CHAPTER ONE

Financial Market and Arbitrage

AGENTS TRADE in financial markets in order to transfer funds across time and states of nature. Transfers across time correspond to saving or borrowing. Transfers across states of nature correspond to hedging or speculation. This chapter introduces a simple and highly idealized model that captures this basic function of financial markets and will also serve as a building block in the dynamic extension of Part II. Within this simple model, we develop the foundations of arbitrage-pricing theory. The mathematical background for Chapters 1 and 2 is contained in the first seven sections of Appendix A.

1.1 Market and Arbitrage

There are two times, labeled zero and one. At time zero there is no uncertainty, while at time one there are K possible **states** that can prevail, labeled $1, 2, \ldots, K$. We treat time zero and each of the K states in an integrated fashion and we refer to them as **spots**. There are therefore $1 + K$ spots, labeled $0, 1, \ldots, K$.

A **cash flow** is a vector of the form $c = (c_0, c_1, \ldots, c_K) \in \mathbb{R}^{1+K}$. Alternatively, a cash flow c can be thought of as the **stochastic process** $(c(0), c(1))$, where $c(0) = c_0$ and $c(1) \in \mathbb{R}^K$ is a **random variable** taking the value $c(1)_k = c_k$ at state k. We assume that each c_k is real-valued, representing a spot-contingent payment in some unit of account. We regard the set of cash flows as an inner-product space with the usual Euclidean inner product:

$$x \cdot y = \sum_{k=0}^{K} x_k y_k.$$

The **Arrow cash flow** corresponding to spot k is denoted $\mathbf{1}^k$ and is defined by

$$\mathbf{1}^k_l = \begin{cases} 1, & \text{if } l = k; \\ 0, & \text{if } l \neq k. \end{cases}$$

In particular, $\mathbf{1}^0 = (1, 0, \ldots, 0)$. The Arrow cash flows correspond to the usual Euclidean basis of $\mathbb{R}^{1+K}$ and therefore every cash flow is a linear combination of Arrow cash flows.

A financial market can be thought of as a set X of net incremental cash flows that can be obtained by trading financial contracts such as bonds, stocks, futures, options and swaps. In this text, we consider perfectly competitive markets, that is, markets in which every trader has negligible market power and therefore takes the terms and prices of contracts as given. Unless otherwise indicated, we also assume that there are no position limits or short sale constraints, there are no transaction costs such as bid-ask spreads and no indivisibilities such as minimum amounts that one can trade in any one contract.

The above informal assumptions motivate the following formal properties of the set of traded cash flows X:

1. $x, y \in X$ implies $x + y \in X$.
2. $x \in X$ and $\alpha \in \mathbb{R}_+$ implies $\alpha x \in X$.
3. $x \in X$ implies $-x \in X$.

Conditions 1 and 2 mean that trades can be combined and arbitrarily scaled. They both hold if and only if X is a convex cone. Condition 3 means that the reverse to every trade is also a possible trade. Conditions 2 and 3 combined imply the possibility of short-selling. A convex cone X satisfies condition 3 if and only if X is a linear subspace, motivating the following formal definition of a market.

Definition 1.1. *A* ***market*** *is a linear subspace X of the set of cash flows* $\mathbb{R}^{1+K}$.

A market X is taken as given throughout this chapter. We call its elements the **traded cash flows** or just **trades**. We are interested in the implications of the assumption that the market X contains no arbitrage opportunities. An arbitrage is a trade that results in an inflow at some spot and an outflow at no spot. For example, if two securities trade at different prices but generate identical future cash flows, a trader can short the relatively more expensive security and buy the relatively cheaper one, generating cash at time zero with no subsequent net cash flow. (The impossibility of this special type of arbitrage is known as the “law of one price” and is further discussed in the exercises.) In a perfectly competitive market there are no impediments to immediately exploiting arbitrage opportunities, which consequently cannot exist in equilibrium. Equilibrium will be formalized in Chapter 3. In this chapter we directly impose the no-arbitrage assumption and explore its consequences.

Definition 1.2. *A cash flow c is an* ***arbitrage*** *if* $0 \neq c \geq 0$. *The market* X *is* ***arbitrage-free*** *if it contains no arbitrage.*

We note that the market X is arbitrage-free if and only if it intersects the positive orthant of $\mathbb{R}^{1+K}$ only at zero, that is, $X \cap \mathbb{R}^{1+K}_{+} = \{0\}$.

Trades are often implemented by spot or forward trading of some asset. Formally, we model an **asset** as a random variable $D \in \mathbb{R}^K$, representing a time-one payoff. The scalar S is a **spot price** of the asset D if $(-S, D) \in X$. The scalar F is a **forward price** of the asset D if $(0, D - F\mathbf{1}) \in X$. A **unit discount bond** is the asset $\mathbf{1}$, that is, the asset that pays one at every state. A **risk-free discount factor** is a spot price of the unit discount bond, that is, any scalar ρ such that $(-\rho, \mathbf{1}) \in X$. In an arbitrage-free market an asset has at most one spot price and at most one forward price. In particular, there can be at most one risk-free discount factor, which is necessarily strictly positive.

Proposition 1.3. *Suppose the market is arbitrage-free and* ρ *is a (necessarily unique) risk-free discount factor. If* S *is the spot price of an asset* D *and* F *is the forward price of* D, *then* $S = F\rho$.

Proof. Adding up the trades $(-S, D), (0, F\mathbf{1} - D), (F\rho, -F\mathbf{1}) \in X$ implies $(F\rho - S)\mathbf{1}^0 \in X$. Since $\mathbf{1}^0 \notin X$, it follows that $S = F\rho$. ■

Proposition 1.3 exemplifies a typical arbitrage argument. The three trades of the proof represent what is known as a **cash-and-carry arbitrage**: buy the asset in the spot market, sell it forward and borrow the present value of the forward payment. If $S < F\rho$, then the result is the positive inflow $F\rho - S$ at time zero, with no subsequent cash flow. If $S > F\rho$, a **reverse cash-and-carry arbitrage** is achieved by reversing the above trades. Trading constraints limit the arbitrage argument. For example, if the potential arbitrageur has no current inventory of the asset and cannot sell the asset short in the spot market, then reverse cash-and-carry arbitrage is not possible and therefore the possibility that $S > F\rho$ cannot be excluded. We return to this issue in Section 1.7.

1.2 Present Value and State Prices

A dual approach to arbitrage pricing is based on the following notion of present value.

Definition 1.4. *A **present-value function** is a linear function of the form* $\Pi : \mathbb{R}^{1+K} \to \mathbb{R}$ *with the following three properties:*

1. $\Pi(x) \leq 0$ *for every* $x \in X$.
2. $\Pi(c) > 0$ *for every arbitrage cash flow* c.
3. $\Pi(\mathbf{1}^0) = 1$.

The first restriction on Π can be thought of as an expression of the perfect competition assumption, which implies that there cannot be a net trade of strictly positive value. Since X is a linear subspace, condition 1 is equivalent to $\Pi(x) = 0$ for all $x \in X$. In Section 1.7 we relax the assumption that X is a linear subspace to allow for trading constraints, in which case the present value of a traded cash flow can be strictly negative. The second restriction on Π expresses the assumption that an arbitrage is valuable to every agent and therefore must be assigned a positive value by any present-value rule. The first two restrictions on Π together rule out arbitrage trades. The last restriction on Π is merely a normalization.

A simple example of the use of a present-value function in deriving arbitrage restrictions follows.

Example 1.5. *Suppose* Π *is a present-value function and* S *is the spot price of an asset* D. *Applying the restriction* $\Pi(x) = 0$ *to the trade* $x = (-S, D) = -S\mathbf{1}^0 + (0, D)$ *and using the linearity of* Π *and the fact that* $\Pi(\mathbf{1}^0) = 1$, *we find* $S = \Pi(0, D)$. *In particular, if* ρ *is a risk-free discount factor, then* $\rho = \Pi(0, \mathbf{1}) > 0$. *If* F *is the forward price of the asset* D, *setting the present value of the trade* $(0, F\mathbf{1} - D)$ *to zero results in* $F\Pi(0, \mathbf{1}) = \Pi(0, D)$. *Combining the equations for* S, ρ *and* F, *we recover the restriction* $S = \rho F$ *of Proposition 1.3.*

The above example proves Proposition 1.3 under the additional assumption that a present-value function exists. We will see shortly, however, that this assumption is a consequence of the no-arbitrage condition.

Present-value functions are conveniently represented in terms of state prices.

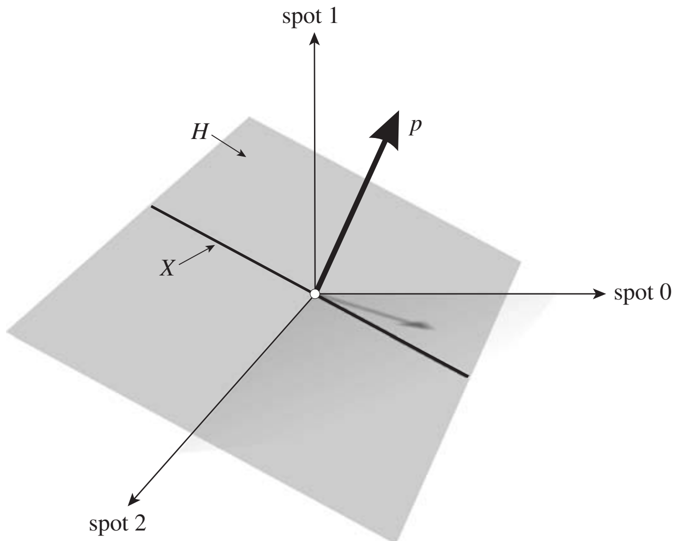

Figure 1.1 A state-price vector.

Definition 1.6. *A* ***state-price vector*** *is any vector of the form*

$$p = (p_0, p_1, \ldots, p_K) \in \mathbb{R}^{1+K}_{++}$$

such that $p \cdot x \leq 0$ for all $x \in X$. A state-price vector p ***represents*** *the present-value function*

$$\Pi(c) = \frac{p \cdot c}{p_0} = c_0 + \sum_{k=1}^{K} \frac{p_k}{p_0} c_k, \quad c \in \mathbb{R}^{1+K}. \tag{1.1}$$

While this definition will apply more generally in Section 1.7, in the current context X is a linear subspace and therefore a state-price vector p is necessarily orthogonal to X (see Figure 1.1).

Any given present-value function Π is represented by the state-price vector $\bar{p}$, where $\bar{p}_k = \Pi(\mathbf{1}^k)$. Mathematically, $\bar{p}$ is the Riesz representation of Π (see Proposition A.8). Any other state-price vector p representing Π satisfies $p = p_0 \bar{p}$ and therefore

$$\frac{p_k}{p_l} = \frac{\Pi(\mathbf{1}^k)}{\Pi(\mathbf{1}^l)}, \quad k, l \in \{0, 1, \ldots, K\}.$$

In words, state prices represent the relative present value of Arrow cash flows.

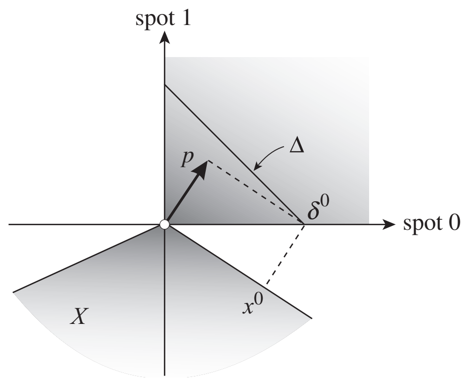

Figure 1.2 Proof of the first fundamental theorem of asset pricing.

To visualize the duality between the no-arbitrage condition and state pricing, consider any nonzero vector p in $\mathbb{R}^{1+K}$ and the orthogonal hyperplane H it defines (see Figure 1.1). Let $H_{++} = \{y : p \cdot y > 0\}$ be the open half-space on the side of H containing p, and let $H_{-} = \{x : p \cdot x \leq 0\}$ be the closed half-space on the other side of H. The nonzero vector p is a state-price vector if and only if the market X is included in H_{-} and the set of all arbitrage cash flows is included in H_{++}. Intuitively, the existence of such a strictly separating hyperplane is equivalent to the nonintersection of the two convex sets being separated, which is exactly the no-arbitrage condition. This is formally proved below using the projection theorem.[1]

Theorem 1.7 (First Fundamental Theorem of Asset Pricing). *A present-value function exists if and only if X is arbitrage-free.*

Proof. Although we assumed that X is a linear subspace, the proof uses only the assumption that X is a closed convex cone. (This generality will be utilized in Proposition 1.14 and Section 1.7.) If Π is a present-value function, then $\Pi(c) > 0$ for every arbitrage c, while

[1] The separating hyperplane theorem (Corollary A.30), whose proof is also based on the projection theorem, implies only the weak separation of an arbitrage-free market X and the positive orthant. The fact that the positive orthant contains no lines is important for strict separation, since two intersecting lines cannot be strictly separated.

$\Pi(x) \leq 0$ for every $x \in X$. Therefore, X must be arbitrage-free. Conversely, suppose X is arbitrage-free and let

$$\Delta = \left\{ x \in \mathbb{R}_+^{1+K} : \sum_{k=0}^{K} x_k = 1 \right\}.$$

Since X is arbitrage-free, $X \cap \Delta = \emptyset$. By Proposition A.19, there exists a $(x^0, \delta^0) \in X \times \Delta$ that minimizes the Euclidean distance $\|x - \delta\|$ over all $(x, \delta) \in X \times \Delta$. Note that x^0 is the projection of δ^0 on X and δ^0 is the projection of x^0 on Δ. By Theorem A.21, $-p = x^0 - \delta^0$ supports X at x^0 and therefore $p \cdot x \leq p \cdot x^0$ for all $x \in X$. Since X is a cone, it follows that $p \cdot x^0 = 0$ and $p \cdot x \leq 0$ for all $x \in X$. Similarly, $p = \delta^0 - x^0$ supports Δ at δ^0, implying that $p \cdot \delta \geq p \cdot \delta^0$ for all $\delta \in \Delta$. Since $p \cdot x^0 = 0$, $p \cdot \delta^0 = p \cdot p > 0$. Therefore, $p \cdot \delta > 0$ for all $\delta \in \Delta$, which implies that p is strictly positive. A corresponding present-value function is defined by (1.1). ■

1.3 Market Completeness and Dominant Choice

A present-value function need not be unique. For example, in Figure 1.1 the shaded plane H is orthogonal to the state-price vector p and is associated with a present-value function relative to the market X. The plane H can be rotated around the line X and as long as it does not cut into the positive orthant, it defines a whole range of present-value functions. On the other hand, if X were the whole plane H, a present-value function would be unique.

To characterize uniqueness of a present-value function more generally, we introduce the notion of market completeness.

Definition 1.8. *A cash flow m is* ***marketed*** *if there exists some $w \in \mathbb{R}$ such that $m - w\mathbf{1}^0 \in X$. The market X is* ***complete*** *if every cash flow is marketed, and* ***incomplete*** *otherwise.*

Recall that a cash flow x is traded if and only if $x \in X$. A traded cash flow is therefore marketed, but the converse need not be true. We let M denote the set of marketed cash flows associated with the market X. Geometrically, M is the linear subspace spanned by X and $\mathbf{1}^0$. The market X is complete if and only if $M = \mathbb{R}^{1+K}$. Assuming X is arbitrage-free, it follows that the market X is complete if and only if it is a hyperplane, meaning that its orthogonal subspace is one-dimensional, a condition that clearly implies the uniqueness of a present-value function. More generally, the present value of any marketed cash flow is uniquely defined, even if the market is incomplete.

Proposition 1.9. *If Π is a present-value function and m is a marketed cash flow, then $\Pi(m)$ is the unique value of w such that $m - w\mathbf{1}^0 \in X$.*

Proof. If $m - w\mathbf{1}^0 \in X$, then $\Pi(m - w\mathbf{1}^0) = 0$ and therefore $w = w\Pi(\mathbf{1}^0) = \Pi(m)$. ■

Theorem 1.10 (Second Fundamental Theorem of Asset Pricing). *A present-value function is unique if and only if the market is complete.*

Proof. If the market is complete, then every cash flow is marketed, and the uniqueness of present values follows from the last proposition (or the above geometric argument). Conversely, suppose the market is incomplete and $\Pi^0(x) = p^0 \cdot x$ is a present-value function, where $p^0 \in \mathbb{R}^{1+K}_{++}$. Fix any nonmarketed cash flow c. By Corollary A.25, $c = m + n$, where $m \in M$ and n is nonzero and orthogonal to $M = \operatorname{span}(X, \mathbf{1}^0)$. Let $\beta \in \mathbb{R}$ be small enough so that $p^\beta = p^0 + \beta n \in \mathbb{R}^{1+K}_{++}$. Since n is orthogonal to X, $p^\beta \cdot x = 0$ for all $x \in X$, and since n is orthogonal to $\mathbf{1}^0$, $p^\beta \cdot \mathbf{1}^0 = p^0 \cdot \mathbf{1}^0 = 1$. Therefore, $\Pi^\beta(x) = p^\beta \cdot x$ is a present-value function and $\Pi^\beta(c) = \Pi^0(c) + \beta n \cdot n \neq \Pi^0(c)$. ■

A complete arbitrage-free market has the remarkable property that it removes the subjectivity of the optimality of a cash flow choice within any given set. To elaborate, consider the problem of selecting a cash flow δ out of a given set $\mathcal{D}$ of cash flows. In the absence of a market, the optimal choice generally depends on the preferences and endowment of the agent making the choice, as will be discussed in Chapter 3. Suppose now that the market X is also available.

Definition 1.11. *A cash flow $\delta^* \in \mathcal{D}$ is **dominant** (in $\mathcal{D}$ given X) if for any $\delta \in \mathcal{D}$, there exists some $x \in X$ such that $\delta^* + x \geq \delta$.*

A dominant choice is optimal for any two agents who do not dislike additional income at any spot. For $i \in \{1, 2\}$, suppose agent i finds δ^i optimal in $\mathcal{D}$. If $\delta^* \in \mathcal{D}$ is dominant, then there exist trades $x^1, x^2 \in X$ such that $\delta^* + x^i \geq \delta^i$. Agent i is therefore at least as well off selecting δ^* instead of δ^i and at the same time entering the trade x^i. In this sense, both agents agree on the optimality of δ^*, even though the way they use the market to transform δ^* can differ. Given a complete arbitrage-free market, a dominant choice is one that maximizes present value.

Proposition 1.12. *Suppose the market X is complete and Π is a present-value function. For any set of cash flows $\mathcal{D}$, the cash flow $\delta^* \in \mathcal{D}$ is dominant in $\mathcal{D}$ if and only if* $\Pi(\delta^*) = \max\{\Pi(\delta) : \delta \in \mathcal{D}\}$.

Proof. Suppose δ^* is dominant. For any $\delta \in \mathcal{D}$, we can write $\delta^* \geq \delta + x$ for some $x \in X$. Taking present values, $\Pi(\delta^*) \geq \Pi(\delta + x) = \Pi(\delta)$. Conversely, suppose δ^* maximizes present value in $\mathcal{D}$ and $\delta \in \mathcal{D}$. Since X is complete, we can write $\delta^* = \Pi(\delta^*)\mathbf{1}^0 + y^*$ and $\delta = \Pi(\delta)\mathbf{1}^0 + y$ for some $y^*, y \in X$. Therefore, $\delta^* \geq \Pi(\delta)\mathbf{1}^0 + y^* = \delta + x$, where $x = y^* - y \in X$, confirming the dominance of δ^*. ■

Corollary 1.13. *Suppose the set of cash flows $\mathcal{D}$ is compact and the market X is complete and arbitrage-free. Then a dominant choice in $\mathcal{D}$ exists.*

A generalization of the last proposition that does not require the market X to be complete follows.

Proposition 1.14. *Suppose the market X is arbitrage-free. For any set of cash flows $\mathcal{D}$, the cash flow $\delta^* \in \mathcal{D}$ is dominant in $\mathcal{D}$ if and only if* $\Pi(\delta^*) = \max\{\Pi(\delta) : \delta \in \mathcal{D}\}$ *for every present-value function* Π.

Proof. The "only if" part follows exactly as for Proposition 1.12. Conversely, suppose $\delta^* \in \mathcal{D}$ is not dominant in $\mathcal{D}$ and therefore there exists some $\delta \in \mathcal{D}$ such that $\delta^* - \delta + x \notin \mathbb{R}_+^{1+K}$ for all $x \in X$. Let $x^* = \delta^* - \delta$. The set

$$X^* = \{x + \alpha x^* : x \in X,\ \alpha \in \mathbb{R}_+\}$$

is a closed convex cone that contains no arbitrage. The proof of the first fundamental theorem of asset pricing implies the existence of a vector $p \in \mathbb{R}^{1+K}_{++}$ such that $p \cdot x \leq 0$ for every $x \in X^*$. Such a vector p is a state-price vector that satisfies $p \cdot x^* \leq 0$. If $p \cdot x^* = 0$, we can construct a new state-price vector p^ε such that $p^\varepsilon \cdot x^* < 0$. To see how, we project x^* onto X to write $x^* = \bar{x} + n$, where $\bar{x} \in X$ and n is nonzero and orthogonal to X. For any small enough scalar $\varepsilon > 0$, $p^\varepsilon = p - \varepsilon n$ is a state-price vector. If $p \cdot x^* = 0$, then $p^\varepsilon \cdot x^* = -\varepsilon n \cdot n < 0$. We conclude that for some $\varepsilon \geq 0$, the present-value function Π defined by p^ε is such that $\Pi(x^*) < 0$ and therefore $\Pi(\delta^*) < \Pi(\delta)$. ■

1.4 Probabilistic Representations of Value

Arbitrage arguments rely on an assumed set of possible states but not the likelihood that one assigns to these states. Nevertheless, probabilistic representations of present value, introduced below, are methodologically useful, because they relate the valuation problem to a powerful set of available probabilistic tools. The benefit of these tools becomes clearer in dynamic extensions of the theory.

For the remainder of this chapter, we take as given a **strictly positive probability** P, defined as any vector $P \in \mathbb{R}^K_{++}$ whose elements add up to one. While in applications P_k typically represents an economic agent's or an econometrician's prior belief that state k will occur, the manipulations that follow apply for any choice of a reference strictly positive probability P. Given any random variables $x, y \in \mathbb{R}^K$, we denote their usual probabilistic averages relative to P as follows:

- (expectation) $\mathbb{E}[x] = \mathbb{E}x = \sum_{k=1}^K x_k P_k$.
- (covariance) $\text{cov}[x, y] = \mathbb{E}[(x - \mathbb{E}x)(y - \mathbb{E}y)] = \mathbb{E}[xy] - \mathbb{E}x\mathbb{E}y$.
- (variance) $\text{var}[x] = \text{cov}[x, x]$.
- (standard deviation) $\text{stdev}[x] = \sqrt{\text{var}[x]}$.
- (correlation coefficient) $\text{corr}[x, y] = \text{cov}[x, y]/(\text{stdev}[x]\text{stdev}[y])$.

We continue with the reference market X taken as given.

Definition 1.15. *A **state-price density (SPD)** is a vector of the form* $\pi = (\pi_0, \pi_1, \ldots, \pi_K)$ *such that* $p = (\pi_0, \pi_1 P_1, \ldots, \pi_K P_K)$ *is a state-price vector.*

The present-value function ***represented*** *by π is the present-value function represented by p.*

Like any vector in $\mathbb{R}^{1+K}$, we can equivalently regard an SPD π as a stochastic process $\pi = (\pi(0), \pi(1))$, where $\pi(0) = \pi_0$ and $\pi(1)_k = \pi_k$ for $k = 1, \ldots, K$. The present-value function Π represented by an SPD π can therefore be written as

$$\Pi(c) = c(0) + \mathbb{E}\left[\frac{\pi(1)}{\pi(0)} c(1)\right]. \tag{1.2}$$

Our earlier discussion of the relationship between state-price vectors and present-value functions implies that a present-value function can be represented by an SPD that is unique up to positive scaling.

The **risk-free discount factor implied** by the SPD π is

$$\rho = \mathbb{E}\left[\frac{\pi(1)}{\pi(0)}\right], \tag{1.3}$$

which is the price of a unit discount bond if one is traded. In terms of ρ, the present-value equation (1.2) can be expressed as

$$\Pi(c) = c(0) + \rho\mathbb{E}[c(1)] + \operatorname{cov}\left[\frac{\pi(1)}{\pi(0)}, c(1)\right]. \tag{1.4}$$

Note that risks are not priced based on their variance. For example, suppose $\bar{c} = (c(0), \mathbb{E}c(1))$ and $\operatorname{cov}[\pi(1), c(1)] > 0$. Then $\Pi(c) > \Pi(\bar{c})$, even though c is riskier than $\bar{c}$ in the sense of higher variance. Intuitively, while c is variable, it tends to pay more at spots where one unit of account is more highly valued.

Another common way of representing present-value functions is in terms of equivalent martingale measures.

Definition 1.16. *An* ***equivalent martingale measure (EMM)*** *is a probability Q such that $p = (1, \rho Q_1, \rho Q_2, \ldots, \rho Q_K)$ is a state-price vector for some scalar ρ, in which case (Q, ρ) is an* ***EMM-discount pair****. The present-value function* ***represented*** *by (Q, ρ) is the present-value function represented by p.*

By letting $\mathbb{E}^Q$ denote the expectation operator relative to the probability Q, the present-value function Π represented by an EMM-discount pair (Q, ρ) can be written as

$$\Pi(c) = c(0) + \rho\mathbb{E}^Q[c(1)]. \tag{1.5}$$

Any present-value function is represented by a unique EMM-discount pair. The fundamental theorems of asset pricing can therefore be restated as: *The market is arbitrage-free if and only if an EMM exists. An EMM-discount pair is unique if and only if the market is complete.*

State-price densities and EMM-discount pairs are matched by equations (1.3) and

$$\frac{\pi(1)}{\pi(0)} = \rho \frac{dQ}{dP}, \tag{1.6}$$

where dQ/dP is the **density** of Q with respect to P, defined as the random variable taking the value Q_k/P_k at state k. If (Q, ρ) is an EMM-discount pair representing the present-value function Π, then equation (1.6) defines the unique SPD representation of Π given any positive value of $\pi(0)$. Conversely, suppose π is an SPD representing Π and let ρ be its implied risk-free discount factor. Then (Q, ρ), where Q is defined by (1.6), is the unique EMM-discount pair that represents Π.

The forward price of a traded asset D is given in terms of an EMM Q by

$$F = \mathbb{E}^Q[D] = \mathbb{E}D + \text{cov}\left[\frac{dQ}{dP}, D\right]. \tag{1.7}$$

The expectation operator $\mathbb{E}^Q$ relative to an EMM Q can therefore be interpreted as a forward-pricing operator. For instance, suppose D is an **Arrow security**, meaning that there exists some state k such that $D_k = 1$ and $D_l = 0$ for $l \neq k$. If (Q, ρ) is an EMM-discount pair, then the forward price of D is Q_k. The Arrow security can be thought of as an insurance contract against the occurrence of state k. The probability Q_k is the forward premium of this insurance contract.

The term equivalent martingale measure comes from probability theory. Two probabilities are **equivalent** if they assign zero probability to the same events. In our simple context, an EMM Q is equivalent to P since both are assumed to be strictly positive. The term **martingale measure** reflects the fact that given an EMM-discount pair (Q, ρ) and an asset D with spot price S, the discounted price process $(S, \rho D)$ is a **martingale** relative to the probability Q, which in our simple context means that $S = \mathbb{E}^Q[\rho D]$. The role of martingales will become clearer in Part II.

Pricing in terms of an EMM Q is also known as **risk-neutral pricing**, the probability Q is known as a **risk-neutral probability**, and the expectation $\mathbb{E}^Q$ as a **risk-neutral expectation**. The basic idea is that in a fictitious world in which beliefs coincide with Q, the present value of a time-one payoff is its expected value discounted as if it were a sure payment, as in equation (1.5). The economic interpretation of Q is, however, as a list of forward premia, not beliefs.

1.5 Financial Contracts and Portfolios

In a financial market in which every traded cash flow is customized to fit a particular trader's individual needs, finding a suitable counterparty and enforcing the resulting contract can be expensive. This consideration has led to the creation of standardized contracts that are traded in highly competitive and liquid markets. Traders can combine standardized contracts in ways that best approximate their individual cash flow needs. In this section we model the implementation of an ideal competitive financial market through the trading of (financial) contracts.

Formally, a **contract** is any stochastic process $\upsilon = (\upsilon(0), \upsilon(1)) \in \mathbb{R} \times \mathbb{R}^K$, with $\upsilon(t)$ interpreted as the time-t market value of the contract. For any $\alpha \in \mathbb{R}$, a trader can enter into α contracts resulting in an incremental cash flow $\alpha(-\upsilon(0), \upsilon(1))$. A positive value of α corresponds to a **long position** in the contract, while a negative value of α corresponds to a **short position**. We say that a contract υ is **traded** in the market X if $(-\upsilon(0), \upsilon(1)) \in X$.

Throughout this section, we fix the reference contracts $V_1, \ldots, V_J$ and we define the matrix

$$V = \begin{pmatrix} V_1 \\ \vdots \\ V_J \end{pmatrix} = \begin{pmatrix} V_1(0) & V_1(1) \\ \vdots & \vdots \\ V_J(0) & V_J(1) \end{pmatrix} = (V(0), V(1)). \tag{1.8}$$

V can equivalently be viewed as a column vector of stochastic processes or as a $J \times (1+K)$ matrix with V_j as its jth row. A **portfolio** is a column vector $\theta = (\theta_1, \ldots, \theta_J)' \in \mathbb{R}^J$, where θ_j represents a position in contract V_j. A portfolio θ **generates** the cash flow x defined by

$$x(0) = -\theta' V(0) \quad \text{and} \quad x(1) = \theta' V(1). \tag{1.9}$$

The market **implemented** by the contracts $V_1, \ldots, V_J$ is denoted and defined by

$$X(V_1, \ldots, V_J) = X(V) = \{x : x \text{ is generated by some } \theta \in \mathbb{R}^J\}. \tag{1.10}$$

The **synthetic contract generated** by the portfolio θ is the contract

$$V^\theta = \theta' V.$$

Introducing a synthetic contract does not change the implemented market:

$$X(V^{\theta}, V_1, \ldots, V_J) = X(V_1, \ldots, V_J), \quad \text{for all } \theta \in \mathbb{R}^J. \tag{1.11}$$

The portfolio θ **replicates** the contract υ if $\upsilon(1) = V^{\theta}(1)$. If the market $X(\upsilon, V_1, \ldots, V_J)$ is arbitrage-free[2] and θ replicates υ, then $\upsilon(0) = V^{\theta}(0)$, in which case υ is said to have been priced by arbitrage.

Example 1.17. *Suppose that $J = 3$ and*

$$V_1 = (\rho, \mathbf{1}), \quad V_2 = (S, D), \quad V_3 = (0, D - F\mathbf{1}).$$

The three contracts implement, respectively, default-free borrowing and lending, spot trading of asset D and forward trading of asset D. Each of these contracts can be replicated by a portfolio in the other two. If $X(V)$ is arbitrage-free, any of these replication relationships leads to equation $S = F\rho$ of Proposition 1.3. If $S = F\rho$, then each one of the three contracts is a synthetic contract in the other two.

Example 1.18. *Suppose the arbitrage-free market X is implemented by spot trading in the J assets forming the rows of the $J \times K$ matrix $D = (D_1', \ldots, D_J')'$. That means $X = X(V)$, where $V(1) = D$ and $V(0) = S$ for a column vector $S \in \mathbb{R}^J$ whose jth entry is the spot price of asset D_j. If $D_k = \theta' D$ for some replicating portfolio θ with $\theta_k = 0$, then $S_k = \theta' S$. The analogous discussion applies to a forward market in the J assets.*

The contracts $V_1, \ldots, V_J$ are **independent** if they are linearly independent as vectors in $\mathbb{R}^{1+K}$. (This notion of independence is therefore unrelated to any underlying probability and should not be confused with stochastic independence.) If $X = X(V)$, the corresponding set of marketed cash flows is given by

$$M = \{(w, \theta' V(1)) : w \in \mathbb{R},\ \theta \in \mathbb{R}^J\}.$$

Viewing $V(1)$ as a $J \times K$ matrix, we note that the dimensionality of M is $1+ \operatorname{rank}(V(1))$. The market X is complete if and only if $M = \mathbb{R}^{1+K}$. This shows that X is complete if and only if $J \geq K$ and $V(1)$ is full rank. We conclude the section with some related observations.

[2] As verified in Exercise 9, here and throughout this section the no-arbitrage assumption can be weakened to the law of one price.

Proposition 1.19. *Suppose $X(V)$ is arbitrage-free. Then the rank of $V \in \mathbb{R}^{J\times(1+K)}$ is the same as the rank of $V(1) \in \mathbb{R}^{J\times K}$. In particular, the contracts $V_1, \ldots, V_J$ are independent if and only if $V_1(1), \ldots, V_J(1)$ are linearly independent as vectors in $\mathbb{R}^K$.*

Proof. Suppose V_* is a submatrix of V obtained by isolating a set of independent contracts. If θ_* is a portfolio in those contracts and $\theta_*' V_*(1) = 0$, then $\theta_*' V_*(0) = 0$ and therefore $\theta_*' V_* = 0$. Since V_* has independent rows, it follows that $\theta_* = 0$, proving that $V_*(1)$ also has independent rows. This proves that $\text{rank}(V(1)) \geq \text{rank}(V)$. The remaining claims are immediate. ■

Corollary 1.20. *An arbitrage-free market is complete if and only if it can be implemented by K independent contracts.*

1.6 Returns

Empirical and theoretical results in the asset pricing literature are often reported in terms of returns rather than prices and payoffs. Returns have the advantage of being independent of the unit of account and the size of the initial investment, allowing easy comparison across different contracts. On the other hand, returns are not defined for contracts with zero initial cash flow such as forward contracts. In this section we extend our discussion of arbitrage pricing focusing on returns rather than cash flows.

We use the term "return" to mean cumulative return throughout; that is, a return is a payoff divided by the corresponding initial investment. Formally, the **return** of a cash flow x such that $x(0) \neq 0$ is defined to be the ratio $-x(1)/x(0)$. Fixing the reference market X, we denote the corresponding set of all **traded returns** by

$$\mathcal{R} = \left\{ -\frac{x(1)}{x(0)} : x \in X,\ x(0) \neq 0 \right\}.$$

An exercise shows that $\mathcal{R}$ is a linear manifold.

Example 1.21. *Suppose X is implemented by spot trading in the assets $D = (D_1', \ldots, D_J')'$, with corresponding spot-price vector $S = (S_1, \ldots, S_J)'$, where $S_j \neq 0$ for all j. We let $R_j = D_j/S_j$ denote the **return of asset** j and we define the $J \times K$ matrix $R = (R_1', \ldots, R_J')'$. Portfolio return calculations can* *be carried out in terms of asset returns alone. Given any portfolio* θ *such that* $\theta' S \neq 0$, *we define the column vector of* ***portfolio weights*** $\psi = (\psi_1, \ldots, \psi_J)'$, *where* $\psi_j = \theta_j S_j / \theta' S$. *The corresponding portfolio return is* $\psi' R = \theta' D / \theta' S$. *The set of traded returns is therefore*

$$\mathcal{R} = \{\psi' R : \psi_1 + \cdots + \psi_J = 1, \psi \in \mathbb{R}^{J \times 1}\},$$

which can be visualized as the linear manifold through the points $R_1, \ldots, R_J$.

Proposition 1.22. *Suppose* $\mathcal{R} \neq \emptyset$. *A strictly positive stochastic process* π *is an SPD if and only if*

$$\mathbb{E}\left[\frac{\pi(1)}{\pi(0)} R\right] = 1 \quad \text{for all } R \in \mathcal{R}. \tag{1.12}$$

Similarly, a pair of a strictly positive probability Q *and a positive scalar* ρ *is an EMM-discount pair if and only if*

$$\mathbb{E}^Q[R] = \frac{1}{\rho} \quad \text{for all } R \in \mathcal{R}. \tag{1.13}$$

Proof. Suppose (1.12) holds. Given any $x \in X$, we wish to verify that

$$\pi(0)x(0) + \mathbb{E}[\pi(1)x(1)] = 0. \tag{1.14}$$

If $x(0) \neq 0$, equation (1.14) follows by rearranging (1.12) for $R = -x(1)/x(0)$. Suppose now that $x(0) = 0$. Since $\mathcal{R} \neq \emptyset$, there exists some $x^0 \in X$ such that $x^0(0) \neq 0$, which implies that $y = x + x^0 \in X$ and $y(0) \neq 0$. Subtracting equation (1.14) with x^0 in place of x from equation (1.14) with y in place of x, we obtain (1.14). This proves that π is an SPD. The converse is immediate. The proposition's last part is obtained from the first one with Q in place of P and the process $(1, \rho)$ in place of π. ■

Equation (1.13) states that the expectation of every traded return under the EMM is the risk-free return, a fact that motivates the alternative term "risk-neutral probability" for an EMM.

Given a reference risk-free discount factor ρ, the **risk premium** of a traded return R is the difference $\mathbb{E}R - (1/\rho)$. The following proposition relates the risk premium of a traded return to its covariance with a state-price density, or an EMM density, in a way that is consistent with our earlier interpretation of these covariances as a measure of the market price of risk. The proposition also shows that although the standard deviation of a traded return does not explain its risk premium, the ratio of absolute risk premium to standard deviation is bounded above by the standard deviation of an EMM density.

Proposition 1.23. *Suppose π is an SPD with implied risk-free discount factor ρ and corresponding EMM Q. Then for every $R \in \mathcal{R}$,*

$$\mathbb{E}R - \frac{1}{\rho} = -\frac{1}{\rho}\operatorname{cov}\left[\frac{\pi(1)}{\pi(0)}, R\right] = -\operatorname{cov}\left[\frac{dQ}{dP}, R\right], \tag{1.15}$$

and provided R has positive variance,

$$\frac{|\mathbb{E}R - (1/\rho)|}{\operatorname{stdev}[R]} \leq \frac{1}{\rho}\operatorname{stdev}\left[\frac{\pi(1)}{\pi(0)}\right] = \operatorname{stdev}\left[\frac{dQ}{dP}\right].$$

Proof. The definition of a covariance implies

$$\operatorname{cov}\left[\frac{\pi(1)}{\pi(0)}, R\right] = \mathbb{E}\left[\frac{\pi(1)}{\pi(0)}R\right] - \mathbb{E}\left[\frac{\pi(1)}{\pi(0)}\right]\mathbb{E}[R] = 1 - \rho\mathbb{E}[R].$$

Rearranging gives the first claimed equation in (1.15), while the second one follows from equation (1.6). To show the claimed inequality, let

$$\varrho = \operatorname{corr}\left[\frac{\pi(1)}{\pi(0)}, R\right].$$

Then (1.15) can be restated as

$$\frac{\mathbb{E}R - (1/\rho)}{\operatorname{stdev}[R]} = -\frac{\varrho}{\rho}\operatorname{stdev}\left[\frac{\pi(1)}{\pi(0)}\right] = -\varrho\operatorname{stdev}\left[\frac{dQ}{dP}\right].$$

By the Cauchy-Schwarz inequality, $|\varrho| \leq 1$, and the result follows. ■

1.7 Trading Constraints

Trading constraints generally weaken the pricing implications of the no-arbitrage assumption. This section introduces the role of trading constraints with a simple generalization of this chapter's main market model for which the first fundamental theorem of asset pricing remains valid.

We analyze the arbitrage opportunities of a reference trader in the market implemented by the contracts $V_1, \ldots, V_J$ under the constraint that the trader's portfolio must lie in the set $\Theta \subseteq \mathbb{R}^J$. The trader's current positions are given by the portfolio $\theta^0 \in \Theta$. For any incremental portfolio θ, we define the corresponding generated cash flow x as before by (1.9). The set of feasible incremental cash flows for this trader is

$$X = \{x : x \text{ is generated by some } \theta \text{ such that } \theta^0 + \theta \in \Theta\}. \tag{1.16}$$

From the trader's perspective, X is the market in the informal sense we have used the term, but clearly X need not be a linear subspace and can depend on the trader's initial positions θ^0. Some examples of the constraint set Θ follow. Examples involving margin requirements and bid-ask spreads are given in Exercises 17 and 18, respectively.

Example 1.24 (Missing Markets). *Suppose that contracts indexed in the nonempty set $A \subseteq \{1, \ldots, J\}$ cannot be traded. The corresponding portfolio constraint set is*

$$\Theta = \{\theta \in \mathbb{R}^J : \theta_j = 0 \textit{ for all } j \in A\}.$$

In this case no modification of our earlier theory is required, since X is a linear subspace. Assuming the contracts $V_1, \ldots, V_J$ are independent, X is an incomplete market. Market incompleteness is a form of trading constraint.

Example 1.25 (Short-Sale Constraints). *Suppose that contracts indexed in the set $A \subseteq \{1, \ldots, J\}$ cannot be sold short. The corresponding portfolio constraint set is*

$$\Theta = \{\theta \in \mathbb{R}^J : \theta_j \geq 0 \textit{ for all } j \in A\}.$$

In this case the market X is a function of the reference trader's initial portfolio θ^0. For a simple illustration of a short-sale constraint, consider the setting of Example 1.17 with the assumption that the asset cannot be sold short in the spot market ($A = \{2\}$). As explained at the end of Section 1.1, reverse cash-and-carry arbitrage requires the selling of the asset in the spot market. Given the short-sale constraint, an arbitrage-free market X implies that $S = F\rho$ if $\theta_2^0 > 0$, but the strict inequality $S > F\rho$ cannot be ruled out by an arbitrage argument if $\theta_2^0 = 0$. Even if $\theta_2^0 > 0$, the scale of reverse cash-and-carry arbitrage is limited by the trader's initial position θ_2^0, in contrast to an unconstrained market where any arbitrage can be scaled arbitrarily.

We will formulate a generalized version of the first fundamental theorem of asset pricing that applies with the type of constraints discussed above and in the exercises. For this purpose, we extend the definition of a market.

Definition 1.26. *A **constrained market** is a closed convex set of cash flows $X \subseteq \mathbb{R}^{1+K}$ such that $0 \in X$ and for some $\varepsilon > 0$,*

$$x \in X \text{ and } 0 < \|x\| < \varepsilon \quad \text{implies} \quad \frac{\varepsilon}{\|x\|}\, x \in X. \tag{1.17}$$

Here $\|x\|$ can be taken to be the Euclidean norm of x, although, for reasons that are explained in Section A.5, Definition 1.26 does not depend on the norm choice. Condition (1.17) states that every nonzero trade whose norm is less than ε can be scaled up so that its norm equals ε.

The preceding examples are all instances of a constrained market specification, as a consequence of the following observation.

Proposition 1.27. *Suppose $\Theta \subseteq \mathbb{R}^J$ is a finite intersection of closed half-spaces and $\theta^0 \in \Theta$. Then equation* (1.16) *defines a constrained market X.*

As before, X is arbitrage-free if and only if $X \cap \mathbb{R}_+^{1+K} = \{0\}$. Definition 1.4 of a present-value function and its various representations in Section 1.4 also apply relative to a constrained market X. Whereas in the unconstrained case the present value of any traded cash flow is zero, the present value of a traded cash flow in a constrained market can be strictly negative. The first fundamental theorem of asset pricing remains valid.

Theorem 1.28. *For a constrained market X, a present-value function exists if and only if X is arbitrage-free.*

Proof. Let $C = \{kx : x \in X,\ k \in \mathbb{R}_+\}$ be the cone generated by X. One can easily check that X is arbitrage-free if and only if $C \cap \mathbb{R}_+^{1+K} = \{0\}$. Clearly, C is convex. If it is also closed, then the result follows from the proof of Theorem 1.7 (where X was assumed to be a closed convex cone). Given the $\varepsilon > 0$ of condition (1.17), let $B = \{x \in \mathbb{R}^{1+K} : \|x\| \leq \varepsilon\}$. The cone C is closed if and only if $C \cap B$ is closed (why?). Since X and B are closed, we prove the closure of C by showing that $C \cap B = X \cap B$. Clearly, $X \cap B \subseteq C \cap B$. Conversely, suppose $y \in C \cap B$ and therefore $y = kx$ for some $x \in X$ and $k \in \mathbb{R}_+$. If $k \leq 1$, then y is a convex combination of 0 and x. If $k > 1$, then y is a convex combination of x and $\varepsilon x/\|x\|$, which is an element of X by condition (1.17). In either case, y is a convex combination of elements of X and therefore $y \in X \cap B$. ■

1.8 Exercises

1. In the discussion leading to the definition of the market X as a linear subspace, conditions 2 and 3 combined are said to imply the possibility of short selling. Does condition 3 ($x \in X \implies -x \in X$) alone imply the possibility of short selling? Explain.
2. (a) Verify the claims of Example 1.17.

   (b) Show that every complete arbitrage-free market can be implemented by trading in a unit discount bond and a set of forward markets in Arrow securities. How many such contracts are needed?
3. A **call option** (resp. **put option**) on asset $D \in \mathbb{R}^K$ with strike $L \in \mathbb{R}$ is the asset $(D - L\mathbf{1})^+$ (resp. $(L\mathbf{1} - D)^+$). Consider an arbitrage-free market X that allows unrestricted spot trading of a unit discount bond, the asset D as well as a call and a put on D, both with strike L. The respective spot prices are denoted ρ, S, S_c and S_p. Express the difference $S_c - S_p$ in terms of ρ, S and L. (The resulting identity is known as put-call parity.) Show your result in two ways: (a) by constructing a suitable arbitrage as a consequence of the violation of the claimed relationship, and (b) by using the existence of a present-value function.
4. Suppose there are only two states ($K = 2$) and consider a market that is implemented by spot trading of an asset $D \in \mathbb{R}^2$ with spot price $S > 0$, a unit discount bond with corresponding discount factor $\rho > 0$, and a call option on the asset D with strike L, which is the asset $(D - L\mathbf{1})^+$. The option's spot price, known as its premium, is S_c. The possible values of D are $D_1 = (1 + u)S$ in state one and $D_2 = (1 + d)S$ in state two, where $u > d > -1$.

   (a) Derive necessary and sufficient conditions on the parameters for the market implemented by spot trading on the stock and the bond to be arbitrage-free. Assume these conditions are satisfied for the remainder of this question.

   (b) Show that the market implemented by spot trading in the stock and the bond is complete and compute the corresponding state-price vectors.

   (c) Compute the arbitrage price S_c of the call using a state-price vector.

(d) Rederive the arbitrage price of the call by pricing a portfolio in the stock and the bond that replicates the call option.

5. Show that if the market is arbitrage-free and the cash flow $(0, \mathbf{1})$ is not marketed, then there are two state-price densities implying different risk-free discount factors.

6. Suppose the market X is arbitrage-free.

   (a) Show that a cash flow c is marketed if and only if there exists a scalar w such that $\Pi(c) = w$ for every present-value function Π.

   (b) Show that $x \in X$ if and only if $\Pi(x) = 0$ for every present-value function Π.

7. This exercise shows that for a complete market, the proof of the first fundamental theorem of asset pricing simplifies significantly. Suppose X is a complete arbitrage-free market and let $p = \mathbf{1}^0 - x^0$, where x^0 is the projection of $\mathbf{1}^0$ on X. Show that $p_0 > 0$ and verify that the linear functional $\Pi(c) = p_0^{-1}(p \cdot c)$ is a present-value function. Finally, give a second proof by showing the converse to Proposition 1.9 for an arbitrage-free complete market.

8. Consider an "option," defined as a set $\mathcal{D}$ of cash flows, out of which the option owner must choose exactly one cash flow. Given an arbitrage-free reference market X, suppose that the option $\mathcal{D}$ contains a dominant choice δ^*. Suppose further that δ^* is marketed and therefore δ^* has a uniquely defined present value p^*.

   (a) Suppose the option $\mathcal{D}$ can be bought for a premium p, meaning that any cash flow of the form $-p\mathbf{1}^0 + \delta$, where $\delta \in \mathcal{D}$, is available and can be combined with any trade in X. Show that no arbitrage can be created in this manner if and only if $p \geq p^*$.

   (b) Suppose the option $\mathcal{D}$ can be sold (or written) for a premium p, meaning that some cash flow of the form $p\mathbf{1}^0 - \delta$, where $\delta \in \mathcal{D}$, is available and can be combined with any trade in X. Note that δ is selected by the option buyer and it need not be the dominant choice. Show that a seller of the option $\mathcal{D}$ has no arbitrage available for any choice by the option buyer if and only if $p \leq p^*$.

9. The market X satisfies the **law of one price** if $x(1) = y(1)$ implies $x(0) = y(0)$, for all $x, y \in X$.

   (a) Show that the law of one price is equivalent to the condition $\mathbf{1}^0 \notin X$ and is therefore a consequence of the no-arbitrage assumption. Is a market that satisfies the law of one price necessarily arbitrage-free?

(b) Verify that Proposition 1.3, Example 1.17, Proposition 1.19 and Corollary 1.20 remain valid if the arbitrage-free assumption is replaced by the law-of-one-price assumption.

(c) Let us call a **linear valuation rule** any linear functional $\Pi : \mathbb{R}^{1+K} \to \mathbb{R}$ such that $\Pi(x) \leq 0$ for all $x \in X$ and $\Pi(\mathbf{1}^0) = 1$; that is, a linear valuation rule is a present-value function without the strict positivity requirement. Show that the market satisfies the law of one price if and only if there exists a linear valuation rule.

For the remainder of this exercise, assume that the market satisfies the law of one price.

(d) Show that if the market is complete, there exists a unique linear valuation rule.

(e) Suppose the market X is incomplete. Show that given any nonmarketed cash flow c and any scalar α, there exists some linear valuation rule that assigns the value α to c.

10. Given a market X, prove that $\mathbf{1}^0 \notin X$ and $(0, \mathbf{1}) \notin X$ if and only if there exists a vector p such that $p_0 > 0$, $\sum_{k=1}^{K} p_k > 0$ and $p \cdot x \leq 0$ for all $x \in X$.

11. Let H be a finite-dimensional vector space with the inner product $(\cdot \mid \cdot)$. Suppose $C \subseteq H$ is a closed convex cone such that $C \cap (-C) = \{0\}$, where $-C = \{-x : x \in C\}$. Then there exists a nonzero vector p such that $(p \mid x) > 0$ for all nonzero $x \in C$. Prove this claim by generalizing the argument used to show the first fundamental theorem of asset pricing. Show that the latter is a special case. *Hint:* The role of the set Δ can be played by the convex hull of the set $C \cap \{x : \|x\| = 1\}$.

12. Define an inner product under which an SPD π such that $\pi(0) = 1$ is the Riesz representation of the present-value function it represents. Appendix A shows that in a finite-dimensional inner product space every linear functional has a unique Riesz representation. Use this fact to conclude that every present-value function can be represented by a unique, up to positive scaling, SPD. Finally, express your inner product as a quadratic form; that is, define a positive definite symmetric matrix Q such that $(x \mid y) = xQy'$, where the cash flows x and y are viewed as row vectors.

13. (a) Show that any present-value function is represented by a unique EMM-discount pair.

(b) Show that the second equation in (1.7) holds for any probability Q (not necessarily an EMM). Use this fact to show the equivalence of the present-value representations (1.4) and (1.5).

14. Show that the set of traded returns $\mathcal{R}$ is a linear manifold.

15. In the context of Section 1.5, show the following:

(a) The contracts $V_1, \ldots, V_J$ are traded in a market X if and only if $X(V) \subseteq X$.

(b) $X(V)$ is equal to the intersection of all markets in which all of the contracts $V_1, \ldots, V_J$ are traded.

(c) Equality (1.11) and Corollary 1.20.

16. Does Theorem 1.28 remain valid if condition (1.17) is omitted from the definition of a constrained market? If yes, prove the stronger result. If not, give a counterexample and explain what step of the proof of Theorem 1.28 is no longer valid.

17. (Margin Requirements) This exercise models a simple example of a margin requirement, which is a type of collateral constraint. For each $j \in \{1, \ldots, J\}$, let $V_j = (S_j, D_j)$ for some asset D_j and spot price $S_j > 0$. The first asset is a unit discount bond ($D_1 = \mathbf{1}$), while the remaining assets are risky. The margin requirement is that the combined value of all long risky-asset positions plus the (possibly negative) value of the discount-bond position must be at least equal to half the total value of long risky-asset positions plus one and a half times the amount raised by short selling risky assets.

(a) Show that corresponding portfolio constraint set can be written as

$$\Theta = \left\{ \alpha \in \mathbb{R}^J : \sum_{j=1}^{J} \alpha_j S_j \geq \frac{1}{2} \sum_{j=2}^{J} |\alpha_j| S_j \right\},$$

and verify that Theorem 1.28 applies in this context.

(b) For a simple illustration of how the margin constraint weakens the arbitrage-pricing argument, suppose that $J = 3$, $S_1 = S_2 = 1$, $S_3 = 1 + \delta > 1$ and $D_2 = D_3$. Suppose a trader with no initial holdings of assets two or three ($\theta_2^0 = \theta_3^0 = 0$) attempts to arbitrage the difference in price of the two identical assets by purchasing a portfolio $\theta = (\theta_1, \theta_2, \theta_3)$, where $\theta_2 = \beta > 0$ and $\theta_3 = -\beta$. Show that θ generates an arbitrage and satisfies the margin requirement if and only if

$$\beta\delta \geq \theta_1 \geq 0 \quad \text{and} \quad \theta_1^0 + \theta_1 \geq \beta\left(1 + \frac{3\delta}{2}\right).$$

The arbitrage is possible if and only if the trader has some initial capital $\theta_1^0 > 0$, in which case the size of the arbitrage is limited by the initial capital.

18. (Bid-Ask Spreads) This exercise models bid-ask spreads by regarding the purchase and the sale of an asset as separate contracts on which short positions are not possible. (The approach is limited to the single-period case.) A **long spot position** in asset D is the contract $(-S_a, D)$, the scalar S_a representing the **ask spot price** of the asset. A **short spot position** in asset D is the contract $(S_b, -D)$, the scalar S_b representing the **bid spot price** of the asset. In particular, a long spot position in a unit discount bond $(-\rho_a, \mathbf{1})$ implements default-free lending with interest rate $r_l = (1/\rho_a) - 1$, and a short position in a unit discount bond $(\rho_b, -\mathbf{1})$ implements default-free borrowing with interest rate $r_b = (1/\rho_b) - 1$. Bid-ask spreads in the forward market for an asset D are modeled analogously through the contracts $(0, D - F_a\mathbf{1})$ and $(0, F_b\mathbf{1} - D)$, representing **long** and **short forward positions**, respectively. The prices F_a and F_b are the **ask** and **bid forward prices** of the asset, respectively.

(a) Show that in an arbitrage-free market the following restrictions must hold:

$$r_l \leq r_b, \quad \rho_b \leq \rho_a, \quad S_b \leq F_a\rho_a \quad \text{and} \quad \rho_b F_b \leq S_a. \tag{1.18}$$

Can the size of the spreads $r_b - r_l$, $\rho_a - \rho_b$, $S_a - S_b$ and $F_a - F_b$ be limited by an arbitrage argument? (In actual dealership markets, bid-ask spreads are limited by dealer competition. Even in a perfectly competitive dealership market, however, bid-ask spreads are positive to compensate dealers for carrying an inventory and for trading with potentially better informed traders.)

(b) Show that if (Q, ρ) is an EMM-discount pair, then

$$\rho_b \leq \rho \leq \rho_a, \quad S_b \leq \rho\mathbb{E}^Q[D] \leq S_a \quad \text{and} \quad F_b \leq \mathbb{E}^Q[D] \leq F_a.$$

Use these inequalities to recover the pricing restrictions (1.18). Does Theorem 1.28 apply here and how?

(c) Explain how restrictions (1.18) can be improved under various assumptions on the initial positions of a potential arbitrageur.

1.9 Notes

The simple but powerful idea that a contingent payoff can be thought of as a basket of what we called “Arrow securities” is due to Arrow ((1953); translated in English in Arrow (1963)). Identifying an Arrow security with a commodity, also known as an Arrow-Debreu commodity in this context, was the key step in extending classical competitive analysis to financial markets, as further explained in Chapter 3. Pricing through arbitrage arguments first achieved prominence in financial theory in the seminal arguments of Modigliani and Miller (1958), Merton (1973b) and Black and Scholes (1973).

The equivalence of the no-arbitrage condition and the existence of a present-value function, commonly referred to as the first fundamental theorem of asset pricing, is due to Ross (1978b) for the case of a finite uncertainty model. As we saw, mathematically the result amounts to the strict separation of convex cones. Assuming the market is implemented by some finite set of contracts (possibly under constraints), the finite-dimensional fundamental theorem of asset pricing is an example of the so-called theorems of the alternative in convex analysis, a textbook account of which can be found, for example, in Chapter 1 of Stoer and Witzgall (1970). The idea of risk-neutral pricing already appears in Arrow (1970) and Drèze (1971), and it is exploited in option pricing by Cox and Ross (1976). Harrison and Kreps (1979), who coined the term equivalent martingale measure, established clearly the relationship between positive linear pricing and the martingale property of properly discounted prices, as explained in Part II. An extension with bid-ask spreads was given in Jouini and Kallal (1995).

The fundamental theorem of asset pricing was extended to infinite-dimensional spaces by Kreps (1981) in a strict-separation result that was independently shown in the mathematics literature by Yan (1980) and is known as the Kreps-Yan theorem. A remarkably simple to state result for the case of a market generated by finitely many assets (in discrete time) and infinitely many states is due to Dalang, Morton, and Willinger (1990), with simplified proofs given by Schachermayer (1992) and Kabanov and

Kramkov (1994). This line of research continued in mathematics, with notable contributions by Delbaen and Schachermayer (1994, 1998, 2006).

Exercise 1.4 is the single-period case of the binomial option pricing model first proposed by Cox, Ross and Rubinstein (1979), Rendleman and Bartter (1979) and Sharpe (1978). The model will be revisited in Chapter 5.