Chapter One

The Single Period Binomial Model

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1.1 INTRODUCTION

Derivatives pricing and investment management seem to have little in common. Even at the organizational level, they belong to two quite separate parts of financial markets. The so-called sell side, represented mainly by the investment banks, among other things offers derivatives products to their customers. Some of them are wealth managers, belonging to the so-called buy side of financial markets.

So far, the only universally accepted method of derivative pricing is based upon the idea of risk replication. Models have been developed which allow for perfect replication of option payoffs via implementation of a replicating and self-financing strategy. We call such models complete. The option price is calculated as the cost of this replication. Adjustments to the price are later made to cover for risks due to the unrealistic representation of reality.

More accurate description of the market is given by the so-called incomplete models in which not all risk in a derivative product can be eliminated by dynamic hedging. However, this potential model advantage is hampered by another difficulty. Namely, the concept of price for a derivative contract is not uniquely defined. Many

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approaches have been proposed and extensively studied; however, until now no clear consensus has emerged.

On the other side of the spectrum of financial markets there are wealth managers. They have developed their own methodology for implementation of their investment decisions. They may use derivative products to improve their performance; however, their focus is on investment strategy with a view to optimize returns rather than on risk replication. Therefore, it should not come as a surprise that the models they use are very different from the models used in derivatives pricing.

The main aim of this chapter is to work toward convergence of the methodologies used in these apparently quite distant areas. The idea is to associate the concept of price for a derivative contract with a rather natural, to a wealth manager, constraint, that is, maximization of expected utility of wealth. We choose to work with exponential utility and a very simple model structure, namely, the classical single period binomial model. We do so in order to eliminate all technical difficulties, explain the fundamental ideas and compare them with the classical arbitrage free theory, and concentrate exclusively on the most important links between the two areas.

The chapter is organized as follows. In the next section we introduce and analyze in detail the single period binomial model. In particular, we derive an intuitively appealing formula for the indifference price of a general claim. Then, we study the various properties of the indifference price and exhibit the connection with convex risk measures. The link with the classical methodology of pricing by replication is analyzed next. It turns out that the analogue of the so-called delta retains its natural interpretation as the sensitivity of the price with respect to the movement of the instrument used for hedging. Moreover, it also appears that the components of risk that are left unhedged, in our incomplete model setup, have zero value from the perspective of valuation by indifference.

Another important observation about the nature of pricing by the indifference is exposed in the subsection dedicated to relative pricing. Namely, this type of pricing scheme is relative to the agent’s portfolio in contrast to the arbitrage-free pricing scheme which is relative to the market portfolio. When interpreted this way, the indifference valuation can be viewed as linear, while, of course, when seen as a functional over a set of random variables it is not.

Going deeper into the comparisons with the pricing by arbitrage, we then investigate the issue of unit choice and the necessary consistency with the static no-arbitrage constraint. We show, in particular, that in order to eliminate static arbitrage one needs to relate the risk aversion parameter with the unit of wealth. To our knowledge, this is the first time that modeling issues pertinent to consistency across units have been identified and addressed.

To accommodate more general situations, we allow for the risk aversion to be modified according to our local in time views about anticipated performance of the traded securities. Specifically, we study the case when the risk aversion parameter depends on the future value of the traded stock. It turns out that the pricing rule retains its intuitive form. Moreover, the associated value function exhibits an interesting relationship between the risk tolerance (the reciprocal of risk aversion) at the end and at the beginning of a time period. Namely, the end of a period of risk tolerance
can be viewed as an option payoff, and the consistent risk tolerance for the beginning of the period is its arbitrage-free price.

Motivated by the general need for absence of static arbitrage, and by the above observation in particular, we conclude introducing the notions of the utility normalization and the concepts of the backward and forward utilities.

1.2 THE INCOMPLETE MODEL

We introduce a simple one-period binomial model with one riskless and two risky assets, of which only one is traded. By construction, the model is incomplete and our aim is to develop a coherent approach for investment management and derive from it a pricing methodology for derivative contracts. Optimal investment management is based on maximization of expected utility of wealth. There are a number of constraints we want to impose on our investment decision process and on the derivatives valuation method. To mention just two, we want our investment decisions not to depend on units in which the wealth is expressed. This is mainly because we also need to ensure that our pricing method is consistent with the absence of arbitrage and that it is also numeraire independent. We also want our pricing concept to have a clear intuitive meaning, so an effort is made to interpret the results and, whenever possible, to draw analogies with the classical arbitrage-free theory of complete markets.

1.2.1 Indifference Price Representation

Consider a single period model in a market environment with one riskless and two risky assets. The riskless asset is assumed to offer zero interest rate. Only one of the traded assets can be traded, taken to be a stock. The current values of the traded and nontraded risky assets are denoted, respectively, by $S_0$ and $Y_0$. At the end of the period $T$, the value of the traded asset is $S_T$ with $S_T = S_0 \xi$, where the random variable $\xi = \xi^d, \xi^n$ and $0 < \xi^d < 1 < \xi^n$. Similarly, the value of the nontraded asset $Y_T$ satisfies $Y_T = Y_0 \eta$, with $\eta = \eta^d, \eta^n$, with $\eta^d < \eta^n$ ($Y_0, Y_T \neq 0$).

We introduce randomness into our single-period model by means of the probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$, where $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\mathbb{P}$ is a probability measure on the $\sigma-$algebra $\mathcal{F}_T = 2^{\Omega}$ of all subsets of $\Omega$. For each $i = 1, \ldots, 4$, we assume that $p_i = \mathbb{P}\{\omega_i\} > 0$ and we model the upwards and the downwards movement of the two risky assets $S_T$ and $Y_T$ by setting their values as follows:

$$S_T(\omega_1) = S_0 \xi^d, \quad Y_T(\omega_1) = Y_0 \eta^d \quad S_T(\omega_3) = S_0 \xi^n, \quad Y_T(\omega_3) = Y_0 \eta^n$$

$$S_T(\omega_2) = S_0 \xi^d, \quad Y_T(\omega_2) = Y_0 \eta^d \quad S_T(\omega_4) = S_0 \xi^n, \quad Y_T(\omega_4) = Y_0 \eta^n.$$

The measure $\mathbb{P}$ represents the so-called historical measure.

Observe that the $\sigma$-algebra $\mathcal{F}_T$ coincides with the $\sigma$-algebra $\mathcal{F}_T^{(S,Y)}$ generated by the random variables $S_T$ and $Y_T$. In what follows we will also need the $\sigma$-algebra $\mathcal{F}_T^S$ generated exclusively by the random variable $S_T$.

Consider a portfolio consisting of $\alpha$ shares of the traded asset and the amount $\beta$ invested in the riskless one. Its current value $X_0 = x$ is equal to $\beta + \alpha S_0 = x$, while
its wealth $X_T$, at the end of the period $[0, T]$, is given by
\[ X_T = \beta + \alpha S_T = x + \alpha (S_T - S_0). \] (1.1)

Now introduce a claim, settling at time $T$ and yielding payoff $C_T$. In pricing of $C_T$, we need to specify our risk preferences. We choose to work with the exponential utility
\[ U(x) = -e^{-\gamma x}, \quad x \in \mathbb{R} \quad \text{and} \quad \gamma > 0. \] (1.2)

Optimality of investments, which will ultimately yield the indifference price of the claim, is examined via the value function
\[ V_{C_T}(x) = \sup_a \mathbb{E}_P \left( -e^{-\gamma (X_T - C_T)} \right) \] (1.3)
\[ = e^{-\gamma x} \sup_a \mathbb{E}_P \left( -e^{-\gamma (S_T - S_0) + \gamma C_T} \right). \]

Below, we recall the definition of indifference prices.

**Definition 1.1** The indifference price of the claim $C_T = c(S_T, Y_T)$ is defined as the amount $v(C_T)$ for which the two value functions $V_{C_T}$ and $V^0$, defined in (1.3) and corresponding, respectively, to the claims $C_T$ and 0, coincide. Namely, $v(C_T)$ is the amount which satisfies
\[ V^0(x) = V_{C_T}(x + v(C_T)) \] (1.4)
for all initial wealth levels $x \in \mathbb{R}$.

Looking at the classical arbitrage-free pricing theory, we recall that derivative valuation has two fundamental components which do not depend on specific model assumptions. Namely, the price is obtained as a linear functional of the (discounted) payoff representable via the (unique) risk neutral equivalent martingale measure.

Our goal is to understand how these two components, namely, the linear valuation operator and the risk neutral pricing measure, change when markets become incomplete. In the context of pricing by indifference, we will look for a valuation functional and a naturally related pricing measure under which the price is given as
\[ v(C_T) = \mathcal{E}_Q(C_T). \] (1.5)

Before we determine the fundamental features that $\mathcal{E}$ and $\mathcal{Q}$ should have, let us look at some representative cases.

**Examples:**

i) First, we consider a claim of the form $C_T = c(S_T)$. Intuitively, the indifference price should coincide with the arbitrage-free price, for there is no risk that cannot be hedged. Indeed, one can construct a nested complete one-period binomial model and show that
\[ v(c(S_T)) = \mathbb{E}_Q^+(c(S_T)), \] (1.6)
with \( \mathbb{Q}^* \) being the relevant risk neutral measure. The indifference price mechanism reduces to the arbitrage-free one and any effect on preferences dissipates.

ii) Next, we look at a claim of the form \( C_T = c(Y_T) \) and assume for simplicity that the random variables \( S_T \) and \( Y_T \) are independent under the measure \( \mathbb{P} \). In this case, intuitively, the presence of the traded asset should \textit{not} affect the price. Indeed, working directly with the value function (1.3) and Definition 1.1, it is straightforward to deduce that

\[
v(c(Y_T)) = \frac{1}{\gamma} \log \mathbb{E}_{\mathbb{P}}(e^{\gamma c(Y_T)}).
\]

(1.7)

The indifference price coincides with the classical actuarial valuation principle, the so-called \textit{certainty equivalent} value, which is nonlinear in the payoff and uses as pricing measure the historical one.

iii) Finally, we examine a claim of the form \( C_T = c_1(S_T) + c_2(Y_T) \). One could be, wrongly, tempted to price \( C_T \) by first pricing \( c_1(S_T) \) by arbitrage, next pricing \( c_2(Y_T) \) by certainty equivalent, and adding the results. Intuitively, this should work when \( S_T \) and \( Y_T \) are independent. However, this cannot possibly work under strong dependence between the two variables, for example, when \( Y_T \) is a function of \( S_T \). In general,

\[
v(c_1(S_T) + c_2(Y_T)) \neq \mathbb{E}_{\mathbb{Q}^*}(c_1(S_T)) + \frac{1}{\gamma} \log \mathbb{E}_{\mathbb{P}}(e^{\gamma c_2(Y_T)}).
\]

The above illustrative examples indicate certain fundamental characteristics \( E \) and \( \mathbb{Q} \) should have. First of all, we observe that a \textit{nonlinear} valuation functional must be sought. Clearly, any effort to represent indifference prices as expected payoffs under an appropriately chosen universal measure should be abandoned. Indeed, \textit{no linear pricing mechanism can be compatible with the concept of indifference based valuation as defined in (1.4)}. Note that this fundamental observation comes in contrast to the central direction of existing approaches in incomplete models that yield prices as expected payoffs under an optimally chosen measure.

We also see that risk preferences may affect the valuation device given their inherent role in price specification. However, intuitively speaking, we would prefer to specify the pricing measure \textit{independently} on the risk preferences. Finally, the pricing measure and the valuation device should ideally be the same for all claims to be priced.

The next proposition yields the indifference price in the desired form (1.5).

**Proposition 1.1** Let \( \mathbb{Q} \) be a measure under which the traded asset is a martingale and, at the same time, the conditional distribution of the nontraded asset, given the traded one, is preserved with respect to the historical measure \( \mathbb{P} \), i.e.,

\[
\mathbb{Q}(Y_T|S_T) = \mathbb{P}(Y_T|S_T).
\]

(1.8)

Let \( C_T = c(S_T, Y_T) \) be the claim to be priced under exponential preferences with risk aversion coefficient \( \gamma \). Then, the indifference price of \( C_T \) is given by

\[
v(C_T) = \mathbb{E}_{\mathbb{Q}}(C_T) = \mathbb{E}_{\mathbb{Q}} \left( \frac{1}{\gamma} \log \mathbb{E}_{\mathbb{Q}}(e^{\gamma C_T}|S_T) \right).
\]

(1.9)
Proof. We prove the above result by constructing the indifference price via its definition (1.4). We start with the specification of the value functions $V^0$ and $V^{C_T}$. We represent the payoff $C_T$ as a random variable defined on $\Omega$ with values $C_T(\omega_i) = c_i \in \mathbb{R}$, for $i = 1, \ldots, 4$. Elementary arguments lead to

$$V^{C_T}(x) = e^{-\gamma x} \sup_{\alpha} (-e^{-\gamma \alpha S_0(\xi^u - 1)}(e^{\gamma c_1} p_1 + e^{\gamma c_2} p_2) - e^{-\gamma \alpha S_0(\xi^d - 1)}(e^{\gamma c_3} p_3 + e^{\gamma c_4} p_4)).$$

Maximizing over $\alpha$ leads to the optimal number of shares $\alpha^{C_T,*}$, given by

$$\alpha^{C_T,*} = \frac{1}{\gamma S_0(\xi^u - \xi^d)} \log \left( \frac{\xi^u - 1}{\xi^d} \right) \left( \frac{p_1 + p_2}{1 - \xi^d} \right) (p_3 + p_4) + \frac{1}{\gamma S_0(\xi^u - \xi^d)} \log \left( \frac{e^{\gamma c_1} p_1 + e^{\gamma c_2} p_2}{e^{\gamma c_3} p_3 + e^{\gamma c_4} p_4} \right) \left( p_3 + p_4 \right).$$

Further straightforward, albeit tedious, calculations yield

$$V^{C_T}(x) = -e^{-\gamma x} \frac{1}{q^q(1 - q)^{1-q}} (e^{\gamma c_1} p_1 + e^{\gamma c_2} p_2)^q (e^{\gamma c_3} p_3 + e^{\gamma c_4} p_4)^{1-q},$$

where

$$q = \frac{1 - \xi^d}{\xi^u - \xi^d}.$$  

For $C_T = 0$, the value function takes the form

$$V^0(x) = -e^{-\gamma x} \left( \frac{p_1 + p_2}{q} \right)^q \left( \frac{p_3 + p_4}{1 - q} \right)^{1-q}.$$  

From the definition of the indifference price (1.4) and the representations (1.11), (1.13) of the relevant value functions, it follows that

$$v(C_T) = q - \log \frac{e^{\gamma c_1} p_1 + e^{\gamma c_2} p_2}{p_1 + p_2} + (1 - q) - \log \frac{e^{\gamma c_3} p_3 + e^{\gamma c_4} p_4}{p_3 + p_4}. \quad (1.14)$$

Next, we show that the above price admits the probabilistic representation (1.9). We first consider the terms involving the historical probabilities in (1.14) and we note that they can be actually written in terms of the conditional historical expectations, namely,

$$\frac{e^{\gamma c_1} p_1 + e^{\gamma c_2} p_2}{p_1 + p_2} = \mathbb{E}_\omega (e^{\gamma C_T}|A)$$

and

$$\frac{e^{\gamma c_3} p_3 + e^{\gamma c_4} p_4}{p_3 + p_4} = \mathbb{E}_\omega (e^{\gamma C_T}|A^c),$$

where $A = \{\omega_1, \omega_2\} = \{\omega: S_T(\omega) = S_0 \xi^u\}$. It is important to observe that conditioning is taken with respect to the terminal values of the traded asset.

We continue with the specification of the pricing measure defined in (1.12). For this, we denote (with a slight abuse of notation) by $q_1, q_2, q_3, q_4$ the elementary
probabilities of the sought measure $\mathbb{Q}$. Straightforward calculations yield that
\[ q_1 + q_2 = q, \]  
with $q$ as in (1.12). In order to compute the quantity $q_1$, we look at the conditional historical probability of $\{Y_T = Y_0\eta^n\}$, given $\{S_T = S_0\xi^n\}$, and we impose (1.8), yielding
\[ \frac{p_1}{p_1 + p_2} = \frac{q_1}{q}. \]

The probabilities $q_2$, $q_3$ and $q_4$, computed in a similar manner, are written below in a concise form:
\[ q_i = q \frac{p_i}{p_1 + p_2}, \quad i = 1, 2 \quad \text{and} \quad q_i = (1 - q) \frac{p_i}{p_3 + p_4}, \quad i = 3, 4. \]  
(1.16)

It follows easily that the nonlinear terms in (1.14) can be compiled as
\[ \frac{1}{\gamma} \left( \log \frac{e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2}{p_1 + p_2} \right) I_A + \frac{1}{\gamma} \left( \log \frac{e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4}{p_3 + p_4} \right) I_{A^c} \]
\[ \quad = \frac{1}{\gamma} \log \mathbb{E}_\mathbb{P} (e^{\gamma c_T} | A) I_A + \frac{1}{\gamma} \log \mathbb{E}_\mathbb{P} (e^{\gamma c_T} | A^c) I_{A^c} \]
\[ \quad = \frac{1}{\gamma} \log \mathbb{E}_\mathbb{Q} (e^{\gamma c_T} | S_T). \]

Therefore, taking the expectation with respect to $\mathbb{Q}$ yields
\[ \mathbb{E}_\mathbb{Q} \left( \frac{1}{\gamma} \log \mathbb{E}_\mathbb{Q} (e^{\gamma c_T} | S_T) \right) \]
\[ = \mathbb{E}_\mathbb{Q} \left( \frac{1}{\gamma} \left( \log \frac{e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2}{p_1 + p_2} \right) I_A + \frac{1}{\gamma} \left( \log \frac{e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4}{p_3 + p_4} \right) I_{A^c} \right) \]
\[ = q \frac{1}{\gamma} \log \frac{e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2}{p_1 + p_2} + (1 - q) \frac{1}{\gamma} \log \frac{e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4}{p_3 + p_4} = \nu(C_T), \]
where we used (1.14) to conclude. \hfill \square

We next discuss the key ingredients and highlight the intuitively natural features of the probabilistic pricing formula (1.9).

**INTERPRETATION OF THE INDIFFERENCE PRICE:**

Valuation is done via a two-step nonlinear procedure and under a single pricing measure.

i) Valuation procedure: In the first step, risk preferences are injected into the valuation process. The original derivative payoff is being distorted to the preference adjusted payoff, to be called the conditional certainty equivalent
\[ \tilde{C}_T = \frac{1}{\gamma} \log \mathbb{E}_\mathbb{Q} (e^{\gamma c_T} | S_T). \]  
(1.17)

This new payoff has actuarial-type characteristics and reflects the weight that risk aversion carries in the utility-based methodology. However, the certainty equivalent
rule is not applied in a naive way. Indeed, we do not consider any classical actuarial type functional, since

\[ \hat{C}_T \neq \frac{1}{\gamma} \log \mathbb{E}_\theta(e^{\gamma C_T}) \quad \text{and} \quad \hat{C}_T \neq \frac{1}{\gamma} \log \mathbb{E}_Q(e^{\gamma C_T}). \]

In the second step, the pricing procedure picks up arbitrage-free pricing characteristics. It prices the preference adjusted payoff \( \hat{C}_T \), dependent only on \( S_T \), through an arbitrage-free method. The same pricing measure is being used in both steps.

The price is then given by

\[ v(C_T) = \mathbb{E}_Q(C_T) = \mathbb{E}_Q(\hat{C}_T). \]

It is important to observe that the two steps are not interchangeable and of entirely different natures. The first step prices in a nonlinear way as opposed to the second step that uses linear, arbitrage-free, valuation principles. In a sense, this is entirely justifiable: \( \text{the unhedgeable risks are identified, isolated, and priced in the first step, and, thus, the remaining risks become hedgeable.} \)

One should then use a nonlinear valuation device for the unhedgeable risks and linear pricing for the hedgeable ones. A natural consequence of this is that risk preferences enter exclusively in the conditional certainty equivalent term, the only term related to unhedgeable risks. Both steps are generic and valid for any payoff.

\( ii) \ Pricing \ measure: \) One pricing measure is used throughout. Its essential role is not to alter the conditional distribution of risks, given the ones we can trade, from their respective historical values.

Naturally, there is no dependence on the payoff. The most interesting part, however, is its independence on risk preferences. This universality is expected and quite pleasing. It follows from the way we identified the pricing measure, via (1.8), a selection criterion that is obviously independent of any risk attitude. Finally, the distorted payoff \( \hat{C}_T \) can be computed under both the historical and the pricing measure; indeed, we have

\[ \hat{C}_T = \frac{1}{\gamma} \log \mathbb{E}_Q(e^{\gamma C_T} | S_T) = \frac{1}{\gamma} \log \mathbb{E}_\theta(e^{\gamma C_T} | S_T). \]

The remainder of this section is dedicated to a comparison of our representation of the indifference prices and of the associated value functions with the well-known representations obtained by Rouge and El Karoui in [237] and Delbaen et al. in [81] (see also Kabanov and Stricker [146]). The technical arguments are not difficult and therefore the discussion is provided in a casual fashion. The conclusions are presented in Proposition 1.3.

In the aforementioned works, it has been established that the indifference price solves a stochastic optimization problem. The objective therein is to maximize, over all martingale measures, the expected payoff of the claim, reduced by a (relative) entropic penalty term (see (1.20) below). This representation is a direct result of the choice of exponential preferences and of the duality approach used on the primary expected utility problem. Details on the duality approach will be presented in the Chapter 9 by Elliott and Van der Hoek.
A martingale measure that naturally arises in this analysis is the so-called minimal relative entropy measure, denoted, for the moment, by $\hat{Q}$. It is defined as the minimizer of the relative entropy, namely,

$$H(\hat{Q}|\mathbb{P}) = \min_{Q \in \mathcal{Q}_e} H(Q|\mathbb{P}),$$

(1.18)

where

$$H(Q|\mathbb{P}) = \mathbb{E}_\mathbb{P} \left( \frac{dQ}{d\mathbb{P}} \ln \frac{dQ}{d\mathbb{P}} \right).$$

(1.19)

For an extensive study of this measure, we refer to the work of Frittelli [96,97]. Under general model assumptions, the following result was established by Rouge and El Karoui in [237] and Delbaen et al. in [81].

**Proposition 1.2** The indifference price $\nu(C_T)$ is given by

$$\nu(C_T) = \sup_{Q \in \mathcal{Q}_e} \left( \mathbb{E}_Q(C_T) - \frac{1}{\gamma} (H(Q|\mathbb{P}) - H(\hat{Q}|\mathbb{P})) \right),$$

(1.20)

where $\mathcal{Q}_e$ is the set of martingale measures equivalent to $\mathbb{P}$.

The above formula has several attractive features. It is valid for general models and arbitrary payoffs. The entropic penalty directly quantifies the effects of incompleteness on the prices. The formula also exposes the limiting behavior of the price as the investor becomes risk neutral, namely, as $\gamma$ converges to zero. Finally, it highlights, in an intuitively pleasing way, the monotonicity of the price in terms of risk preferences and its convergence to the arbitrage-free price as the market becomes complete.

This representation has, however, some shortcomings. It provides the price via a new optimization problem, a fact that does not allow for a universal analogue to its arbitrage-free counterpart. It also yields a pricing measure that has the undesirable feature of depending on the specific payoff. Moreover, the price formula (1.20) considerably obstructs the analysis and study of certain important aspects of indifference valuation, as, for example, its numeraire independence and its generalization when risk preferences become stochastic. It also provides limited intuition for the construction of the relevant risk monitoring strategies and the associated payoff decomposition formulas.

We start our comparative analysis by exploring the relation between the pricing measure $\mathbb{Q}$ used in (1.9), and of the minimal entropy measure $\hat{Q}$ appearing in (1.20). We can readily see that the two measures coincide. In fact, consider the relative entropy (1.19) and look at its minimizers. For the simple model at hand, if $\hat{Q}$ is an arbitrary martingale measure defined by the elementary probabilities $\hat{q}_i$, $i = 1, \ldots, 4$, then

$$H(\hat{Q}|\mathbb{P}) = \sum_{i=1}^{4} \hat{q}_i \log \frac{\hat{q}_i}{p_i}. $$
Simple calculations yield that the minimizing elementary probabilities, say \( \tilde{q}_i, i = 1, \ldots, 4 \), are given by

\[
\tilde{q}_i = q \frac{p_i}{p_1 + p_2}, \quad i = 1, 2 \quad \text{and} \quad \tilde{q}_i = (1 - q) \frac{p_i}{p_3 + p_4}, \quad i = 3, 4
\]

and, thus, they are equal to the \( q_i, i = 1, \ldots, 4 \) of \( \hat{Q} \) (see (1.16)).

Therefore,

\[
H(Q|P) = H(\hat{Q}|P) = \sum_{i=1}^{4} q_i \log \frac{q_i}{p_i}. \tag{1.21}
\]

We observe that in the static incomplete model studied herein, there is an additional martingale measure, denoted by \( \hat{Q} \), which coincides with \( Q \). Specifically, \( \hat{Q} \) is the minimal martingale measure defined as the minimizer of

\[
H(\hat{Q}|P) = \min_{Q \in \mathcal{Q}} \mathbb{E}_P \left( - \ln \frac{dQ}{dP} \right).
\]

This measure (see, among others, [249]) appears frequently in risk minimization in incomplete markets.

It is important to recall that, in general, the measures \( \hat{Q} \) and \( \hat{Q} \) do not coincide. Herein, even though \( \hat{Q} = \hat{Q} \), we formulate all relevant arguments in terms of the minimal relative entropy measure to preserve the connection with the existing work on indifference valuation under exponential criteria.

We also observe that the minimal relative entropy measure is not a maximizer in the pricing formula (1.20). Indeed, if this were the case, the indifference price would have been the expected value of the payoff under \( \hat{Q} \), an obviously incorrect conclusion. This can happen only if the market is complete in which case, the minimal relative entropy measure coincides with the unique risk-neutral one and the indifference price reduces to the arbitrage-free price.

We can use the above observations to deduce alternative formulas for the involved value functions (1.3). These representations, first produced by Delbaen et al. in [81], are interesting on their own right. As we will see in subsequent sections, they offer valuable insights for the specification of the dynamic risk preferences and are instrumental in the construction of indifference prices in more complex model environments.

To this end, we first observe that

\[
- \log \left( \frac{p_1 + p_2}{q} \right)^q \left( \frac{p_1 + p_4}{1 - q} \right)^{1-q} = \sum_{i=1}^{4} q_i \log \frac{q_i}{p_i},
\]

which, in view of (1.21), implies that the left-hand side represents the minimal relative entropy. Combining this with (1.13) yields the representation (1.22). This structural result is intuitively pleasing. It reflects how risk preferences are dynamically adjusted via the optimal investments. In fact, the value function \( V^0 \) is directly obtained from the terminal utility \( U \) by a mere translation of the wealth argument.

In a sense, the entropy \( H(Q|P) \) represents the wealth value adjustment due to the magnitude of the opportunities offered in the market.
A similar representation can be derived for the value function $V^{C_T}$; see (1.23) below. It follows directly from the definitions of the indifference price and the value functions; see, respectively, (1.3) and (1.4). Formula (1.23) shows that $V^{C_T}$ can be obtained from the terminal utility through two wealth adjustments, one that is related to the indifference price and the other, already appearing in the absence of the claim, reflecting the magnitude of investment opportunities and market incompleteness. We summarize the above results below.

**Proposition 1.3** Let $\nu(C_T)$ be the indifference price of the claim, $\mathbb{Q}$ the pricing measure introduced in (1.8), and $H(Q|\mathbb{P})$ its associated relative entropy (cf. (1.21)).

i) The minimal relative entropy measure $\hat{\mathbb{Q}}$ satisfies

$$\mathbb{Q} \equiv \hat{\mathbb{Q}}.$$  

ii) The value functions $V^0$ and $V^{C_T}$ are represented, respectively, by

$$V^0(x) = -e^{-\gamma x - H(Q|\mathbb{P})} = U \left( x + \frac{1}{\gamma} H(Q|\mathbb{P}) \right)$$  

and

$$V^{C_T}(x) = -e^{-\gamma x - H(Q|\mathbb{P}) + \gamma \nu(C_T)}$$

$$= U \left( x + \frac{1}{\gamma} H(Q|\mathbb{P}) - \nu(C_T) \right),$$

with $U$ as in (1.2).

iii) The indifference price satisfies

$$\nu(C_T) = \sup_{Q \in \mathcal{Q}_{\nu}} \left( E_Q(C_T) - \frac{1}{\gamma} (H(Q|\mathbb{P}) - H(Q|\mathbb{P})) \right) = E_Q(C_T),$$

where the nonlinear pricing functional $E_Q$ is given by

$$E_Q(C_T) = E_Q \left( \frac{1}{\gamma} \log E_Q(e^{\gamma S_T}|S_T) \right).$$

**1.2.2 Properties of the Indifference Prices**

The previous analysis produced the nonlinear price representation

$$\nu(C_T) = E_Q(C_T) = E_Q(\tilde{C}_T),$$

where the preference adjusted payoff $\tilde{C}_T$ is the conditional certainty equivalent (cf. (1.17)),

$$\tilde{C}_T = \frac{1}{\gamma} \log E_Q(e^{\gamma S_T}|S_T),$$

and the pricing measure $\hat{\mathbb{Q}}$ is given by (1.8). This pricing formula yields a direct constitutive analogue to the linear pricing rule of the complete models.

Our next task is to explore the structural properties of the indifference prices, their behavior with respect to various inputs as well as their differences and similarities to the arbitrage-free prices.
Throughout we occasionally adopt the notation $v(C_T; \gamma)$. This is done for clarity and it is omitted whenever there is no ambiguity. Moreover, to ease the analysis and the presentation, it is also assumed that $C_T \geq 0 \mathbb{P}$-a.s. This assumption is easily removed at the expense of more tedious arguments.

i) Behavior with Respect to the Risk Aversion Coefficient

While risk preferences are not affecting the arbitrage-free prices due to perfect risk replication, they represent an indispensable element of indifference prices. Indeed, the risk aversion coefficient $\gamma$ appears in the construction of $\hat{C}_T$. It is through this conditional preference adjusted payoff that the indifference valuation mechanism extracts and valuates the underlying unhedgeable risks.

**Proposition 1.4** The function

$$
\gamma \rightarrow v(C_T; \gamma) = \mathbb{E}_Q \left( \frac{1}{\gamma} \log \mathbb{E}_Q(e^{\gamma C_T}|S_T) \right)
$$

from $\mathbb{R}_+$ into $\mathbb{R}$ is increasing and continuous. Moreover, if for all claims $C_T$ we have

$$
v(C_T; \gamma) = v(C_T; 1),
$$

then $\gamma = 1$.

**Proof.** Continuity follows directly from the formula and the properties of conditional expectation. To establish monotonicity, let us assume that $0 < \gamma_1 < \gamma_2$. Then, Holder’s inequality yields

$$
\mathbb{E}_Q(e^{\gamma_1 C_T}|S_T) \leq (\mathbb{E}_Q(e^{\gamma_2 C_T}|S_T))^{\gamma_1/\gamma_2}
$$

and, in turn,

$$
\frac{1}{\gamma_1} \log \mathbb{E}_Q(e^{\gamma_1 C_T}|S_T) \leq \frac{1}{\gamma_2} \log \mathbb{E}_Q(e^{\gamma_2 C_T}|S_T).
$$

Taking expectation, with respect to the pricing measure $\mathbb{Q}$, we deduce the first statement.

To establish the second assertion, we assume, without loss of generality, that $p_2 \neq 0$. Then, we consider claims of the form $C_T(\omega_i) = c_1, C_T(\omega_i) = 0$, $i = 2, 3, 4$, and we observe that (1.24) leads to

$$
\frac{1}{\gamma} \log \frac{e^{\gamma c_1} p_1 + p_2}{p_1 + p_2} = \log \frac{e^{c_1} p_1 + p_2}{p_1 + p_2}
$$

for all $c_1$. To conclude, it suffices to differentiate both sides with respect to $c_1$ and rearrange terms.

**Proposition 1.5** The following limiting relations hold:

$$
\lim_{\gamma \rightarrow 0^+} v(C_T; \gamma) = \mathbb{E}_Q(C_T),
$$

$$
\lim_{\gamma \rightarrow \infty} v(C_T; \gamma) = \mathbb{E}_Q\|C_T\|_{L^\infty(\Omega; S_T)},
$$

(1.25)

(1.26)
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**Proof.** To show (1.25), we pass to the limit, as \( \gamma \to 0 \), in the price formula (1.14),

\[
\nu(C_T; \gamma) = q \frac{1}{\gamma} \log \left( \frac{e^{\gamma c_1} p_1 + e^{\gamma c_2} p_2}{p_1 + p_2} \right) + (1 - q) \frac{1}{\gamma} \log \left( \frac{e^{\gamma c_3} p_3 + e^{\gamma c_4} p_4}{p_3 + p_4} \right)
\]

(1.27)

to obtain

\[
\lim_{\gamma \to 0} \nu(C_T; \gamma) = q \left( \frac{p_1 c_1}{p_1 + p_2} + \frac{p_2 c_2}{p_1 + p_2} \right) + (1 - q) \left( \frac{p_3 c_3}{p_3 + p_4} + \frac{p_4 c_4}{p_3 + p_4} \right).
\]

On the other hand, by the properties of the pricing measure, we have

\[
q \frac{p_i}{p_1 + p_2} = q_i, \; i = 1, 2 \quad \text{and} \quad (1 - q) \frac{p_i}{p_3 + p_4} = q_i, \; i = 3, 4,
\]

and, in turn,

\[
\lim_{\gamma \to 0} \nu(C_T; \gamma) = \sum_{i=1}^{4} q_i c_i.
\]

To establish (1.26), we pass to the limit as \( \gamma \to \infty \) in (1.27). We readily get that

\[
\lim_{\gamma \to \infty} \nu(C_T) = q \max(c_1, c_2) + (1 - q) \max(c_3, c_4),
\]

and the statement follows. \( \square \)

**Proposition 1.6** The indifference price satisfies

\[
\lim_{\gamma \to 0} \frac{\partial \nu(C_T; \gamma)}{\partial \gamma} = \frac{1}{2} \mathbb{E}_Q(Var_Q(C_T|S_T)), \quad (1.28)
\]

and thus,

\[
\nu(C_T; \gamma) = \mathbb{E}_Q(C_T) + \frac{1}{2} \gamma \mathbb{E}_Q(Var_Q(C_T|S_T)) + o(\gamma). \quad (1.29)
\]

**Proof.** We only show (1.28), since (1.29) is an easy consequence. We first differentiate \( \nu(C_T; \gamma) \) with respect to \( \gamma \), obtaining

\[
\frac{\partial \nu(C_T; \gamma)}{\partial \gamma} = \mathbb{E}_Q \left( \frac{-1}{\gamma^2} \log \mathbb{E}_Q(e^{\gamma C_T}|S_T) + \frac{1}{\gamma} \frac{\mathbb{E}_Q(C_T e^{\gamma C_T}|S_T)}{\mathbb{E}_Q(e^{\gamma C_T}|S_T)} \right)
\]

\[
= \frac{1}{\gamma} \left( \mathbb{E}_Q \left( \frac{\mathbb{E}_Q(C_T e^{\gamma C_T}|S_T)}{\mathbb{E}_Q(e^{\gamma C_T}|S_T)} \right) - \nu(C_T; \gamma) \right).
\]
Therefore,

\[
\lim_{\gamma \to 0} \frac{\partial \nu(C_T; \gamma)}{\partial \gamma} = \lim_{\gamma \to 0} \frac{1}{\gamma} \left( \mathbb{E}_Q \left( \frac{\mathbb{E}_Q(C_T e^\gamma C_T | S_T)}{\mathbb{E}_Q(e^{\gamma C_T} | S_T)} \right) - \nu(C_T; \gamma) \right)
\]

\[
= \lim_{\gamma \to 0} \mathbb{E}_Q \left( \frac{\mathbb{E}_Q(C_T^2 e^{\gamma C_T} | S_T) \mathbb{E}_Q(e^{\gamma C_T} | S_T) - \left( \mathbb{E}_Q(C_T e^\gamma C_T | S_T) \right)^2}{\left( \mathbb{E}_Q(e^{\gamma C_T} | S_T) \right)^2} \right)
\]

\[
- \lim_{\gamma \to 0} \frac{\partial \nu(C_T; \gamma)}{\partial \gamma},
\]

and, thus,

\[
\lim_{\gamma \to 0} \frac{\partial \nu(C_T; \gamma)}{\partial \gamma} = \frac{1}{2} \mathbb{E}_Q(C_T^2 | S_T) - \left( \mathbb{E}_Q(C_T | S_T) \right)^2).
\]

(\(\square\))

**Proposition 1.7** The indifference price is consistent with the no-arbitrage principle, namely, for all \(\gamma > 0\),

\[
\inf_{Q \in Q_e} \mathbb{E}_Q(C_T) \leq \nu(C_T; \gamma) \leq \sup_{Q \in Q_e} \mathbb{E}_Q(C_T),
\]

where \(Q_e\) is the set of martingale measures that are equivalent to \(P\).

**Proof.** We assume, without loss of generality, that \(c_1 < c_2\) and that \(c_3 < c_4\). The monotonicity of the price with respect to risk aversion implies

\[
\lim_{\gamma \to 0} \nu(C_T; \gamma) \leq \nu(C_T; \gamma) \leq \lim_{\gamma \to \infty} \nu(C_T; \gamma)
\]

and, in turn, that

\[
\mathbb{E}_Q(C_T) \leq \nu(C_T; \gamma) \leq \mathbb{E}_Q \|C_T\|_{L^\infty_{|S_T|}}.
\]

Taking the infimum over all martingale measures yields

\[
\inf_{Q \in Q} \mathbb{E}_Q(C_T) \leq \mathbb{E}_Q(C_T) \leq \nu(C_T; \gamma),
\]

and the left-hand side of (1.30) follows. We next observe that

\[
\mathbb{E}_Q \|C_T\|_{L^\infty_{|S_T|}} = \mathbb{E}_{\tilde{Q}}(C_T),
\]

where the martingale measure \(\tilde{Q}\) has elementary probabilities

\[
\tilde{Q}(\omega_1) = 0, \quad \tilde{Q}(\omega_2) = q, \quad \tilde{Q}(\omega_3) = 0, \quad \tilde{Q}(\omega_4) = 1 - q.
\]

Observing that

\[
\nu(C_T; \gamma) \leq \mathbb{E}_Q(C_T) \leq \sup_{Q \in Q} \mathbb{E}_Q(C_T),
\]

we conclude. \(\square\)
ii) Behavior with Respect to Payoffs

We first explore the monotonicity, convexity, and scaling behavior of the indifference prices. We note that in the next two propositions, all inequalities among payoffs and their prices hold both under the historical and the pricing measures $\mathbb{P}$ and $\mathbb{Q}$. Since these two measures are equivalent, we skip any measure-specific notation for the ease of the presentation.

**Proposition 1.8** The indifference price is a nondecreasing and convex function of the claim’s payoff, namely,

$$\text{if } C_T^1 \leq C_T^2 \text{ then } v(C_T^1) \leq v(C_T^2),$$  

(1.31)

and, for $\alpha \in (0, 1)$,

$$v(\alpha C_T^1 + (1 - \alpha) C_T^2) \leq \alpha v(C_T^1) + (1 - \alpha) v(C_T^2).$$  

(1.32)

**Proof.** Inequality (1.31) follows directly from formula (1.9). To establish (1.32), we apply Holder’s inequality to obtain

$$\mathbb{E}_Q \left( \frac{1}{\gamma} \log \mathbb{E}_Q (e^{\gamma (\alpha C_T^1 + (1 - \alpha) C_T^2)} | S_T) \right) \leq \mathbb{E}_Q \left( \frac{1}{\gamma} \log ((\mathbb{E}_Q(e^{\gamma C_T^1} | S_T))^\alpha (\mathbb{E}_Q(e^{\gamma C_T^2} | S_T))^{1 - \alpha}) \right)$$

$$= \alpha \mathbb{E}_Q \left( \frac{1}{\gamma} \log \mathbb{E}_Q (e^{\gamma C_T^1} | S_T) \right) + (1 - \alpha) \mathbb{E}_Q \left( \frac{1}{\gamma} \log \mathbb{E}_Q (e^{\gamma C_T^2} | S_T) \right),$$

and the result follows. \hfill \Box

**Proposition 1.9** The indifference price satisfies

$$v(\alpha C_T) \leq \alpha v(C_T) \text{ for } \alpha \in (0, 1)$$  

(1.33)

and

$$v(\alpha C_T) \geq \alpha v(C_T) \text{ for } \alpha \geq 1.$$  

(1.34)

**Proof.** To show (1.33) we observe

$$v(\alpha C_T) = \mathbb{E}_Q \left( \frac{1}{\gamma} \log \mathbb{E}_Q (e^{\gamma \alpha C_T} | S_T) \right)$$

$$= \alpha \mathbb{E}_Q \left( \frac{1}{\gamma^{\tilde{\gamma}}} \log \mathbb{E}_Q (e^{\gamma^{\tilde{\gamma}} C_T} | S_T) \right),$$

where, for $\alpha \in (0, 1)$,

$$\tilde{\gamma} = \alpha \gamma < \gamma.$$

Using the monotonicity of the price with respect to risk aversion, we conclude. Inequality (1.34) follows by the same argument. \hfill \Box

The following result highlights an important property of the indifference price operator. We see that any hedgeable risk is scaled out from the nonlinear part of the
pricing rule, and it is priced directly by arbitrage. **Hedgeable risks do not differ from their conditional certainty equivalent payoffs.** In this sense, we say that the pricing operator has the property of additive\(^3\) invariance with respect to hedgeable risks.

Note that this property is stronger, and more intuitive, than requiring mere translation invariance with respect to constant risks.

**Proposition 1.10**  
The indifferent pricing operator is additively invariant with respect to hedgeable risks, namely, if \( C_T = C^1_T + C^2_T \), with \( C^1_T = C^1(S_T) \) and \( C^2_T = C^2(S_T, Y_T) \), then

\[
\nu(C_T) = \mathbb{E}_Q(C^1(S_T) + C^2(S_T, Y_T)) = \mathbb{E}_Q(C^1(S_T)) + \nu(C^2(S_T, Y_T)).
\]  
(1.35)

**Proof.** The price formula (1.9), together with the measurability properties of \( C^1(S_T) \), yields

\[
\nu(C_T) = \mathbb{E}_Q \left( \frac{1}{\gamma} \log \mathbb{E}_Q(e^{\gamma(C^1(S_T) + C^2(S_T, Y_T))} | S_T) \right) = \mathbb{E}_Q(C^1(S_T)) + \mathbb{E}_Q \left( \frac{1}{\gamma} \log \mathbb{E}_Q(e^{\gamma C^2(S_T, Y_T)} | S_T) \right) = \mathbb{E}_Q(C^1_T) + \nu(C^2_T).
\]

The above property yields the following conclusions for two extreme cases.

**Special Cases:**

i) Let \( C_T = C^1(S_T) + C^2(S_T, Y_T) \) with \( Y_T \) depending functionally on \( S_T \). The payoff \( C^2_T \) is then \( \mathcal{F}^2_T \)-measurable and, therefore,

\[
C^2(S_T, Y_T) = \tilde{C}^2(S_T, Y_T).
\]

Combining the above with (1.35) implies

\[
\nu(C_T) = \mathbb{E}_Q(C^1(S_T)) + \nu(C^2(S_T, Y_T)) = \mathbb{E}_Q(C^1(S_T)) + \mathbb{E}_Q(C^2(S_T, Y_T)).
\]

The indifference price simplifies to the arbitrage-free one, and the pricing measure coincides with the unique risk neutral measure.

ii) Let \( C_T = C^1(S_T) + C^2(Y_T) \) with \( Y_T \) and \( S_T \) independent under \( \mathbb{P} \). Then,

\[
\tilde{C}^2(Y_T) = \frac{1}{\gamma} \log \mathbb{E}_Q(e^{\gamma C^2(Y_T)} | S_T) = \frac{1}{\gamma} \log \mathbb{E}_{\mathbb{P}}(e^{\gamma C^2(Y_T)}).
\]

The indifference price of \( C_T \) consists of the arbitrage-free price of the first claim plus the traditional actuarial certainty equivalent price of the second,

\[
\nu(C_T) = \mathbb{E}_Q(C^1(S_T)) + \frac{1}{\gamma} \log \mathbb{E}_{\mathbb{P}}(e^{\gamma C^2(Y_T)}).
\]

\(^3\)The authors would like to thank Patrick Cheridito for suggesting this terminology.
We finish this section presenting the link between the indifference pricing functional \( \nu \) and the so-called convex risk measures (see, among others, [89], [90], and [208]).

**Definition 1.2** The mapping \( \rho : \mathcal{F}_T \to \mathbb{R} \) is called a convex risk measure if it satisfies the following conditions for all \( C^1_T, C^2_T \in \mathcal{F}_T \).

- Convexity: \( \rho(\alpha C^1_T + (1-\alpha)C^2_T) \leq \alpha \rho(C^1_T) + (1-\alpha)\rho(C^2_T), \) \( 0 \leq \alpha \leq 1 \).
- Monotonicity: If \( C^1_T \leq C^2_T \), then \( \rho(C^1_T) \geq \rho(C^2_T) \).
- Translation invariance: If \( m \in \mathbb{R} \), then \( \rho(C^1_T + m) = \rho(C^1_T) - m \).

For any \( C_T \in \mathcal{F}_T \) define
\[
\rho(C_T) = \nu(-C_T) = \mathbb{E}_Q \left( \frac{1}{Y} \log \mathbb{E}_Q (e^{-\nu C_T} | \mathcal{F}_T) \right) .
\] (1.36)

**Proposition 1.11** The mapping \( \rho \) given in (1.36) defines a convex risk measure.

*Proof.* All conditions follow trivially from the properties of the indifference price discussed earlier. \( \square \)

Note that the number \( \nu(C_T) \) represents the indifference value of the payoff \( C_T \), while the number \( \rho(C_T) = \nu(-C_T) \) is usually interpreted as a capital requirement imposed by a supervising body for accepting the position \( C_T \). It is interesting to observe that the concept of indifference value, deduced from the desire to behave optimally as an investor, is in the above sense consistent with a method that may be used to determine the capital amount, for a position to be acceptable by a supervising body.

### 1.2.3 Risk Monitoring Strategies

We now turn our attention to the important issue of managing risk generated by the derivative contract. In complete markets, the payoff is reproduced by the associated self-financing and replicating portfolio. Consequently, any risk associated with the claim is eliminated.

For the model at hand, any \( \mathcal{F}^S_T \)-measurable claim \( C_T \) is replicable and the familiar representation formula
\[
C_T = \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0} (S_T - S_0)
\] (1.37)
holds, with
\[
\nu(C_T) = \mathbb{E}_Q(C_T) \quad \text{and} \quad \frac{\partial \nu(C_T)}{\partial S_0} = \frac{\partial \mathbb{E}_Q(C_T)}{\partial S_0} .
\] (1.38)

The indifference price coincides with the arbitrage-free price, and its spatial derivative yields the so-called delta.

When the market is incomplete, however, perfect replication is not viable and a payoff representation similar to the above cannot be obtained. However, one may still seek a constitutive analogue to (1.37).
We recall that the indifference price was produced via comparison of the optimal investment decisions with and without the claim in consideration. We should therefore base our study on the analysis of the relevant optimal portfolios, and the relation between the indifference price and the optimal wealth levels they generate. We start with an auxiliary structural result for the optimal policies of the underlying maximal expected utility problems (1.3).

**Proposition 1.12** Let \( v(C_T) \) be the indifference price of the claim \( C_T \) and \( H(Q|\mathbb{P}) \) as in (1.21). The optimal number of shares \( \alpha^{C_T,*} \) in the optimal investment problem (1.3) is given by

\[
\alpha^{C_T,*} = \alpha^{0,*} + \frac{\partial v(C_T)}{\partial S_0},
\]

where

\[
\alpha^{0,*} = -\frac{1}{\gamma} \frac{\partial H(Q|\mathbb{P})}{\partial S_0}
\]

represents the number of shares held optimally in the absence of the claim. Both optimal controls \( \alpha^{C_T,*} \) and \( \alpha^{0,*} \) are wealth independent.

**Proof.** We first recall that \( \alpha^{C_T,*} \) was provided in (1.10), rewritten below for convenience:

\[
\alpha^{C_T,*} = \frac{1}{\gamma S_0(\xi^u - \xi^d)} \log \frac{(\xi^u - 1)(p_1 + p_2)}{(1 - \xi^d)(p_3 + p_4)}
+ \frac{1}{\gamma S_0(\xi^u - \xi^d)} \log \frac{(e^{\gamma c_1} p_1 + e^{\gamma c_2} p_2)(p_3 + p_4)}{(e^{\gamma c_3} p_3 + e^{\gamma c_4} p_4)(p_1 + p_2)}.
\]

When the claim is not taken into account, one can easily deduce, by setting \( c_i = 0, \ i = 1, \ldots, 4 \) above, that the corresponding optimal policy \( \alpha^{0,*} \) equals

\[
\alpha^{0,*} = \frac{1}{\gamma S_0(\xi^u - \xi^d)} \log \frac{(\xi^u - 1)(p_1 + p_2)}{(1 - \xi^d)(p_3 + p_4)}
= \frac{1}{\gamma (S^u - S^d)} \log \frac{(1 - q)(p_1 + p_2)}{q(p_3 + p_4)}.
\]

Using that

\[
\frac{\partial q}{\partial S_0} = \frac{\partial}{\partial S_0} \left( \frac{S_0 - S^d}{S^u - S^d} \right) = \frac{1}{S^u - S^d}
\]

and differentiating the entropy expression (1.21) gives

\[
\frac{\partial H(Q|\mathbb{P})}{\partial S_0} = -\log \left( \frac{(1 - q)(p_1 + p_2)}{q(p_3 + p_4)} \right) \frac{\partial q}{\partial S_0}.
\]

Differentiating, in turn, the price formula (1.9) gives

\[
\frac{\partial v(C_T)}{\partial S_0} = \frac{1}{\gamma S_0(\xi^u - \xi^d)} \log \frac{(e^{\gamma c_1} p_1 + e^{\gamma c_2} p_2)(p_3 + p_4)}{(e^{\gamma c_3} p_3 + e^{\gamma c_4} p_4)(p_1 + p_2)}
\]

which, combined with the expressions for \( \alpha^{C_T,*} \) and \( \alpha^{0,*} \) yields (1.39). □
Next, we consider the optimal wealth variables $X^{C^r,*}$ and $X^{0,*}$ representing, respectively, the agent’s optimal wealth with and without the claim. In the first case, the agent starts with initial wealth $x + v(C_T)$ and buys $a^{C^r,*}$ shares of stock. If the claim is not taken into account, the investor starts with $x$ and follows the strategy $a^{0,*}$. In other words,

$$X^{C^r,*}_T = x + v(C_T) + a^{C^r,*}(S_T - S_0)$$

(1.42)

and

$$X^{0,*}_T = x + a^{0,*}(S_T - S_0).$$

(1.43)

We now introduce two important quantities that will help us produce a meaningful decomposition of the claim’s payoff. They are the residual optimal wealth and the residual risk, denoted respectively by $L$ and $R$. These notions were first introduced by the authors in Musiela and Zariphopoulou [195, 196]; see also [198].

**Definition 1.3** The residual optimal wealth process is defined as

$$L_t = X^{C^r,*}_t - X^{0,*}_t \quad \text{for } t = 0, T.$$

(1.44)

In a complete model environment, the residual optimal wealth coincides with the value of the perfectly replicating portfolio. It is therefore a martingale under the unique risk neutral measure, and it generates the claim’s payoff at expiration.

When the market is incomplete, however, several interesting observations can be made. The residual terminal optimal wealth $L_T$ reproduces the claim only partially. In addition, it is an $\mathcal{F}^S_T$-measurable variable and retains its martingale property under all martingale measures. Its most important property, however, is that it coincides with the conditional certainty equivalent. This fact will play an instrumental role in two directions, namely, in the identification of the replicable part of the claim and in the specification of the risk monitoring policy.

**Proposition 1.13** The residual optimal wealth process satisfies

$$L_0 = v(C_T)$$

(1.45)

and

$$L_T = v(C_T) + \frac{\partial v(C_T)}{\partial S_0}(S_T - S_0).$$

(1.46)

Moreover, $L_T$ coincides with the conditional certainty equivalent,

$$L_T = \tilde{C}_T.$$

(1.47)

Finally, the process $L_t$ is a martingale under all equivalent martingale measures,

$$\mathbb{E}_Q(L_T) = L_0 = v(C_T) \quad \text{for } Q \in \mathcal{Q}_e.$$

(1.48)

**Proof.** Assertions (1.45) and (1.46) follow easily from Definition 1.3, the optimal wealth representations (1.42) and (1.43) and the relation (1.39) between the optimal policies.
To show (1.47), we first recall that
\[ \nu(C_T) = \mathbb{E}_Q(\tilde{C}_T), \] (1.49)
which, in view of (1.46) yields,
\[ L_T = \mathbb{E}_Q(\tilde{C}_T) + \frac{\partial \mathbb{E}_Q(\tilde{C}_T)}{\partial S_0} (S_T - S_0). \]

The claim \( \tilde{C}_T \), however, is \( \mathcal{F}_T \)-measurable and, thus, replicable. Its arbitrage-free decomposition is
\[ \tilde{C}_T = \mathbb{E}_Q(\tilde{C}_T) + \frac{\partial \mathbb{E}_Q(\tilde{C}_T)}{\partial S_0} (S_T - S_0), \]
and the identity (1.47) follows.

The martingale property (1.48) is an easy consequence of (1.46).

**Definition 1.4** The indifference price process \( \nu_t(C_T), t = 0, T \) is defined as
\[ \nu_0(C_T) = \nu(C_T) \text{ and } \nu_T(C_T) = C_T. \] (1.50)

**Definition 1.5** The residual risk process \( R_t, t = 0, T \), is defined as the difference between the indifference price and the residual optimal wealth, namely,
\[ R_t = \nu_t(C_T) - L_t, \]
i.e.,
\[ R_0 = \nu(C_T) - L_0 \text{ and } R_T = C_T - L_T. \] (1.51)

If perfect replication is viable, the residual risk is zero throughout and its notion degenerates. In general, it represents the component of the claim that is not replicable, given that risks which can be hedged have been already extracted optimally according to our utility criteria. As such, the residual risk should not generate any additional conditional certainty equivalent part nor should it, in consequence, acquire any additional indifference value.

**Proposition 1.14** The residual risk process has the following properties:

i) It satisfies
\[ R_0 = 0 \] (1.52)
and
\[ R_T = C_T - \tilde{C}_T. \] (1.53)

ii) Its conditional certainty equivalent is zero,
\[ \hat{R}_T = 0. \] (1.54)

iii) Its indifference price is zero,
\[ \nu(R_T) = 0. \] (1.55)
iv) It is a supermartingale under the pricing measure $\mathbb{Q}$.

\[ \mathbb{E}_\mathbb{Q}(R_T) \leq R_0 = 0. \] (1.56)

v) Its expected, under the historical measure $\mathbb{P}$, certainty equivalent is zero,

\[ \frac{1}{\gamma} \log \mathbb{E}_\mathbb{P}(e^{\gamma R_T}) = 0. \] (1.57)

Proof. Part (i) follows readily from the definition of the residual risk and the properties of $L_t$, $t = 0, T$.

To show (ii), we apply directly the definition of the conditional certainty equivalent. This, together with the measurability of $\tilde{C}_T$, yields

\[ \tilde{R}_T = \frac{1}{\gamma} \log \mathbb{E}_\mathbb{Q}(e^{\gamma (C_T - \tilde{C}_T)} | S_T) \]
\[ = \frac{1}{\gamma} \log \mathbb{E}_\mathbb{Q}(e^{\gamma C_T} | S_T) - \tilde{C}_T \]
\[ = \tilde{C}_T - \tilde{C}_T = 0. \]

Parts (iii) and (iv) are immediate consequences of (1.49), (1.53), and (1.54).

To establish (1.57), we recall that

\[ \tilde{R}_T = \frac{1}{\gamma} \log \mathbb{E}_\mathbb{Q}(e^{\gamma R_T} | S_T) = \frac{1}{\gamma} \log \mathbb{E}_\mathbb{P}(e^{\gamma R_T} | S_T), \]

where we used (1.8). Using (1.54) and taking the expectation under $\mathbb{P}$ yields the result. \qed

Being a supermartingale, the residual risk can be decomposed according to the Doob-Meyer decomposition. The related components can be easily retrieved and are presented below.

**Proposition 1.15** The supermartingale $R_t$, for $t = 0, T$, admits the Doob-Meyer decomposition

\[ R_t = R_t^m + R_t^d, \]

where

\[ R_0^m = 0 \quad \text{and} \quad R_T^m = R_T - E_\mathbb{Q}(R_T), \] (1.58)

and

\[ R_0^d = 0 \quad \text{and} \quad R_T^d = E_\mathbb{Q}(R_T). \] (1.59)

The component $R_t^m$ is an $\mathcal{F}_T^{(S,Y)}$-martingale under $\mathbb{Q}$, while $R_t^d$ is decreasing and adapted to the trivial filtration $\mathcal{F}_0^{(S,Y)}$.

We are now ready to provide the payoff decomposition result. This result is central in the study of risks associated with the indifference valuation method since it provides in a direct manner the constitutive analogue of the arbitrage-free payoff decomposition (1.37).
**Theorem 1.6** Let $\tilde{C}_T$ and $R_T$ be, respectively, the conditional certainty equivalent and the residual risk associated with the claim $C_T$. Let also $R^m_T$ and $R^d_T$ be the elements of the Doob-Meyer decomposition (1.58) and (1.59).

Define the process $M^C_t$, for $t = 0, T$, by

$$M^C_0 = \nu(C_T) \quad \text{and} \quad M^C_T = \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0}(S_T - S_0).$$

(1.60)

**i)** The claim $C_T$ admits the unique, under $Q$, payoff decomposition

$$C_T = \tilde{C}_T + R_T = \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0}(S_T - S_0) + R_T$$

(1.61)

$$= M^C_T + R^m_T + R^d_T.$$

**ii)** The indifference price process $\nu_t$, defined in (1.50), is an $\mathcal{F}^{(S,Y)}_T$-supermartingale under $Q$. It admits the unique decomposition

$$\nu_t(C_T) = M_t + R^d_t,$$

(1.62)

where

$$M_t = M^C_t + R^m_t.$$ 

The components $M_t$ and $R^d_t$ represent, respectively, the associated martingale and the non-increasing parts of the price process $\nu_t$.

From the application point of view, one may think of $R_T$ and its moments as natural variables for the quantification of errors associated with the risk monitoring policy. As the proposition below shows, the expected error obtains a rather intuitive form. It is proportional to the risk aversion and to the expected conditional variance of the nontraded risks. Naturally, both the expectation and the conditional variance need to be considered under the pricing measure $Q$.

**Proposition 1.16** The expected residual risk satisfies

$$\mathbb{E}_Q(R_T) = -\frac{1}{2} \gamma \mathbb{E}_Q(\text{Var}_Q(C_T|S_T)) + o(\gamma)$$

and

$$\mathbb{E}_Q(R_T) = -\frac{1}{2} \gamma \mathbb{E}_Q(\text{Var}_Q(R_T|S_T)) + o(\gamma).$$

**Proof.** The proof follows from (1.53), yielding

$$\mathbb{E}_Q(R_T) = \mathbb{E}_Q(C_T) - \mathbb{E}_Q(\tilde{C}_T),$$

and the approximation formula (1.29). The second equality is obvious. \[\square\]
1.2.4 Relative Indifference Prices

From the previous analysis, we can deduce that the indifference price is not a linear function of the claim’s payoff, namely, for $\alpha \neq 0, 1$,

$$v(\alpha C_T) \neq \alpha v(C_T). \quad (1.63)$$

Indeed, as it was established in Proposition 1.9, if $\alpha > 1$ (resp. $\alpha < 1$), the indifference price is a superhomogeneous (resp. subhomogeneous) function of $C_T$.

Following simple arguments, we easily conclude that if two payoffs, say $C_T^1$ and $C_T^2$, are considered, the indifference price functional is nonadditive, namely,

$$v(C_T^1 + C_T^2) \neq v(C_T^1) + v(C_T^2). \quad (1.64)$$

Extending these arguments to the case of multiple payoffs, we obtain that for $N$ with $N > 1$, payoffs

$$v \left( \sum_{i=1}^N C_T^i \right) \neq \sum_{i=1}^N v(C_T^i). \quad (1.65)$$

The nonadditive behavior of indifference prices is a direct consequence of the nonlinear character of the indifference valuation mechanism. Naturally, this nonlinear characteristic is inherited to the associated risk monitoring strategies. The nonadditivity property is perhaps the one that most differentiates the indifference prices and the relevant risk monitoring strategies from their complete market counterparts.

This might then look as a serious deficiency of the indifference valuation approach both for the theoretical as well as the practical point of view. However, it should be noted that the aggregate valuation of the above claims was considered as if the individual risks were priced in isolation. In practice, risks and projects need to be valued and hedged relative to already undertaken risks. In complete markets, perfect risk elimination makes this relative risk positioning redundant. But, when risks cannot be eliminated, one should develop a methodology that would quantify and price the incoming incremental risks, while taking into account the existing unhedgeable risk exposure.

These considerations lead us to the relative indifference valuation concept. The notion of relative indifference price was first introduced by the authors in Musiela and Zariphopoulou [196] and was recently further developed in Stoikov’s stochastic volatility model [256].

**Definition 1.7** Let $C_T^1 = C^1(S_T, Y_T)$ and $C_T^2 = C^2(S_T, Y_T)$ be two claims that have indifference prices $v(C_T^1)$ and $v(C_T^2)$. Let $V^{C_T^1}$, $V^{C_T^2}$ and $V^{C_T^1+C_T^2}$ be the value functions (1.3) corresponding to claims $C_T^1$, $C_T^2$ and $C_T^1 + C_T^2$.

The relative indifference prices $v(C_T^2/C_T^1)$ and $v(C_T^1/C_T^2)$ are defined, respectively, as the amounts satisfying

$$V^{C_T^1}(x) = V^{C_T^1+C_T^2}(x + v(C_T^2/C_T^1)) \quad (1.66)$$
and
\[ V^{C_1^T}(x) = V^{C_1^T + C_2^T}(x + v(C_1^T/C_2^T)) \]

for all wealth levels \( x \in \mathbb{R} \).

As the following result yields, when a new claim is being priced relatively to an already incorporated risk exposure, the associated indifference prices become linear.

**Proposition 1.17** Assume that the claims \( C_1^T = C_1^T(S_T, Y_T) \) and \( C_2^T = C_2^T(S_T, Y_T) \) have indifference prices \( v(C_1^T) \) and \( v(C_2^T) \) and relative indifference prices \( v(C_1^T/C_2^T) \) and \( v(C_2^T/C_1^T) \).

Then, the indifference price of the claim with payoff \( C_T = C_1^T + C_2^T \) satisfies
\[ v(C_T) = v(C_1^T) + v(C_2^T/C_1^T) \] (1.68)

and
\[ v(C_T) = v(C_2^T) + v(C_1^T/C_2^T). \]

**Proof.** We only show the first statement since the second follows by analogous arguments. For this, we recall the representation formula (1.23), which yields, respectively,
\[ V^{C_1^T}(x) = -e^{-\gamma x - H(Q[P] + v(C_1^T))} \]

and
\[ V^{C_1^T + C_2^T}(x) = -e^{-\gamma x - H(Q[P] + v(C_1^T/C_2^T))}. \]

Moreover, the same formula together with the definition of the relative indifference price \( v(C_1^T/C_2^T) \) implies that
\[ V^{C_1^T}(x) = -e^{-\gamma x - H(Q[P] + v(C_1^T))} \]
\[ = -e^{-\gamma(x + v(C_2^T/C_1^T) - H(Q[P] + v(C_1^T/C_2^T))} \]
\[ = V^{C_1^T + C_2^T}(x + v(C_2^T/C_1^T)) \]

for all wealth levels. Equating the exponents yields (1.68). \( \square \)

The following results are immediate consequences of the above.

**Corollary 1.8** The indifference prices \( v(C_1^T) \) and \( v(C_2^T) \) and their relative counterparts \( v(C_1^T/C_2^T) \) and \( v(C_2^T/C_1^T) \), satisfy
\[ v(C_1^T) - v(C_2^T) = v(C_1^T/C_2^T) - v(C_2^T/C_1^T). \]

**Corollary 1.9** The indifference price of the claim \( C_T = C_1^T + C_2^T \) is given by
\[ v(C_1^T + C_2^T) = \frac{1}{2}(v(C_1^T) + v(C_2^T)) + \frac{1}{2}(v(C_1^T/C_2^T) + v(C_2^T/C_1^T)). \]
Moreover,
\[
\nu(C_1^T + C_2^T) - (\nu(C_1^T) + \nu(C_2^T)) = \frac{1}{2} (\nu(C_1^T/C_2^T) + \nu(C_2^T/C_1^T)) - \frac{1}{2} (\nu(C_1^T) + \nu(C_2^T)).
\]

The latter formula yields the error emerging from the nonadditive character of the indifference price. This error may vanish in certain cases, as the examples below demonstrate. These examples were discussed in detail in Section 1.2.2 in the context of the additive invariance property of indifference prices. They refer to the special cases of a complete and a fully incomplete market setting.

**Special Cases Revisited:**

i) Let \( C_T = C_1^T + C_2^T \) with \( C_1^T = C^1(S_T, Y_T) \) and \( C_2^T = C^2(S_T) \). Property (1.35) implies that the price is additive. In fact,
\[
\nu(C_T) = \nu(C_1^T) + \nu(C_2^T)
\]
with \( \nu(C_2^T) = E_Q(C^2(S_T)) \). Proposition 1.17 then yields that
\[
\nu(C_2^T/C_1^T) = \nu(C_2^T).
\]
If, additionally, \( Y_T \) depends functionally on \( S_T \), then we easily deduce that
\[
\nu(C_T) = E_Q(C_1^T) + E_Q(C_2^T)
\]
and, in turn, that
\[
\nu(C_1^T/C_2^T) = \nu(C_1^T) \quad and \quad \nu(C_2^T/C_1^T) = \nu(C_2^T).
\]

ii) Let \( C_T = C^1(S_T) + C^2(Y_T) \) with \( Y_T \) and \( S_T \) be independent under \( P \). Then, it was shown that the price behaves additively, namely,
\[
\nu(C_T) = \nu(C_1^T) + \nu(C_2^T)
\]
with
\[
\nu(C_1^T) = E_Q(C_1^T) \quad and \quad \nu(C_2^T) = \frac{1}{\gamma} \log E_P(e^{\gamma C^2(Y_T)}).
\]
Proposition 1.17, in turn, implies
\[
\nu(C_1^T/C_2^T) = \nu(C_1^T) \quad and \quad \nu(C_2^T/C_1^T) = \nu(C_2^T).
\]

The above examples demonstrate that the relative indifference prices reduce to the classical ones if the relevant risks are either fully replicable or independent from the traded ones.

### 1.2.5 Wealths, Preferences, and Numeraire

The results of the previous sections were derived under the assumptions of zero interest rate and constant risk aversion. In this case, the wealths at the beginning and the end of a time period are expressed in a comparable unit (spot or forward),
and, thus, the possible dependence of the underlying optimization problems on the unit choice is not apparent.

Below we analyze this question by looking first at the relationship between the spot and forward units. Then, we consider a state-dependent risk aversion coefficient in order to cover other cases of numeraire. In particular, we consider the stock itself as a numeraire and show that the indifference prices can be made numeraire independent and consistent with the static no-arbitrage constraint if the appropriate dependence across units is built into the risk preference structure.

\( i ) \) Indifference Prices in Spot and Forward Units

Consider the one-period model, introduced in Section 1.2.1, of a market with a riskless bond and two risky assets, of which only one is traded. The dynamics of the risky assets remain unchanged, but we now allow for a nonzero riskless rate. The price of the riskless asset, therefore, satisfies \( B_0 = 1 \) and \( B_T = 1 + r \) with \( \xi^d \leq 1 + r \leq \xi^u \).

Because of the nonzero riskless rate, the price formula (1.9) cannot be directly applied. In order to produce meaningful prices, one needs to be consistent with the units in which the quantities that are used in price specification are expressed. For the case at hand, we will consider the valuation problem in spot and in forward units and will force the price to become independent of the unit choice.

We start with the formulation of the indifference price problem in spot units.

Consider a portfolio consisting of \( \alpha \) shares of stock and the amount \( \beta \) invested in the riskless asset. Its current value is given by \( \beta + \alpha S_0 = x \), where \( x \) represents the agent’s initial wealth, \( X_0 = x \). Expressed in spot units, that is discounted to time 0, its spot terminal wealth \( X^s_T \) satisfies

\[
X^s_T = x + \alpha \left( \frac{S_T}{1 + r} - S_0 \right). \tag{1.69}
\]

The investor’s utility is taken to be exponential with constant absolute risk aversion coefficient \( \gamma^s \). It is important to note that for the utility to be well defined, this coefficient needs to be expressed in the reciprocal of the spot unit. Optimality of investments will be carried out through the relevant value function, in spot units, given by

\[
V^{s, C_T}(x) = \sup_{\alpha} E_p \left( -e^{-\gamma^s} \left( \frac{X^s_T - C_T}{1 + r} \right) \right). \tag{1.70}
\]

Note that the option payoff \( C_T \) is also discounted from time \( T \) to time 0. The following definition is a natural extension of Definition 1.1.

**Definition 1.10** The indifference price, in spot units, of the claim \( C_T \) is defined as the amount \( v^s(C_T) \) for which the two spot value functions \( V^{s, C_T} \) and \( V^{s, 0} \), defined in (1.70) and corresponding to claims \( C_T \) and 0, coincide. Namely, it is the amount \( v^s(C_T) \) satisfying

\[
V^{s, 0}(x) = V^{s, C_T}(x + v^s(C_T)) \tag{1.71}
\]

for any initial wealth levels \( x \in \mathbb{R} \).
**Proposition 1.18** Let \( Q^x \) be the measure such that

\[
    \mathbb{E}_{Q^x} \left( \frac{S_T}{1+r} \right) = S_0
\]

and

\[
    Q^x(Y_T|S_T) = \mathbb{P}(Y_T|S_T). \quad (1.72)
\]

Moreover, let \( C_T = c(S_T, Y_T) \) be the claim to be priced, in spot units, with spot risk aversion coefficient \( \gamma^x \). Then, the spot indifference price is given by

\[
    v^x(C_T) = \mathbb{E}_{Q^x} \left( \frac{C_T}{1+r} \right) = \mathbb{E}_{Q^x} \left( \frac{1}{\gamma^x} \log \mathbb{E}_{Q^x} \left( e^{\gamma^x C_T | S_T} \right) \right). \quad (1.73)
\]

**Proof.** Working along similar arguments to the ones used in the proof of Proposition 1.1, we first establish that the spot value functions, \( V^{x,0} \) and \( V^{x,C_T} \), are given by

\[
    V^{x,0}(x) = -e^{-\gamma^x} \frac{1}{(q^x)^{\xi^x}(1-q^x)^{1-q^x}} (p_1 + p_2)^{\xi^x} (p_3 + p_4)^{1-q^x}
\]

and

\[
    V^{x,C_T}(x) = -e^{-\gamma^x} \frac{1}{(q^x)^{\xi^x}(1-q^x)^{1-q^x}} (e^{\gamma^x \xi^x p_1 + e^{\gamma^x \xi^x p_2}}) q^x
\]

\[
    \times (e^{\gamma^x \xi^x p_3 + e^{\gamma^x \xi^x p_4}})^{1-q^x},
\]

where

\[
    q^x = \frac{(1+r) - \xi^d}{\xi^u - \xi^d}. \quad (1.74)
\]

Applying Definition 1.10 gives

\[
    v^x(C_T) = q^x \left( \frac{1}{\gamma^x} \log \frac{e^{\gamma^x \xi^x p_1 + e^{\gamma^x \xi^x p_2}}}{p_1 + p_2} \right)
\]

\[
    + (1-q^x) \left( \frac{1}{\gamma^x} \log \frac{e^{\gamma^x \xi^x p_3 + e^{\gamma^x \xi^x p_4}}}{p_3 + p_4} \right). \quad (1.75)
\]

Straightforward calculations yield that the spot pricing measure \( Q^x \) has elementary probabilities, denoted by \( q^x_i \), \( i = 1, \ldots, 4 \),

\[
    q^x_i = q^x \frac{p_i}{p_1 + p_2}, \quad i = 1, 2 \quad \text{and} \quad q^x_i = (1-q^x) \frac{p_i}{p_3 + p_4}, \quad i = 3, 4. \quad (1.76)
\]

We introduce the conditional certainty equivalent in spot units

\[
    \tilde{C}^x_T = \frac{1}{\gamma^x} \log \mathbb{E}_{Q^x} (e^{\gamma^x C_T | S_T}).
\]

Equation (1.75) then yields

\[
    v^x(C_T) = \mathbb{E}_{Q^x} (\tilde{C}^x_T),
\]

and (1.73) follows. \( \square \)
We next analyze the indifference valuation of $C_T$ assuming that all relevant prices, risk preferences, and value functions are expressed in the reciprocal of the forward unit. To this end, we consider the forward terminal wealth

$$X_T^f = X_T^f (1 + r)$$

$$= x(1 + r) + \alpha (S_T - S_0(1 + r))$$

$$= f + \alpha (F_T - F_0),$$  \hspace{1cm} (1.77)

where $f = x(1 + r)$ is the forward value of the current wealth. Moreover, $F_0 = S_0(1 + r)$ and $F_T = S_T$ is the forward stock price process. Implicitly, we assume existence of the forward market for the risky traded asset $S$, and hence of the quoted prices $F_0$ and $F_T$, for it can be replicated by trading in this market. The corresponding forward value function is

$$V^{f,C_T} (f) = \sup_a E_P (-e^{-\gamma f(X_T^f - C_T)}).$$  \hspace{1cm} (1.78)

The risk aversion coefficient $\gamma^f$ is naturally expressed in forward units.

**Definition 1.11** The indifference price, in forward units, of the claim $C_T$ is defined as the amount $\nu^f (C_T)$ for which the two forward value functions $V^{f,C_T}$ and $V^{f,0}$, defined in (1.78) and corresponding to claims $C_T$ and 0, coincide. Namely, it is the amount $\nu^f (C_T)$ satisfying

$$V^{f,0} (f) = V^{f,C_T} (f + \nu^f (C_T))$$  \hspace{1cm} (1.79)

for any initial wealth levels $f \in \mathbb{R}$.

**Proposition 1.19** Let $Q^f$ be a measure under which

$$E_{Q^f} (F_T) = F_0$$

and

$$Q^f (Y_T | F_T) = \mathbb{P}(Y_T | F_T).$$  \hspace{1cm} (1.80)

Then

$$Q^f = Q^s.$$  \hspace{1cm} (1.81)

Let $C_T$ be the claim to be priced under exponential preferences with forward risk aversion coefficient $\gamma^f$. Then, the indifference price in forward units of $C_T$ is given by

$$\nu^f (C_T) = \mathcal{E}_{Q^f} (C_T)$$

$$= E_{Q^f} \left( \frac{1}{\gamma^f} \log E_{Q^f} (e^{\gamma^f C_T} | F_T) \right).$$  \hspace{1cm} (1.82)

**Proof.** Given the deterministic interest rate assumption, the fact that the measures $Q^f$ and $Q^s$ coincide is obvious. We next observe that the forward value function
The single period binomial model can be written as
\[
V^{f,C_T}(f) = \sup_{\alpha} \mathbb{E}(e^{-\gamma^f(X^f_T-C_T)})
\]
\[
= \sup_{\alpha} \mathbb{E}(e^{-\gamma^f(s(1+r)+\alpha(S_T-S_0(1+r))-C_T)})
\]
\[
= \sup_{\alpha} \mathbb{E}(e^{-\gamma^f(s+\alpha(\frac{S_T}{1+r}-S_0)-\frac{C_T}{1+r})}),
\]
with \(\tilde{\gamma} = \gamma^f(1+r)\). Therefore, \(V^{f,C_T}\), and in turn \(V^{0,C_T}\), can be directly retrieved from their forward counterparts. The rest of the proof follows easily and it is therefore omitted. \(\Box\)

For the rest of the analysis, we denote by \(\mathbb{Q}\) the common spot and forward pricing measure.

We are now ready to investigate when the spot and forward indifference prices are consistent with the static no-arbitrage condition and independent of the units, spot or forward, chosen in the supporting investment optimization problem. The result below gives the necessary and sufficient conditions on the spot and forward risk aversion coefficients.

**Proposition 1.20** The indifference prices, expressed in spot and forward units, are consistent with the no-arbitrage condition, that is,
\[
\nu^f(C_T) = (1+r)\nu^s(C_T),
\]
if and only if the spot and forward risk aversion coefficients satisfy
\[
\gamma^f = \frac{\nu^s}{1+r}.
\]

**Proof.** We first show that if (1.84) holds, then (1.83) follows. Recalling (1.73) and (1.82), we deduce that, if (1.84) holds, then \(\nu^f(C_T)\) can be written as
\[
\nu^f(C_T) = (1+r)\mathbb{E}_\mathbb{Q}\left(\frac{1}{\nu^s} \log \mathbb{E}_\mathbb{Q}(e^{\nu^s C_T} | S_T)\right),
\]
and one direction of the statement follows. We remind the reader that \(\mathbb{Q} = \mathbb{Q}^s = \mathbb{Q}^f\).

We next show that for (1.83) to hold for all \(C_T\) we must have (1.84). Indeed, if the consistency relationship (1.83) holds, then, for all claims \(C_T\),
\[
\frac{1}{1+r} \mathbb{E}_\mathbb{Q}\left(\frac{1}{\nu^f} \log \mathbb{E}_\mathbb{Q}(e^{\nu^f C_T} | S_T)\right) = \mathbb{E}_\mathbb{Q}\left(\frac{1}{\nu^s} \log \mathbb{E}_\mathbb{Q}(e^{\nu^s C_T} | S_T)\right),
\]
and, in turn,
\[
\mathbb{E}_\mathbb{Q}\left(\frac{1}{\nu^f} \log \mathbb{E}_\mathbb{Q}(e^{\nu^f C_T} | S_T)\right) = \mathbb{E}_\mathbb{Q}\left(\frac{1+r}{\nu^s} \log \mathbb{E}_\mathbb{Q}(e^{\nu^s C_T} | S_T)\right).
\]
The statement then follows from straightforward rescaling arguments and (1.24). \(\Box\)

We continue with a representation result for the spot and forward value functions.

We recall that the spot and forward pricing measures reduce to the same measure \(\mathbb{Q}\), which, therefore, has elementary probabilities \(q_i, i = 1, \ldots, 4\) given in (1.76).
Working along similar arguments to the ones used in Section 1.2.1 we can easily establish the following result.

**Proposition 1.21** Let \( H(Q|P) \) be as in (1.21). Then, the value functions \( V^{\omega,C_T} \) and \( V^{f,C_T} \) are given by

\[
V^{\omega,C_T}(x) = -e^{-\gamma^\omega(x-v^\omega(C_T)) - H(Q|P)} = U^\omega \left( x - v^\omega(C_T) + \frac{1}{\gamma^\omega} H(Q|P) \right),
\]

\[
V^{f,C_T}(f) = -e^{-\gamma^f(f-v^f(C_T)) - H(Q|P)} = U^f \left( f - v^f(C_T) + \frac{1}{\gamma^f} H(Q|P) \right),
\]

where \( U^\omega(x) = -e^{-\gamma^\omega x} \) and \( U^f(f) = -e^{-\gamma^f f} \) represent the spot and forward utility functions.

Recall that the argument \( x \) in \( U^\omega(x) \) and in \( V^{\omega,C_T}(x) \) is expressed in the spot units, while the same argument \( f \) in \( U^f(f) \) and \( V^{f,C_T}(f) \) is expressed in the forward units. Therefore, the utility and the value functions represent the same utility and value, independently on the units in which the relevant optimization problems are solved, if and only if (1.84) holds.

More generally, the indifference-based valuation as well as the associated optimal investment problems can be formulated and solved in a numeraire-independent fashion provided the appropriate relations are built into the preference structure. In fact, these problems can be analyzed without making any reference to a unit by optimizing over unitless quantities like \( \gamma^\omega X_T^\omega \) or \( \gamma^f X_T^f \).

**ii) Indifference Prices and State-Dependent Preferences**

Before we proceed with the specification of the price and the conditions for numeraire independence, we extend our previous setup to the case of the random risk aversion coefficient. Specifically, we assume that it is a function of the states of the traded asset. We may then conveniently represent the risk aversion at time \( T \) as the \( \mathcal{F}_T^3 \)-measurable random variable \( \gamma_T = \gamma(S_T) \) taking the values

\[
\gamma^u = \gamma(S_0 \xi^u) \quad \text{and} \quad \gamma^d = \gamma(S_0 \xi^d),
\]

when the events \( \{\omega : S_T(\omega) = S_0 \xi^u \} = \{\omega^1, \omega^2\} \) and \( \{\omega : S_T(\omega) = S_0 \xi^d \} = \{\omega^3, \omega^4\} \) occur.

Clearly, the risk aversion \( \gamma_T \) is expressed in the unit that is the reciprocal of the wealth \( X_T \) unit. Alternatively, one may think of the risk tolerance

\[
\delta_T = \frac{1}{\gamma_T}, \tag{1.85}
\]

which is obviously expressed in the units of wealth at time \( T \).

Here we assume the same model as in the previous section and choose to work with spot units, so \( X_T = X_T^\omega \) as in (1.69).

We have mentioned already that representations of the indifference prices under minimal model assumptions have been derived via duality arguments. These results can be extended even to the cases when the risk aversion coefficient is random. The
related pricing formulas, however, take a form that reveals limited insights about the numeraire and units effects. For this, we seek an alternative price representation, provided below, which, to the best of our knowledge, is new.

We also adopt the notation \( v(C_T; \gamma_T) \), \( V^{C_T}(x; \gamma_T) \) and \( V^0(x; \gamma_T) \) for the price and the relevant value functions, so that the random nature of risk preferences is conveniently highlighted.

**Proposition 1.22** Assume that the risk aversion coefficient \( \gamma_T \) is of the form \( \gamma_T = \gamma(S_T) \). Let \( \mathbb{Q} \) be the measure defined in (1.72) and let \( C_T = c(Y_T, S_T) \) be a claim to be priced under exponential utility with risk aversion coefficient \( \gamma_T \). Then, the indifference price of \( C_T \) is given by

\[
v(C_T; \gamma_T) = \mathbb{E}_{\mathbb{Q}} \left( \frac{1}{\gamma_T} \log \mathbb{E}_{\mathbb{Q}} (e^{\gamma_T \frac{C_T}{T+rT}} | S_T) \right).
\]

**Proof.** In order to construct the indifference price, we need to compute the value functions \( V^{C_T}(x; \gamma_T) \) and \( V^0(x; \gamma_T) \). We recall that

\[
v^{C_T}(x; \gamma_T) = \sup_a \mathbb{E}_P \left( -e^{-\gamma_T (x+\alpha \left( \frac{\xi}{T+rT} - S_0 \right) - \frac{C_T}{T+rT}} \right) \quad (1.86)
\]

and we introduce the notation

\[
\beta^u = \gamma^u \left( \frac{\xi^u}{1+r} - 1 \right) \quad \text{and} \quad \beta^d = \gamma^d \left( 1 - \frac{\xi^d}{1+r} \right). \quad (1.87)
\]

Further calculations yield

\[
V^{C_T}(x; \gamma_T) = \sup_a \phi(a),
\]

where

\[
\phi(a) = e^{-a S_0 \beta^u} \left( e^{-\gamma^u (x-\frac{\xi^u}{T+rT})} p_1 + e^{-\gamma^u (x-\frac{\xi^u}{T+rT})} p_2 \right) - e^{-a S_0 \beta^d} \left( e^{-\gamma^d (x-\frac{\xi^d}{T+rT})} p_3 + e^{-\gamma^d (x-\frac{\xi^d}{T+rT})} p_4 \right),
\]

with \( \beta^u \) and \( \beta^d \) given in (1.87).

Differentiating with respect to \( \alpha \) yields that the maximum occurs at

\[
\alpha^{C_T, *} = \frac{1}{S_0 (\beta^u + \beta^d)} \log \frac{\beta^u e^{-\gamma^u x} \left( e^{\gamma^u \frac{\xi^u}{T+rT}} p_1 + e^{\gamma^u \frac{\xi^u}{T+rT}} p_2 \right)}{\beta^d e^{-\gamma^d x} \left( e^{\gamma^d \frac{\xi^d}{T+rT}} p_3 + e^{\gamma^d \frac{\xi^d}{T+rT}} p_4 \right)}. \quad (1.88)
\]

Calculating the terms \( -e^{-a^{C_T, *} S_0 \beta^u} \) and \( -e^{-a^{C_T, *} S_0 \beta^d} \) gives

\[
e^{-a^{C_T, *} S_0 \beta^u} = \left( \frac{\beta^u}{\beta^d} \right)^{\frac{\beta^d}{\beta^u (\beta^u - \beta^d)}} \left( \frac{e^{\beta^u \left( \frac{\xi^u}{T+rT} \right)} p_1 + e^{\beta^u \left( \frac{\xi^u}{T+rT} \right)} p_2}{e^{\beta^d \left( \frac{\xi^d}{T+rT} \right)} p_3 + e^{\beta^d \left( \frac{\xi^d}{T+rT} \right)} p_4} \right)^{\frac{\beta^u}{\beta^u + \beta^d}} - \frac{\beta^u}{\beta^u + \beta^d},
\]
and

\[ e^{-\alpha C_T \cdot S_0 \beta^d} = \left( \frac{\beta^u}{\beta^d} \right)^{-\frac{\mu^d}{\mu^u + \mu^d}} e^{\frac{\mu^u}{\mu^u + \mu^d}} \left( \frac{\epsilon^{u_{11}} p_1 + \epsilon^{u_{22}} p_2}{\epsilon^{d_{11}} p_3 + \epsilon^{d_{22}} p_4} \right)^{-\frac{\rho^d}{\rho^u + \rho^d}}. \]

It then follows, after tedious but routine calculations that

\[
V^C_T(x; \gamma_T) = \phi(\alpha^{C_T \cdot a})
\]

\[
= - \left( \left( \frac{\beta^u}{\beta^d} \right)^{-\frac{\mu^d}{\mu^u + \mu^d}} + \left( \frac{\beta^u}{\beta^d} \right)^{\frac{\rho^d}{\rho^u + \rho^d}} \right) e^{\frac{\rho^d}{\rho^u + \rho^d}} x
\]

\[
\times \left( e^{\nu_{11}^u \tau_{11}^u} p_1 + e^{\nu_{22}^u \tau_{22}^u} p_2 \right)^{\frac{\rho^d}{\rho^u + \rho^d}} \left( e^{\nu_{11}^d \tau_{11}^d} p_3 + e^{\nu_{22}^d \tau_{22}^d} p_4 \right)^{\frac{\rho^d}{\rho^u + \rho^d}}.
\]

(1.89)

Substituting \( C_T = 0 \) in turn implies

\[
V^0(x; \gamma_T) = - \left( \left( \frac{\beta^u}{\beta^d} \right)^{-\frac{\mu^d}{\mu^u + \mu^d}} + \left( \frac{\beta^u}{\beta^d} \right)^{\frac{\rho^d}{\rho^u + \rho^d}} \right) e^{\frac{\rho^d}{\rho^u + \rho^d}} x
\]

\[
\times (p_1 + p_2)^{\frac{\rho^d}{\rho^u + \rho^d}} (p_3 + p_4)^{\frac{\rho^d}{\rho^u + \rho^d}}.
\]

(1.90)

Using (1.89), (1.90), and (1.4) (cf. Definition 1.1), we get

\[
v(C_T; \gamma_T) = \left( \frac{\beta^u \gamma^d + \beta^d \gamma^u}{\beta^u + \beta^d} \right)^{-1}
\]

\[
\times \left( \frac{\beta^d}{\beta^u + \beta^d} \log \frac{e^{\nu_{11}^u \tau_{11}^u} p_1 + e^{\nu_{22}^u \tau_{22}^u} p_2}{p_1 + p_2}
\]

\[
+ \frac{\beta^u}{\beta^u + \beta^d} \log \frac{e^{\nu_{11}^d \tau_{11}^d} p_3 + e^{\nu_{22}^d \tau_{22}^d} p_4}{p_3 + p_4} \right).
\]

(1.91)

We next observe that

\[
\frac{\gamma^u \beta^d}{\beta^u \gamma^d + \beta^d \gamma^u} = \frac{(1 + r) - \xi^d}{\xi^u - \xi^d}
\]

\[= \frac{S_0(1 + r) - S^d}{S^u - S^d} = q, \]

(1.92)

and

\[
\frac{\gamma^d \beta^u}{\beta^u \gamma^d + \beta^d \gamma^u} = \frac{\xi^u - (1 + r)}{\xi^u - \xi^d}
\]

\[= \frac{S^u - S_0(1 + r)}{S^u - S^d} = 1 - q, \]

(1.93)
where $S^u = S_0 \xi^u$ and $S^d = S_0 \xi^d$. The above equalities combined with (1.91) yield
\begin{equation}
\nu(C_T; \gamma_T) = q \frac{1}{\gamma^u} \log \frac{e^{\gamma^u \frac{c_1}{r_T}} p_1 + e^{\gamma^u \frac{c_2}{r_T}} p_2}{p_1 + p_2} + (1 - q) \frac{1}{\gamma^d} \log \frac{e^{\gamma^d \frac{c_3}{r_T}} p_3 + e^{\gamma^d \frac{c_4}{r_T}} p_4}{p_3 + p_4}. \tag{1.93}
\end{equation}

Following the arguments developed in the proof of Proposition 1.2 we see that
\begin{equation}
\frac{e^{\gamma^u \frac{c_1}{r_T}} p_1 + e^{\gamma^u \frac{c_2}{r_T}} p_2}{p_1 + p_2} = \mathbb{E}_Q(e^{\gamma_T \frac{C_T}{r_T}} | S_T = S^u) \tag{1.94}
\end{equation}
and
\begin{equation}
\frac{e^{\gamma^d \frac{c_3}{r_T}} p_3 + e^{\gamma^d \frac{c_4}{r_T}} p_4}{p_3 + p_4} = \mathbb{E}_Q(e^{\gamma_T \frac{C_T}{r_T}} | S_T = S^d). \tag{1.95}
\end{equation}

Finally, using the equalities (1.94) and (1.95) and the expression in (1.93), we easily conclude. \hfill \Box

**Proposition 1.23** Assume that the risk aversion coefficient $\gamma_T$ is of the form $\gamma_T = \gamma(S_T)$ and let $\delta_T$ be the risk tolerance coefficient introduced in (1.85). Let $\mathbb{Q}$ be the measure defined in (1.72) and let $C_T = c(Y_T, S_T)$ be a claim to be priced under exponential utility with risk aversion coefficient $\gamma_T$.

Then, the value functions $V^0(x; \gamma_T)$ and $V^C_T(x; \gamma_T)$, defined in (1.86), admit the following representations
\begin{equation}
V^0(x; \gamma_T) = -\exp \left( -x \frac{1}{\mathbb{E}_Q(\delta_T)} \right) \tag{1.96}
\end{equation}
\begin{equation}
V^C_T(x; \gamma_T) = -\exp \left( -x - \nu(C_T; \gamma_T) \frac{1}{\mathbb{E}_Q(\delta_T)} \right), \tag{1.97}
\end{equation}
where
\begin{equation}
\nu(C_T; \gamma_T) = \mathbb{E}_Q \left( \frac{1}{\gamma_T} \log \mathbb{E}_Q(e^{\gamma_T \frac{C_T}{r_T}} | S_T) \right) \tag{1.98}
\end{equation}
and
\begin{equation}
\frac{d\mathbb{Q}^*}{d\mathbb{Q}}(\omega) = \frac{\delta_T(\omega)}{\mathbb{E}_Q(\delta_T)}. \tag{1.99}
\end{equation}

**Proof.** To show (1.96) and (1.97), we work with formulas (1.89), and (1.90), interpreting appropriately the involved quantities.

We first observe that
\begin{align*}
\frac{\beta^u \gamma^d + \beta^d \gamma^u}{\beta^u + \beta^d} &= \left( \frac{\beta^u + \beta^d}{\beta^u \gamma^d + \beta^d \gamma^u} \right)^{-1} \\
&= \frac{1}{\gamma^u} (1 - q) + \frac{1}{\gamma^d} q = \frac{1}{\mathbb{E}_Q(\delta_T)}
\end{align*}
and, that
\[
\frac{\beta^u}{\beta^d} = \frac{\gamma^u(1-q)}{\gamma^d q}, \quad \frac{\beta^u}{\beta^d} = \frac{(\gamma^d)^{-1}(1-q)}{E_Q(\delta_T)}, \quad \frac{\beta^d}{\beta^u} = \frac{(\gamma^u)^{-1}q}{E_Q(\delta_T)}.
\]

Combining the above, we obtain
\[
V^0(x; \gamma_T) = -e^{-\frac{x}{E_Q(\delta_T)}} \left( \frac{p_1 + p_2}{q^*} \right)^{\frac{q^*}{p_3 + p_4}} \left( \frac{p_3 + p_4}{1-q^*} \right)^{1-q^*}
= -e^{-\frac{x}{E_Q(\delta_T)}} \cdot \gamma(\gamma^u(1+r)).
\]

Herein, the measure \(Q^*\) is defined by
\[
q_i^* = q^* \frac{p_i}{p_1 + p_2}, \quad i = 1, 2 \quad \text{and} \quad q_i^* = (1-q^*) \frac{p_i}{p_3 + p_4}, \quad i = 3, 4,
\]
with
\[
q^* = \frac{(\gamma^u)^{-1}q}{E_Q(\delta_T)}, \quad \text{(1.100)}
\]

where \(q\) given in (1.92).

Note that the emerging measure \(Q^*\) satisfies
\[
E_{Q^*}(\gamma_T(S_T - S_0(1+r))) = 0 \quad \text{(1.101)}
\]

and gives the same conditional distribution of \(Y_T\) given \(S_T\) as the measure \(P\).

Alternatively, one can define \(Q^*\) by its Radon-Nikodym density with respect to the pricing measure \(Q\). Namely,
\[
\frac{dQ^*}{dQ}(\omega_i) = \frac{q^*}{q} = \frac{(\gamma^u)^{-1}}{E_Q(\delta_T)}, \quad i = 1, 2
\]

and
\[
\frac{dQ^*}{dQ}(\omega_i) = \frac{1-q^*}{1-q} = \frac{(\gamma^d)^{-1}}{E_Q(\delta_T)}, \quad i = 3, 4.
\]

To complete the proof, it remains to show (1.97). For this, it suffices to observe that the terms appearing in (1.89), containing the payoff, can be written as
\[
\left( \frac{e^\nu_{\omega_1} \gamma_T p_1 + e^\nu_{\omega_2} \gamma_T p_2}{p_1 + p_2} \right)^{\frac{\beta^d}{\beta^u + \beta^d}} \left( \frac{e^\nu_{\omega_3} \gamma_T p_3 + e^\nu_{\omega_4} \gamma_T p_4}{p_3 + p_4} \right)^{\frac{\beta^u}{\beta^u + \beta^d}}
= \exp \left( \frac{\nu(C_T; \gamma_T)}{E_Q(\delta_T)} \right),
\]

where we used (1.98). Formula (1.97) then follows from the latter equality and assertions (1.89) and (1.96). \(\square\)

We note that even though both measures, \(Q\) and \(Q^*\), have minimal, relative to \(P\), entropy, they have different martingales properties. Namely, under \(Q\),
\[
E_Q(S_T - S_0(1+r)) = 0,
\]
while, under $\mathbb{Q}^*$,

$$
\mathbb{E}_{\mathbb{Q}^*}(\gamma_T(S_T - S_0(1 + r))) = 0.
$$

We finish this section with an important decomposition result for the writer’s optimal policy.

**Proposition 1.24** The writer’s optimal investment policy $\alpha_{C^\mathbb{Q}^*}$ (cf. (1.88)) admits the following decomposition:

$$
\alpha_{C^\mathbb{Q}^*} = \alpha_{C^\mathbb{Q}^*}^0 + \alpha_{C^\mathbb{Q}^*}^1 + \alpha_{C^\mathbb{Q}^*}^2,
$$

where

$$
\alpha_{C^\mathbb{Q}^*}^0 = -\frac{\partial H(\mathbb{Q}^* | \mathbb{P})}{\partial S_0} \mathbb{E}_{\mathbb{Q}^*}(\delta_T), \quad \alpha_{C^\mathbb{Q}^*}^1 = \frac{\partial \log \mathbb{E}_{\mathbb{Q}^*}(\delta_T)}{\partial S_0}
$$

and

$$
\alpha_{C^\mathbb{Q}^*}^2 = \mathbb{E}_{\mathbb{Q}^*}(\delta_T) \frac{\nu(C_T; \gamma_T)}{\mathbb{E}_{\mathbb{Q}^*}(\delta_T)}.
$$

**Proof.** We first recall that

$$
\alpha_{C^\mathbb{Q}^*} = \frac{1}{S_0(\beta^u + \beta^d)} \log \frac{\beta^u e^{-\gamma^u s} \left( e^{\gamma^u t_1} p_1 + e^{\gamma^u t_2} p_2 \right)}{\beta^d e^{-\gamma^d s} \left( e^{\gamma^d t_3} p_3 + e^{\gamma^d t_4} p_4 \right)}.
$$

We then write $\alpha_{C^\mathbb{Q}^*}$ as

$$
\alpha_{C^\mathbb{Q}^*} = \alpha_{C^\mathbb{Q}^*}^0 + \alpha_{C^\mathbb{Q}^*}^1 + \alpha_{C^\mathbb{Q}^*}^2,
$$

where

$$
\alpha_{C^\mathbb{Q}^*}^0 = \frac{1}{S_0(\beta^u + \beta^d)} \log \frac{\beta^u (p_1 + p_2)}{\beta^d (p_3 + p_4)},
$$

$$
\alpha_{C^\mathbb{Q}^*}^1 = \frac{1}{S_0(\beta^u + \beta^d)} \log \frac{e^{-\gamma^u s}}{e^{-\gamma^d s}},
$$

and

$$
\alpha_{C^\mathbb{Q}^*}^2 = \frac{1}{S_0(\beta^u + \beta^d)} \log \frac{\left( e^{\gamma^u t_1} p_1 + e^{\gamma^u t_2} p_2 \right) (p_3 + p_4)}{\left( e^{\gamma^d t_3} p_3 + e^{\gamma^d t_4} p_4 \right) (p_1 + p_2)}.
$$

We will next represent the various quantities in terms of the measures $\mathbb{Q}$ and $\mathbb{Q}^*$. Recall that

$$
q = \frac{S_0(1 + r) - S^d}{S^u - S^d}, \quad \frac{\partial q}{\partial S_0} = \frac{1 + r}{S^u - S^d},
$$

and

$$
\frac{\beta^d}{\beta^u + \beta^d} = \frac{(\gamma^u)^{-1} q}{\mathbb{E}_{\mathbb{Q}^*}(\delta_T)} = q^*,
$$

$$
\frac{\beta^u}{\beta^u + \beta^d} = \frac{(\gamma^d)^{-1} (1 - q)}{\mathbb{E}_{\mathbb{Q}^*}(\delta_T)} = 1 - q^*.
$$
It follows, trivially, that
\[
\log \frac{\beta^u(p_1 + p_2)}{\beta^d(p_3 + p_4)} = \log \frac{(1-q^*)(p_1 + p_2)}{q^*(p_3 + p_4)}.
\]

Moreover, the relative entropy \( H(Q^*|P) \) is given by
\[
H(Q^*|P) = q^* \log \frac{q^*}{p_1 + p_2} + (1-q^*) \log \frac{1-q^*}{p_3 + p_4},
\]
and, hence,
\[
\frac{\partial H(Q^*|P)}{\partial S_0} = - \left( \frac{\partial q^*}{\partial S_0} \right) \log \frac{(1-q^*)(p_1 + p_2)}{q^*(p_3 + p_4)}.
\]

The sensitivity of \( q^* \) to \( S_0 \) can be easily calculated. Indeed, we get
\[
\frac{\partial q^*}{\partial S_0} = \frac{\partial q}{\partial S_0} \frac{(\gamma^u)^{-1}(\gamma^d)^{-1}}{E_Q(\delta_T)}.
\]

Also, because the coefficient \( S_0(\beta^u + \beta^d) \) can be written as
\[
\frac{1}{S_0(\beta^u + \beta^d)} = \frac{\partial q}{\partial S_0} \frac{(\gamma^u)^{-1}(\gamma^d)^{-1}}{E_Q(\delta_T)},
\]
we get
\[
\frac{1}{S_0(\beta^u + \beta^d)} = \frac{\partial q^*}{\partial S_0} E_Q(\delta_T).
\]

The above formulas imply that
\[
\alpha^{0,*} = - \frac{\partial H(Q^*|P)}{\partial S_0} E_Q(\delta_T).
\]

Moving to the second term, \( \alpha^{*,1} \), we notice that
\[
\alpha^{*,1} = - \frac{1}{S_0(\beta^u + \beta^d)}(\gamma^u - \gamma^d)x.
\]

Moreover, because,
\[
\frac{\partial \log E_Q(\delta_T)}{\partial S_0} = \frac{\partial \log E_Q(\delta_T)}{\partial q} \frac{\partial q}{\partial S_0}
\]
\[
= \frac{1}{E_Q(\delta_T)} ((\gamma^u)^{-1} - (\gamma^d)^{-1}) \frac{\partial q}{\partial S_0}
\]
\[
= - \frac{1}{S_0(\beta^u + \beta^d)}(\gamma^u - \gamma^d),
\]
we easily obtain that
\[
\alpha^{*,1} = \frac{\partial \log E_Q(\delta_T)}{\partial S_0} x.
\]
Obviously, the last term, $\alpha^{C^r,*}$, has to do with the indifference price sensitivity. Indeed, after rather tedious calculations, we obtain

\[ \alpha^{C^r,*} = (E_Q(\delta_T)) \frac{\partial q^*}{\partial S_0} \frac{\partial}{\partial q^*} \left( E_Q^* \left( \log E_Q^* \left( e^{\gamma T C_T} | S_T \right) \right) \right) \]

\[ = E_Q(\delta_T) \frac{\partial}{\partial S_0} \left( E_Q^* \left( \log E_Q^* \left( e^{\gamma T C_T} | S_T \right) \right) \right) \]

\[ = E_Q(\delta_T) \frac{\partial}{\partial S_0} \left( \frac{\nu(C_T; \gamma T)}{E_Q(\delta_T)} \right), \]

and the proof is complete. \( \square \)

The above propositions demonstrate the effects of the random nature of risk aversion on the form of the value function, the indifference price, and the optimal policy. The appearance of the expected value of the risk tolerance, the reciprocal of risk aversion, seems to indicate that this quantity, rather than risk aversion, is more natural from the structural and interpretation points of view.

Moreover, two interesting new features appear. First of all, the optimal policy $\alpha^{C^r,*}$ is no longer independent of the initial wealth. However, the hedging demand $\alpha^{2,*}$, coming from the presence of a derivative contract, remains independent on the initial wealth. So does the shares amount $\alpha^0$, which is aiming to benefit from the opportunities created by the differences in the probabilities allocated to the outcomes by the historical measure $\mathbb{P}$ and by $\mathbb{Q}^*$. The number of shares $\alpha^{1,*}$ depends linearly on the initial wealth $x$, with the slope $\frac{\partial \log E_Q(\delta_T)}{\partial S_0}$ representing the relative sensitivity of the current risk tolerance to the changes in the stock price. The second interesting and new feature is the sensitivity $\frac{\partial \nu(C_T; \gamma_T)}{\partial S_0} E_Q(\delta_T)$ of the option price expressed in a unitless fashion, i.e., relatively to the current risk tolerance.

### iii) Indifference Prices and General Numeraires

Recall that the wealth $X_T$ at time $T$, given in (1.69), is expressed in the spot units. Observe that if the stock price is taken as the numeraire, the wealth will be expressed in the number of shares of stock and not in a dollar amount.

Specifically, the terminal wealth $X_T^S$ is given by

\[ X_T^S = \frac{x}{S_T} + \alpha \left( \frac{1}{1+r} - \frac{S_0}{S_T} \right) \]

and the current wealth, equal to the number of shares at time 0, by

\[ X_0^S = \frac{x}{S_0} = x^S. \]

Note that $X_T$ is discounted to time 0, and hence $X_T^S$ is the time 0 equivalent of the number of shares held in the portfolio at time $T$.

The related value function is given by

\[ V^{S,C_T}(x^S) = \sup_{\sigma} \mathbb{E}_P \left( -e^{-\gamma S(T)} \left( x_T^S - \frac{C_T}{S_T(T)} \right) \right), \]

where $\gamma^S(S_T)$ represents the risk aversion associated with this unit.
The following definition is a direct extension of Definition 1.1. Notice that in the current unit framework, the *indifference price is expressed in number of shares* and not as a dollar amount.

**Definition 1.12** The indifference price of the claim $C_T$ is defined as the number of shares $v^S(C_T)$ for which the two value functions $V^{S,C_T}$ and $V^{S,0}$, defined in (1.103) and corresponding to claims $C_T$ and 0, coincide. Namely, it is the number of stock shares $v^S(C_T)$ satisfying

$$V^{S,0}(x^S) = V^{S,C_T}(x^S + v^S(C_T)) \quad (1.104)$$

for any initial number of shares $x^S \in \mathbb{R}$.

**Proposition 1.25** Let $\mathbb{Q}^S$ be a measure under which the discounted, by the traded asset, riskless bond, $B_t/S_t$, $t = 0, T$ is a martingale and, at the same time, the conditional distribution of the nontraded asset, given the traded one, is preserved with respect to the historical measure $\mathbb{P}$, i.e.,

$$\mathbb{Q}^S(Y_T|S_T) = \mathbb{P}(Y_T|S_T). \quad (1.105)$$

Let $C_T = c(S_T, Y_T)$ be the claim to be priced under exponential preferences with state-dependent risk aversion coefficient $\gamma^S(S_T)$. Then, the indifference price of $C_T$, quoted in the number of shares of stock, is

$$v^S(C_T) = \mathbb{E}_{\mathbb{Q}^S} \left( \frac{1}{\gamma^S(S_T)} \log \mathbb{E}_{\mathbb{Q}^S} \left( e^{\gamma^S(S_T) \mathbb{E}_{\mathbb{P}}(C_T | S_T)} \right) \right). \quad (1.106)$$

**Proof.** We start with the specification of the measure $\mathbb{Q}^S$. We recall that, given the choice of numeraire, the martingale in consideration is $B_t^S = \frac{B_t}{S_t}$, where $B_t$ and $S_t$ stand, respectively, for the original bond and stock process. We denote by $q_i^S$, $i = 1, \ldots, 4$ the elementary probabilities of $\mathbb{Q}^S$. Simple calculations yield that

$$q_i^S = q^S \frac{p_i}{p_1 + p_2}, \quad i = 1, 2 \quad \text{and} \quad q_i^S = (1 - q^S) \frac{p_i}{p_3 + p_4}, \quad i = 3, 4,$$

where

$$q^S = \left( \frac{1}{\xi^d} - \frac{1}{1 + r} \right) \frac{\xi^n \xi^d}{\xi^n - \xi^d}.$$

Alternatively, the measure $\mathbb{Q}^S$ can be defined by its Radon-Nikodym density with respect to $\mathbb{Q}$, namely,

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \frac{S_T}{(1 + r)S_0}.$$

Indeed, we have

$$\mathbb{E}_{\mathbb{Q}^S} \left( \frac{1 + r}{S_T} \right) = \mathbb{E}_{\mathbb{Q}} \left( \frac{S_T}{(1 + r)S_0} \frac{1 + r}{S_T} \right) = \frac{1}{S_0}.$$

We next observe that the value function $V^{S,C_T}$ (cf. (1.103)) can be written as

$$V^{S,C_T}(x^S) = \sup_a \mathbb{E}_{\mathbb{P}} \left( -e^{-\lambda_T \left( x + a \left( \frac{S_T}{(1 + r)S_0} - C_T \right) \right)} \right).$$
where

\[ \lambda_T = \frac{\gamma^S(S_T)}{S_T}. \] (1.107)

Working as in the proof of Proposition 1.23, we get

\[ V^{S,C_T}(x^S) = V^{C_T}(x; \lambda_T) \]

\[ = - \exp \left( - \frac{x - v(C_T; \lambda_T)}{E_Q(\lambda_T^{-1})} \right), \]

where

\[ \frac{d\tilde{Q}}{dQ} = \frac{\lambda_T^{-1}}{E_Q(\lambda_T^{-1})}. \]

The wealth argument \( x \) in \( V^{C_T}(x; \lambda_T) \) as well as the price

\[ v(C_T; \lambda_T) = E_Q \left( \frac{1}{\lambda_T} \log E_Q \left( e^{\gamma^T \frac{\epsilon_T}{1+r} | S_T} \right) \right) \]

are expressed in the spot units and, hence, for all claims \( C_T \) we have

\[ v(C_T; \gamma_T) = v(C_T; \lambda_T). \]

Considering, as in Proposition 1.4, claims of the form \( C_T^1(\omega_1) = c_1, C_T^1(\omega_2) = 0, \]
\( i = 2, 3, 4 \) and \( C_T^2(\omega_3) = c_5, C_T^2(\omega_4) = 0, i = 1, 2, 4 \), we get

\[ \gamma_T = \lambda_T. \]

Consequently, the risk aversion \( \gamma^S(S_T) \) associated with the numeraire \( S \) must satisfy

\[ \gamma^S(S_T) = \gamma_T S_T. \]

We recall, however, that for any payoff, say \( G \), dependent only on \( S_T \),

\[ E_Q \left( \frac{G(S_T)}{1+r} \right) = S_0 E_{Q^S} \left( \frac{G(S_T)}{S_T} \right). \]

Therefore,

\[ v(C_T; \gamma_T) = E_Q \left( \frac{1}{\gamma_T} \log E_Q \left( e^{\gamma^T \frac{\epsilon_T}{1+r} | S_T} \right) \right) \]

\[ = (1 + r) S_0 E_{Q^S} \left( \frac{1}{\gamma_T S_T} \log E_Q \left( e^{\gamma^T \frac{\epsilon_T}{1+r} | S_T} \right) \right) \]

\[ = (1 + r) S_0 E_{Q^S} \left( \frac{1}{\gamma^S(S_T)} \log E_{Q^S} \left( e^{\gamma^T(S_T) \frac{\epsilon_T}{1+r} | S_T} \right) \right). \]

The statement then follows because the quantity (cf. (1.98))

\[ \frac{v(C_T; \gamma_T)}{(1 + r) S_0} = E_{Q^S} \left( \frac{1}{\gamma^S(S_T)} \log E_{Q^S} \left( e^{\gamma^T(S_T) \frac{\epsilon_T}{1+r} | S_T} \right) \right) \]

is the indifference price quoted in the equivalent number of shares.

\[ \square \]
1.2.6 Functions of Value and Utility

The analysis on numeraires in the previous section exposed the important fact that the arguments of the functions of value and utility can be arbitrary as long as we are careful with the units in which the relevant quantities are expressed. Specifically, we saw that in order to refer to the same value and utility, independently on the arguments, and, also, in order to eliminate static arbitrage opportunities from our model, one needs to ensure that the risk aversion multiplied by wealth represents the same quantity independently of the wealth units. This produced necessary and sufficient conditions on risk preferences in spot and forward units, and for the case of general numeraires. In other words, natural properties of indifference prices imposed a certain structure on preferences and, in turn, on the involved utility and value functions.

As one moves to a multiperiod setup and to more complex dynamic payoffs, additional price considerations are expected to impose further structural properties on the supporting utilities and value functions. As a consequence, the concept of utility, as a static expression of risk attitude, will need to be put in the correct dynamic context in order to ultimately generate consistent prices across times, units, etc.

Even though the single-period framework impedes us from exposing such issues, we, nevertheless, provide below some motivational observations. For convenience, we fix the benchmark risk aversion parameter $\gamma_T = \gamma(S_T)$ as representing our aversion to risk associated with wealth expressed in the spot units, i.e., discounted to the current time. Whenever convenient, we also use the corresponding risk tolerance $\delta_T = \gamma_T^{-1}$ (cf. (1.85)).

From the previous analysis, the utility and value function in the single period Merton problem ([187]) can be written as

$$U(x; \delta_T) = -e^{-\frac{x}{\delta_T}}$$

and

$$V^0(x; \delta_T) = -e^{-\frac{x}{E_Q[\delta_T]} - H(Q^*|P)}, \quad (1.108)$$

where

$$\frac{dQ^*}{dQ} = \frac{\delta_T}{E_Q(\delta_T)},$$

and $Q$ as in (1.8).

We recall that indifference prices are built via the values of the relevant investment opportunities and that these values are generated by the preference structure and the assumptions on the market. In order, therefore, to build consistent pricing systems, one needs to correctly specify the investor’s preferences across times and to understand the interplay between utilities and value functions.

To gain some intuition, let us simply look at how zero wealth is being valued at the beginning and at the end of the trading horizon. Setting $x = 0$ in (1.108) yields

$$V^0(0; \delta_T) = -e^{-H(Q^*|P)}.$$
This is the value of zero wealth at the beginning of the period, as measured by the value function. Hence, it depends on the entropy term \( H(Q^*|P) \) and thus on the model.

On the other hand, the utility of zero wealth at the end of the time period \( T \), as measured by the utility function, equals

\[
U_T(0; \delta_T) = -1.
\]

We then say that the utility value of zero wealth is \(-1\).\(^4\)

Alternatively, the investor may also want to associate \(-1\) to the value of zero wealth at the beginning of the period, thus making it independent of the model. This can be easily achieved by normalizing the utility function (1.108) accordingly. Indeed, let

\[
\hat{U}(x; \delta_T) = -e^{-(\frac{1}{T} + H(Q^*|P))}
\]

represent the utility of wealth \( x \) at time \( T \). Obviously, at the beginning of the period, the associated value function \( \bar{V}^0(x; \delta_T) \) satisfies

\[
\bar{V}^0(x; \delta_T) = e^{H(Q^*|P)} V^0(x; \delta_T) = -e^{-(\frac{1}{T} + H(Q^*|P))}.
\]

The corresponding values of zero wealth then become

\[
\hat{U}(0; \delta_T) = -e^{H(Q^*|P)},
\]

and

\[
\bar{V}^0(0; \delta_T) = e^{H(Q^*|P)} V^0(0; \delta_T) = -1.
\]

The above considerations might look pedantic in the single-period setting. However, in the multiperiod case, one needs to reconcile single-period and multiperiod concepts of the value and utility functions. Indeed, when dealing concurrently with investment problems over multiple investments horizons, one needs to identify the utility function at a given horizon with the value function obtained by solving the optimal investment problem over the next time period.

Together, the functions of utility and value lead to the natural concept of a dynamic utility or of a term structure of utilities. Such a utility is a function of wealth, the investment horizon, and the market model. In the above simple case, the dynamic utility for the beginning and the end of period, as given by the functions

\[
U^F(x, 0) = -e^{-(\frac{1}{T} + H(Q^*|P))}, \quad U^F(x, T) = -e^{-(\frac{1}{T} + H(Q^*|P))},
\]

is normalized at the beginning of the period. We call it the forward utility.

On the other hand, the dynamic utility, given by

\[
U^B(x, 0) = -e^{-(\frac{1}{T} + H(Q^*|P))}, \quad U^B(x, T, T) = -e^{-(\frac{1}{T})},
\]

is normalized at the end of the time period. We call it the backward utility.

\(^4\)An investor may prefer to associate 0, rather than \(-1\), utility value with zero wealth. This requirement is easily met by adding the normalizing constant 1 to \( U_T(x; \delta_T) \). We choose not to do it because this does not change anything in our analysis and lengthens many expressions.