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Jacob Lurie: Higher Topos Theory

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Chapter One

An Overview of Higher Category Theory

This chapter is intended as a general introduction to higher category theory. We begin with what we feel is the most intuitive approach to the subject using *topological categories*. This approach is easy to understand but difficult to work with when one wishes to perform even simple categorical constructions. As a remedy, we will introduce the more suitable formalism of ∞ -*categories* (called *weak Kan complexes* in [10] and *quasi-categories* in [43]), which provides a more convenient setting for adaptations of sophisticated category-theoretic ideas. Our goal in §1.1.1 is to introduce both approaches and to explain why they are equivalent to one another. The proof of this equivalence will rely on a crucial result (Theorem 1.1.5.13) which we will prove in §2.2.

Our second objective in this chapter is to give the reader an idea of how to work with the formalism of ∞ -categories. In §1.2, we will establish a vocabulary which includes ∞ -categorical analogues (often direct generalizations) of most of the important concepts from ordinary category theory. To keep the exposition brisk, we will postpone the more difficult proofs until later chapters of this book. Our hope is that, after reading this chapter, a reader who does not wish to be burdened with the details will be able to understand (at least in outline) some of the more conceptual ideas described in Chapter 5 and beyond.

1.1 FOUNDATIONS FOR HIGHER CATEGORY THEORY

1.1.1 Goals and Obstacles

Recall that a *category* \mathcal{C} consists of the following data:

- (1) A collection $\{X, Y, Z, \dots\}$ whose members are the *objects* of \mathcal{C} . We typically write $X \in \mathcal{C}$ to indicate that X is an object of \mathcal{C} .
- (2) For every pair of objects $X, Y \in \mathcal{C}$, a set $\text{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms* from X to Y . We will typically write $f : X \rightarrow Y$ to indicate that $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and say that f is a *morphism from X to Y* .
- (3) For every object $X \in \mathcal{C}$, an *identity morphism* $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$.
- (4) For every triple of objects $X, Y, Z \in \mathcal{C}$, a composition map

$$\text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z).$$

Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we will usually denote the image of the pair (f, g) under the composition map by gf or $g \circ f$.

These data are furthermore required to satisfy the following conditions, which guarantee that composition is unital and associative:

- (5) For every morphism $f : X \rightarrow Y$, we have $\text{id}_Y \circ f = f = f \circ \text{id}_X$ in $\text{Hom}_{\mathcal{C}}(X, Y)$.
- (6) For every triple of composable morphisms

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z,$$

we have an equality $h \circ (g \circ f) = (h \circ g) \circ f$ in $\text{Hom}_{\mathcal{C}}(W, Z)$.

The theory of categories has proven to be a valuable organization tool in many areas of mathematics. Mathematical structures of virtually any type can be viewed as the objects of a suitable category \mathcal{C} , where the morphisms in \mathcal{C} are given by structure-preserving maps. There is a veritable legion of examples of categories which fit this paradigm:

- The category Set whose objects are sets and whose morphisms are maps of sets.
- The category Grp whose objects are groups and whose morphisms are group homomorphisms.
- The category Top whose objects are topological spaces and whose morphisms are continuous maps.
- The category Cat whose objects are (small) categories and whose morphisms are functors. (Recall that a functor F from \mathcal{C} to \mathcal{D} is a map which assigns to each object $C \in \mathcal{C}$ another object $FC \in \mathcal{D}$, and to each morphism $f : C \rightarrow C'$ in \mathcal{C} a morphism $F(f) : FC \rightarrow FC'$ in \mathcal{D} , so that $F(\text{id}_C) = \text{id}_{FC}$ and $F(g \circ f) = F(g) \circ F(f)$.)
- ...

In general, the existence of a morphism $f : X \rightarrow Y$ in a category \mathcal{C} reflects some relationship that exists between the objects $X, Y \in \mathcal{C}$. In some contexts, these relationships themselves become basic objects of study and can be fruitfully organized into categories:

Example 1.1.1.1. Let Grp be the category whose objects are groups and whose morphisms are group homomorphisms. In the theory of groups, one is often concerned only with group homomorphisms *up to conjugacy*. The relation of conjugacy can be encoded as follows: for every pair of groups $G, H \in \text{Grp}$, there is a category $\text{Map}(G, H)$ whose objects are group homomorphisms from G to H (that is, elements of $\text{Hom}_{\text{Grp}}(G, H)$), where a morphism from $f : G \rightarrow H$ to $f' : G \rightarrow H$ is an element $h \in H$ such that $hf(g)h^{-1} = f'(g)$ for all $g \in G$. Note that two group homomorphisms $f, f' : G \rightarrow H$ are conjugate if and only if they are isomorphic when viewed as objects of $\text{Map}(G, H)$.

Example 1.1.1.2. Let X and Y be topological spaces and let $f_0, f_1 : X \rightarrow Y$ be continuous maps. Recall that a *homotopy* from f_0 to f_1 is a continuous map $f : X \times [0, 1] \rightarrow Y$ such that $f|_{X \times \{0\}}$ coincides with f_0 and $f|_{X \times \{1\}}$ coincides with f_1 . In algebraic topology, one is often concerned not with the category Top of topological spaces but with its *homotopy category*: that is, the category obtained by identifying those pairs of morphisms $f_0, f_1 : X \rightarrow Y$ which are homotopic to one another. For many purposes, it is better to do something a little bit more sophisticated: namely, one can form a category $\text{Map}(X, Y)$ whose objects are continuous maps $f : X \rightarrow Y$ and whose morphisms are given by (homotopy classes of) homotopies.

Example 1.1.1.3. Given a pair of categories \mathcal{C} and \mathcal{D} , the collection of all functors from \mathcal{C} to \mathcal{D} is itself naturally organized into a category $\text{Fun}(\mathcal{C}, \mathcal{D})$, where the morphisms are given by *natural transformations*. (Recall that, given a pair of functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a natural transformation $\alpha : F \rightarrow G$ is a collection of morphisms $\{\alpha_C : F(C) \rightarrow G(C)\}_{C \in \mathcal{C}}$ which satisfy the following condition: for every morphism $f : C \rightarrow C'$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(C') \\ \downarrow \alpha_C & & \downarrow \alpha_{C'} \\ G(C) & \xrightarrow{G(f)} & G(C') \end{array}$$

commutes in \mathcal{D} .)

In each of these examples, the objects of interest can naturally be organized into what is called a *2-category* (or *bicategory*): we have not only a collection of objects and a notion of morphisms between objects but also a notion of morphisms between morphisms, which are called *2-morphisms*. The vision of higher category theory is that there should exist a good notion of n -category for all $n \geq 0$ in which we have not only objects, morphisms, and 2-morphisms but also k -morphisms for all $k \leq n$. Finally, in some sort of limit we might hope to obtain a theory of ∞ -categories, where there are morphisms of all orders.

Example 1.1.1.4. Let X be a topological space and $0 \leq n \leq \infty$. We can extract an n -category $\pi_{\leq n}X$ (roughly) as follows. The objects of $\pi_{\leq n}X$ are the points of X . If $x, y \in X$, then the morphisms from x to y in $\pi_{\leq n}X$ are given by continuous paths $[0, 1] \rightarrow X$ starting at x and ending at y . The 2-morphisms are given by homotopies of paths, the 3-morphisms by homotopies between homotopies, and so forth. Finally, if $n < \infty$, then two n -morphisms of $\pi_{\leq n}X$ are considered to be the same if and only if they are homotopic to one another.

If $n = 0$, then $\pi_{\leq n}X$ can be identified with the set π_0X of path components of X . If $n = 1$, then our definition of $\pi_{\leq n}X$ agrees with the usual definition for the fundamental groupoid of X . For this reason, $\pi_{\leq n}X$ is often called the *fundamental n -groupoid* of X . It is called an *n -groupoid* (rather than a mere

n -category) because every k -morphism of $\pi_{\leq k}X$ has an inverse (at least up to homotopy).

There are many approaches to realizing the theory of higher categories. We might begin by defining a 2-category to be a “category enriched over Cat .” In other words, we consider a collection of objects together with a *category* of morphisms $\text{Hom}(A, B)$ for any two objects A and B and composition *functors* $c_{ABC} : \text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$ (to simplify the discussion, we will ignore identity morphisms for a moment). These functors are required to satisfy an associative law, which asserts that for any quadruple (A, B, C, D) of objects, the diagram

$$\begin{array}{ccc} \text{Hom}(A, B) \times \text{Hom}(B, C) \times \text{Hom}(C, D) & \longrightarrow & \text{Hom}(A, C) \times \text{Hom}(C, D) \\ \downarrow & & \downarrow \\ \text{Hom}(A, B) \times \text{Hom}(B, D) & \longrightarrow & \text{Hom}(A, D) \end{array}$$

commutes; in other words, one has an *equality* of functors

$$c_{ACD} \circ (c_{ABC} \times 1) = c_{ABD} \circ (1 \times c_{BCD})$$

from $\text{Hom}(A, B) \times \text{Hom}(B, C) \times \text{Hom}(C, D)$ to $\text{Hom}(A, D)$. This leads to the definition of a *strict 2-category*.

At this point, we should object that the definition of a strict 2-category violates one of the basic philosophical principles of category theory: one should never demand that two functors F and F' be equal to one another. Instead one should postulate the existence of a natural isomorphism between F and F' . This means that the associative law should not take the form of an equation but of additional structure: a collection of isomorphisms $\gamma_{ABCD} : c_{ACD} \circ (c_{ABC} \times 1) \simeq c_{ABD} \circ (1 \times c_{BCD})$. We should further demand that the isomorphisms γ_{ABCD} be functorial in the quadruple (A, B, C, D) and satisfy certain higher associativity conditions, which generalize the “Pentagon axiom” described in §A.1.3. After formulating the appropriate conditions, we arrive at the definition of a *weak 2-category*.

Let us contrast the notions of strict 2-category and weak 2-category. The former is easier to define because we do not have to worry about the higher associativity conditions satisfied by the transformations γ_{ABCD} . On the other hand, the latter notion seems more natural if we take the philosophy of category theory seriously. In this case, we happen to be lucky: the notions of strict 2-category and weak 2-category turn out to be equivalent. More precisely, any weak 2-category is equivalent (in the relevant sense) to a strict 2-category. The choice of definition can therefore be regarded as a question of aesthetics.

We now plunge onward to 3-categories. Following the above program, we might define a *strict 3-category* to consist of a collection of objects together with strict 2-categories $\text{Hom}(A, B)$ for any pair of objects A and B , together with a strictly associative composition law. Alternatively, we could seek a definition of *weak 3-category* by allowing $\text{Hom}(A, B)$ to be a weak

2-category, requiring associativity only up to natural 2-isomorphisms, which satisfy higher associativity laws up to natural 3-isomorphisms, which in turn satisfy still higher associativity laws of their own. Unfortunately, it turns out that these notions are *not* equivalent.

Both of these approaches have serious drawbacks. The obvious problem with weak 3-categories is that an explicit definition is extremely complicated (see [33], where a definition is given along these lines), to the point where it is essentially unusable. On the other hand, strict 3-categories have the problem of not being the correct notion: most of the weak 3-categories which occur in nature are not equivalent to strict 3-categories. For example, the fundamental 3-groupoid of the 2-sphere S^2 cannot be described using the language of strict 3-categories. The situation only gets worse (from either point of view) as we pass to 4-categories and beyond.

Fortunately, it turns out that major simplifications can be introduced if we are willing to restrict our attention to ∞ -categories in which most of the higher morphisms are invertible. From this point forward, we will use the term (∞, n) -category to refer to ∞ -categories in which all k -morphisms are invertible for $k > n$. The ∞ -categories described in Example 1.1.1.4 (when $n = \infty$) are all $(\infty, 0)$ -categories. The converse, which asserts that every $(\infty, 0)$ -category has the form $\pi_{\leq \infty} X$ for some topological space X , is a generally accepted principle of higher category theory. Moreover, the ∞ -groupoid $\pi_{\leq \infty} X$ encodes the entire homotopy type of X . In other words, $(\infty, 0)$ -categories (that is, ∞ -categories in which *all* morphisms are invertible) have been extensively studied from another point of view: they are essentially the same thing as “spaces” in the sense of homotopy theory, and there are many equivalent ways to describe them (for example, we can use CW complexes or simplicial sets).

Convention 1.1.1.5. We will sometimes refer to $(\infty, 0)$ -categories as ∞ -groupoids and $(\infty, 2)$ -categories as ∞ -bicatagories. Unless we specify otherwise, the generic term “ ∞ -category” will refer to an $(\infty, 1)$ -category.

In this book, we will restrict our attention almost entirely to the theory of ∞ -categories (in which we have only invertible n -morphisms for $n \geq 2$). Our reasons are threefold:

- (1) Allowing noninvertible n -morphisms for $n > 1$ introduces a number of additional complications to the theory at both technical and conceptual levels. As we will see throughout this book, many ideas from category theory generalize to the ∞ -categorical setting in a natural way. However, these generalizations are not so straightforward if we allow noninvertible 2-morphisms. For example, one must distinguish between strict and lax fiber products, even in the setting of “classical” 2-categories.
- (2) For the applications studied in this book, we will not need to consider (∞, n) -categories for $n > 2$. The case $n = 2$ is of some relevance

because the collection of (small) ∞ -categories can naturally be viewed as a (large) ∞ -bicategory. However, we will generally be able to exploit this structure in an ad hoc manner without developing any general theory of ∞ -bicategories.

- (3) For $n > 1$, the theory of (∞, n) -categories is most naturally viewed as a special case of *enriched* (higher) category theory. Roughly speaking, an n -category can be viewed as a category enriched over $(n-1)$ -categories. As we explained above, this point of view is inadequate because it requires that composition satisfies an associative law up to equality, while in practice the associativity holds only up to isomorphism or some weaker notion of equivalence. In other words, to obtain the correct definition we need to view the collection of $(n-1)$ -categories as an n -category, not as an ordinary category. Consequently, the naive approach is circular: though it does lead to a good theory of n -categories, we can make sense of it only if the theory of n -categories is already in place.

Thinking along similar lines, we can view an (∞, n) -category as an ∞ -category which is *enriched over* $(\infty, n-1)$ -categories. The collection of $(\infty, n-1)$ -categories is itself organized into an (∞, n) -category $\text{Cat}_{(\infty, n-1)}$, so at a first glance this definition suffers from the same problem of circularity. However, because the associativity properties of composition are required to hold up to *equivalence*, rather than up to arbitrary natural transformation, the noninvertible k -morphisms in $\text{Cat}_{(\infty, n-1)}$ are irrelevant for $k > 1$. One can define an (∞, n) -category to be a category enriched over $\text{Cat}_{(\infty, n-1)}$, where the latter is regarded as an ∞ -category by discarding noninvertible k -morphisms for $2 \leq k \leq n$. In other words, the naive inductive definition of higher category theory is reasonable *provided that we work in the ∞ -categorical setting from the outset*. We refer the reader to [75] for a definition of n -categories which follows this line of thought.

The theory of *enriched* ∞ -categories is a useful and important one but will not be treated in this book. Instead we refer the reader to [50] for an introduction using the same language and formalism we employ here.

Though we will not need a theory of (∞, n) -categories for $n > 1$, the case $n = 1$ is the main subject matter of this book. Fortunately, the above discussion suggests a definition. Namely, an ∞ -category \mathcal{C} should consist of a collection of objects and an ∞ -groupoid $\text{Map}_{\mathcal{C}}(X, Y)$ for every pair of objects $X, Y \in \mathcal{C}$. These ∞ -groupoids can be identified with topological spaces, and should be equipped with an associative composition law. As before, we are faced with two choices as to how to make this precise: do we require associativity on the nose or only up to (coherent) homotopy? Fortunately, the answer turns out to be irrelevant: as in the theory of 2-categories, any ∞ -category with a coherently associative multiplication can be replaced by

an equivalent ∞ -category with a strictly associative multiplication. We are led to the following:

Definition 1.1.1.6. A *topological category* is a category which is enriched over \mathcal{CG} , the category of compactly generated (and weakly Hausdorff) topological spaces. The category of topological categories will be denoted by Cat_{top} .

More explicitly, a topological category \mathcal{C} consists of a collection of objects together with a (compactly generated) topological space $\text{Map}_{\mathcal{C}}(X, Y)$ for any pair of objects $X, Y \in \mathcal{C}$. These mapping spaces must be equipped with an associative composition law given by continuous maps

$$\text{Map}_{\mathcal{C}}(X_0, X_1) \times \text{Map}_{\mathcal{C}}(X_1, X_2) \times \cdots \times \text{Map}_{\mathcal{C}}(X_{n-1}, X_n) \rightarrow \text{Map}_{\mathcal{C}}(X_0, X_n)$$

(defined for all $n \geq 0$). Here the product is taken in the category of compactly generated topological spaces.

Remark 1.1.1.7. The decision to work with compactly generated topological spaces, rather than arbitrary spaces, is made in order to facilitate the comparison with more combinatorial approaches to homotopy theory. This is a purely technical point which the reader may safely ignore.

It is possible to use Definition 1.1.1.6 as a foundation for higher category theory: that is, to *define* an ∞ -category to be a topological category. However, this approach has a number of technical disadvantages. We will describe an alternative (though equivalent) formalism in the next section.

1.1.2 ∞ -Categories

Of the numerous formalizations of higher category theory, Definition 1.1.1.6 is the quickest and most transparent. However, it is one of the most difficult to actually work with: many of the basic constructions of higher category theory give rise most naturally to $(\infty, 1)$ -categories for which the composition of morphisms is associative only up to (coherent) homotopy (for several examples of this phenomenon, we refer the reader to §1.2). In order to remain in the world of topological categories, it is necessary to combine these constructions with a “straightening” procedure which produces a strictly associative composition law. Although it is always possible to do this (see Theorem 2.2.5.1), it is much more technically convenient to work from the outset within a more flexible theory of $(\infty, 1)$ -categories. Fortunately, there are many candidates for such a theory, including the theory of Segal categories ([71]), the theory of complete Segal spaces ([64]), and the theory of model categories ([40], [38]). To review all of these notions and their interrelationships would involve too great a digression from the main purpose of this book. However, the frequency with which we will encounter sophisticated categorical constructions necessitates the use of *one* of these more efficient approaches. We will employ the theory of *weak Kan complexes*, which goes

back to Boardman-Vogt ([10]). These objects have subsequently been studied more extensively by Joyal ([43], [44]), who calls them *quasi-categories*. We will simply call them ∞ -*categories*.

To get a feeling for what an ∞ -category \mathcal{C} should be, it is useful to consider two extreme cases. If *every* morphism in \mathcal{C} is invertible, then \mathcal{C} is equivalent to the fundamental ∞ -groupoid of a topological space X . In this case, higher category theory reduces to classical homotopy theory. On the other hand, if \mathcal{C} has no nontrivial n -morphisms for $n > 1$, then \mathcal{C} is equivalent to an ordinary category. A general formalism must capture the features of both of these examples. In other words, we need a class of mathematical objects which can behave both like categories and like topological spaces. In §1.1.1, we achieved this by “brute force”: namely, we directly amalgamated the theory of topological spaces and the theory of categories by considering topological categories. However, it is possible to approach the problem more directly using the theory of *simplicial sets*. We will assume that the reader has some familiarity with the theory of simplicial sets; a brief review of this theory is included in §A.2.7, and a more extensive introduction can be found in [32].

The theory of simplicial sets originated as a combinatorial approach to homotopy theory. Given any topological space X , one can associate a simplicial set $\text{Sing } X$, whose n -simplices are precisely the continuous maps $|\Delta^n| \rightarrow X$, where $|\Delta^n| = \{(x_0, \dots, x_n) \in [0, 1]^{n+1} \mid x_0 + \dots + x_n = 1\}$ is the standard n -simplex. Moreover, the topological space X is *determined*, up to weak homotopy equivalence, by $\text{Sing } X$. More precisely, the singular complex functor $X \mapsto \text{Sing } X$ admits a left adjoint, which carries every simplicial set K to its *geometric realization* $|K|$. For every topological space X , the counit map $|\text{Sing } X| \rightarrow X$ is a weak homotopy equivalence. Consequently, if one is only interested in studying topological spaces up to weak homotopy equivalence, one might as well work with simplicial sets instead.

If X is a topological space, then the simplicial set $\text{Sing } X$ has an important property, which is captured by the following definition:

Definition 1.1.2.1. Let K be a simplicial set. We say that K is a *Kan complex* if, for any $0 \leq i \leq n$ and any diagram of solid arrows

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

there exists a dotted arrow as indicated rendering the diagram commutative. Here $\Lambda_i^n \subseteq \Delta^n$ denotes the i th horn, obtained from the simplex Δ^n by deleting the interior and the face opposite the i th vertex.

The singular complex of any topological space X is a Kan complex: this follows from the fact that the horn $|\Lambda_i^n|$ is a retract of the simplex $|\Delta^n|$ in the category of topological spaces. Conversely, any Kan complex K “behaves like” a space: for example, there are simple combinatorial recipes for

extracting homotopy groups from K (which turn out to be isomorphic to the homotopy groups of the topological space $|K|$). According to a theorem of Quillen (see [32] for a proof), the singular complex and geometric realization provide mutually inverse equivalences between the homotopy category of CW complexes and the homotopy category of Kan complexes.

The formalism of simplicial sets is also closely related to category theory. To any category \mathcal{C} , we can associate a simplicial set $N(\mathcal{C})$ called the *nerve* of \mathcal{C} . For each $n \geq 0$, we let $N(\mathcal{C})_n = \text{Map}_{\text{Set}_\Delta}(\Delta^n, N(\mathcal{C}))$ denote the set of all functors $[n] \rightarrow \mathcal{C}$. Here $[n]$ denotes the linearly ordered set $\{0, \dots, n\}$, regarded as a category in the obvious way. More concretely, $N(\mathcal{C})_n$ is the set of all composable sequences of morphisms

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} C_n$$

having length n . In this description, the face map d_i carries the above sequence to

$$C_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} C_n$$

while the degeneracy s_i carries it to

$$C_0 \xrightarrow{f_1} \dots \xrightarrow{f_i} C_i \xrightarrow{\text{id}_{C_i}} C_i \xrightarrow{f_{i+1}} C_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} C_n.$$

It is more or less clear from this description that the simplicial set $N(\mathcal{C})$ is just a fancy way of encoding the structure of \mathcal{C} as a category. More precisely, we note that the category \mathcal{C} can be recovered (up to isomorphism) from its nerve $N(\mathcal{C})$. The objects of \mathcal{C} are simply the *vertices* of $N(\mathcal{C})$: that is, the elements of $N(\mathcal{C})_0$. A morphism from C_0 to C_1 is given by an edge $\phi \in N(\mathcal{C})_1$ with $d_1(\phi) = C_0$ and $d_0(\phi) = C_1$. The identity morphism from an object C to itself is given by the degenerate simplex $s_0(C)$. Finally, given a diagram $C_0 \xrightarrow{\phi} C_1 \xrightarrow{\psi} C_2$, the edge of $N(\mathcal{C})$ corresponding to $\psi \circ \phi$ may be uniquely characterized by the fact that there exists a 2-simplex $\sigma \in N(\mathcal{C})_2$ with $d_2(\sigma) = \phi$, $d_0(\sigma) = \psi$, and $d_1(\sigma) = \psi \circ \phi$.

It is not difficult to characterize those simplicial sets which arise as the nerve of a category:

Proposition 1.1.2.2. *Let K be a simplicial set. Then the following conditions are equivalent:*

- (1) *There exists a small category \mathcal{C} and an isomorphism $K \simeq N(\mathcal{C})$.*
- (2) *For each $0 < i < n$ and each diagram*

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & K \\ \downarrow & \nearrow & \uparrow \\ \Delta^n & & \end{array}$$

there exists a unique dotted arrow rendering the diagram commutative.

Proof. We first show that (1) \Rightarrow (2). Let K be the nerve of a small category \mathcal{C} and let $f_0 : \Lambda_i^n \rightarrow K$ be a map of simplicial sets, where $0 < i < n$. We wish to show that f_0 can be extended uniquely to a map $f : \Delta^n \rightarrow K$. For $0 \leq k \leq n$, let $X_k \in \mathcal{C}$ be the image of the vertex $\{k\} \subseteq \Lambda_i^n$. For $0 < k \leq n$, let $g_k : X_{k-1} \rightarrow X_k$ be the morphism in \mathcal{C} determined by the restriction $f_0|_{\Delta^{\{k-1,k\}}}$. The composable chain of morphisms

$$X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \dots \xrightarrow{g_n} X_n$$

determines an n -simplex $f : \Delta^n \rightarrow K$. We will show that f is the desired solution to our extension problem (the uniqueness of this solution is evident: if $f' : \Delta^n \rightarrow K$ is any other map with $f'|_{\Lambda_i^n} = f_0$, then f' must correspond to the same chain of morphisms in \mathcal{C} , so that $f' = f$). It will suffice to prove the following for every $0 \leq j \leq n$:

($*_j$) If $j \neq i$, then

$$f|_{\Delta^{\{0,\dots,j-1,j+1,\dots,n\}}} = f_0|_{\Delta^{\{0,\dots,j-1,j+1,\dots,n\}}}.$$

To prove ($*_j$), it will suffice to show that f and f_0 have the same restriction to $\Delta^{\{k,k'\}}$, where k and k' are adjacent elements of the linearly ordered set $\{0, \dots, j-1, j+1, \dots, n\} \subseteq [n]$. If k and k' are adjacent in $[n]$, then this follows by construction. In particular, ($*$) is automatically satisfied if $j = 0$ or $j = n$. Suppose instead that $k = j-1$ and $k' = j+1$, where $0 < j < n$. If $n = 2$, then $j = 1 = i$ and we obtain a contradiction. We may therefore assume that $n > 2$, so that either $j-1 > 0$ or $j+1 < n$. Without loss of generality, $j-1 > 0$, so that $\Delta^{\{j-1,j+1\}} \subseteq \Delta^{\{1,\dots,n\}}$. The desired conclusion now follows from ($*_0$).

We now prove the converse. Suppose that the simplicial set K satisfies (2); we claim that K is isomorphic to the nerve of a small category \mathcal{C} . We construct the category \mathcal{C} as follows:

- (i) The objects of \mathcal{C} are the vertices of K .
- (ii) Given a pair of objects $x, y \in \mathcal{C}$, we let $\text{Hom}_{\mathcal{C}}(x, y)$ denote the collection of all edges $e : \Delta^1 \rightarrow K$ such that $e|_{\{0\}} = x$ and $e|_{\{1\}} = y$.
- (iii) Let x be an object of \mathcal{C} . Then the identity morphism id_x is the edge of K defined by the composition

$$\Delta^1 \rightarrow \Delta^0 \xrightarrow{e} K.$$

- (iv) Let $f : x \rightarrow y$ and $g : y \rightarrow z$ be morphisms in \mathcal{C} . Then f and g together determine a map $\sigma_0 : \Lambda_1^2 \rightarrow K$. In view of condition (2), the map σ_0 can be extended uniquely to a 2-simplex $\sigma : \Delta^2 \rightarrow K$. We define the composition $g \circ f$ to be the morphism from x to z in \mathcal{C} corresponding to the edge given by the composition

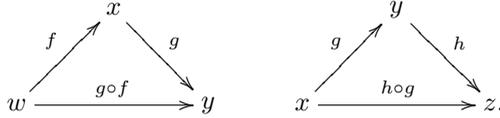
$$\Delta^1 \simeq \Delta^{\{0,2\}} \subseteq \Delta^2 \xrightarrow{\sigma} K.$$

We first claim that \mathcal{C} is a category. To prove this, we must verify the following axioms:

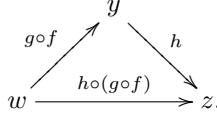
- (a) For every object $y \in \mathcal{C}$, the identity id_y is a unit with respect to composition. In other words, for every morphism $f : x \rightarrow y$ in \mathcal{C} and every morphism $g : y \rightarrow z$ in \mathcal{C} , we have $\text{id}_y \circ f = f$ and $g \circ \text{id}_y = g$. These equations are “witnessed” by the 2-simplices $s_1(f), s_0(g) \in \text{Hom}_{\text{Set}_\Delta}(\Delta^2, K)$.
- (b) Composition is associative. That is, for every sequence of composable morphisms

$$w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z,$$

we have $h \circ (g \circ f) = (h \circ g) \circ f$. To prove this, let us first choose 2-simplices σ_{012} and σ_{123} as indicated below:



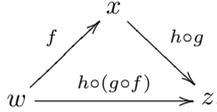
Now choose a 2-simplex σ_{023} corresponding to a diagram



These three 2-simplices together define a map $\tau_0 : \Lambda_2^3 \rightarrow K$. Since K satisfies condition (2), we can extend τ_0 to a 3-simplex $\tau : \Delta^3 \rightarrow K$. The composition

$$\Delta^2 \simeq \Delta^{\{0,1,3\}} \subseteq \Delta^3 \xrightarrow{\tau} K$$

corresponds to the diagram



which witnesses the associativity axiom $h \circ (g \circ f) = (h \circ g) \circ f$.

It follows that \mathcal{C} is a well-defined category. By construction, we have a canonical map of simplicial sets $\phi : K \rightarrow N\mathcal{C}$. To complete the proof, it will suffice to show that ϕ is an isomorphism. We will prove, by induction on $n \geq 0$, that ϕ induces a bijection $\text{Hom}_{\text{Set}_\Delta}(\Delta^n, K) \rightarrow \text{Hom}_{\text{Set}_\Delta}(\Delta^n, N\mathcal{C})$. For $n = 0$ and $n = 1$, this is obvious from the construction. Assume therefore that $n \geq 2$ and choose an integer i such that $0 < i < n$. We have a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\text{Set}_\Delta}(\Delta^n, K) & \longrightarrow & \text{Hom}_{\text{Set}_\Delta}(\Delta^n, N\mathcal{C}) \\ \downarrow & & \downarrow \\ \text{Hom}_{\text{Set}_\Delta}(\Lambda_i^n, K) & \longrightarrow & \text{Hom}_{\text{Set}_\Delta}(\Lambda_i^n, N\mathcal{C}). \end{array}$$

Since K and \mathbf{NC} both satisfy (2) (for \mathbf{NC} , this follows from the first part of the proof), the vertical maps are bijective. It will therefore suffice to show that the lower horizontal map is bijective, which follows from the inductive hypothesis. \square

We note that condition (2) of Proposition 1.1.2.2 is very similar to Definition 1.1.2.1. However, it is different in two important respects. First, it requires the extension condition only for *inner* horns Λ_i^n with $0 < i < n$. Second, the asserted condition is stronger in this case: not only does any map $\Lambda_i^n \rightarrow K$ extend to the simplex Δ^n , but the extension is unique.

Remark 1.1.2.3. It is easy to see that it is not reasonable to expect condition (2) of Proposition 1.1.2.2 to hold for *outer* horns Λ_i^n where $i \in \{0, n\}$. Consider, for example, the case where $i = n = 2$ and where K is the nerve of a category \mathcal{C} . Giving a map $\Lambda_2^2 \rightarrow K$ corresponds to supplying the solid arrows in the diagram

$$\begin{array}{ccc} & C_1 & \\ \text{---} \nearrow & & \searrow \text{---} \\ C_0 & \xrightarrow{\quad} & C_2, \end{array}$$

and the extension condition would amount to the assertion that one could always find a dotted arrow rendering the diagram commutative. This is true in general only when the category \mathcal{C} is a *groupoid*.

We now see that the notion of a simplicial set is a flexible one: a simplicial set K can be a good model for an ∞ -groupoid (if K is a Kan complex) or for an ordinary category (if it satisfies the hypotheses of Proposition 1.1.2.2). Based on these observations, we might expect that some more general class of simplicial sets could serve as models for ∞ -categories in general.

Consider first an arbitrary simplicial set K . We can try to envision K as a generalized category whose objects are the vertices of K (that is, the elements of K_0) and whose morphisms are the edges of K (that is, the elements of K_1). A 2-simplex $\sigma : \Delta^2 \rightarrow K$ should be thought of as a diagram

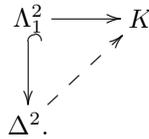
$$\begin{array}{ccc} & Y & \\ \phi \nearrow & & \searrow \psi \\ X & \xrightarrow{\quad \theta \quad} & Z \end{array}$$

together with an identification (or homotopy) between θ and $\psi \circ \phi$ which witnesses the “commutativity” of the diagram. (In higher category theory, commutativity is not merely a condition: the homotopy $\theta \simeq \psi \circ \phi$ is an additional datum.) Simplices of larger dimension may be thought of as verifying the commutativity of certain higher-dimensional diagrams.

Unfortunately, for a general simplicial set K , the analogy outlined above is not very strong. The essence of the problem is that, though we may refer to the 1-simplices of K as morphisms, there is in general no way to compose

them. Taking our cue from the example of $N(\mathcal{C})$, we might say that a morphism $\theta : X \rightarrow Z$ is a composition of morphisms $\phi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ if there exists a 2-simplex $\sigma : \Delta^2 \rightarrow K$ as in the diagram indicated above. We must now consider two potential difficulties: the desired 2-simplex σ may not exist, and if it does exist it may not be unique, so that we have more than one choice for the composition θ .

The existence requirement for σ can be formulated as an extension condition on the simplicial set K . We note that a composable pair of morphisms (ψ, ϕ) determines a map of simplicial sets $\Lambda_1^2 \rightarrow K$. Thus, the assertion that σ can always be found may be formulated as an extension property: any map of simplicial sets $\Lambda_1^2 \rightarrow K$ can be extended to Δ^2 , as indicated in the following diagram:



The uniqueness of θ is another matter. It turns out to be unnecessary (and unnatural) to require that θ be uniquely determined. To understand this point, let us return to the example of the fundamental groupoid of a topological space X . This is a category whose objects are the points $x \in X$. The morphisms between a point $x \in X$ and a point $y \in X$ are given by continuous paths $p : [0, 1] \rightarrow X$ such that $p(0) = x$ and $p(1) = y$. Two such paths are considered to be equivalent if there is a homotopy between them. Composition in the fundamental groupoid is given by concatenation of paths. Given paths $p, q : [0, 1] \rightarrow X$ with $p(0) = x$, $p(1) = q(0) = y$, and $q(1) = z$, the composite of p and q should be a path joining x to z . There are many ways of obtaining such a path from p and q . One of the simplest is to define

$$r(t) = \begin{cases} p(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ q(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

However, we could just as well use the formula

$$r'(t) = \begin{cases} p(3t) & \text{if } 0 \leq t \leq \frac{1}{3} \\ q\left(\frac{3t-1}{2}\right) & \text{if } \frac{1}{3} \leq t \leq 1 \end{cases}$$

to define the composite path. Because the paths r and r' are homotopic to one another, it does not matter which one we choose.

The situation becomes more complicated if we try to think 2-categorically. We can capture more information about the space X by considering its *fundamental 2-groupoid*. This is a 2-category whose objects are the points of X , whose morphisms are paths between points, and whose 2-morphisms are given by homotopies between paths (which are themselves considered modulo homotopy). In order to have composition of morphisms unambiguously defined, we would have to choose some formula once and for all. Moreover,

there is no particularly compelling choice; for example, neither of the formulas written above leads to a strictly associative composition law.

The lesson to learn from this is that in higher-categorical situations, we should not necessarily ask for a uniquely determined composition of two morphisms. In the fundamental groupoid example, there are many choices for a composite path, but all of them are homotopic to one another. Moreover, in keeping with the philosophy of higher category theory, *any* path which is homotopic to the composite should be just as good as the composite itself. From this point of view, it is perhaps more natural to view composition as a relation than as a function, and this is very efficiently encoded in the formalism of simplicial sets: a 2-simplex $\sigma : \Delta^2 \rightarrow K$ should be viewed as “evidence” that $d_0(\sigma) \circ d_2(\sigma)$ is homotopic to $d_1(\sigma)$.

Exactly what conditions on a simplicial set K will guarantee that it behaves like a higher category? Based on the above argument, it seems reasonable to require that K satisfy an extension condition with respect to certain horn inclusions Λ_i^n , as in Definition 1.1.2.1. However, as we observed in Remark 1.1.2.3, this is reasonable only for the inner horns where $0 < i < n$, which appear in the statement of Proposition 1.1.2.2.

Definition 1.1.2.4. An ∞ -category is a simplicial set K which has the following property: for any $0 < i < n$, any map $f_0 : \Lambda_i^n \rightarrow K$ admits an extension $f : \Delta^n \rightarrow K$.

Definition 1.1.2.4 was first formulated by Boardman and Vogt ([10]). They referred to ∞ -categories as *weak Kan complexes*, motivated by the obvious analogy with Definition 1.1.2.1. Our terminology places more emphasis on the analogy with the characterization of ordinary categories given in Proposition 1.1.2.2: we require the same extension conditions but drop the uniqueness assumption.

Example 1.1.2.5. Any Kan complex is an ∞ -category. In particular, if X is a topological space, then we may view its singular complex $\text{Sing } X$ as an ∞ -category: this is one way of defining the fundamental ∞ -groupoid $\pi_{\leq \infty} X$ of X introduced informally in Example 1.1.1.4.

Example 1.1.2.6. The nerve of any category is an ∞ -category. We will occasionally abuse terminology by identifying a category \mathcal{C} with its nerve $\mathbf{N}(\mathcal{C})$; by means of this identification, we may view ordinary category theory as a special case of the study of ∞ -categories.

The weak Kan condition of Definition 1.1.2.4 leads to a very elegant and powerful version of higher category theory. This theory has been developed by Joyal in [43] and [44] (where simplicial sets satisfying the condition of Definition 1.1.2.4 are called *quasi-categories*) and will be used throughout this book.

Notation 1.1.2.7. Depending on the context, we will use two different notations in connection with simplicial sets. When emphasizing their role as

∞ -categories, we will often denote them by calligraphic letters such as \mathcal{C} , \mathcal{D} , and so forth. When casting simplicial sets in their different (though related) role as representatives of homotopy types, we will employ capital Roman letters. To avoid confusion, we will also employ the latter notation when we wish to contrast the theory of ∞ -categories with some other approach to higher category theory, such as the theory of topological categories.

1.1.3 Equivalences of Topological Categories

We have now introduced two approaches to higher category theory: one based on topological categories and one based on simplicial sets. These two approaches turn out to be equivalent to one another. However, the equivalence itself needs to be understood in a higher-categorical sense. We take our cue from classical homotopy theory, in which we can take the basic objects to be either topological spaces or simplicial sets. It is not true that every Kan complex is isomorphic to the singular complex of a topological space or that every CW complex is homeomorphic to the geometric realization of a simplicial set. However, both of these statements become true if we replace the words “isomorphic to” by “homotopy equivalent to.” We would like to formulate a similar statement regarding our approaches to higher category theory. The first step is to find a concept which replaces homotopy equivalence. If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor between topological categories, under what circumstances should we regard F as an equivalence (so that \mathcal{C} and \mathcal{D} really represent the same higher category)?

The most naive answer is that F should be regarded as an equivalence if it is an isomorphism of topological categories. This means that F induces a bijection between the objects of \mathcal{C} and the objects of \mathcal{D} , and a homeomorphism $\text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{\mathcal{D}}(F(X), F(Y))$ for every pair of objects $X, Y \in \mathcal{C}$. However, it is immediately obvious that this condition is far too strong; for example, in the case where \mathcal{C} and \mathcal{D} are ordinary categories (which we may view also as topological categories where all morphism sets are endowed with the discrete topology), we recover the notion of an isomorphism between categories. This notion does not play an important role in category theory. One rarely asks whether or not two categories are isomorphic; instead, one asks whether or not they are equivalent. This suggests the following definition:

Definition 1.1.3.1. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between topological categories is a *strong equivalence* if it is an equivalence in the sense of enriched category theory. In other words, F is a strong equivalence if it induces homeomorphisms $\text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{\mathcal{D}}(F(X), F(Y))$ for every pair of objects $X, Y \in \mathcal{C}$, and every object of \mathcal{D} is isomorphic (in \mathcal{D}) to $F(X)$ for some $X \in \mathcal{C}$.

The notion of strong equivalence between topological categories has the virtue that, when restricted to ordinary categories, it reduces to the usual notion of equivalence. However, it is still not the right definition: for a pair

of objects X and Y of a higher category \mathcal{C} , the morphism space $\text{Map}_{\mathcal{C}}(X, Y)$ should itself be well-defined only up to homotopy equivalence.

Definition 1.1.3.2. Let \mathcal{C} be a topological category. The *homotopy category* $\text{h}\mathcal{C}$ is defined as follows:

- The objects of $\text{h}\mathcal{C}$ are the objects of \mathcal{C} .
- If $X, Y \in \mathcal{C}$, then we define $\text{Hom}_{\text{h}\mathcal{C}}(X, Y) = \pi_0 \text{Map}_{\mathcal{C}}(X, Y)$.
- Composition of morphisms in $\text{h}\mathcal{C}$ is induced from the composition of morphisms in \mathcal{C} by applying the functor π_0 .

Example 1.1.3.3. Let \mathcal{C} be the topological category whose objects are CW complexes, where $\text{Map}_{\mathcal{C}}(X, Y)$ is the set of continuous maps from X to Y , equipped with the (compactly generated version of the) compact-open topology. We will denote the homotopy category of \mathcal{C} by \mathcal{H} and refer to \mathcal{H} as the *homotopy category of spaces*.

There is a second construction of the homotopy category \mathcal{H} which will play an important role in what follows. First, we must recall a bit of terminology from classical homotopy theory.

Definition 1.1.3.4. A map $f : X \rightarrow Y$ between topological spaces is said to be a *weak homotopy equivalence* if it induces a bijection $\pi_0 X \rightarrow \pi_0 Y$, and if for every point $x \in X$ and every $i \geq 1$, the induced map of homotopy groups

$$\pi_i(X, x) \rightarrow \pi_i(Y, f(x))$$

is an isomorphism.

Given a space $X \in \mathcal{CG}$, classical homotopy theory ensures the existence of a CW complex X' equipped with a weak homotopy equivalence $\phi : X' \rightarrow X$. Of course, X' is not uniquely determined; however, it is unique up to canonical homotopy equivalence, so that the assignment

$$X \mapsto [X] = X'$$

determines a functor $\theta : \mathcal{CG} \rightarrow \mathcal{H}$. By construction, θ carries weak homotopy equivalences in \mathcal{CG} to isomorphisms in \mathcal{H} . In fact, θ is universal with respect to this property. In other words, we may describe \mathcal{H} as the category obtained from \mathcal{CG} by formally inverting all weak homotopy equivalences. This is one version of Whitehead's theorem, which is usually stated as follows: every weak homotopy equivalence between CW complexes admits a homotopy inverse.

We can now improve upon Definition 1.1.3.2 slightly. We first observe that the functor $\theta : \mathcal{CG} \rightarrow \mathcal{H}$ preserves products. Consequently, we can apply the construction of Remark A.1.4.3 to convert any topological category \mathcal{C} into a category enriched over \mathcal{H} . We will denote this \mathcal{H} -enriched category by $\text{h}\mathcal{C}$ and refer to it as the *homotopy category* of \mathcal{C} . More concretely, the homotopy category $\text{h}\mathcal{C}$ may be described as follows:

- (1) The objects of $\mathbf{h}\mathcal{C}$ are the objects of \mathcal{C} .
- (2) For $X, Y \in \mathcal{C}$, we have

$$\mathrm{Map}_{\mathbf{h}\mathcal{C}}(X, Y) = [\mathrm{Map}_{\mathcal{C}}(X, Y)].$$

- (3) The composition law on $\mathbf{h}\mathcal{C}$ is obtained from the composition law on \mathcal{C} by applying the functor $\theta : \mathcal{C}\mathcal{G} \rightarrow \mathcal{H}$.

Remark 1.1.3.5. If \mathcal{C} is a topological category, we have now defined $\mathbf{h}\mathcal{C}$ in two different ways: first as an ordinary category and later as a category enriched over \mathcal{H} . These two definitions are compatible with one another in the sense that $\mathbf{h}\mathcal{C}$ (regarded as an ordinary category) is the underlying category of $\mathbf{h}\mathcal{C}$ (regarded as an \mathcal{H} -enriched category). This follows immediately from the observation that for every topological space X , there is a canonical bijection $\pi_0 X \simeq \mathrm{Map}_{\mathcal{J}\mathcal{C}}(*, [X])$.

If \mathcal{C} is a topological category, we may imagine that $\mathbf{h}\mathcal{C}$ is the object which is obtained by forgetting the topological morphism spaces of \mathcal{C} and remembering only their (weak) homotopy types. The following definition codifies the idea that these homotopy types should be “all that really matter.”

Definition 1.1.3.6. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between topological categories. We will say that F is a *weak equivalence*, or simply an *equivalence*, if the induced functor $\mathbf{h}\mathcal{C} \rightarrow \mathbf{h}\mathcal{D}$ is an equivalence of \mathcal{H} -enriched categories.

More concretely, a functor F is an equivalence if and only if the following conditions are satisfied:

- For every pair of objects $X, Y \in \mathcal{C}$, the induced map

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \rightarrow \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$$

is a weak homotopy equivalence of topological spaces.

- Every object of \mathcal{D} is isomorphic in $\mathbf{h}\mathcal{D}$ to $F(X)$ for some $X \in \mathcal{C}$.

Remark 1.1.3.7. A morphism $f : X \rightarrow Y$ in \mathcal{D} is said to be an *equivalence* if the induced morphism in $\mathbf{h}\mathcal{D}$ is an isomorphism. In general, this is much weaker than the condition that f be an isomorphism in \mathcal{D} ; see Proposition 1.2.4.1.

It is Definition 1.1.3.6 which gives the correct notion of equivalence between topological categories (at least, when one is using them to describe higher category theory). We will agree that all relevant properties of topological categories are invariant under this notion of equivalence. We say that two topological categories are *equivalent* if there is an equivalence between them, or more generally if there is a chain of equivalences joining them. Equivalent topological categories should be regarded as interchangeable for all relevant purposes.

Remark 1.1.3.8. According to Definition 1.1.3.6, a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence if and only if the induced functor $\mathrm{h}\mathcal{C} \rightarrow \mathrm{h}\mathcal{D}$ is an equivalence. In other words, the homotopy category $\mathrm{h}\mathcal{C}$ (regarded as a category which is enriched over \mathcal{H}) is an invariant of \mathcal{C} which is sufficiently powerful to detect equivalences between ∞ -categories. This should be regarded as analogous to the more classical fact that the homotopy groups $\pi_i(X, x)$ of a CW complex X are homotopy invariants which detect homotopy equivalences between CW complexes (by Whitehead's theorem). However, it is important to remember that $\mathrm{h}\mathcal{C}$ does not determine \mathcal{C} up to equivalence, just as the homotopy type of a CW complex is not determined by its homotopy groups.

1.1.4 Simplicial Categories

In the previous sections we introduced two very different approaches to the foundations of higher category theory: one based on topological categories, the other on simplicial sets. In order to prove that they are equivalent to one another, we will introduce a third approach which is closely related to the first but shares the combinatorial flavor of the second.

Definition 1.1.4.1. A *simplicial category* is a category which is enriched over the category Set_Δ of simplicial sets. The category of simplicial categories (where morphisms are given by simplicially enriched functors) will be denoted by Cat_Δ .

Remark 1.1.4.2. Every simplicial category can be regarded as a simplicial object in the category Cat . Conversely, a simplicial object of Cat arises from a simplicial category if and only if the underlying simplicial set of objects is constant.

Like topological categories, simplicial categories can be used as models of higher category theory. If \mathcal{C} is a simplicial category, then we will generally think of the simplicial sets $\mathrm{Map}_{\mathcal{C}}(X, Y)$ as encoding homotopy types or ∞ -groupoids.

Remark 1.1.4.3. If \mathcal{C} is a simplicial category with the property that each of the simplicial sets $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is an ∞ -category, then we may view \mathcal{C} itself as a kind of ∞ -bicategory. We will not use this interpretation of simplicial categories in this book. Usually we will consider only *fibrant* simplicial categories; that is, simplicial categories for which the mapping objects $\mathrm{Map}_{\mathcal{C}}(X, Y)$ are Kan complexes.

The relationship between simplicial categories and topological categories is easy to describe. Let Set_Δ denote the category of simplicial sets and \mathcal{CG} the category of compactly generated Hausdorff spaces. We recall that there exists a pair of adjoint functors

$$\mathrm{Set}_\Delta \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\mathrm{Sing}} \\ \xrightarrow{\quad} \end{array} \mathcal{CG}$$

which are called the *geometric realization* and *singular complex* functors, respectively. Both of these functors commute with finite products. Consequently, if \mathcal{C} is a simplicial category, we may define a topological category $|\mathcal{C}|$ in the following way:

- The objects of $|\mathcal{C}|$ are the objects of \mathcal{C} .
- If $X, Y \in \mathcal{C}$, then $\text{Map}_{|\mathcal{C}|}(X, Y) = |\text{Map}_{\mathcal{C}}(X, Y)|$.
- The composition law for morphisms in $|\mathcal{C}|$ is obtained from the composition law on \mathcal{C} by applying the geometric realization functor.

Similarly, if \mathcal{C} is a topological category, we may obtain a simplicial category $\text{Sing } \mathcal{C}$ by applying the singular complex functor to each of the morphism spaces individually. The singular complex and geometric realization functors determine an adjunction between Cat_{Δ} and Cat_{top} . This adjunction should be understood as determining an equivalence between the theory of simplicial categories and the theory of topological categories. This is essentially a formal consequence of the fact that the geometric realization and singular complex functors determine an equivalence between the homotopy theory of topological spaces and the homotopy theory of simplicial sets. More precisely, we recall that a map $f : S \rightarrow T$ of simplicial sets is said to be a *weak homotopy equivalence* if the induced map $|S| \rightarrow |T|$ of topological spaces is a weak homotopy equivalence. A theorem of Quillen (see [32] for a proof) asserts that the unit and counit morphisms

$$S \rightarrow \text{Sing } |S|$$

$$|\text{Sing } X| \rightarrow X$$

are weak homotopy equivalences for every (compactly generated) topological space X and every simplicial set S . It follows that the category obtained from $\mathcal{C}\mathcal{G}$ by inverting weak homotopy equivalences (of spaces) is equivalent to the category obtained from Set_{Δ} by inverting weak homotopy equivalences. We use the symbol \mathcal{H} to denote either of these (equivalent) categories.

If \mathcal{C} is a simplicial category, we let $\text{h}\mathcal{C}$ denote the \mathcal{H} -enriched category obtained by applying the functor $\text{Set}_{\Delta} \rightarrow \mathcal{H}$ to each of the morphism spaces of \mathcal{C} . We will refer to $\text{h}\mathcal{C}$ as the *homotopy category of \mathcal{C}* . We note that this is the same notation that was introduced in §1.1.3 for the homotopy category of a topological category. However, there is little risk of confusion: the above remarks imply the existence of canonical isomorphisms

$$\text{h}\mathcal{C} \simeq \text{h}|\mathcal{C}|$$

$$\text{h}\mathcal{D} \simeq \text{hSing } \mathcal{D}$$

for every simplicial category \mathcal{C} and every topological category \mathcal{D} .

Definition 1.1.4.4. A functor $\mathcal{C} \rightarrow \mathcal{C}'$ between simplicial categories is an *equivalence* if the induced functor $\text{h}\mathcal{C} \rightarrow \text{h}\mathcal{C}'$ is an equivalence of \mathcal{H} -enriched categories.

In other words, a functor $\mathcal{C} \rightarrow \mathcal{C}'$ between simplicial categories is an equivalence if and only if the geometric realization $|\mathcal{C}| \rightarrow |\mathcal{C}'|$ is an equivalence of topological categories. In fact, one can say more. It follows easily from the preceding remarks that the unit and counit maps

$$\begin{aligned} \mathcal{C} &\rightarrow \text{Sing}|\mathcal{C}| \\ |\text{Sing} \mathcal{D}| &\rightarrow \mathcal{D} \end{aligned}$$

induce *isomorphisms* between homotopy categories. Consequently, if we are working with topological or simplicial categories *up to equivalence*, we are always free to replace a simplicial category \mathcal{C} by $|\mathcal{C}|$ or a topological category \mathcal{D} by $\text{Sing} \mathcal{D}$. In this sense, the notions of topological category and simplicial category are equivalent, and either can be used as a foundation for higher category theory.

1.1.5 Comparing ∞ -Categories with Simplicial Categories

In §1.1.4, we introduced the theory of simplicial categories and explained why (for our purposes) it is equivalent to the theory of topological categories. In this section, we will show that the theory of simplicial categories is also closely related to the theory of ∞ -categories. Our discussion requires somewhat more elaborate constructions than were needed in the previous sections; a reader who does not wish to become bogged down in details is urged to skip ahead to §1.2.1.

We will relate simplicial categories with simplicial sets by means of the *simplicial nerve functor*

$$N : \text{Cat}_\Delta \rightarrow \text{Set}_\Delta,$$

originally introduced by Cordier (see [16]). The nerve of an ordinary category \mathcal{C} is characterized by the formula

$$\text{Hom}_{\text{Set}_\Delta}(\Delta^n, N(\mathcal{C})) = \text{Hom}_{\text{Cat}}([n], \mathcal{C});$$

here $[n]$ denotes the linearly ordered set $\{0, \dots, n\}$ regarded as a category. This definition makes sense also when \mathcal{C} is a simplicial category but is clearly not very interesting: it makes no use of the simplicial structure on \mathcal{C} . In order to obtain a more interesting construction, we need to replace the ordinary category $[n]$ by a suitable “thickening,” a simplicial category which we will denote by $\mathfrak{C}[\Delta^n]$.

Definition 1.1.5.1. Let J be a finite nonempty linearly ordered set. The simplicial category $\mathfrak{C}[\Delta^J]$ is defined as follows:

- The objects of $\mathfrak{C}[\Delta^J]$ are the elements of J .
- If $i, j \in J$, then

$$\text{Map}_{\mathfrak{C}[\Delta^J]}(i, j) = \begin{cases} \emptyset & \text{if } j < i \\ N(P_{i,j}) & \text{if } i \leq j. \end{cases}$$

Here $P_{i,j}$ denotes the partially ordered set $\{I \subseteq J : (i, j \in I) \wedge (\forall k \in I)[i \leq k \leq j]\}$.

- If $i_0 \leq i_1 \leq \dots \leq i_n$, then the composition

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0, i_1) \times \dots \times \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_{n-1}, i_n) \rightarrow \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0, i_n)$$

is induced by the map of partially ordered sets

$$P_{i_0, i_1} \times \dots \times P_{i_{n-1}, i_n} \rightarrow P_{i_0, i_n}$$

$$(I_1, \dots, I_n) \mapsto I_1 \cup \dots \cup I_n.$$

In order to help digest Definition 1.1.5.1, let us analyze the structure of the topological category $|\mathfrak{C}[\Delta^n]|$. The objects of this category are elements of the set $[n] = \{0, \dots, n\}$. For each $0 \leq i \leq j \leq n$, the topological space $\mathrm{Map}_{|\mathfrak{C}[\Delta^n]|}(i, j)$ is homeomorphic to a cube; it may be identified with the set of all functions $p : \{k \in [n] : i \leq k \leq j\} \rightarrow [0, 1]$ which satisfy $p(i) = p(j) = 1$. The morphism space $\mathrm{Map}_{|\mathfrak{C}[\Delta^n]|}(i, j)$ is empty when $j < i$, and composition of morphisms is given by concatenation of functions.

Remark 1.1.5.2. Let us try to understand better the simplicial category $\mathfrak{C}[\Delta^n]$ and its relationship to the ordinary category $[n]$. These categories have the same objects: the elements of $\{0, \dots, n\}$. In the category $[n]$, there is a unique morphism $q_{ij} : i \rightarrow j$ whenever $i \leq j$. By virtue of the uniqueness, these elements satisfy $q_{jk} \circ q_{ij} = q_{ik}$ for $i \leq j \leq k$.

In the simplicial category $\mathfrak{C}[\Delta^n]$, there is a vertex $p_{ij} \in \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j)$ for each $i \leq j$, given by the element $\{i, j\} \in P_{ij}$. We note that $p_{jk} \circ p_{ij} \neq p_{ik}$ (except in degenerate cases where $i = j$ or $j = k$). Instead, the collection of all compositions

$$p_{i_n i_{n-1}} \circ p_{i_{n-1} i_{n-2}} \circ \dots \circ p_{i_1 i_0},$$

where $i = i_0 < i_1 < \dots < i_{n-1} < i_n = j$ constitute all of the different vertices of the cube $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j)$. The weak contractibility of $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j)$ expresses the idea that although these compositions do not coincide, they are all canonically homotopic to one another. We observe that there is a (unique) functor $\mathfrak{C}[\Delta^n] \rightarrow [n]$ which is the identity on objects. This functor is an equivalence of simplicial categories. We can summarize the situation informally as follows: the simplicial category $\mathfrak{C}[\Delta^n]$ is a thickened version of $[n]$ where we have dropped the strict associativity condition

$$q_{jk} \circ q_{ij} = q_{ik}$$

and instead have imposed associativity only up to (coherent) homotopy. (We can formulate this idea more precisely by saying that $\mathfrak{C}[\Delta^\bullet]$ is a cofibrant replacement for $[\bullet]$ with respect to a suitable model structure on the category of cosimplicial objects of Cat_Δ .)

The construction $J \mapsto \mathfrak{C}[\Delta^J]$ is functorial in J , as we now explain.

Definition 1.1.5.3. Let $f : J \rightarrow J'$ be a monotone map between linearly ordered sets. The simplicial functor $\mathfrak{C}[f] : \mathfrak{C}[\Delta^J] \rightarrow \mathfrak{C}[\Delta^{J'}]$ is defined as follows:

- For each object $i \in \mathfrak{C}[\Delta^J]$, $\mathfrak{C}[f](i) = f(i) \in \mathfrak{C}[\Delta^{J'}]$.
- If $i \leq j$ in J , then the map $\text{Map}_{\mathfrak{C}[\Delta^J]}(i, j) \rightarrow \text{Map}_{\mathfrak{C}[\Delta^{J'}]}(f(i), f(j))$ induced by f is the nerve of the map

$$P_{i,j} \rightarrow P_{f(i),f(j)}$$

$$I \mapsto f(I).$$

Remark 1.1.5.4. Using the notation of Remark 1.1.5.2, we note that Definition 1.1.5.3 has been rigged so that the functor $\mathfrak{C}[f]$ carries the vertex $p_{ij} \in \text{Map}_{\mathfrak{C}[\Delta^J]}(i, j)$ to the vertex $p_{f(i)f(j)} \in \text{Map}_{\mathfrak{C}[\Delta^{J'}]}(f(i), f(j))$.

It is not difficult to check that the construction described in Definition 1.1.5.3 is well-defined, and compatible with composition in f . Consequently, we deduce that \mathfrak{C} determines a functor

$$\Delta \rightarrow \text{Cat}_\Delta$$

$$\Delta^n \mapsto \mathfrak{C}[\Delta^n],$$

which we may view as a cosimplicial object of Cat_Δ .

Definition 1.1.5.5. Let \mathcal{C} be a simplicial category. The *simplicial nerve* $N(\mathcal{C})$ is the simplicial set described by the formula

$$\text{Hom}_{\text{Set}_\Delta}(\Delta^n, N(\mathcal{C})) = \text{Hom}_{\text{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

If \mathcal{C} is a topological category, we define the *topological nerve* $N(\mathcal{C})$ of \mathcal{C} to be the simplicial nerve of $\text{Sing } \mathcal{C}$.

Remark 1.1.5.6. If \mathcal{C} is a simplicial (topological) category, we will often abuse terminology by referring to the simplicial (topological) nerve of \mathcal{C} simply as the *nerve* of \mathcal{C} .

Warning 1.1.5.7. Let \mathcal{C} be a simplicial category. Then \mathcal{C} can be regarded as an ordinary category by ignoring all simplices of positive dimension in the mapping spaces of \mathcal{C} . The simplicial nerve of \mathcal{C} does *not* coincide with the nerve of this underlying ordinary category. Our notation is therefore potentially ambiguous. We will adopt the following convention: whenever \mathcal{C} is a simplicial category, $N(\mathcal{C})$ will denote the *simplicial* nerve of \mathcal{C} unless we specify otherwise. Similarly, if \mathcal{C} is a topological category, then the topological nerve of \mathcal{C} does not generally coincide with the nerve of the underlying category; the notation $N(\mathcal{C})$ will be used to indicate the topological nerve unless otherwise specified.

Example 1.1.5.8. Any ordinary category \mathcal{C} may be considered as a simplicial category by taking each of the simplicial sets $\text{Hom}_{\mathcal{C}}(X, Y)$ to be *constant*. In this case, the set of simplicial functors $\mathfrak{C}[\Delta^n] \rightarrow \mathcal{C}$ may be identified with the set of functors from $[n]$ into \mathcal{C} . Consequently, the simplicial nerve of \mathcal{C} agrees with the ordinary nerve of \mathcal{C} as defined in §1.1.2. Similarly, the ordinary nerve of \mathcal{C} can be identified with the topological nerve of \mathcal{C} , where \mathcal{C} is regarded as a topological category with discrete morphism spaces.

In order to get a feel for what the nerve of a topological category \mathcal{C} looks like, let us explicitly describe its low-dimensional simplices:

- The 0-simplices of $N(\mathcal{C})$ may be identified with the objects of \mathcal{C} .
- The 1-simplices of $N(\mathcal{C})$ may be identified with the morphisms of \mathcal{C} .
- To give a map from the boundary of a 2-simplex into $N(\mathcal{C})$ is to give a diagram (not necessarily commutative)

$$\begin{array}{ccc}
 & Y & \\
 f_{XY} \nearrow & & \searrow f_{YZ} \\
 X & \xrightarrow{f_{XZ}} & Z.
 \end{array}$$

To give a 2-simplex of $N(\mathcal{C})$ having this specified boundary is equivalent to giving a path from f_{XZ} to $f_{YZ} \circ f_{XY}$ in $\text{Map}_{\mathcal{C}}(X, Z)$.

The category Cat_{Δ} of simplicial categories admits (small) colimits. Consequently, by formal nonsense, the functor $\mathfrak{C} : \Delta \rightarrow \text{Cat}_{\Delta}$ extends uniquely (up to unique isomorphism) to a colimit-preserving functor $\text{Set}_{\Delta} \rightarrow \text{Cat}_{\Delta}$, which we will denote also by \mathfrak{C} . By construction, the functor \mathfrak{C} is left adjoint to the simplicial nerve functor N . For each simplicial set S , we can view $\mathfrak{C}[S]$ as the simplicial category “freely generated” by S : every n -simplex $\sigma : \Delta^n \rightarrow S$ determines a functor $\mathfrak{C}[\Delta^n] \rightarrow \mathfrak{C}[S]$, which we can think of as a homotopy coherent diagram $[n] \rightarrow \mathfrak{C}[S]$.

Example 1.1.5.9. Let A be a partially ordered set. The simplicial category $\mathfrak{C}[N A]$ can be constructed using the following generalization of Definition 1.1.5.1:

- The objects of $\mathfrak{C}[N A]$ are the elements of A .
- Given a pair of elements $a, b \in A$, the simplicial set $\text{Map}_{\mathfrak{C}[N A]}(a, b)$ can be identified with $N P_{a,b}$, where $P_{a,b}$ denotes the collection of linearly ordered subsets $S \subseteq A$ with least element a and largest element b , partially ordered by inclusion.
- Given a sequence of elements $a_0, \dots, a_n \in A$, the composition map

$$\text{Map}_{\mathfrak{C}[N A]}(a_0, a_1) \times \cdots \times \text{Map}_{\mathfrak{C}[N A]}(a_{n-1}, a_n) \rightarrow \text{Map}_{\mathfrak{C}[N A]}(a_0, a_n)$$

is induced by the map of partially ordered sets

$$\begin{aligned}
 P_{a_0, a_1} \times \cdots \times P_{a_{n-1}, a_n} &\rightarrow P_{a_0, a_n} \\
 (S_1, \dots, S_n) &\mapsto S_1 \cup \cdots \cup S_n.
 \end{aligned}$$

Proposition 1.1.5.10. *Let \mathcal{C} be a simplicial category having the property that, for every pair of objects $X, Y \in \mathcal{C}$, the simplicial set $\text{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex. Then the simplicial nerve $N(\mathcal{C})$ is an ∞ -category.*

Proof. We must show that if $0 < i < n$, then $N(\mathcal{C})$ has the right extension property with respect to the inclusion $\Lambda_i^n \subseteq \Delta^n$. Rephrasing this in the language of simplicial categories, we must show that \mathcal{C} has the right extension property with respect to the simplicial functor $\mathfrak{C}[\Lambda_i^n] \rightarrow \mathfrak{C}[\Delta^n]$. To prove this, we make use of the following observations concerning $\mathfrak{C}[\Lambda_i^n]$, which we view as a simplicial subcategory of $\mathfrak{C}[\Delta^n]$:

- The objects of $\mathfrak{C}[\Lambda_i^n]$ are the objects of $\mathfrak{C}[\Delta^n]$: that is, elements of the set $[n]$.
- For $0 \leq j \leq k \leq n$, the simplicial set $\text{Map}_{\mathfrak{C}[\Lambda_i^n]}(j, k)$ coincides with $\text{Map}_{\mathfrak{C}[\Delta^n]}(j, k)$ unless $j = 0$ and $k = n$ (note that this condition fails if $i = 0$ or $i = n$).

Consequently, every extension problem

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{F} & N(\mathcal{C}) \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

is equivalent to

$$\begin{array}{ccc} \text{Map}_{\mathfrak{C}[\Lambda_i^n]}(0, n) & \longrightarrow & \text{Map}_{\mathcal{C}}(F(0), F(n)) \\ \downarrow & \nearrow & \\ \text{Map}_{\mathfrak{C}[\Delta^n]}(0, n) & & \end{array}$$

Since the simplicial set on the right is a Kan complex by assumption, it suffices to verify that the left vertical map is anodyne. This follows by inspection: the simplicial set $\text{Map}_{\mathfrak{C}[\Delta^n]}(0, n)$ can be identified with the cube $(\Delta^1)^{\{1, \dots, n-1\}}$. Under this identification, $\text{Map}_{\mathfrak{C}[\Lambda_i^n]}(0, n)$ corresponds to the simplicial subset of $(\Delta^1)^{\{1, \dots, n-1\}}$ obtained by removing the interior of the cube together with one of its faces. \square

Remark 1.1.5.11. The proof of Proposition 1.1.5.10 actually provides a slightly stronger result: if $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor between simplicial categories which induces Kan fibrations $\text{Map}_{\mathcal{C}}(C, C') \rightarrow \text{Map}_{\mathcal{D}}(F(C), F(C'))$ for every pair of objects $C, C' \in \mathcal{C}$, then the associated map $N(\mathcal{C}) \rightarrow N(\mathcal{D})$ is an inner fibration of simplicial sets (see Definition 2.0.0.3).

Corollary 1.1.5.12. *Let \mathcal{C} be a topological category. Then the topological nerve $N(\mathcal{C})$ is an ∞ -category.*

Proof. This follows immediately from Proposition 1.1.5.10 (note that the singular complex of any topological space is a Kan complex). \square

We now cite the following theorem, which will be proven in §2.2.4 and refined in §2.2.5:

Theorem 1.1.5.13. *Let \mathcal{C} be a topological category and let $X, Y \in \mathcal{C}$ be objects. Then the counit map*

$$|\mathrm{Map}_{\mathcal{C}[\mathbf{N}(\mathcal{C})]}(X, Y)| \rightarrow \mathrm{Map}_{\mathcal{C}}(X, Y)$$

is a weak homotopy equivalence of topological spaces.

Using Theorem 1.1.5.13, we can explain why the theory of ∞ -categories is equivalent to the theory of topological categories (or equivalently, simplicial categories). The adjoint functors \mathbf{N} and $|\mathcal{C}[\bullet]|$ are not mutually inverse equivalences of categories. However, they *are* homotopy inverse to one another. To make this precise, we need to introduce a definition.

Definition 1.1.5.14. Let S be a simplicial set. The *homotopy category* $\mathbf{h}S$ is defined to be the homotopy category $\mathbf{h}\mathcal{C}[S]$ of the simplicial category $\mathcal{C}[S]$. We will often view $\mathbf{h}S$ as a category enriched over the homotopy category \mathcal{H} of spaces via the construction of §1.1.4: that is, for every pair of vertices $x, y \in S$, we have $\mathrm{Map}_{\mathbf{h}S}(x, y) = [\mathrm{Map}_{\mathcal{C}[S]}(x, y)]$. A map $f : S \rightarrow T$ of simplicial sets is a *categorical equivalence* if the induced map $\mathbf{h}S \rightarrow \mathbf{h}T$ is an equivalence of \mathcal{H} -enriched categories.

Remark 1.1.5.15. In [44], Joyal uses the term “weak categorical equivalence” for what we have called a categorical equivalence, and reserves the term “categorical equivalence” for a stronger notion of equivalence.

Remark 1.1.5.16. We have introduced the term “categorical equivalence,” rather than simply “equivalence” or “weak equivalence,” in order to avoid confusing the notion of categorical equivalence of simplicial sets with the (more classical) notion of weak homotopy equivalence of simplicial sets.

Remark 1.1.5.17. It is immediate from the definition that $f : S \rightarrow T$ is a categorical equivalence if and only if $\mathcal{C}[S] \rightarrow \mathcal{C}[T]$ is an equivalence (of simplicial categories) if and only if $|\mathcal{C}[S]| \rightarrow |\mathcal{C}[T]|$ is an equivalence (of topological categories).

We now observe that the adjoint functors $(|\mathcal{C}[\bullet]|, \mathbf{N})$ determine an equivalence between the theory of simplicial sets (up to categorical equivalence) and that of topological categories (up to equivalence). In other words, for any topological category \mathcal{C} the counit map $|\mathcal{C}[\mathbf{N}(\mathcal{C})]| \rightarrow \mathcal{C}$ is an equivalence of topological categories, and for any simplicial set S the unit map $S \rightarrow \mathbf{N}|\mathcal{C}[S]|$ is a categorical equivalence of simplicial sets. In view of Remark 1.1.5.17, the second assertion is a formal consequence of the first. Moreover, the first assertion is merely a reformulation of Theorem 1.1.5.13.

Remark 1.1.5.18. The reader may at this point object that we have obtained a comparison between the theory of topological categories and the theory of simplicial sets but that not every simplicial set is an ∞ -category. However, every simplicial set is categorically equivalent to an ∞ -category. In fact, Theorem 1.1.5.13 implies that every simplicial set S is categorically equivalent to the nerve of the topological category $|\mathcal{C}[S]|$, which is an ∞ -category (Corollary 1.1.5.12).

1.2 THE LANGUAGE OF HIGHER CATEGORY THEORY

One of the main goals of this book is to demonstrate that many ideas from classical category theory can be adapted to the setting of higher categories. In this section, we will survey some of the simplest examples.

1.2.1 The Opposite of an ∞ -Category

If \mathcal{C} is an ordinary category, then the opposite category \mathcal{C}^{op} is defined in the following way:

- The objects of \mathcal{C}^{op} are the objects of \mathcal{C} .
- For $X, Y \in \mathcal{C}$, we have $\text{Hom}_{\mathcal{C}^{op}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$. Identity morphisms and composition are defined in the obvious way.

This definition generalizes without change to the setting of topological or simplicial categories. Adapting this definition to the setting of ∞ -categories requires a few additional words. We may define more generally the *opposite* of a simplicial set S as follows: for any finite nonempty linearly ordered set J , we set $S^{op}(J) = S(J^{op})$, where J^{op} denotes the same set J endowed with the opposite ordering. More concretely, we have $S_n^{op} = S_n$, but the face and degeneracy maps on S^{op} are given by the formulas

$$(d_i : S_n^{op} \rightarrow S_{n-1}^{op}) = (d_{n-i} : S_n \rightarrow S_{n-1})$$

$$(s_i : S_n^{op} \rightarrow S_{n+1}^{op}) = (s_{n-i} : S_n \rightarrow S_{n+1}).$$

The formation of opposite categories is fully compatible with all of the constructions we have introduced for passing back and forth between different models of higher category theory.

It is clear from the definition that a simplicial set S is an ∞ -category if and only if its opposite S^{op} is an ∞ -category: for $0 < i < n$, S has the extension property with respect to the horn inclusion $\Lambda_i^n \subseteq \Delta^n$ if and only if S^{op} has the extension property with respect to the horn inclusion $\Lambda_{n-i}^n \subseteq \Delta^n$.

1.2.2 Mapping Spaces in Higher Category Theory

If X and Y are objects of an ordinary category \mathcal{C} , then one has a well-defined set $\text{Hom}_{\mathcal{C}}(X, Y)$ of morphisms from X to Y . In higher category theory, one has instead a morphism *space* $\text{Map}_{\mathcal{C}}(X, Y)$. In the setting of topological or simplicial categories, this morphism space (either a topological space or a simplicial set) is an inherent feature of the formalism. It is less obvious how to define $\text{Map}_{\mathcal{C}}(X, Y)$ in the setting of ∞ -categories. However, it is at least clear what to do on the level of the homotopy category.

Definition 1.2.2.1. Let S be a simplicial set containing vertices x and y and let \mathcal{H} denote the homotopy category of spaces. We define $\text{Map}_S(x, y) =$

$\text{Map}_{\text{h}S}(x, y) \in \mathcal{H}$ to be the object of \mathcal{H} representing the space of maps from x to y in S . Here $\text{h}S$ denotes the homotopy category of S regarded as a \mathcal{H} -enriched category (Definition 1.1.5.14).

Warning 1.2.2.2. Let S be a simplicial set. The notation $\text{Map}_S(X, Y)$ has two *very* different meanings. When X and Y are vertices of S , then our notation should be interpreted in the sense of Definition 1.2.2.1, so that $\text{Map}_S(X, Y)$ is an object of \mathcal{H} . If X and Y are objects of $(\text{Set}_\Delta)_{/S}$, then we instead let $\text{Map}_S(X, Y)$ denote the simplicial mapping object

$$Y^X \times_{S^X} \{\phi\} \in \text{Set}_\Delta,$$

where ϕ denotes the structural morphism $X \rightarrow S$. We trust that it will be clear from the context which of these two definitions applies in a given situation.

We now consider the following question: given a simplicial set S containing a pair of vertices x and y , how can we compute $\text{Map}_S(x, y)$? We have defined $\text{Map}_S(x, y)$ as an object of the homotopy category \mathcal{H} , but for many purposes it is important to choose a simplicial set M which represents $\text{Map}_S(x, y)$. The most obvious candidate for M is the simplicial set $\text{Map}_{\mathcal{C}[S]}(x, y)$. The advantages of this definition are that it works in all cases (that is, S does not need to be an ∞ -category) and comes equipped with an associative composition law. However, the construction of the simplicial set $\text{Map}_{\mathcal{C}[S]}(x, y)$ is quite complicated. Furthermore, $\text{Map}_{\mathcal{C}[S]}(x, y)$ is usually not a Kan complex, so it can be difficult to extract algebraic invariants like homotopy groups even when a concrete description of its simplices is known.

In order to address these shortcomings, we will introduce another simplicial set which represents the homotopy type $\text{Map}_S(x, y) \in \mathcal{H}$, at least when S is an ∞ -category. We define a new simplicial set $\text{Hom}_S^R(x, y)$, the space of *right morphisms* from x to y , by letting $\text{Hom}_{\text{Set}_\Delta}(\Delta^n, \text{Hom}_S^R(x, y))$ denote the set of all $z : \Delta^{n+1} \rightarrow S$ such that $z|_{\Delta^{\{n+1\}}} = y$ and $z|_{\Delta^{\{0, \dots, n\}}}$ is a constant simplex at the vertex x . The face and degeneracy operations on $\text{Hom}_S^R(x, y)_n$ are defined to coincide with corresponding operations on S_{n+1} .

We first observe that when S is an ∞ -category, $\text{Hom}_S^R(x, y)$ really is a “space”:

Proposition 1.2.2.3. *Let \mathcal{C} be an ∞ -category containing a pair of objects x and y . The simplicial set $\text{Hom}_{\mathcal{C}}^R(x, y)$ is a Kan complex.*

Proof. It is immediate from the definition that if \mathcal{C} is a ∞ -category, then $M = \text{Hom}_{\mathcal{C}}^R(x, y)$ satisfies the Kan extension condition for every horn inclusion $\Lambda_i^n \subseteq \Delta^n$, where $0 < i \leq n$. This implies that M is a Kan complex (Proposition 1.2.5.1). \square

Remark 1.2.2.4. If S is a simplicial set and $x, y, z \in S_0$, then there is no obvious composition law

$$\text{Hom}_S^R(x, y) \times \text{Hom}_S^R(y, z) \rightarrow \text{Hom}_S^R(x, z).$$

We will later see that if S is an ∞ -category, then there is a composition law which is well-defined up to a contractible space of choices. The absence of a canonical choice for a composition law is the main drawback of $\mathrm{Hom}_S^{\mathrm{R}}(x, y)$ in comparison with $\mathrm{Map}_{\mathcal{C}[S]}(x, y)$. The main goal of §2.2 is to show that if S is an ∞ -category, then there is a (canonical) isomorphism between $\mathrm{Hom}_S^{\mathrm{R}}(x, y)$ and $\mathrm{Map}_{\mathcal{C}[S]}(x, y)$ in the homotopy category \mathcal{H} . In particular, we will conclude that $\mathrm{Hom}_S^{\mathrm{R}}(x, y)$ represents $\mathrm{Map}_S(x, y)$ whenever S is an ∞ -category.

Remark 1.2.2.5. The definition of $\mathrm{Hom}_S^{\mathrm{R}}(x, y)$ is not self-dual: that is, $\mathrm{Hom}_S^{\mathrm{R}}(x, y) \neq \mathrm{Hom}_S^{\mathrm{R}}(y, x)$ in general. Instead, we define $\mathrm{Hom}_S^{\mathrm{L}}(x, y) = \mathrm{Hom}_S^{\mathrm{R}}(y, x)^{\mathrm{op}}$, so that $\mathrm{Hom}_S^{\mathrm{L}}(x, y)_n$ is the set of all $z \in S_{n+1}$ such that $z|\Delta^{\{0\}} = x$ and $z|\Delta^{\{1, \dots, n+1\}}$ is the constant simplex at the vertex y .

Although the simplicial sets $\mathrm{Hom}_S^{\mathrm{L}}(x, y)$ and $\mathrm{Hom}_S^{\mathrm{R}}(x, y)$ are generally not isomorphic to one another, they are homotopy equivalent whenever S is an ∞ -category. To prove this, it is convenient to define a third, self-dual, space of morphisms: let $\mathrm{Hom}_S(x, y) = \{x\} \times_S S^{\Delta^1} \times_S \{y\}$. In other words, to give an n -simplex of $\mathrm{Hom}_S(x, y)$, one must give a map $f : \Delta^n \times \Delta^1 \rightarrow S$ such that $f|\Delta^n \times \{0\}$ is constant at x and $f|\Delta^n \times \{1\}$ is constant at y . We observe that there exist natural inclusions

$$\mathrm{Hom}_S^{\mathrm{R}}(x, y) \hookrightarrow \mathrm{Hom}_S(x, y) \hookrightarrow \mathrm{Hom}_S^{\mathrm{L}}(x, y),$$

which are induced by retracting the cylinder $\Delta^n \times \Delta^1$ onto certain maximal-dimensional simplices. We will later show (Corollary 4.2.1.8) that these inclusions are homotopy equivalences provided that S is an ∞ -category.

1.2.3 The Homotopy Category

For every ordinary category \mathcal{C} , the nerve $\mathrm{N}(\mathcal{C})$ is an ∞ -category. Informally, we can describe the situation as follows: the nerve functor is a fully faithful inclusion from the bicategory of categories to the ∞ -bicategory of ∞ -categories. Moreover, this inclusion has a left adjoint:

Proposition 1.2.3.1. *The nerve functor $\mathrm{Cat} \rightarrow \mathrm{Set}_\Delta$ is right adjoint to the functor $\mathrm{h} : \mathrm{Set}_\Delta \rightarrow \mathrm{Cat}$, which associates to every simplicial set S its homotopy category $\mathrm{h}S$ (here we ignore the \mathcal{H} -enrichment of $\mathrm{h}S$).*

Proof. Let us temporarily distinguish between the nerve functor $\mathrm{N} : \mathrm{Cat} \rightarrow \mathrm{Set}_\Delta$ and the simplicial nerve functor $\mathrm{N}' : \mathrm{Cat}_\Delta \rightarrow \mathrm{Set}_\Delta$. These two functors are related by the fact that N can be written as a composition

$$\mathrm{Cat} \subseteq \mathrm{Cat}_\Delta \xrightarrow{\mathrm{N}'} \mathrm{Set}_\Delta.$$

The functor $\pi_0 : \mathrm{Set}_\Delta \rightarrow \mathrm{Set}$ is a left adjoint to the inclusion functor $\mathrm{Set} \rightarrow \mathrm{Set}_\Delta$, so the functor

$$\mathrm{Cat}_\Delta \rightarrow \mathrm{Cat}$$

$$\mathcal{C} \mapsto \mathrm{h}\mathcal{C}$$

is left adjoint to i . It follows that $N = N' \circ i$ has a left adjoint, given by the composition

$$\text{Set}_\Delta \xrightarrow{\mathcal{C}[\bullet]} \text{Cat}_\Delta \xrightarrow{h} \text{Cat},$$

which coincides with the homotopy category functor $h : \text{Set}_\Delta \rightarrow \text{Cat}$ by definition. \square

Remark 1.2.3.2. The formation of the homotopy category is literally left adjoint to the inclusion $\text{Cat} \subseteq \text{Cat}_\Delta$. The analogous assertion is not quite true in the setting of topological categories because the functor $\pi_0 : \mathcal{CG} \rightarrow \text{Set}$ is a left adjoint only when restricted to locally path-connected spaces.

Warning 1.2.3.3. If \mathcal{C} is a simplicial category, then we do not necessarily expect that $h\mathcal{C} \simeq hN(\mathcal{C})$. However, this is always the case when \mathcal{C} is *fibrant* in the sense that every simplicial set $\text{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex.

Remark 1.2.3.4. If S is a simplicial set, Joyal ([44]) refers to the category hS as the *fundamental category* of S . This is motivated by the observation that if S is a Kan complex, then hS is the fundamental groupoid of S in the usual sense.

Our objective for the remainder of this section is to obtain a more explicit understanding of the homotopy category hS of a simplicial set S . Proposition 1.2.3.1 implies that hS admits the following presentation by generators and relations:

- The objects of hS are the vertices of S .
- For every edge $\phi : \Delta^1 \rightarrow S$, there is a morphism $\overline{\phi}$ from $\phi(0)$ to $\phi(1)$.
- For each $\sigma : \Delta^2 \rightarrow S$, we have $\overline{d_0(\sigma)} \circ \overline{d_2(\sigma)} = \overline{d_1(\sigma)}$.
- For each vertex x of S , the morphism $\overline{s_0 x}$ is the identity id_x .

If S is an ∞ -category, there is a much more satisfying construction of the category hS . We will describe this construction in detail since it nicely illustrates the utility of the weak Kan condition of Definition 1.1.2.4.

Let \mathcal{C} be an ∞ -category. We will construct a category $\pi(\mathcal{C})$ (which we will eventually show to be equivalent to the homotopy category $h\mathcal{C}$). The objects of $\pi(\mathcal{C})$ are the vertices of \mathcal{C} . Given an edge $\phi : \Delta^1 \rightarrow \mathcal{C}$, we shall say that ϕ has *source* $C = \phi(0)$ and *target* $C' = \phi(1)$ and write $\phi : C \rightarrow C'$. For each object C of \mathcal{C} , we let id_C denote the degenerate edge $s_0(C) : C \rightarrow C$.

Let $\phi : C \rightarrow C'$ and $\phi' : C \rightarrow C'$ be a pair of edges of \mathcal{C} having the same source and target. We will say that ϕ and ϕ' are *homotopic* if there is a 2-simplex $\sigma : \Delta^2 \rightarrow \mathcal{C}$, which we depict as follows:

$$\begin{array}{ccc} & C' & \\ \phi \nearrow & & \searrow \text{id}_{C'} \\ C & \xrightarrow{\phi'} & C' \end{array}$$

In this case, we say that σ is a *homotopy* between ϕ and ϕ' .

Proposition 1.2.3.5. *Let \mathcal{C} be an ∞ -category and let C and C' be objects of $\pi(\mathcal{C})$. Then the relation of homotopy is an equivalence relation on the edges joining C to C' .*

Proof. Let $\phi : \Delta^1 \rightarrow \mathcal{C}$ be an edge. Then $s_1(\phi)$ is a homotopy from ϕ to itself. Thus homotopy is a reflexive relation.

Suppose next that $\phi, \phi', \phi'' : C \rightarrow C'$ are edges with the same source and target. Let σ be a homotopy from ϕ to ϕ' , and σ' a homotopy from ϕ to ϕ'' . Let $\sigma'' : \Delta^2 \rightarrow \mathcal{C}$ denote the constant map at the vertex C' . We have a commutative diagram

$$\begin{array}{ccc} \Lambda_3^3 & \xrightarrow{(\sigma'', \bullet, \sigma', \sigma)} & \mathcal{C} \\ \downarrow & \dashrightarrow \tau & \\ \Delta^3 & & \end{array}$$

Since \mathcal{C} is an ∞ -category, there exists a 3-simplex $\tau : \Delta^3 \rightarrow \mathcal{C}$ as indicated by the dotted arrow in the diagram. It is easy to see that $d_1(\tau)$ is a homotopy from ϕ' to ϕ'' .

As a special case, we can take $\phi = \phi''$; we then deduce that the relation of homotopy is symmetric. It then follows immediately from the above that the relation of homotopy is also transitive. \square

Remark 1.2.3.6. The definition of homotopy that we have given is not evidently self-dual; in other words, it is not immediately obvious that a homotopic pair of edges $\phi, \phi' : C \rightarrow C'$ of an ∞ -category \mathcal{C} remain homotopic when regarded as edges in the opposite ∞ -category \mathcal{C}^{op} . To prove this, let σ be a homotopy from ϕ to ϕ' and consider the commutative diagram

$$\begin{array}{ccc} \Lambda_2^3 & \xrightarrow{(\sigma, s_1\phi, \bullet, s_0\phi)} & \mathcal{C} \\ \downarrow & \dashrightarrow \tau & \\ \Delta^3 & & \end{array}$$

The assumption that \mathcal{C} is an ∞ -category guarantees a 3-simplex τ rendering the diagram commutative. The face $d_2\tau$ may be regarded as a homotopy from ϕ' to ϕ in \mathcal{C}^{op} .

We can now define the morphism sets of the category $\pi(\mathcal{C})$: given vertices X and Y of \mathcal{C} , we let $\text{Hom}_{\pi(\mathcal{C})}(X, Y)$ denote the set of homotopy classes of edges $\phi : X \rightarrow Y$ in \mathcal{C} . For each edge $\phi : \Delta^1 \rightarrow \mathcal{C}$, we let $[\phi]$ denote the corresponding morphism in $\pi(\mathcal{C})$.

We define a composition law on $\pi(\mathcal{C})$ as follows. Suppose that X, Y , and Z are vertices of \mathcal{C} and that we are given edges $\phi : X \rightarrow Y$, $\psi : Y \rightarrow Z$. The pair (ϕ, ψ) determines a map $\Lambda_1^2 \rightarrow \mathcal{C}$. Since \mathcal{C} is an ∞ -category, this map extends to a 2-simplex $\sigma : \Delta^2 \rightarrow \mathcal{C}$. We now define $[\psi] \circ [\phi] = [d_1\sigma]$.

Proposition 1.2.3.7. *Let \mathcal{C} be an ∞ -category. The composition law on $\pi(\mathcal{C})$ is well-defined. In other words, the homotopy class $[\psi] \circ [\phi]$ does not depend on the choice of ψ representing $[\psi]$, the choice of ϕ representing $[\phi]$, or the choice of the 2-simplex σ .*

Proof. We begin by verifying the independence of the choice of σ . Suppose that we are given two 2-simplices $\sigma, \sigma' : \Delta^2 \rightarrow \mathcal{C}$, satisfying

$$d_0\sigma = d_0\sigma' = \psi$$

$$d_2\sigma = d_2\sigma' = \phi.$$

Consider the diagram

$$\begin{array}{ccc} \Lambda^3 & \xrightarrow{(s_1\psi, \bullet, \sigma', \sigma)} & \mathcal{C} \\ \downarrow & \dashrightarrow \tau & \\ \Delta^3 & & \end{array}$$

Since \mathcal{C} is an ∞ -category, there exists a 3-simplex τ as indicated by the dotted arrow. It follows that $d_1\tau$ is a homotopy from $d_1\sigma$ to $d_1\sigma'$.

We now show that $[\psi] \circ [\phi]$ depends only on ψ and ϕ only up to homotopy. In view of Remark 1.2.3.6, the assertion is symmetric with respect to ψ and ϕ ; it will therefore suffice to show that $[\psi] \circ [\phi]$ does not change if we replace ϕ by a morphism ϕ' which is homotopic to ϕ . Let σ be a 2-simplex with $d_0\sigma = \psi$ and $d_2\sigma = \phi$, and let σ' be a homotopy from ϕ to ϕ' . Consider the diagram

$$\begin{array}{ccc} \Lambda^3 & \xrightarrow{(s_0\psi, \bullet, \sigma, \sigma')} & \mathcal{C} \\ \downarrow & \dashrightarrow \tau & \\ \Delta^3 & & \end{array}$$

Again, the hypothesis that \mathcal{C} is an ∞ -category guarantees the existence of a 3-simplex τ as indicated in the diagram. Let $\sigma'' = d_1\tau$. Then $[\psi] \circ [\phi'] = [d_1\sigma'']$. But $d_1\sigma = d_1\sigma''$ by construction, so that $[\psi] \circ [\phi] = [\psi] \circ [\phi']$, as desired. \square

Proposition 1.2.3.8. *If \mathcal{C} is an ∞ -category, then $\pi(\mathcal{C})$ is a category.*

Proof. Let C be a vertex of \mathcal{C} . We first verify that $[\text{id}_C]$ is an identity with respect to the composition law on $\pi(\mathcal{C})$. For every edge $\phi : C' \rightarrow C$ in \mathcal{C} , the 2-simplex $s_1(\phi)$ verifies the equation

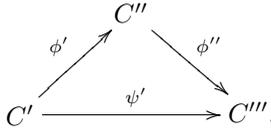
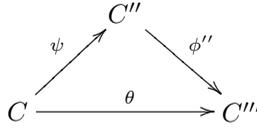
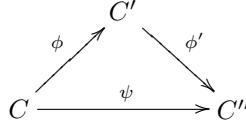
$$[\text{id}_C] \circ [\phi] = [\phi].$$

This proves that id_C is a left identity; the dual argument (Remark 1.2.3.6) shows that $[\text{id}_C]$ is a right identity.

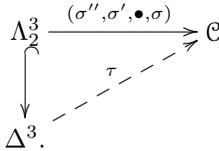
The only other thing we need to check is the associative law for composition in $\pi(\mathcal{C})$. Suppose we are given a composable sequence of edges

$$C \xrightarrow{\phi} C' \xrightarrow{\phi'} C'' \xrightarrow{\phi''} C'''.$$

Choose 2-simplices $\sigma, \sigma', \sigma'' : \Delta^2 \rightarrow \mathcal{C}$ corresponding to diagrams



respectively. Then $[\phi'] \circ [\phi] = [\psi]$, $[\phi'''] \circ [\psi] = [\theta]$, and $[\phi'''] \circ [\phi'] = [\psi']$. Consider the diagram



Since \mathcal{C} is an ∞ -category, there exists a 3-simplex τ rendering the diagram commutative. Then $d_2(\tau)$ verifies the equation $[\psi'] \circ [\phi] = [\theta]$, so that

$$([\phi'''] \circ [\phi']) \circ [\phi] = [\theta] = [\phi'''] \circ [\psi] = [\phi'''] \circ ([\phi'] \circ [\phi]),$$

as desired. □

We now show that if \mathcal{C} is an ∞ -category, then $\pi(\mathcal{C})$ is naturally equivalent (in fact, isomorphic) to $\mathbf{h}\mathcal{C}$.

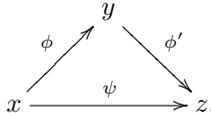
Proposition 1.2.3.9. *Let \mathcal{C} be an ∞ -category. There exists a unique functor $F : \mathbf{h}\mathcal{C} \rightarrow \pi(\mathcal{C})$ with the following properties:*

- (1) *On objects, F is the identity map.*
- (2) *For every edge ϕ of \mathcal{C} , $F(\overline{\phi}) = [\phi]$.*

Moreover, F is an isomorphism of categories.

Proof. The existence and uniqueness of F follows immediately from our presentation of $\mathbf{h}\mathcal{C}$ by generators and relations. It is obvious that F is bijective on objects and surjective on morphisms. To complete the proof, it will suffice to show that F is faithful.

We first show that every morphism $f : x \rightarrow y$ in $\mathbf{h}\mathcal{C}$ may be written as $\overline{\phi}$ for some $\phi \in \mathcal{C}$. Since the morphisms in $\mathbf{h}\mathcal{C}$ are generated by morphisms having the form $\overline{\phi}$ under composition, it suffices to show that the set of such morphisms contains all identity morphisms and is stable under composition. The first assertion is clear since $\overline{s_0 x} = \text{id}_x$. For the second, we note that if $\phi : x \rightarrow y$ and $\phi' : y \rightarrow z$ are composable edges, then there exists a 2-simplex $\sigma : \Delta^2 \rightarrow \mathcal{C}$ which we may depict as follows:



Thus $\overline{\phi'} \circ \overline{\phi} = \overline{\psi}$.

Now suppose that $\phi, \phi' : x \rightarrow y$ are such that $[\phi] = [\phi']$; we wish to show that $\overline{\phi} = \overline{\phi'}$. By definition, there exists a homotopy $\sigma : \Delta^2 \rightarrow \mathcal{C}$ joining ϕ and ϕ' . The existence of σ entails the relation

$$\text{id}_y \circ \overline{\phi} = \overline{\phi'}$$

in the homotopy category $\mathbf{h}\mathcal{S}$, so that $\overline{\phi} = \overline{\phi'}$, as desired. \square

1.2.4 Objects, Morphisms, and Equivalences

As in ordinary category theory, we may speak of *objects* and *morphisms* in a higher category \mathcal{C} . If \mathcal{C} is a topological (or simplicial) category, these should be understood literally as the objects and morphisms in the underlying category of \mathcal{C} . We may also apply this terminology to ∞ -categories (or even more general simplicial sets): if S is a simplicial set, then the *objects* of S are the vertices $\Delta^0 \rightarrow S$, and the *morphisms* of S are the edges $\Delta^1 \rightarrow S$. A morphism $\phi : \Delta^1 \rightarrow S$ is said to have *source* $X = \phi(0)$ and *target* $Y = \phi(1)$; we will often denote this by writing $\phi : X \rightarrow Y$. If $X : \Delta^0 \rightarrow S$ is an object of S , we will write $\text{id}_X = s_0(X) : X \rightarrow X$ and refer to this as the *identity morphism* of X .

If $f, g : X \rightarrow Y$ are two morphisms in a higher category \mathcal{C} , then f and g are *homotopic* if they determine the same morphism in the homotopy category $\mathbf{h}\mathcal{C}$. In the setting of ∞ -categories, this coincides with the notion of homotopy introduced in the previous section. In the setting of topological categories, this simply means that f and g lie in the same path component of $\text{Map}_{\mathcal{C}}(X, Y)$. In either case, we will sometimes indicate this relationship between f and g by writing $f \simeq g$.

A morphism $f : X \rightarrow Y$ in an ∞ -category \mathcal{C} is said to be an *equivalence* if it determines an isomorphism in the homotopy category $\mathbf{h}\mathcal{C}$. We say that X and Y are *equivalent* if there is an equivalence between them (in other words, if they are isomorphic as objects of $\mathbf{h}\mathcal{C}$).

If \mathcal{C} is a topological category, then the requirement that a morphism $f : X \rightarrow Y$ be an equivalence is quite a bit weaker than the requirement that f be an isomorphism. In fact, we have the following:

Proposition 1.2.4.1. *Let $f : X \rightarrow Y$ be a morphism in a topological category. The following conditions are equivalent:*

- (1) *The morphism f is an equivalence.*
- (2) *The morphism f has a homotopy inverse $g : Y \rightarrow X$: that is, a morphism g such that $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$.*
- (3) *For every object $Z \in \mathcal{C}$, the induced map $\text{Map}_{\mathcal{C}}(Z, X) \rightarrow \text{Map}_{\mathcal{C}}(Z, Y)$ is a homotopy equivalence.*
- (4) *For every object $Z \in \mathcal{C}$, the induced map $\text{Map}_{\mathcal{C}}(Z, X) \rightarrow \text{Map}_{\mathcal{C}}(Z, Y)$ is a weak homotopy equivalence.*
- (5) *For every object $Z \in \mathcal{C}$, the induced map $\text{Map}_{\mathcal{C}}(Y, Z) \rightarrow \text{Map}_{\mathcal{C}}(X, Z)$ is a homotopy equivalence.*
- (6) *For every object $Z \in \mathcal{C}$, the induced map $\text{Map}_{\mathcal{C}}(Y, Z) \rightarrow \text{Map}_{\mathcal{C}}(X, Z)$ is a weak homotopy equivalence.*

Proof. It is clear that (2) is merely a reformulation of (1). We will show that (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1); the implications (2) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1) follow using the same argument.

To see that (2) implies (3), we note that if g is a homotopy inverse to f , then composition with g gives a map $\text{Map}_{\mathcal{C}}(Z, Y) \rightarrow \text{Map}_{\mathcal{C}}(Z, X)$ which is homotopy inverse to composition with f . It is clear that (3) implies (4). Finally, if (4) holds, then we note that X and Y represent the same functor on $\text{h}\mathcal{C}$ so that f induces an isomorphism between X and Y in $\text{h}\mathcal{C}$. \square

Example 1.2.4.2. Let \mathcal{C} be the category of CW complexes which we regard as a topological category by endowing each of the sets $\text{Hom}_{\mathcal{C}}(X, Y)$ with the (compactly generated) compact open topology. A pair of objects $X, Y \in \mathcal{C}$ are equivalent (in the sense defined above) if and only if they are homotopy equivalent (in the sense of classical topology).

If \mathcal{C} is an ∞ -category (topological category, simplicial category), then we shall write $X \in \mathcal{C}$ to mean that X is an object of \mathcal{C} . We will generally understand that all meaningful properties of objects are invariant under equivalence. Similarly, all meaningful properties of morphisms are invariant under homotopy and under composition with equivalences.

In the setting of ∞ -categories, there is a very useful characterization of equivalences which is due to Joyal.

Proposition 1.2.4.3 (Joyal [44]). *Let \mathcal{C} be an ∞ -category and $\phi : \Delta^1 \rightarrow \mathcal{C}$ a morphism of \mathcal{C} . Then ϕ is an equivalence if and only if, for every $n \geq 2$ and every map $f_0 : \Lambda_0^n \rightarrow \mathcal{C}$ such that $f_0|_{\Delta^{\{0,1\}}} = \phi$, there exists an extension of f_0 to Δ^n .*

The proof requires some ideas which we have not yet introduced and will be given in §2.1.2.

1.2.5 ∞ -Groupoids and Classical Homotopy Theory

Let \mathcal{C} be an ∞ -category. We will say that \mathcal{C} is an ∞ -groupoid if the homotopy category $\mathrm{h}\mathcal{C}$ is a groupoid: in other words, if every morphism in \mathcal{C} is an equivalence. In §1.1.1, we asserted that the theory of ∞ -groupoids is equivalent to classical homotopy theory. We can now formulate this idea in a very precise way:

Proposition 1.2.5.1 (Joyal [43]). *Let \mathcal{C} be a simplicial set. The following conditions are equivalent:*

- (1) *The simplicial set \mathcal{C} is an ∞ -category, and its homotopy category $\mathrm{h}\mathcal{C}$ is a groupoid.*
- (2) *The simplicial set \mathcal{C} satisfies the extension condition for all horn inclusions $\Lambda_i^n \subseteq \Delta^n$ for $0 \leq i < n$.*
- (3) *The simplicial set \mathcal{C} satisfies the extension condition for all horn inclusions $\Lambda_i^n \subseteq \Delta^n$ for $0 < i \leq n$.*
- (4) *The simplicial set \mathcal{C} is a Kan complex; in other words, it satisfies the extension condition for all horn inclusions $\Lambda_i^n \subseteq \Delta^n$ for $0 \leq i \leq n$.*

Proof. The equivalence (1) \Leftrightarrow (2) follows immediately from Proposition 1.2.4.3. Similarly, the equivalence (1) \Leftrightarrow (3) follows by applying Proposition 1.2.4.3 to \mathcal{C}^{op} . We conclude by observing that (4) \Leftrightarrow (2) \wedge (3). \square

Remark 1.2.5.2. The assertion that we can identify ∞ -groupoids with spaces is less obvious in other formulations of higher category theory. For example, suppose that \mathcal{C} is a topological category whose homotopy category $\mathrm{h}\mathcal{C}$ is a groupoid. For simplicity, we will assume furthermore that \mathcal{C} has a single object X . We may then identify \mathcal{C} with the topological monoid $M = \mathrm{Hom}_{\mathcal{C}}(X, X)$. The assumption that $\mathrm{h}\mathcal{C}$ is a groupoid is equivalent to the assumption that the discrete monoid $\pi_0 M$ is a group. In this case, one can show that the unit map $M \rightarrow \Omega BM$ is a weak homotopy equivalence, where BM denotes the classifying space of the topological monoid M . In other words, up to equivalence, specifying \mathcal{C} (together with the object X) is equivalent to specifying the space BM (together with its base point).

Informally, we might say that the inclusion functor i from Kan complexes to ∞ -categories exhibits the ∞ -category of (small) ∞ -groupoids as a full subcategory of the ∞ -bicategory of (small) ∞ -categories. Conversely, every ∞ -category \mathcal{C} has an “underlying” ∞ -groupoid, which is obtained by discarding the noninvertible morphisms of \mathcal{C} :

Proposition 1.2.5.3 ([44]). *Let \mathcal{C} be an ∞ -category. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the largest simplicial subset of \mathcal{C} having the property that every edge of \mathcal{C}' is an equivalence in \mathcal{C} . Then \mathcal{C}' is a Kan complex. It may be characterized by the following universal property: for any Kan complex K , the induced map $\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathcal{C}') \rightarrow \mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathcal{C})$ is a bijection.*

Proof. It is straightforward to check that \mathcal{C}' is an ∞ -category. Moreover, if f is a morphism in \mathcal{C}' , then f has a homotopy inverse $g \in \mathcal{C}$. Since g is itself an equivalence in \mathcal{C} , we conclude that g belongs to \mathcal{C}' and is therefore a homotopy inverse to f in \mathcal{C}' . In other words, every morphism in \mathcal{C}' is an equivalence, so that \mathcal{C}' is a Kan complex by Proposition 1.2.5.1. To prove the last assertion, we observe that if K is an ∞ -category, then any map of simplicial sets $\phi : K \rightarrow \mathcal{C}$ carries equivalences in K to equivalences in \mathcal{C} . In particular, if K is a Kan complex, then ϕ factors (uniquely) through \mathcal{C}' . \square

We can describe the situation of Proposition 1.2.5.3 by saying that \mathcal{C}' is the largest Kan complex contained in \mathcal{C} . The functor $\mathcal{C} \mapsto \mathcal{C}'$ is right adjoint to the inclusion functor from Kan complexes to ∞ -categories. It is easy to see that this right adjoint is an invariant notion: that is, a categorical equivalence of ∞ -categories $\mathcal{C} \rightarrow \mathcal{D}$ induces a homotopy equivalence $\mathcal{C}' \rightarrow \mathcal{D}'$ of Kan complexes.

Remark 1.2.5.4. It is easy to give analogous constructions in the case of topological or simplicial categories. For example, if \mathcal{C} is a topological category, then we can define \mathcal{C}' to be another topological category with the same objects as \mathcal{C} , where $\mathrm{Map}_{\mathcal{C}'}(X, Y) \subseteq \mathrm{Map}_{\mathcal{C}}(X, Y)$ is the subspace consisting of equivalences in $\mathrm{Map}_{\mathcal{C}}(X, Y)$, equipped with the subspace topology.

Remark 1.2.5.5. We will later introduce a relative version of the construction described in Proposition 1.2.5.3, which applies to certain families of ∞ -categories (Corollary 2.4.2.5).

Although the inclusion functor from Kan complexes to ∞ -categories does not literally have a left adjoint, it does have such an in a higher-categorical sense. This left adjoint is computed by any “fibrant replacement” functor (for the usual model structure) from Set_Δ to itself, for example, the functor $S \mapsto \mathrm{Sing}|S|$. The unit map $u : S \rightarrow \mathrm{Sing}|S|$ is always a weak homotopy equivalence but generally not a categorical equivalence. For example, if S is an ∞ -category, then u is a categorical equivalence if and only if S is a Kan complex. In general, $\mathrm{Sing}|S|$ may be regarded as the ∞ -groupoid obtained from S by freely adjoining inverses to all the morphisms in S .

Remark 1.2.5.6. The inclusion functor i and its homotopy-theoretic left adjoint may also be understood using the formalism of *localizations of model categories*. In addition to its usual model category structure, the category Set_Δ of simplicial sets may be endowed with the *Joyal model structure*, which we will define in §2.2.5. These model structures have the same cofibrations (in both cases, the cofibrations are simply the monomorphisms of simplicial sets). However, the Joyal model structure has fewer weak equivalences

(categorical equivalences rather than weak homotopy equivalences) and consequently more fibrant objects (all ∞ -categories rather than only Kan complexes). It follows that the usual homotopy theory of simplicial sets is a localization of the homotopy theory of ∞ -categories. The identity functor from Set_Δ to itself determines a Quillen adjunction between these two homotopy theories, which plays the role of i and its left adjoint.

1.2.6 Homotopy Commutativity versus Homotopy Coherence

Let \mathcal{C} be an ∞ -category (topological category, simplicial category). To a first approximation, working in \mathcal{C} is like working in its homotopy category $\text{h}\mathcal{C}$: up to equivalence, \mathcal{C} and $\text{h}\mathcal{C}$ have the same objects and morphisms. The main difference between $\text{h}\mathcal{C}$ and \mathcal{C} is that in \mathcal{C} one must not ask whether or not morphisms are *equal*; instead one should ask whether or not they are *homotopic*. If so, the homotopy itself is an additional datum which we will need to consider. Consequently, the notion of a commutative diagram in $\text{h}\mathcal{C}$, which corresponds to a *homotopy commutative* diagram in \mathcal{C} , is quite unnatural and usually needs to be replaced by the more refined notion of a *homotopy coherent* diagram in \mathcal{C} .

To understand the problem, let us suppose that $F : \mathcal{J} \rightarrow \mathcal{H}$ is a functor from an ordinary category \mathcal{J} into the homotopy category of spaces \mathcal{H} . In other words, F assigns to each object $X \in \mathcal{J}$ a space (say, a CW complex) $F(X)$, and to each morphism $\phi : X \rightarrow Y$ in \mathcal{J} a continuous map of spaces $F(\phi) : F(X) \rightarrow F(Y)$ (well-defined up to homotopy), such that $F(\phi \circ \psi)$ is homotopic to $F(\phi) \circ F(\psi)$ for any pair of composable morphisms ϕ, ψ in \mathcal{J} . In this situation, it may or may not be possible to *lift* F to an actual functor \tilde{F} from \mathcal{J} to the ordinary category of topological spaces such that \tilde{F} induces a functor $\mathcal{J} \rightarrow \mathcal{H}$ which is naturally isomorphic to F . In general, there are obstructions to both the existence and the uniqueness of the lifting \tilde{F} , even up to homotopy. To see this, let us suppose for a moment that \tilde{F} exists, so that there exist homotopies $k_\phi : \tilde{F}(\phi) \simeq F(\phi)$. These homotopies determine *additional* data on F : namely, one obtains a canonical homotopy $h_{\phi, \psi}$ from $F(\phi \circ \psi)$ to $F(\phi) \circ F(\psi)$ by composing

$$F(\phi \circ \psi) \simeq \tilde{F}(\phi \circ \psi) = \tilde{F}(\phi) \circ \tilde{F}(\psi) \simeq F(\phi) \circ F(\psi).$$

The functor F to the homotopy category \mathcal{H} should be viewed as a first approximation to \tilde{F} ; we obtain a second approximation when we take into account the homotopies $h_{\phi, \psi}$. These homotopies are not arbitrary: the associativity of composition gives a relationship between $h_{\phi, \psi}$, $h_{\psi, \theta}$, $h_{\phi, \psi \circ \theta}$, and $h_{\phi \circ \psi, \theta}$, for a composable triple of morphisms (ϕ, ψ, θ) in \mathcal{J} . This relationship may be formulated in terms of the existence of a certain higher homotopy, which is once again canonically determined by \tilde{F} (and the homotopies k_ϕ). To obtain the next approximation to \tilde{F} , we should take these higher homotopies into account and formulate the associativity properties that *they* enjoy, and so on. Roughly speaking, a *homotopy coherent* diagram in \mathcal{C} is a functor $F : \mathcal{J} \rightarrow \text{h}\mathcal{C}$ together with all of the extra data that would be

available if we were able to lift F to a functor $\tilde{F} : \mathcal{J} \rightarrow \mathcal{C}$.

The distinction between homotopy commutativity and homotopy coherence is arguably the *main* difficulty in working with higher categories. The idea of homotopy coherence is simple enough and can be made precise in the setting of a general topological category. However, the amount of data required to specify a homotopy coherent diagram is considerable, so the concept is quite difficult to employ in practical situations.

Remark 1.2.6.1. Let \mathcal{J} be an ordinary category and let \mathcal{C} be a topological category. Any functor $F : \mathcal{J} \rightarrow \mathcal{C}$ determines a homotopy coherent diagram in \mathcal{C} (with all of the homotopies involved being constant). For many topological categories \mathcal{C} , the converse fails: not every homotopy-coherent diagram in \mathcal{C} can be obtained in this way, even up to equivalence. In these cases, it is the notion of *homotopy coherent* diagram which is fundamental; a homotopy coherent diagram should be regarded as “just as good” as a strictly commutative diagram for ∞ -categorical purposes. As evidence for this, we remark that given an equivalence $\mathcal{C}' \rightarrow \mathcal{C}$, a strictly commutative diagram $F : \mathcal{J} \rightarrow \mathcal{C}$ cannot always be lifted to a strictly commutative diagram in \mathcal{C}' ; however, it can always be lifted (up to equivalence) to a homotopy coherent diagram in \mathcal{C}' .

One of the advantages of working with ∞ -categories is that the definition of a homotopy coherent diagram is easy to formulate. We can simply define a homotopy coherent diagram in an ∞ -category \mathcal{C} to be a map of simplicial sets $f : N(\mathcal{J}) \rightarrow \mathcal{C}$. The restriction of f to simplices of low dimension encodes the induced map on homotopy categories. Specifying f on higher-dimensional simplices gives precisely the “coherence data” that the above discussion calls for.

Remark 1.2.6.2. Another possible approach to the problem of homotopy coherence is to restrict our attention to simplicial (or topological) categories \mathcal{C} in which every homotopy coherent diagram is equivalent to a strictly commutative diagram. For example, this is always true when \mathcal{C} arises from a simplicial model category (Proposition 4.2.4.4). Consequently, in the framework of model categories, it is possible to ignore the theory of homotopy coherent diagrams and work with strictly commutative diagrams instead. This approach is quite powerful, particularly when combined with the observation that every simplicial category \mathcal{C} admits a fully faithful embedding into a simplicial model category (for example, one can use a simplicially enriched version of the Yoneda embedding). This idea can be used to show that every homotopy coherent diagram in \mathcal{C} can be “straightened” to a commutative diagram, possibly after replacing \mathcal{C} by an equivalent simplicial category (for a more precise version of this statement, we refer the reader to Corollary 4.2.4.7).

1.2.7 Functors Between Higher Categories

The notion of a homotopy coherent diagram in an higher category \mathcal{C} is a special case of the more general notion of a functor $F : \mathcal{J} \rightarrow \mathcal{C}$ between higher categories (specifically, it is the special case in which \mathcal{J} is assumed to be an ordinary category). Just as the collection of all ordinary categories forms a bicategory (with functors as morphisms and natural transformations as 2-morphisms), the collection of all ∞ -categories can be organized into an ∞ -bicategory. In particular, for any ∞ -categories \mathcal{C} and \mathcal{C}' , we expect to be able to construct an ∞ -category $\text{Fun}(\mathcal{C}, \mathcal{C}')$ of functors from \mathcal{C} to \mathcal{C}' .

In the setting of topological categories, the construction of an appropriate mapping object $\text{Fun}(\mathcal{C}, \mathcal{C}')$ is quite difficult. The naive guess is that $\text{Fun}(\mathcal{C}, \mathcal{C}')$ should be a category of topological functors from \mathcal{C} to \mathcal{C}' : that is, functors which induce continuous maps between morphism spaces. However, we saw in §1.2.6 that this notion is generally too rigid, even in the special case where \mathcal{C} is an ordinary category.

Remark 1.2.7.1. Using the language of model categories, one might say that the problem is that not every topological category is *cofibrant*. If \mathcal{C} is a *cofibrant* topological category (for example, if $\mathcal{C} = |\mathfrak{C}[S]|$, where S is a simplicial set), then the collection of topological functors from \mathcal{C} to \mathcal{C}' is large enough to contain representatives for every ∞ -categorical functor from \mathcal{C} to \mathcal{C}' . Most ordinary categories are not cofibrant when viewed as topological categories. More importantly, the property of being cofibrant is not stable under products, so that naive attempts to construct a mapping object $\text{Fun}(\mathcal{C}, \mathcal{C}')$ need not give the correct answer even when \mathcal{C} itself is assumed cofibrant (if \mathcal{C} is cofibrant, then we are guaranteed to have “enough” topological functors $\mathcal{C} \rightarrow \mathcal{C}'$ to represent all functors between the underlying ∞ -categories but not necessarily enough natural transformations between them; note that the product $\mathcal{C} \times [1]$ is usually not cofibrant, even in the simplest nontrivial case where $\mathcal{C} = [1]$.) This is arguably the most important technical disadvantage of the theory of topological (or simplicial) categories as an approach to higher category theory.

The construction of functor categories is much easier to describe in the framework of ∞ -categories. If \mathcal{C} and \mathcal{D} are ∞ -categories, then we can simply define a *functor* from \mathcal{C} to \mathcal{D} to be a map $p : \mathcal{C} \rightarrow \mathcal{D}$ of simplicial sets.

Notation 1.2.7.2. Let \mathcal{C} and \mathcal{D} be simplicial sets. We let $\text{Fun}(\mathcal{C}, \mathcal{D})$ denote the simplicial set $\text{Map}_{\text{Set}_\Delta}(\mathcal{C}, \mathcal{D})$ parametrizing maps from \mathcal{C} to \mathcal{D} . We will use this notation only when \mathcal{D} is an ∞ -category (the simplicial set \mathcal{C} will often, but not always, be an ∞ -category as well). We will refer to $\text{Fun}(\mathcal{C}, \mathcal{D})$ as the *∞ -category of functors from \mathcal{C} to \mathcal{D}* (see Proposition 1.2.7.3 below). We will refer to morphisms in $\text{Fun}(\mathcal{C}, \mathcal{D})$ as *natural transformations* of functors, and equivalences in $\text{Fun}(\mathcal{C}, \mathcal{D})$ as *natural equivalences*.

Proposition 1.2.7.3. *Let K be an arbitrary simplicial set.*

- (1) *For every ∞ -category \mathcal{C} , the simplicial set $\text{Fun}(K, \mathcal{C})$ is an ∞ -category.*

- (2) Let $\mathcal{C} \rightarrow \mathcal{D}$ be a categorical equivalence of ∞ -categories. Then the induced map $\mathrm{Fun}(K, \mathcal{C}) \rightarrow \mathrm{Fun}(K, \mathcal{D})$ is a categorical equivalence.
- (3) Let \mathcal{C} be an ∞ -category and $K \rightarrow K'$ a categorical equivalence of simplicial sets. Then the induced map $\mathrm{Fun}(K', \mathcal{C}) \rightarrow \mathrm{Fun}(K, \mathcal{C})$ is a categorical equivalence.

The proof makes use of the Joyal model structure on Set_Δ and will be given in §2.2.5.

1.2.8 Joins of ∞ -Categories

Let \mathcal{C} and \mathcal{C}' be ordinary categories. We will define a new category $\mathcal{C} \star \mathcal{C}'$, called the *join* of \mathcal{C} and \mathcal{C}' . An object of $\mathcal{C} \star \mathcal{C}'$ is either an object of \mathcal{C} or an object of \mathcal{C}' . The morphism sets are given as follows:

$$\mathrm{Hom}_{\mathcal{C} \star \mathcal{C}'}(X, Y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \text{if } X, Y \in \mathcal{C} \\ \mathrm{Hom}_{\mathcal{C}'}(X, Y) & \text{if } X, Y \in \mathcal{C}' \\ \emptyset & \text{if } X \in \mathcal{C}', Y \in \mathcal{C} \\ * & \text{if } X \in \mathcal{C}, Y \in \mathcal{C}'. \end{cases}$$

Composition of morphisms in $\mathcal{C} \star \mathcal{C}'$ is defined in the obvious way.

The join construction described above is often useful when discussing diagram categories, limits, and colimits. In this section, we will introduce a generalization of this construction to the ∞ -categorical setting.

Definition 1.2.8.1. If S and S' are simplicial sets, then the simplicial set $S \star S'$ is defined as follows: for each nonempty finite linearly ordered set J , we set

$$(S \star S')(J) = \coprod_{J=I \sqcup I'} S(I) \times S'(I'),$$

where the union is taken over all decompositions of J into disjoint subsets I and I' , satisfying $i < i'$ for all $i \in I, i' \in I'$. Here we allow the possibility that either I or I' is empty, in which case we agree to the convention that $S(\emptyset) = S'(\emptyset) = *$.

More concretely, we have

$$(S \star S')_n = S_n \cup S'_n \cup \bigcup_{i+j=n-1} S_i \times S'_j.$$

The join operation endows Set_Δ with the structure of a monoidal category (see §A.1.3). The identity for the join operation is the empty simplicial set $\emptyset = \Delta^{-1}$. More generally, we have natural isomorphisms $\phi_{ij} : \Delta^{i-1} \star \Delta^{j-1} \simeq \Delta^{(i+j)-1}$ for all $i, j \geq 0$.

Remark 1.2.8.2. The operation \star is essentially determined by the isomorphisms ϕ_{ij} , together with its behavior under the formation of colimits: for any fixed simplicial set S , the functors

$$T \mapsto T \star S$$

$$T \mapsto S \star T$$

commute with colimits when regarded as functors from \mathbf{Set}_Δ to the undercategory $(\mathbf{Set}_\Delta)_{S/}$ of simplicial sets *under* S .

Passage to the nerve carries joins of categories into joins of simplicial sets. More precisely, for every pair of categories \mathcal{C} and \mathcal{C}' , there is a canonical isomorphism

$$N(\mathcal{C} \star \mathcal{C}') \simeq N(\mathcal{C}) \star N(\mathcal{C}').$$

(The existence of this isomorphism persists when we allow \mathcal{C} and \mathcal{C}' to be simplicial or topological categories and apply the appropriate generalization of the nerve functor.) This suggests that the join operation on simplicial sets is the appropriate ∞ -categorical analogue of the join operation on categories.

We remark that the formation of joins does not commute with the functor $\mathfrak{C}[\bullet]$. However, the simplicial category $\mathfrak{C}[S \star S']$ contains $\mathfrak{C}[S]$ and $\mathfrak{C}[S']$ as full (topological) subcategories and contains no morphisms from objects of $\mathfrak{C}[S']$ to objects of $\mathfrak{C}[S]$. Consequently, there is unique map $\phi : \mathfrak{C}[S \star S'] \rightarrow \mathfrak{C}[S] \star \mathfrak{C}[S']$ which reduces to the identity on $\mathfrak{C}[S]$ and $\mathfrak{C}[S']$. We will later show that ϕ is an equivalence of simplicial categories (Corollary 4.2.1.4).

We conclude by recording a pleasant property of the join operation:

Proposition 1.2.8.3 (Joyal [44]). *If S and S' are ∞ -categories, then $S \star S'$ is an ∞ -category.*

Proof. Let $p : \Lambda_i^n \rightarrow S \star S'$ be a map, with $0 < i < n$. If p carries Λ_i^n entirely into $S \subseteq S \star S'$ or into $S' \subseteq S \star S'$, then we deduce the existence of an extension of p to Δ^n using the assumption that S and S' are ∞ -categories. Otherwise, we may suppose that p carries the vertices $\{0, \dots, j\}$ into S , and the vertices $\{j + 1, \dots, n\}$ into S' . We may now restrict p to obtain maps

$$\Delta^{\{0, \dots, j\}} \rightarrow S$$

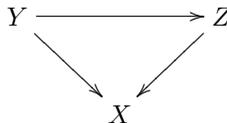
$$\Delta^{\{j+1, \dots, n\}} \rightarrow S',$$

which together determine a map $\Delta^n \rightarrow S \star S'$ extending p . □

Notation 1.2.8.4. Let K be a simplicial set. The *left cone* K^\triangleleft is defined to be the join $\Delta^0 \star K$. Dually, the *right cone* K^\triangleright is defined to be the join $K \star \Delta^0$. Either cone contains a distinguished vertex (belonging to Δ^0), which we will refer to as the *cone point*.

1.2.9 Overcategories and Undercategories

Let \mathcal{C} be an ordinary category and $X \in \mathcal{C}$ an object. The *overcategory* $\mathcal{C}_{/X}$ is defined as follows: the objects of $\mathcal{C}_{/X}$ are morphisms $Y \rightarrow X$ in \mathcal{C} having target X . Morphisms are given by commutative triangles



and composition is defined in the obvious way.

One can rephrase the definition of the overcategory as follows. Let $[0]$ denote the category with a single object possessing only an identity morphism. Then specifying an object $X \in \mathcal{C}$ is equivalent to specifying a functor $x : [0] \rightarrow \mathcal{C}$. The overcategory $\mathcal{C}_{/X}$ may then be described by the following universal property: for any category \mathcal{C}' , we have a bijection

$$\mathrm{Hom}(\mathcal{C}', \mathcal{C}_{/X}) \simeq \mathrm{Hom}_x(\mathcal{C}' \star [0], \mathcal{C}),$$

where the subscript on the right hand side indicates that we consider only those functors $\mathcal{C}' \star [0] \rightarrow \mathcal{C}$ whose restriction to $[0]$ coincides with x .

Our goal in this section is to generalize the construction of overcategories to the ∞ -categorical setting. Let us begin by working in the framework of topological categories. In this case, there is a natural candidate for the relevant overcategory. Namely, if \mathcal{C} is a topological category containing an object X , then the overcategory $\mathcal{C}_{/X}$ (defined as above) has the structure of a topological category where each morphism space $\mathrm{Map}_{\mathcal{C}_{/X}}(Y, Z)$ is topologized as a subspace of $\mathrm{Map}_{\mathcal{C}}(Y, Z)$ (here we are identifying an object of $\mathcal{C}_{/X}$ with its image in \mathcal{C}). This topological category is usually *not* a model for the correct ∞ -categorical slice construction. The problem is that a morphism in $\mathcal{C}_{/X}$ consists of a commutative triangle

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & Z \\ & \searrow & \swarrow \\ & X & \end{array}$$

of objects over X . To obtain the correct notion, we should also allow triangles which commute only up to homotopy.

Remark 1.2.9.1. In some cases, the naive overcategory $\mathcal{C}_{/X}$ is a good approximation to the correct construction: see Lemma 6.1.3.13.

In the setting of ∞ -categories, Joyal has given a much simpler description of the desired construction (see [43]). This description will play a vitally important role throughout this book. We begin by noting the following:

Proposition 1.2.9.2 ([43]). *Let S and K be simplicial sets, and $p : K \rightarrow S$ an arbitrary map. There exists a simplicial set $S_{/p}$ with the following universal property:*

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(Y, S_{/p}) = \mathrm{Hom}_p(Y \star K, S),$$

where the subscript on the right hand side indicates that we consider only those morphisms $f : Y \star K \rightarrow S$ such that $f|_K = p$.

Proof. One defines $(S_{/p})_n$ to be $\mathrm{Hom}_p(\Delta^n \star K, S)$. The universal property holds by definition when Y is a simplex. It holds in general because both sides are compatible with the formation of colimits in Y . \square

Let $p : K \rightarrow S$ be as in Proposition 1.2.9.2. If S is an ∞ -category, we will refer to $S_{/p}$ as an *overcategory* of S or as the *∞ -category of objects of S over p* . The following result guarantees that the operation of passing to overcategories is well-behaved:

Proposition 1.2.9.3. *Let $p : K \rightarrow \mathcal{C}$ be a map of simplicial sets and suppose that \mathcal{C} is an ∞ -category. Then $\mathcal{C}_{/p}$ is an ∞ -category. Moreover, if $q : \mathcal{C} \rightarrow \mathcal{C}'$ is a categorical equivalence of ∞ -categories, then the induced map $\mathcal{C}_{/p} \rightarrow \mathcal{C}'_{/qp}$ is a categorical equivalence as well.*

The proof requires a number of ideas which we have not yet introduced and will be postponed (see Proposition 2.1.2.2 for the first assertion, and §2.4.5 for the second).

Remark 1.2.9.4. Let \mathcal{C} be an ∞ -category. In the particular case where $p : \Delta^n \rightarrow \mathcal{C}$ classifies an n -simplex $\sigma \in \mathcal{C}_n$, we will often write $\mathcal{C}_{/\sigma}$ in place of $\mathcal{C}_{/p}$. In particular, if X is an object of \mathcal{C} , we let $\mathcal{C}_{/X}$ denote the overcategory $\mathcal{C}_{/p}$, where $p : \Delta^0 \rightarrow \mathcal{C}$ has image X .

Remark 1.2.9.5. Let $p : K \rightarrow \mathcal{C}$ be a map of simplicial sets. The preceding discussion can be dualized, replacing $Y \star K$ by $K \star Y$; in this case we denote the corresponding simplicial set by $\mathcal{C}_{p/}$, which (if \mathcal{C} is an ∞ -category) we will refer to as an *undercategory* of \mathcal{C} . In the special case where $K = \Delta^n$ and p classifies a simplex $\sigma \in \mathcal{C}_n$, we will also write $\mathcal{C}_{\sigma/}$ for $\mathcal{C}_{p/}$; in particular, we will write $\mathcal{C}_{X/}$ when X is an object of \mathcal{C} .

Remark 1.2.9.6. If \mathcal{C} is an ordinary category and $X \in \mathcal{C}$, then there is a canonical isomorphism $N(\mathcal{C})_{/X} \simeq N(\mathcal{C}_{/X})$. In other words, the overcategory construction for ∞ -categories can be regarded as a *generalization* of the relevant construction from classical category theory.

1.2.10 Fully Faithful and Essentially Surjective Functors

Definition 1.2.10.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between topological categories (simplicial categories, simplicial sets). We will say that F is *essentially surjective* if the induced functor $hF : h\mathcal{C} \rightarrow h\mathcal{D}$ is essentially surjective (that is, if every object of \mathcal{D} is equivalent to $F(X)$ for some $X \in \mathcal{C}$).

We will say that F is *fully faithful* if hF is a fully faithful functor of \mathcal{H} -enriched categories. In other words, F is fully faithful if and only if, for every pair of objects $X, Y \in \mathcal{C}$, the induced map $\text{Map}_{h\mathcal{C}}(X, Y) \rightarrow \text{Map}_{h\mathcal{D}}(F(X), F(Y))$ is an isomorphism in the homotopy category \mathcal{H} .

Remark 1.2.10.2. Because Definition 1.2.10.1 makes reference only to the homotopy categories of \mathcal{C} and \mathcal{D} , it is invariant under equivalence and under operations which pass between the various models for higher category theory that we have introduced.

Just as in ordinary category theory, a functor F is an equivalence if and only if it is fully faithful and essentially surjective.

1.2.11 Subcategories of ∞ -Categories

Let \mathcal{C} be an ∞ -category and let $(h\mathcal{C})' \subseteq h\mathcal{C}$ be a subcategory of its homotopy category. We can then form a pullback diagram of simplicial sets:

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ N(h\mathcal{C})' & \longrightarrow & N(h\mathcal{C}). \end{array}$$

We will refer to \mathcal{C}' as the *subcategory of \mathcal{C} spanned by $(h\mathcal{C})'$* . In general, we will say that a simplicial subset $\mathcal{C}' \subseteq \mathcal{C}$ is a *subcategory* of \mathcal{C} if it arises via this construction.

Remark 1.2.11.1. We use the term “subcategory,” rather than “sub- ∞ -category,” in order to avoid awkward language. The terminology is not meant to suggest that \mathcal{C}' is itself a category or is isomorphic to the nerve of a category.

In the case where $(h\mathcal{C})'$ is a full subcategory of $h\mathcal{C}$, we will say that \mathcal{C}' is a *full subcategory* of \mathcal{C} . In this case, \mathcal{C}' is determined by the set \mathcal{C}'_0 of those objects $X \in \mathcal{C}$ which belong to \mathcal{C}' . We will then say that \mathcal{C}' is the *full subcategory of \mathcal{C} spanned by \mathcal{C}'_0* .

It follows from Remark 1.2.2.4 that the inclusion $\mathcal{C}' \subseteq \mathcal{C}$ is fully faithful. In general, any fully faithful functor $f : \mathcal{C}'' \rightarrow \mathcal{C}$ factors as a composition

$$\mathcal{C}'' \xrightarrow{f'} \mathcal{C}' \xrightarrow{f''} \mathcal{C},$$

where f' is an equivalence of ∞ -categories and f'' is the inclusion of the full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ spanned by the set of objects $f(\mathcal{C}''_0) \subseteq \mathcal{C}_0$.

1.2.12 Initial and Final Objects

If \mathcal{C} is an ordinary category, then an object $X \in \mathcal{C}$ is said to be *final* if $\text{Hom}_{\mathcal{C}}(Y, X)$ consists of a single element for every $Y \in \mathcal{C}$. Dually, an object $X \in \mathcal{C}$ is *initial* if it is final when viewed as an object of \mathcal{C}^{op} . The goal of this section is to generalize these definitions to the ∞ -categorical setting.

If \mathcal{C} is a topological category, then a candidate definition immediately presents itself: we could ignore the topology on the morphism spaces and consider those objects of \mathcal{C} which are final when \mathcal{C} is regarded as an ordinary category. This requirement is unnaturally strong. For example, the category $\mathcal{C}\mathcal{G}$ of compactly generated Hausdorff spaces has a final object: the topological space $*$, consisting of a single point. However, there are objects of $\mathcal{C}\mathcal{G}$ which are equivalent to $*$ (any contractible space) but not isomorphic to $*$ (and therefore not final objects of $\mathcal{C}\mathcal{G}$, at least in the classical sense). Since any reasonable ∞ -categorical notion is stable under equivalence, we need to find a weaker condition.

Definition 1.2.12.1. Let \mathcal{C} be a topological category (simplicial category, simplicial set). An object $X \in \mathcal{C}$ is *final* if it is final in the homotopy category

$\mathbf{h}\mathcal{C}$, regarded as a category enriched over \mathcal{H} . In other words, X is final if and only if for each $Y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathbf{h}\mathcal{C}}(Y, X)$ is weakly contractible (that is, a final object of \mathcal{H}).

Remark 1.2.12.2. Since Definition 1.2.12.1 makes reference only to the homotopy category $\mathbf{h}\mathcal{C}$, it is invariant under equivalence and under passing between the various models for higher category theory.

In the setting of ∞ -categories, it is convenient to employ a slightly more sophisticated definition, which we borrow from [43].

Definition 1.2.12.3. Let \mathcal{C} be a simplicial set. A vertex X of \mathcal{C} is *strongly final* if the projection $\mathcal{C}_{/X} \rightarrow \mathcal{C}$ is a trivial fibration of simplicial sets.

In other words, a vertex X of \mathcal{C} is strongly final if and only if any map $f_0 : \partial \Delta^n \rightarrow \mathcal{C}$ such that $f_0(n) = X$ can be extended to a map $f : \Delta^n \rightarrow \mathcal{C}$.

Proposition 1.2.12.4. *Let \mathcal{C} be an ∞ -category containing an object Y . The object Y is strongly final if and only if, for every object $X \in \mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}^{\mathbf{R}}(X, Y)$ is contractible.*

Proof. The “only if” direction is clear: the space $\mathrm{Hom}_{\mathcal{C}}^{\mathbf{R}}(X, Y)$ is the fiber of the projection $p : \mathcal{C}_{/Y} \rightarrow \mathcal{C}$ over the vertex X . If p is a trivial fibration, then the fiber is a contractible Kan complex. Since p is a right fibration (Proposition 2.1.2.1), the converse holds as well (Lemma 2.1.3.4). \square

Corollary 1.2.12.5. *Let \mathcal{C} be a simplicial set. Every strongly final object of \mathcal{C} is a final object of \mathcal{C} . The converse holds if \mathcal{C} is an ∞ -category.*

Proof. Let $[0]$ denote the category with a single object and a single morphism. Suppose that Y is a strongly final vertex of \mathcal{C} . Then there exists a retraction of $\mathcal{C}^{\triangleright}$ onto \mathcal{C} carrying the cone point to Y . Consequently, we obtain a retraction of (\mathcal{H} -enriched) homotopy categories from $\mathbf{h}\mathcal{C} \star [0]$ to $\mathbf{h}\mathcal{C}$ carrying the unique object of $[0]$ to Y . This implies that Y is final in $\mathbf{h}\mathcal{C}$, so that Y is a final object of \mathcal{C} .

To prove the converse, we note that if \mathcal{C} is an ∞ -category, then the Kan complex $\mathrm{Hom}_{\mathcal{C}}^{\mathbf{R}}(X, Y)$ represents the homotopy type $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathcal{H}$; by Proposition 1.2.12.4 this space is contractible for all X if and only if Y is strongly final. \square

Remark 1.2.12.6. The above discussion dualizes in an evident way, so that we have a notion of *initial* objects of an ∞ -category \mathcal{C} .

Example 1.2.12.7. Let \mathcal{C} be an ordinary category containing an object X . Then X is a final (initial) object of the ∞ -category $\mathbf{N}(\mathcal{C})$ if and only if it is a final (initial) object of \mathcal{C} in the usual sense.

Remark 1.2.12.8. Definition 1.2.12.3 is only natural in the case where \mathcal{C} is an ∞ -category. For example, if \mathcal{C} is not an ∞ -category, then the collection of strongly final vertices of \mathcal{C} need not be stable under equivalence.

An ordinary category \mathcal{C} may have more than one final object, but any two final objects are uniquely isomorphic to one another. In the setting of ∞ -categories, an analogous statement holds but is slightly more complicated because the word “unique” needs to be interpreted in a homotopy-theoretic sense:

Proposition 1.2.12.9 (Joyal). *Let \mathcal{C} be a ∞ -category and let \mathcal{C}' be the full subcategory of \mathcal{C} spanned by the final vertices of \mathcal{C} . Then \mathcal{C}' either is empty or is a contractible Kan complex.*

Proof. We wish to prove that every map $p : \partial \Delta^n \rightarrow \mathcal{C}'$ can be extended to an n -simplex of \mathcal{C}' . If $n = 0$, this is possible unless \mathcal{C}' is empty. For $n > 0$, the desired extension exists because p carries the n th vertex of $\partial \Delta^n$ to a final object of \mathcal{C} . \square

1.2.13 Limits and Colimits

An important consequence of the distinction between homotopy commutativity and homotopy coherence is that the appropriate notions of limit and colimit in a higher category \mathcal{C} do not coincide with the notions of limit and colimit in the homotopy category $\mathrm{h}\mathcal{C}$ (where limits and colimits often do not exist). Limits and colimits in \mathcal{C} are often referred to as *homotopy limits* and *homotopy colimits* to avoid confusing them with ordinary limits and colimits.

Homotopy limits and colimits can be defined in a topological category, but the definition is rather complicated. We will review a few special cases here and discuss the general definition in the Appendix (§A.2.8).

Example 1.2.13.1. Let $\{X_\alpha\}$ be a family of objects in a topological category \mathcal{C} . A *homotopy product* $X = \prod_\alpha X_\alpha$ is an object of \mathcal{C} equipped with morphisms $f_\alpha : X \rightarrow X_\alpha$ which induce a weak homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(Y, X) \rightarrow \prod_{\alpha} \mathrm{Map}_{\mathcal{C}}(Y, X_{\alpha})$$

for every object $Y \in \mathcal{C}$.

Passing to path components and using the fact that π_0 commutes with products, we deduce that

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, X) \simeq \prod_{\alpha} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, X_{\alpha}),$$

so that any product in \mathcal{C} is also a product in $\mathrm{h}\mathcal{C}$. In particular, the object X is determined up to canonical isomorphism in $\mathrm{h}\mathcal{C}$.

In the special case where the index set is empty, we recover the notion of a final object of \mathcal{C} : an object X for which each of the mapping spaces $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is weakly contractible.

Example 1.2.13.2. Given two morphisms $\pi : X \rightarrow Z$ and $\psi : Y \rightarrow Z$ in a topological category \mathcal{C} , let us define $\mathrm{Map}_{\mathcal{C}}(W, X \times_Z^h Y)$ to be the space consisting of points $p \in \mathrm{Map}_{\mathcal{C}}(W, X)$ and $q \in \mathrm{Map}_{\mathcal{C}}(W, Y)$ together

with a path $r : [0, 1] \rightarrow \text{Map}_{\mathcal{C}}(W, Z)$ joining $\pi \circ p$ to $\psi \circ q$. We endow $\text{Map}_{\mathcal{C}}(W, X \times_{\mathbb{Z}}^h Y)$ with the obvious topology, so that $X \times_{\mathbb{Z}}^h Y$ can be viewed as a presheaf of topological spaces on \mathcal{C} . A *homotopy fiber product for X and Y over Z* is an object of \mathcal{C} which represents this presheaf up to weak homotopy equivalence. In other words, it is an object $P \in \mathcal{C}$ equipped with a point $p \in \text{Map}_{\mathcal{C}}(P, X \times_{\mathbb{Z}}^h Y)$ which induces weak homotopy equivalences $\text{Map}_{\mathcal{C}}(W, P) \rightarrow \text{Map}_{\mathcal{C}}(W, X \times_{\mathbb{Z}}^h Y)$ for every $W \in \mathcal{C}$.

We note that if there exists a fiber product (in the ordinary sense) $X \times_Z Y$ in the category \mathcal{C} , then this ordinary fiber product admits a (canonically determined) map to the homotopy fiber product (if the homotopy fiber product exists). This map need not be an equivalence, but it is an equivalence in many good cases. We also note that a homotopy fiber product P comes equipped with a map to the fiber product $X \times_Z Y$ taken in the category $\text{h}\mathcal{C}$ (if this fiber product exists); this map is usually not an isomorphism.

Remark 1.2.13.3. Homotopy limits and colimits in general may be described in relation to homotopy limits of topological spaces. The homotopy limit X of a diagram of objects $\{X_{\alpha}\}$ in an arbitrary topological category \mathcal{C} is determined, up to equivalence, by the requirement that there exists a natural weak homotopy equivalence

$$\text{Map}_{\mathcal{C}}(Y, X) \simeq \text{holim}\{\text{Map}_{\mathcal{C}}(Y, X_{\alpha})\}.$$

Similarly, the homotopy colimit of the diagram is characterized by the existence of a natural weak homotopy equivalence

$$\text{Map}_{\mathcal{C}}(X, Y) \simeq \text{holim}\{\text{Map}_{\mathcal{C}}(X_{\alpha}, Y)\}.$$

For a more precise discussion, we refer the reader to Remark A.3.3.13.

In the setting of ∞ -categories, limits and colimits are quite easy to define:

Definition 1.2.13.4 (Joyal [43]). Let \mathcal{C} be an ∞ -category and let $p : K \rightarrow \mathcal{C}$ be an arbitrary map of simplicial sets. A *colimit* for p is an initial object of $\mathcal{C}_{p/}$, and a *limit* for p is a final object of $\mathcal{C}_{/p}$.

Remark 1.2.13.5. According to Definition 1.2.13.4, a colimit of a diagram $p : K \rightarrow \mathcal{C}$ is an object of $\mathcal{C}_{p/}$. We may identify this object with a map $\bar{p} : K^{\triangleright} \rightarrow \mathcal{C}$ extending p . In general, we will say that a map $\bar{p} : K^{\triangleright} \rightarrow \mathcal{C}$ is a *colimit diagram* if it is a colimit of $p = \bar{p}|_K$. In this case, we will also abuse terminology by referring to $\bar{p}(\infty) \in \mathcal{C}$ as a *colimit of p* , where ∞ denotes the cone point of K^{\triangleright} .

If $p : K \rightarrow \mathcal{C}$ is a diagram, we will sometimes write $\varinjlim(p)$ to denote a colimit of p (considered either as an object of $\mathcal{C}_{p/}$ or of $\overline{\mathcal{C}}$), and $\varprojlim(p)$ to denote a limit of p (as either an object of $\mathcal{C}_{/p}$ or an object of \mathcal{C}). This notation is slightly abusive since $\varinjlim(p)$ is not uniquely determined by p . This phenomenon is familiar in classical category theory: the colimit of a diagram is not unique but is determined up to canonical isomorphism. In the ∞ -categorical setting, we have a similar uniqueness result: Proposition 1.2.12.9 implies that the collection of candidates for $\varinjlim(p)$, if nonempty, is parametrized by a contractible Kan complex.

Remark 1.2.13.6. In §4.2.4, we will show that Definition 1.2.13.4 agrees with the classical theory of homotopy (co)limits when we specialize to the case where \mathcal{C} is the nerve of a topological category.

Remark 1.2.13.7. Let \mathcal{C} be an ∞ -category, $\mathcal{C}' \subseteq \mathcal{C}$ a full subcategory, and $p : K \rightarrow \mathcal{C}'$ a diagram. Then $\mathcal{C}'_{p/} = \mathcal{C}' \times_{\mathcal{C}} \mathcal{C}_{p/}$. In particular, if p has a colimit in \mathcal{C} and that colimit belongs to \mathcal{C}' , then the same object may be regarded as a colimit for p in \mathcal{C}' .

Let $f : \mathcal{C} \rightarrow \mathcal{C}'$ be a map between ∞ -categories. Let $p : K \rightarrow \mathcal{C}$ be a diagram in \mathcal{C} having a colimit $x \in \mathcal{C}_{p/}$. The image $f(x) \in \mathcal{C}'_{fp/}$ may or may not be a colimit for the composite map $f \circ p$. If it is, we will say that f *preserves* the colimit of the diagram p . Often we will apply this terminology not to a particular diagram p but to some class of diagrams: for example, we may speak of maps f which preserve coproducts, pushouts, or filtered colimits (see §4.4 for a discussion of special classes of colimits). Similarly, we may ask whether or not a map f preserves the limit of a particular diagram or various families of diagrams.

We conclude this section by giving a simple example of a colimit-preserving functor.

Proposition 1.2.13.8. *Let \mathcal{C} be an ∞ -category and let $q : T \rightarrow \mathcal{C}$ and $p : K \rightarrow \mathcal{C}_{/q}$ be two diagrams. Let p_0 denote the composition of p with the projection $\mathcal{C}_{/q} \rightarrow \mathcal{C}$. Suppose that p_0 has a colimit in \mathcal{C} . Then*

- (1) *The diagram p has a colimit in $\mathcal{C}_{/q}$, and that colimit is preserved by the projection $\mathcal{C}_{/q} \rightarrow \mathcal{C}$.*
- (2) *An extension $\tilde{p} : K^\triangleright \rightarrow \mathcal{C}_{/q}$ is a colimit of p if and only if the composition*

$$K^\triangleright \rightarrow \mathcal{C}_{/q} \rightarrow \mathcal{C}$$

is a colimit of p_0 .

Proof. We first prove the “if” direction of (2). Let $\tilde{p} : K^\triangleright \rightarrow \mathcal{C}_{/q}$ be such that the composite map $\tilde{p}_0 : K^\triangleright \rightarrow \mathcal{C}$ is a colimit of p_0 . We wish to show that \tilde{p} is a colimit of p . We may identify \tilde{p} with a map $K \star \Delta^0 \star T \rightarrow \mathcal{C}$. For this, it suffices to show that for any inclusion $A \subseteq B$ of simplicial sets, it is possible to solve the lifting problem depicted in the following diagram:

$$\begin{array}{ccc} (K \star B \star T) \amalg_{K \star A \star T} (K \star \Delta^0 \star A \star T) & \xrightarrow{\cong} & \mathcal{C} \\ \downarrow & \dashrightarrow & \\ K \star \Delta^0 \star B \star T & & \end{array}$$

Because \tilde{p}_0 is a colimit of p_0 , the projection

$$\mathcal{C}_{\tilde{p}_0/} \rightarrow \mathcal{C}_{p_0/}$$

is a trivial fibration of simplicial sets and therefore has the right lifting property with respect to the inclusion $A \star T \subseteq B \star T$.

We now prove (1). Let $\tilde{p}_0 : K^\triangleright \rightarrow \mathcal{C}$ be a colimit of p_0 . Since the projection $\mathcal{C}_{\tilde{p}_0/} \rightarrow \mathcal{C}_{p_0/}$ is a trivial fibration, it has the right lifting property with respect to T : this guarantees the existence of an extension $\tilde{p} : K^\triangleright \rightarrow \mathcal{C}$ lifting \tilde{p}_0 . The preceding analysis proves that \tilde{p} is a colimit of p .

Finally, the “only if” direction of (2) follows from (1) since any two colimits of p are equivalent. \square

1.2.14 Presentations of ∞ -Categories

Like many other types of mathematical structures, ∞ -categories can be described by generators and relations. In particular, it makes sense to speak of a *finitely presented* ∞ -category \mathcal{C} . Roughly speaking, \mathcal{C} is finitely presented if it has finitely many objects and its morphism spaces are determined by specifying a finite number of generating morphisms, a finite number of relations among these generating morphisms, a finite number of relations among the relations, and so forth (a finite number of relations in all).

Example 1.2.14.1. Let \mathcal{C} be the free higher category generated by a single object X and a single morphism $f : X \rightarrow X$. Then \mathcal{C} is a finitely presented ∞ -category with a single object and $\mathrm{Hom}_{\mathcal{C}}(X, X) = \{1, f, f^2, \dots\}$ is infinite and discrete. In particular, we note that the finite presentation of \mathcal{C} does not guarantee finiteness properties of the morphism spaces.

Example 1.2.14.2. If we identify ∞ -groupoids with spaces, then giving a presentation for an ∞ -groupoid corresponds to giving a cell decomposition of the associated space. Consequently, the finitely presented ∞ -groupoids correspond precisely to the finite cell complexes.

Example 1.2.14.3. Suppose that \mathcal{C} is a higher category with only two objects X and Y , that X and Y have contractible endomorphism spaces, and that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is empty. Then \mathcal{C} is completely determined by the morphism space $\mathrm{Hom}_{\mathcal{C}}(Y, X)$, which may be arbitrary. In this case, \mathcal{C} is finitely presented if and only if $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is a finite cell complex (up to homotopy equivalence).

The idea of giving a presentation for an ∞ -category is very naturally encoded in Joyal’s model structure on the category of simplicial sets, which we will discuss in §2.2.4). This model structure can be described as follows:

- The fibrant objects of Set_Δ are precisely the ∞ -categories.
- The weak equivalences in Set_Δ are precisely those maps $p : S \rightarrow S'$ which induce equivalences $\mathcal{C}[S] \rightarrow \mathcal{C}[S']$ of simplicial categories.

If S is an arbitrary simplicial set, we can choose a “fibrant replacement” for S : that is, a categorical equivalence $S \rightarrow \mathcal{C}$, where \mathcal{C} is an ∞ -category. For example, we can take \mathcal{C} to be the nerve of the topological category $|\mathcal{C}[S]|$. The ∞ -category \mathcal{C} is well-defined up to equivalence, and we may regard it as

an ∞ -category “generated by” S . The simplicial set S itself can be thought of as a “blueprint” for building \mathcal{C} . We may view S as generated from the empty (simplicial) set by adjoining nondegenerate simplices. Adjoining a 0-simplex to S has the effect of adding an object to the ∞ -category \mathcal{C} , and adjoining a 1-simplex to S has the effect of adjoining a morphism to \mathcal{C} . Higher-dimensional simplices can be thought of as encoding relations among the morphisms.

1.2.15 Set-Theoretic Technicalities

In ordinary category theory, one frequently encounters categories in which the collection of objects is too large to form a set. Generally speaking, this does not create any difficulties so long as we avoid doing anything which is obviously illegal (such as considering the “category of all categories” as an object of itself).

The same issues arise in the setting of higher category theory and are in some sense even more of a nuisance. In ordinary category theory, one generally allows a category \mathcal{C} to have a proper class of objects but still requires $\text{Hom}_{\mathcal{C}}(X, Y)$ to be a *set* for fixed objects $X, Y \in \mathcal{C}$. The formalism of ∞ -categories treats objects and morphisms on the same footing (they are both simplices of a simplicial set), and it is somewhat unnatural (though certainly possible) to directly impose the analogous condition; see §5.4.1 for a discussion.

There are several means of handling the technical difficulties inherent in working with large objects (in either classical or higher category theory):

- (1) One can employ some set-theoretic device that enables one to distinguish between “large” and “small”. Examples include:
 - Assuming the existence of a sufficient supply of (Grothendieck) universes.
 - Working in an axiomatic framework which allows both sets and *classes* (collections of sets which are possibly too large for themselves to be considered sets).
 - Working in a standard set-theoretic framework (such as Zermelo-Frankel) but incorporating a theory of classes through some ad hoc device. For example, one can define a class to be a collection of sets which is defined by some formula in the language of set theory.
- (2) One can work exclusively with small categories, and mirror the distinction between large and small by keeping careful track of relative sizes.
- (3) One can simply ignore the set-theoretic difficulties inherent in discussing large categories.

Needless to say, approach (2) yields the most refined information. However, it has the disadvantage of burdening our exposition with an additional layer of technicalities. On the other hand, approach (3) will sometimes be inadequate because we will need to make arguments which play off the distinction between a large category and a small subcategory which determines it. Consequently, we shall officially adopt approach (1) for the remainder of this book. More specifically, we assume that for every cardinal κ_0 , there exists a strongly inaccessible cardinal $\kappa \geq \kappa_0$. We then let $\mathcal{U}(\kappa)$ denote the collection of all sets having rank $< \kappa$, so that $\mathcal{U}(\kappa)$ is a *Grothendieck universe*: in other words, $\mathcal{U}(\kappa)$ satisfies all of the usual axioms of set theory. We will refer to a mathematical object as *small* if it belongs to $\mathcal{U}(\kappa)$ (or is isomorphic to such an object), and *essentially small* if it is equivalent (in whatever relevant sense) to a small object. Whenever it is convenient to do so, we will choose another strongly inaccessible cardinal $\kappa' > \kappa$ to obtain a larger Grothendieck universe $\mathcal{U}(\kappa')$ in which $\mathcal{U}(\kappa)$ becomes small.

For example, an ∞ -category \mathcal{C} is essentially small if and only if it satisfies the following conditions:

- The set of isomorphism classes of objects in the homotopy category $\mathrm{h}\mathcal{C}$ has cardinality $< \kappa$.
- For every morphism $f : X \rightarrow Y$ in \mathcal{C} and every $i \geq 0$, the homotopy set $\pi_i(\mathrm{Hom}_{\mathcal{C}}^{\mathbb{R}}(X, Y), f)$ has cardinality $< \kappa$.

For a proof and further discussion, we refer the reader to §5.4.1.

Remark 1.2.15.1. The existence of the strongly inaccessible cardinal κ cannot be proven from the standard axioms of set theory, and the assumption that κ exists cannot be proven consistent with the standard axioms of set theory. However, it should be clear that assuming the existence of κ is merely the most convenient of the devices mentioned above; none of the results proven in this book will depend on this assumption in an essential way.

1.2.16 The ∞ -Category of Spaces

The category of sets plays a central role in classical category theory. The main reason is that *every* category \mathcal{C} is enriched over sets: given a pair of objects $X, Y \in \mathcal{C}$, we may regard $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ as an object of Set . In the higher-categorical setting, the proper analogue of Set is the ∞ -category \mathcal{S} of *spaces*, which we will now introduce.

Definition 1.2.16.1. Let $\mathcal{K}\mathrm{an}$ denote the full subcategory of Set_{Δ} spanned by the collection of Kan complexes. We will regard $\mathcal{K}\mathrm{an}$ as a simplicial category. Let $\mathcal{S} = \mathrm{N}(\mathcal{K}\mathrm{an})$ denote the (simplicial) nerve of $\mathcal{K}\mathrm{an}$. We will refer to \mathcal{S} as the *∞ -category of spaces*.

Remark 1.2.16.2. For every pair of objects $X, Y \in \mathcal{K}\mathrm{an}$, the simplicial set $\mathrm{Map}_{\mathcal{K}\mathrm{an}}(X, Y) = Y^X$ is a Kan complex. It follows that \mathcal{S} is an ∞ -category (Proposition 1.1.5.10).

Remark 1.2.16.3. There are many other ways to construction a suitable “ ∞ -category of spaces.” For example, we could instead define \mathcal{S} to be the (topological) nerve of the category of CW complexes and continuous maps. All that really matters is that we have a ∞ -category which is equivalent to $\mathcal{S} = N(\text{Kan})$. We have selected Definition 1.2.16.1 for definiteness and to simplify our discussion of the Yoneda embedding in §5.1.3.

Remark 1.2.16.4. We will occasionally need to distinguish between large and small spaces. In these contexts, we will let \mathcal{S} denote the ∞ -category of small spaces (defined by taking the simplicial nerve of the category of small Kan complexes), and $\widehat{\mathcal{S}}$ the ∞ -category of large spaces (defined by taking the simplicial nerve of the category of *all* Kan complexes). We observe that \mathcal{S} is a large ∞ -category and that $\widehat{\mathcal{S}}$ is even bigger.