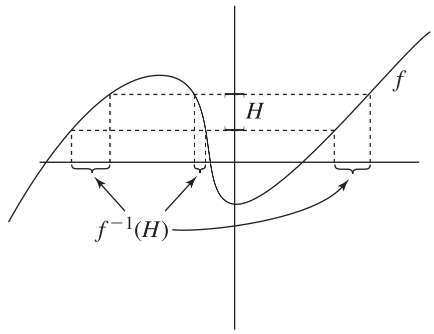


FIGURE 2.6.10

Exercise 2.6.6 Find and prove the analogue of Theorem 2.6.4 when the function f is replaced with a correspondence G, giving examples with the subset relations being proper.

2.6.b Inverse Relations

Inverse relations simply reverse the order in which we consider the axes.

Definition 2.6.7 *Given a relation R between A and B, the **inverse of** R is the relation R^{-1} between B and A defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$. Images of sets under R^{-1} are called **inverse images**.*

The inverse of a function need not be a function, though it will always be a correspondence.

Example 2.6.8 *In general, functions are many-to-one. For example $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$ maps both $+\sqrt{r}$ and $-\sqrt{r}$ to r when $r \geq 0$. In this case, the relation f^{-1}, viewed as a correspondence maps every nonnegative r to $\{-\sqrt{r}, +\sqrt{r}\}$, and maps every negative r to $\emptyset$.*

Example 2.6.9 *Let W be a finite set (of workers) and F a finite set (of firms). A function μ mapping W to $F \cup W$ is a **matching** iff for all w, $[\mu(w) \in F] \vee [\mu(w) = w]$. We interpret $\mu(w) = w$ as the worker w being self-employed or unemployed. For $f \in F$, $\mu^{-1}(f) \subset W$ is the set of people who work at firm f.*

It is well worth the effort to be even more specific for functions.

Definition 2.6.10 *If f is a function from A to B and $H \subset B$, then the **inverse image of** H **under** f, denoted $f^{-1}(H)$, is the subset $\{a \in A | f(a) \in H\}$.* [*See Figure 2.6.10.*]

When $H = \{b\}$ is a one-point set, we write $f^{-1}(b)$ instead of $f^{-1}(\{b\})$.

Exercise 2.6.11 Just to be sure that the notation is clear, prove the following and illustrate the results with pictures: $f^{-1}(H) = \cup_{b \in H} f^{-1}(b)$, $\text{proj}_B^{-1}(H) = A \times H$, and $\text{proj}_A^{-1}(E) = E \times B$.

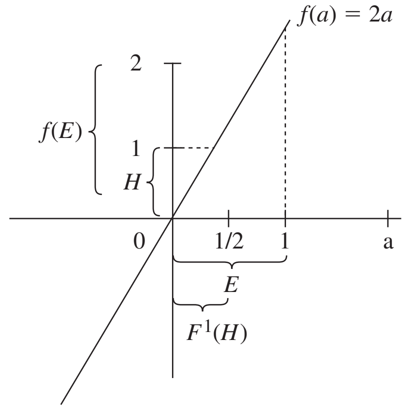


FIGURE 2.6.18 $-f : \mathbb{R} \to \mathbb{R}$ given by $f(a) = 2a$.

Example 2.6.16 *Let $E = \{2, 4, 6, \ldots\}$ be the set of even natural numbers and define $f(n) = 2n$. Then f is one-to-one, $f(\mathbb{N}) = E$, and f^{-1} is a function from E to $\mathbb{N}$.*

Definition 2.6.17 *If Range$(f) = B$, f maps A* ***onto*** *B and we call f a* ***surjection****; f is a* ***bijection*** *if it is one-to-one and onto, that is, if it is both an injection and a surjection, in which case we write $f : A \leftrightarrow B$.*

Note that surjectiveness of a map depends on the set into which the map is defined. For example, if we consider $f(x) = x^2$ as $f : \mathbb{R} \to \mathbb{R}_+$ then it is onto, whereas the same function viewed as $f : \mathbb{R} \to \mathbb{R}$ is not onto.

To summarize:

- injections are one-to-one and map A into B but may not cover all of B;
- surjections put A all over B but may not be one-to-one; and
- bijections from A to B are one-to-one onto functions, which means that their inverse correspondences are functions from B to A.

Example 2.6.18 *Let $E = [0, 1] \subset A = \mathbb{R}$, $H = [0, 1] \subset B = \mathbb{R}$, and $f(a) = 2a$. Range$(f) = \mathbb{R}$ so that f is a surjection; the image set is $f(E) = [0, 2]$; the inverse image set is $f^{-1}(H) = [0, \frac{1}{2}]$; and f is an injection, has inverse $f^{-1}(b) = \frac{1}{2}b$, and as a consequence of being one-to-one and onto is a bijection.* [*See Figure 2.6.18.*]

Exercise 2.6.19 Show that if $f : A \leftrightarrow B$ is a bijection between A and B, then the subset relations in Theorem 2.6.4 (p. 34) hold with equality.

2.6.d Compositions of Functions

If we first apply f to an $a \in A$ to get $b = f(a)$, and then apply g to b, we have a new, composite function, $h(a) = g(f(a))$.