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Jacob Lurie: Higher Topos Theory

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Chapter One

An Overview of Higher Category Theory

This chapter is intended as a general introduction to higher category theory. We begin with what we feel is the most intuitive approach to the subject using *topological categories*. This approach is easy to understand but difficult to work with when one wishes to perform even simple categorical constructions. As a remedy, we will introduce the more suitable formalism of ∞ -*categories* (called *weak Kan complexes* in [10] and *quasi-categories* in [43]), which provides a more convenient setting for adaptations of sophisticated category-theoretic ideas. Our goal in §1.1.1 is to introduce both approaches and to explain why they are equivalent to one another. The proof of this equivalence will rely on a crucial result (Theorem 1.1.5.13) which we will prove in §2.2.

Our second objective in this chapter is to give the reader an idea of how to work with the formalism of ∞ -categories. In §1.2, we will establish a vocabulary which includes ∞ -categorical analogues (often direct generalizations) of most of the important concepts from ordinary category theory. To keep the exposition brisk, we will postpone the more difficult proofs until later chapters of this book. Our hope is that, after reading this chapter, a reader who does not wish to be burdened with the details will be able to understand (at least in outline) some of the more conceptual ideas described in Chapter 5 and beyond.

1.1 FOUNDATIONS FOR HIGHER CATEGORY THEORY

1.1.1 Goals and Obstacles

Recall that a *category* \mathcal{C} consists of the following data:

- (1) A collection $\{X, Y, Z, \dots\}$ whose members are the *objects* of \mathcal{C} . We typically write $X \in \mathcal{C}$ to indicate that X is an object of \mathcal{C} .
- (2) For every pair of objects $X, Y \in \mathcal{C}$, a set $\text{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms* from X to Y . We will typically write $f : X \rightarrow Y$ to indicate that $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and say that f is a *morphism from X to Y* .
- (3) For every object $X \in \mathcal{C}$, an *identity morphism* $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$.
- (4) For every triple of objects $X, Y, Z \in \mathcal{C}$, a composition map
$$\text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z).$$

Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we will usually denote the image of the pair (f, g) under the composition map by gf or $g \circ f$.

These data are furthermore required to satisfy the following conditions, which guarantee that composition is unital and associative:

- (5) For every morphism $f : X \rightarrow Y$, we have $\text{id}_Y \circ f = f = f \circ \text{id}_X$ in $\text{Hom}_{\mathcal{C}}(X, Y)$.
- (6) For every triple of composable morphisms

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z,$$

we have an equality $h \circ (g \circ f) = (h \circ g) \circ f$ in $\text{Hom}_{\mathcal{C}}(W, Z)$.

The theory of categories has proven to be a valuable organization tool in many areas of mathematics. Mathematical structures of virtually any type can be viewed as the objects of a suitable category \mathcal{C} , where the morphisms in \mathcal{C} are given by structure-preserving maps. There is a veritable legion of examples of categories which fit this paradigm:

- The category Set whose objects are sets and whose morphisms are maps of sets.
- The category Grp whose objects are groups and whose morphisms are group homomorphisms.
- The category Top whose objects are topological spaces and whose morphisms are continuous maps.
- The category Cat whose objects are (small) categories and whose morphisms are functors. (Recall that a functor F from \mathcal{C} to \mathcal{D} is a map which assigns to each object $C \in \mathcal{C}$ another object $FC \in \mathcal{D}$, and to each morphism $f : C \rightarrow C'$ in \mathcal{C} a morphism $F(f) : FC \rightarrow FC'$ in \mathcal{D} , so that $F(\text{id}_C) = \text{id}_{FC}$ and $F(g \circ f) = F(g) \circ F(f)$.)
- ...

In general, the existence of a morphism $f : X \rightarrow Y$ in a category \mathcal{C} reflects some relationship that exists between the objects $X, Y \in \mathcal{C}$. In some contexts, these relationships themselves become basic objects of study and can be fruitfully organized into categories:

Example 1.1.1.1. Let Grp be the category whose objects are groups and whose morphisms are group homomorphisms. In the theory of groups, one is often concerned only with group homomorphisms *up to conjugacy*. The relation of conjugacy can be encoded as follows: for every pair of groups $G, H \in \text{Grp}$, there is a category $\text{Map}(G, H)$ whose objects are group homomorphisms from G to H (that is, elements of $\text{Hom}_{\text{Grp}}(G, H)$), where a morphism from $f : G \rightarrow H$ to $f' : G \rightarrow H$ is an element $h \in H$ such that $hf(g)h^{-1} = f'(g)$ for all $g \in G$. Note that two group homomorphisms $f, f' : G \rightarrow H$ are conjugate if and only if they are isomorphic when viewed as objects of $\text{Map}(G, H)$.

Example 1.1.1.2. Let X and Y be topological spaces and let $f_0, f_1 : X \rightarrow Y$ be continuous maps. Recall that a *homotopy* from f_0 to f_1 is a continuous map $f : X \times [0, 1] \rightarrow Y$ such that $f|_{X \times \{0\}}$ coincides with f_0 and $f|_{X \times \{1\}}$ coincides with f_1 . In algebraic topology, one is often concerned not with the category Top of topological spaces but with its *homotopy category*: that is, the category obtained by identifying those pairs of morphisms $f_0, f_1 : X \rightarrow Y$ which are homotopic to one another. For many purposes, it is better to do something a little bit more sophisticated: namely, one can form a category $\text{Map}(X, Y)$ whose objects are continuous maps $f : X \rightarrow Y$ and whose morphisms are given by (homotopy classes of) homotopies.

Example 1.1.1.3. Given a pair of categories \mathcal{C} and \mathcal{D} , the collection of all functors from \mathcal{C} to \mathcal{D} is itself naturally organized into a category $\text{Fun}(\mathcal{C}, \mathcal{D})$, where the morphisms are given by *natural transformations*. (Recall that, given a pair of functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a natural transformation $\alpha : F \rightarrow G$ is a collection of morphisms $\{\alpha_C : F(C) \rightarrow G(C)\}_{C \in \mathcal{C}}$ which satisfy the following condition: for every morphism $f : C \rightarrow C'$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(C') \\ \downarrow \alpha_C & & \downarrow \alpha_{C'} \\ G(C) & \xrightarrow{G(f)} & G(C') \end{array}$$

commutes in \mathcal{D} .)

In each of these examples, the objects of interest can naturally be organized into what is called a *2-category* (or *bicategory*): we have not only a collection of objects and a notion of morphisms between objects but also a notion of morphisms between morphisms, which are called *2-morphisms*. The vision of higher category theory is that there should exist a good notion of n -category for all $n \geq 0$ in which we have not only objects, morphisms, and 2-morphisms but also k -morphisms for all $k \leq n$. Finally, in some sort of limit we might hope to obtain a theory of ∞ -categories, where there are morphisms of all orders.

Example 1.1.1.4. Let X be a topological space and $0 \leq n \leq \infty$. We can extract an n -category $\pi_{\leq n}X$ (roughly) as follows. The objects of $\pi_{\leq n}X$ are the points of X . If $x, y \in X$, then the morphisms from x to y in $\pi_{\leq n}X$ are given by continuous paths $[0, 1] \rightarrow X$ starting at x and ending at y . The 2-morphisms are given by homotopies of paths, the 3-morphisms by homotopies between homotopies, and so forth. Finally, if $n < \infty$, then two n -morphisms of $\pi_{\leq n}X$ are considered to be the same if and only if they are homotopic to one another.

If $n = 0$, then $\pi_{\leq n}X$ can be identified with the set π_0X of path components of X . If $n = 1$, then our definition of $\pi_{\leq n}X$ agrees with the usual definition for the fundamental groupoid of X . For this reason, $\pi_{\leq n}X$ is often called the *fundamental n -groupoid* of X . It is called an *n -groupoid* (rather than a mere

n -category) because every k -morphism of $\pi_{\leq k}X$ has an inverse (at least up to homotopy).

There are many approaches to realizing the theory of higher categories. We might begin by defining a 2-category to be a “category enriched over Cat .” In other words, we consider a collection of objects together with a *category* of morphisms $\text{Hom}(A, B)$ for any two objects A and B and composition *functors* $c_{ABC} : \text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$ (to simplify the discussion, we will ignore identity morphisms for a moment). These functors are required to satisfy an associative law, which asserts that for any quadruple (A, B, C, D) of objects, the diagram

$$\begin{array}{ccc} \text{Hom}(A, B) \times \text{Hom}(B, C) \times \text{Hom}(C, D) & \longrightarrow & \text{Hom}(A, C) \times \text{Hom}(C, D) \\ \downarrow & & \downarrow \\ \text{Hom}(A, B) \times \text{Hom}(B, D) & \longrightarrow & \text{Hom}(A, D) \end{array}$$

commutes; in other words, one has an *equality* of functors

$$c_{ACD} \circ (c_{ABC} \times 1) = c_{ABD} \circ (1 \times c_{BCD})$$

from $\text{Hom}(A, B) \times \text{Hom}(B, C) \times \text{Hom}(C, D)$ to $\text{Hom}(A, D)$. This leads to the definition of a *strict 2-category*.

At this point, we should object that the definition of a strict 2-category violates one of the basic philosophical principles of category theory: one should never demand that two functors F and F' be equal to one another. Instead one should postulate the existence of a natural isomorphism between F and F' . This means that the associative law should not take the form of an equation but of additional structure: a collection of isomorphisms $\gamma_{ABCD} : c_{ACD} \circ (c_{ABC} \times 1) \simeq c_{ABD} \circ (1 \times c_{BCD})$. We should further demand that the isomorphisms γ_{ABCD} be functorial in the quadruple (A, B, C, D) and satisfy certain higher associativity conditions, which generalize the “Pentagon axiom” described in §A.1.3. After formulating the appropriate conditions, we arrive at the definition of a *weak 2-category*.

Let us contrast the notions of strict 2-category and weak 2-category. The former is easier to define because we do not have to worry about the higher associativity conditions satisfied by the transformations γ_{ABCD} . On the other hand, the latter notion seems more natural if we take the philosophy of category theory seriously. In this case, we happen to be lucky: the notions of strict 2-category and weak 2-category turn out to be equivalent. More precisely, any weak 2-category is equivalent (in the relevant sense) to a strict 2-category. The choice of definition can therefore be regarded as a question of aesthetics.

We now plunge onward to 3-categories. Following the above program, we might define a *strict 3-category* to consist of a collection of objects together with strict 2-categories $\text{Hom}(A, B)$ for any pair of objects A and B , together with a strictly associative composition law. Alternatively, we could seek a definition of *weak 3-category* by allowing $\text{Hom}(A, B)$ to be a weak

2-category, requiring associativity only up to natural 2-isomorphisms, which satisfy higher associativity laws up to natural 3-isomorphisms, which in turn satisfy still higher associativity laws of their own. Unfortunately, it turns out that these notions are *not* equivalent.

Both of these approaches have serious drawbacks. The obvious problem with weak 3-categories is that an explicit definition is extremely complicated (see [33], where a definition is given along these lines), to the point where it is essentially unusable. On the other hand, strict 3-categories have the problem of not being the correct notion: most of the weak 3-categories which occur in nature are not equivalent to strict 3-categories. For example, the fundamental 3-groupoid of the 2-sphere S^2 cannot be described using the language of strict 3-categories. The situation only gets worse (from either point of view) as we pass to 4-categories and beyond.

Fortunately, it turns out that major simplifications can be introduced if we are willing to restrict our attention to ∞ -categories in which most of the higher morphisms are invertible. From this point forward, we will use the term (∞, n) -category to refer to ∞ -categories in which all k -morphisms are invertible for $k > n$. The ∞ -categories described in Example 1.1.1.4 (when $n = \infty$) are all $(\infty, 0)$ -categories. The converse, which asserts that every $(\infty, 0)$ -category has the form $\pi_{\leq \infty} X$ for some topological space X , is a generally accepted principle of higher category theory. Moreover, the ∞ -groupoid $\pi_{\leq \infty} X$ encodes the entire homotopy type of X . In other words, $(\infty, 0)$ -categories (that is, ∞ -categories in which *all* morphisms are invertible) have been extensively studied from another point of view: they are essentially the same thing as “spaces” in the sense of homotopy theory, and there are many equivalent ways to describe them (for example, we can use CW complexes or simplicial sets).

Convention 1.1.1.5. We will sometimes refer to $(\infty, 0)$ -categories as ∞ -groupoids and $(\infty, 2)$ -categories as ∞ -bicategories. Unless we specify otherwise, the generic term “ ∞ -category” will refer to an $(\infty, 1)$ -category.

In this book, we will restrict our attention almost entirely to the theory of ∞ -categories (in which we have only invertible n -morphisms for $n \geq 2$). Our reasons are threefold:

- (1) Allowing noninvertible n -morphisms for $n > 1$ introduces a number of additional complications to the theory at both technical and conceptual levels. As we will see throughout this book, many ideas from category theory generalize to the ∞ -categorical setting in a natural way. However, these generalizations are not so straightforward if we allow noninvertible 2-morphisms. For example, one must distinguish between strict and lax fiber products, even in the setting of “classical” 2-categories.
- (2) For the applications studied in this book, we will not need to consider (∞, n) -categories for $n > 2$. The case $n = 2$ is of some relevance

