Chapter 1

Introduction

This book relates the most modern aspects and most recent developments in the theory of planar quasiconformal mappings and their application in conformal geometry, partial differential equations (PDEs) and nonlinear analysis. There are profound applications in such wide-ranging areas as holomorphic dynamical systems, singular integral operators, inverse problems, the geometry of mappings and, more generally, the calculus of variations—all of which are presented here. It is a simply amazing fact that the mathematics that underpins the geometry, structure and dimension of such concepts as Julia sets and limit sets of Kleinian groups, the spaces of moduli of Riemann surfaces, conformal dynamical systems and so forth is the *very same* as that which underpins existence, regularity, singular set structure and so forth for precisely the most important class of equations one meets in physical (and other) applications, namely, second-order divergence-type equations. All these theories are inextricably linked in two dimensions by the theory of quasiconformal mappings.

Because of these and other compelling applications, there has recently been considerable pressure to extend classical results from conformal geometry to more general settings, for instance, to obtain optimal bounds on the existence, regularity and geometric properties of solutions of quasilinear and general nonlinear systems in the plane both in the classical elliptic setting and now in the degenerate elliptic setting. Here one moves from the established theory of quasiconformal mappings, through the theory of weakly quasiregular mappings, and comes to the more general class of Sobolev mappings of finite distortion. This progression is natural as one seeks greater knowledge about the fine properties of these mappings for implementation. Even for such well-known problems as the nonlinear $\overline{\partial}$-problem, we find that precise L^2-bounds lead to a simple and beautiful proof of the extension theorem for holomorphic motions. In the same vein, we use optimal regularity to prove Pucci's conjecture, as well as related precise results to give a solution to Calderón's problem on impedance tomography and also to Painlevé's problem on the size and structure of removable singular sets for solutions to elliptic and degenerate elliptic equations.

These precise results are in a large part due to a new understanding of the relationship between quasiconformal mappings and holomorphic flows on the one hand, and, on the other, precise results on the L^p-invertibility of classes of singular integral operators called Beltrami operators. However, there have been other recent developments in the theory of quasiconformal mappings—notably in the field of analysis on metric spaces principally established by Heinonen and Koskela. These advances could not leave a book such as ours untouched, for they clarify many of the basic facts and the precise hypotheses necessary to prove them and often provide elementary and clear proofs. Thus the reader will find novelty and simplicity here even for the foundations of the theory, which now go back more than half a century.

Another novelty in the approach of this book is the use of many of the significant advances in harmonic analysis made over the last few decades; these include H^1-BMO duality, maximal function estimates, the theory of nonlinear commutators and integral estimates for Jacobian determinants both above and below their natural Sobolev domain of definition, all crucial for our studies of optimal regularity and nonlinear PDEs, as well as the Painlevé problem on removable singularities. The reader will have ample opportunity to see these powerful modern techniques in diverse applications.

1.1 Calculus of Variations, PDEs and Quasiconformal Mappings

The strong interplay among the calculus of variations, partial differential equations and the geometric theory of mappings (which is what this book is all about) has a long and distinguished history—going back at least to d'Alembert who in 1746 first related the derivatives of the real and imaginary part of a complex function in his work on hydrodynamics [51, p. 497]. These equations came to be known as the Cauchy-Riemann equations.

Conservation laws and equations of motion or state in physics and mathematics are described by divergence-type second-order differential equations. This is no accident. It is a fundamental precept of physics that a system acts so as to minimize some action functional—Hamilton's principle of least action. Hamilton's principle applies quite generally to classical fields such as the electromagnetic, gravitational and even quantum fields. We are therefore naturally led to study the minima of energy functionals, regularity of minimizers and other aspects of the calculus of variations. We give a classical problem a review in the next section. Loosely, minima satisfy an associated Euler-Lagrange equation that appears in divergence form as a result of integration by parts in the derivation of the equation. Similar examples appear in continuum mechanics and materials science.

On the other hand, general conservation laws are described as follows. Suppose the flux density of a scalar quantity e, such as density, concentration, temperature or energy, is $q = \mathcal{A}(z, \nabla e)$, a function of the gradient of e. A basic assumption of continuum physics is that the gain of the physical quantity in a domain Ω corresponds to the loss of this quantity across the boundary $\partial\Omega$. Thus

$$\int_{\partial\Omega} q \cdot \nu = \int_{\Omega} f$$

Here f denotes the source density and ν denotes the outer normal. The above identity is called the conservation law with respect to the flux q and leads (we describe how in Section 16.3) to the differential equation

$$\dot{e} + \operatorname{div}(q) = \dot{e} + \operatorname{div}\mathcal{A}(z, \nabla e) = f$$

This is a conservation law for the physical quantity e. In the steady-state case we obtain a second-order equation in divergence form for q.

With so many compelling applications in hand, it is no wonder that there is considerable interest in the topological and analytic properties of the minimizers of various functionals and also in solutions of second-order equations in divergence form. These topological and analytic properties describe, for instance, the flow lines of the field and the structure and size of any singular set.

Let us explain using an elementary example from the calculus of variations how related first- and second-order equations might arise. Consider deforming the unit disk $\mathbb{D}$ to another domain Ω minimizing energy. This was in fact Riemann's approach to his mapping theorem, and which he called the Dirichlet principle. He obtained the desired conformal mapping as an absolute minimizer of the Dirichlet energy. Weierstrass showed Riemann's argument was not generally valid, however Hilbert later ironed out the details—ultimately requiring some regularity of $\partial\Omega$. As this discussion suggests, the example is quite classical, but it's solution contains many key ideas and provides us with some important lessons.

Problem. *Given a simply connected domain* Ω*,*
(a) *find the homeomorphism of minimal energy mapping the disk to* Ω*,*
(b) *find the minimizer subject to prescribed boundary values.*

The energy of a mapping is defined as the Dirichlet integral, so we are asked to find

$$\min_{f:\mathbb{D}\to\Omega} \left\{ \int_{\mathbb{D}} \|Df(z)\|^2 \, dz \right\}, \qquad \|Df\|^2 = |f_x|^2 + |f_y|^2,$$

over all homeomorphisms, with the possible restriction $f|\partial\mathbb{D} = g_o$. In order to solve this problem (if it is possible at all), we should consider the correct function space to start looking for a solution. For the minimum to be finite, we certainly need for there to be some mapping f_0 satisfying the hypotheses (the gradient of f_0 should be square-integrable with correct boundary values). Given this mapping, we can then assume the existence of a sequence tending to the minimum (a *minimizing sequence*). Then comes the difficult problem of proving this sequence has a convergent subsequence whose limit is sufficiently regular to satisfy the hypotheses (thus the need for a priori estimates). For the problem in hand, Hadamard's inequality for matrices $A \in \mathbb{R}^{2\times 2}(\mathbb{C})$ states $\|A\|^2 = \mathrm{tr}\,(A^t A) \geqslant 2 \det A$ and therefore gives the pointwise almost everywhere estimate

$$\|Df(z)\|^2 \geqslant 2\, J(z,f) = 2 \det\, Df(z)$$

(we consider only orientation-preserving homeomorphisms, meaning that the Jacobian $J(z,f) \geqslant 0$ almost everywhere in $\mathbb{D}$.) Then for every homeomorphism of Sobolev class $W^{1,2}(\mathbb{D})$, we have

$$\int_{\mathbb{D}} \|Df\|^2 \geqslant 2 \int_{\mathbb{D}} J(z,f) = 2|f(\mathbb{D})| = 2|\Omega|,$$

providing a lower bound on the minimum. Consequently, if there is to be an absolute minimizer f achieving this lower bound we must have it solving the first(!)-order equation for an absolute minimizer

$$\|Df(z)\|^2 = 2\, J(z,f)$$

Some linear algebra (we have equality in Hadamard's estimate) shows this to be equivalent to

$$D^t f(z)\, Df(z) = J(z,f)\, \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. This is the equation for a conformal mapping, of course—in complex notation this system is the Cauchy-Riemann equations (which points to the virtue of complex notation). Back to our problem, if we prescribe the boundary values and they happen not to be those of a conformal mapping, then a minimizer cannot achieve our a priori lower bound. Another approach is to vary a supposed minimizer f by a parameterized family of homeomorphisms of $\mathbb{D}$ that are the identity near the boundary, say φ_t normalized so $\varphi_0(z) = z$. Since f is a minimizer we must have

$$\frac{d}{dt} \int_{\mathbb{D}} \|D(f \circ \varphi_t)\|^2 \,\Big|_{t=0} = 0,$$

leading to the second-order Euler-Lagrange equation for f, $\mathrm{div}Df = \Delta f = 0$. Thus the minimum should be a harmonic mapping with the given boundary values, and the question boils down to whether our prescribed boundary values g_0 have a harmonic homeomorphic extension to $\mathbb{D}$. The Poisson formula gives a harmonic function, and we are left to discuss the topological properties of this solution. A way forward here is to show that the Jacobian is continuous and does not vanish (so local injectivity) and use the monodromy theorem, but the geometry of the domain and the boundary values must come into play. For instance, without some convexity assumption on Ω the mean value of g_0 may lie outside Ω. It is a classical theorem of Choquet, Kneser and Rado that as soon as Ω is convex, one can solve the posed problem with homeomorphic boundary data and the solution is a smooth diffeomorphism.

We may consider the above problem in more general circumstances. For instance, if $H : \Omega \to \mathbb{R}^{2\times 2}$, symmetric and positive definite, is some measurable function describing some material property of Ω, we could seek to minimize the new energy functional

$$\int_{\mathbb{D}} \langle H(f(z))Df(z), Df(z)\rangle$$

We use Hadamard's inequality in the form

$$\langle \sqrt{H}Df, \sqrt{H}Df\rangle \geqslant 2\sqrt{\det H}\,\det(Df)$$

and, as before, an absolute minimizer must satisfy the nonlinear PDE

$$D^t fH(f)Df = \sqrt{\det H(f)}\,J(z,f)\,\mathbf{I}$$

It is only in two dimensions that such an equation is not overdetermined (this accounts for higher-dimensional rigidity), and we have the possibility of finding a solution in quite reasonable generality. For conformal geometry we are interested in the case $\det H \equiv 1$ yielding the nonlinear Beltrami equation

$$D^t f\, H(f)\, Df = J(z,f)\mathbf{I}$$

If in the above we consider a tensor field $G : \mathbb{D} \to \mathbb{R}^{2\times 2}$, $\det G \equiv 1$, we have

$$D^t f\, Df = J(z,f)\, G,$$

equivalent to a linear (over $\mathbb{C}$) equation called the complex Beltrami equation,

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\,\frac{\partial f}{\partial z}\,,$$

which we will spend quite a bit of time discussing. Finally, if we consider a constrained problem and look for the Euler-Lagrange equation we are quickly led to second-order equations in divergence form (arising from the necessary integration by parts) for the real and imaginary parts of $f = u + iv$,

$$\operatorname{div} G^{-1}\,\nabla u = 0, \qquad \operatorname{div} G^{-1}\,\nabla v = 0,$$

and ultimately to more general second-order equations in divergence form.

There are a few important points we would like to draw from this discussion regarding minima of variational problems:

- Unconstrained or absolute minimizers of variational functionals are likely to satisfy first-order differential equations.
- Constrained or stationary mappings will likely satisfy a second-order differential equation.
- We may well find stationary solutions that are not minimizers. Indeed, there might not be a minimizer within the class of homeomorphisms.

Of course, in the most general setting of multiple connected domains, one would consider minimizers in a given homotopy class of maps between domains, or more generally, homotopy classes of maps between Riemann surfaces. Moreover we would seek to minimize more general functionals. Here we find clear connections with Teichmüller theory, surface topology and so forth.

A significant portion of this book is given over to the study of the equations like those we have discovered above where we will seek existence, uniqueness and optimal regularity and so forth for their solutions—and also for the counterparts to these equations in other settings. Later we shall discuss recent developments in the study of existence and uniqueness properties for mappings between planar domains whose boundary values are prescribed and have the smallest mean distortion—this will bring the relevance of the first example discussed above back into focus because of a surprising connection with harmonic mappings and other surprises as well. Indeed, the analogy here with Teichmüller theory is quite strong. This theory is partly concerned with extremal quasiconformal mappings in a homotopy class. These mappings minimize the L^∞-norm of the distortion. We investigate what happens when the L^1-norm of the distortion is minimized instead. Further, in these studies we will find many new and unexpected phenomena concerning existence, uniqueness and regularity for these extremal problems where the functionals are polyconvex but typically not convex. These seem to differ markedly from phenomena observed when studying multi-well functionals in the calculus of variations. The phenomena observed concerning mappings between annuli present a case in point.

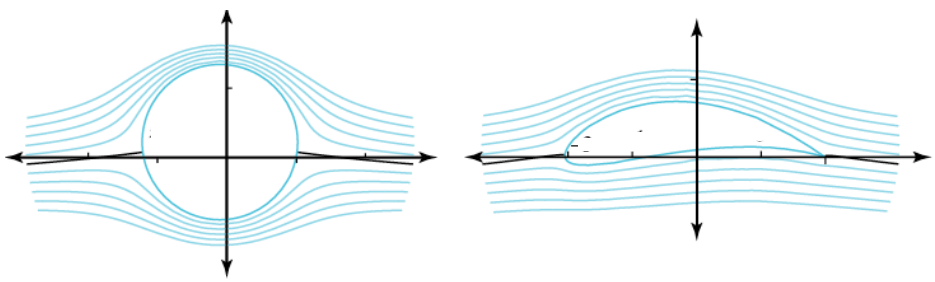
In two dimensions, the methods of complex analysis, conformal geometry and quasiconformal mappings provide powerful techniques, not available in other dimensions, to solve highly nonlinear partial differential equations, especially those in divergence form. Of course the relevance of divergence-type equations to quasiconformal mappings is not new. It has been evident to researchers for at least 70 years, beginning with M.A. Lavrentiev [224], C.B. Morrey [271, 273, 272], R. Caccioppoli [83], L. Bers and L. Nirenberg [54, 56, 57], B. Bojarski [68], Finn [126, 127] and Serrin [325], among many others. In the literature one finds concrete applications in materials science, particularly, nonlinear elasticity, gas flow and fluid flow, and in the calculus of variations going back generations.

One of the primary aims of this book is to give a thorough account of this classical theory from a modern perspective and connect it with the most recent developments.

1.2 Degeneracy

As we have suggested, the equations we consider arise naturally in hydrodynamics, nonlinear elasticity, holomorphic dynamics and several other areas. A good part of this book is concerned with these equations at the extreme limits of regularity and related assumptions on the coefficients. A particular aim is to develop tools to handle these situations where a system of equations might degenerate. Here is a natural example.

In two-dimensional hydrodynamics, the fluid velocity (the gradient of the potential function—see for instance (16.51)—satisfies a Beltrami equation that degenerates as the flow approaches a critical value, the local speed of sound; see (16.54).



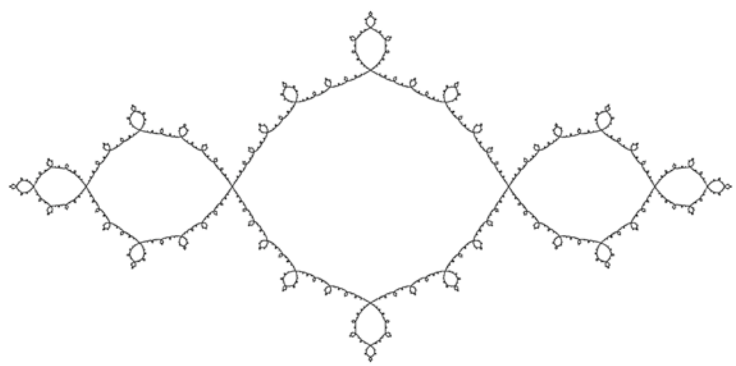
Subsonic fluid flow around a disk and Joukowski aerofoil

What happens as we break the speed of sound? In the 1950s when this was a problem of particular importance, the existence of a shock wave boundary was supposed, presupposing, albeit with good evidence, the particular structure of the singular set; see for instance [100, 144]. On one side of the shock boundary one had an elliptic equation, and on the other a hyperbolic equation. The very early approaches had to assume some degree of analyticity and used various schemes of successive approximations: the Rayleigh-Janzen expansion of the potential function in a power series in the stream Mach number or a modification of this method due to Prandtl and the solution of mixed (transonic) flows by means of power series in the space variables. Perturbative methods were also employed, most based on von Kármán's similarity law for transonic flow [207]. The state of the art as of the late 1950s is described in L. Bers' well-known book [54], and although there has been a great deal of literature on the subject since, most has focused on the study of shock waves in a similar sort of setup (and of course in higher dimensions). In this book we will describe the precise limits of existence and regularity and the structure of the singular set in the degenerate setting—but where there is no shock wave. This allows for isolated points (or even Cantor sets) where one might have degeneracies such as infinite density or pressure. The precise conditions are described in terms of bounded mean oscillation (BMO) bounds on the distortion function of the coefficient - leading to the theory of mappings with exponentially integrable distortion. This was first realized by G. David [103], and here we present substantial sharpening and refinement of these early results. When applied in the setting described above, this theory shows the topological properties of the streamlines and so forth to be the same as those for subsonic incompressible flows (really the Stoilow factorization theorem showing that these mappings are topologically equivalent to analytic mappings).

Thus a significant problem addressed in this book is to see how to relax the classical assumptions on the Beltrami equations making them uniformly elliptic, so as to study the nonuniformly elliptic (that is, degenerate elliptic) setting and yet save as much of the theory as possible.

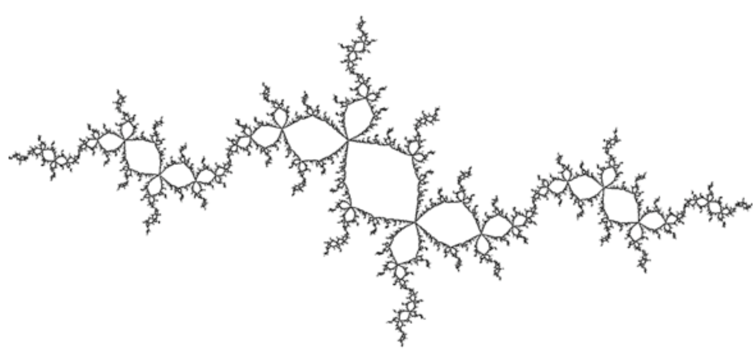
1.3 Holomorphic Dynamical Systems

There are two basic examples of holomorphic dynamical systems. First, is the classical Fatou-Julia theory of iteration of rational mappings of the sphere, [90, 123, 203, 262]. Hardly anyone has not seen the beautiful pictures [293] of the Mandelbrot set and associated Julia sets of quadratic mappings.



The Julia set of a quadratic polynomial.

The theory of quasiconformal maps has played a key role in the study of these conformal dynamical systems ever since D. Sullivan, A. Douady and their coauthors introduced them to the theory [108, 111, 237, 257, 341]. Ideas such as quasiconformal surgery show how one conformal dynamical system can be constructed from another.



The Julia set after quasiconformal surgery: grafted with Douady's rabbit.

A crucial discovery for us, which underpins a good deal of our approach in this book, is the concept of holomorphic motions intoduced by R. Mañé, P. Sad and D. Sullivan [237] and the subsequent conjectures on the extension of these motions by Sullivan and W.P. Thurston [342] and the solution by Z. Słodkowski [329]. This discovery really shows the notions of holomorphically parameterized flows and quasiconformal mappings to be inextricably linked.

In this book we will provide tools that allow one to study the structure, dimension and other properties of Julia sets. As far as the question of degeneracy goes, we will see that the Julia set of $\lambda z + z^2$ is a $\frac{1+|\lambda|}{1-|\lambda|}$-quasicircle if $|\lambda| < 1$ (equivalently for z^2+c when $c = \lambda/2 - \lambda^2/4$ lies in the primary component of the Mandelbrot set) and degeneracy occurs as $|\lambda| \to 1$. The dynamical systems obtained are quasiconformally equivalent on hyperbolic components of parameter space. The intriguing question of what happens as $|\lambda| \to 1$ (or generally moves to the boundary of a hyperbolic component) and the uniform bounds in the theory of quasiconformal mappings are lost is still to some measure unresolved. Haïssinsky has shown, using David's work, that for real $\lambda \nearrow 1$ ($c \nearrow \frac{1}{4}$) the sequence of Julia sets converges to a Jordan curve—the cauliflower Julia set—that is the image of the unit circle under a mapping of exponentially integrable distortion.

The second classical example of quasiconformal mappings being applied in conformal dynamical systems is the way they arise naturally in the study of Kleinian groups; through Teichmüller theory and moduli spaces. The modern approach goes back to Bers' seminal work on simultaneous uniformization [53] and Ahlfors' use of quasiconformal mappings in proving geometric finiteness [5]. The key idea again is that in moduli space the Kleinian groups in question are quasiconformally equivalent. What happens as one goes to the boundary and considers, for instance, degenerating sequences of quasi-fuchsian groups? Again we loose the uniform estimates needed in the classical theory of quasiconformal mappings and need to analyze a degenerate situation. We hope that mappings of finite distortion may play a future role in the analytic understanding of these questions.

1.4 Elliptic Operators and the Beurling Transform

The types of first-order equations $\mathcal{L} f = 0$ we have seen above have evolved from study of the Cauchy-Riemann operators,

$$\mathcal{L}_1 f = \frac{\partial}{\partial \bar{z}} f, \qquad \mathcal{L}_2 f = \frac{\partial}{\partial z} f$$

The solutions to $\mathcal{L}_i f = 0$, $i = 1, 2$, represent analytic and anti-analytic functions. A quantatative distinction between these two classes of mappings is that the former are orientation-preserving and the later orientation-reversing (or positive versus negative Jacobian determinant). In fact, this topological dichotomy of solutions applies to *all* first-order elliptic PDEs in the complex plane. The continuous deformation of a general elliptic system $\mathcal{L} f = 0$, perhaps by varying the coefficients, will never change the orientation of solutions unless ellipticity is violated at some moment.

This idea leads to the homotopy classification of all first-order elliptic systems and the corresponding differential operators into the two classes represented by the Cauchy-Riemann equations $\mathcal{L}_1 f = 0$ and its dual $\mathcal{L}_2 f = 0$. The fundamental connection between these classes is made via the Beurling transform, about which we will have much to say. It is a singular integral operator $\mathcal{S}$ of Calderón-Zygmund type bounded in $L^p(\mathbb{C})$, $1 < p < \infty$. It is the remarkable property

$$\mathcal{S} \circ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} : C_0^\infty(\mathbb{C}) \to C_0^\infty(\mathbb{C})$$

intertwining the Cauchy-Riemann operators that makes it so important in the L^p-theory of elliptic operators. There are six homotopy classes of second-order elliptic operators in the complex plane, or three equivalence classes of elliptic equations, comprising of combinations of the z- and $\bar{z}$-derivatives. The most important of these is the complex Laplace equation $\frac{\partial^2}{\partial z \partial \bar{z}} f = 0$. Notice that for this equation the “factors” $\frac{\partial}{\partial \bar{z}}$ and $\frac{\partial}{\partial z}$ come from different homotopy classes.

For the other two classes, the first-order factors come from the same homotopy class and this partly explains why the equations and their solutions have significantly different features. For instance, the Fredholm alternative fails; as an example, the equation $f_{\bar{z}\bar{z}} = 0$, $f|\partial\mathbb{D} = 0$, admits the uncountable family of solutions $f(z) = (1 - |z|^2)h(z)$, where h is holomorphic and continuous in the closed unit disk.

A major innovation in this book is the study of second-order PDEs of divergence form in the complex plane,

$$\mathrm{div}\mathcal{A}(z, \nabla u) = 0 \tag{1.1}$$

when $\mathcal{A}$ is only supposed δ-monotone, with *no additional regularity assumption.* Here we are still able to obtain a reduction to a first-order system for the complex gradient $f = u_z$ of a solution and show that it is a quasiregular mapping, if $\mathcal{A}$ is spatially independent. In this way the significant results we obtain for such mappings apply to show that f has good regularity and nice topological properties which the solution u then inherits.

Another important approach is via the duality given by the Hodge * operator. Here we reduce the $\mathcal{A}$-harmonic equation (1.1) to the first-order system

$$-\mathcal{A}(z, \nabla u) = *\nabla v$$

for a function v called the $\mathcal{A}$-harmonic conjugate of u. This approach is particularly useful when $\mathcal{A}(z, \nabla u) = A(z)\nabla u$ for some measurable $A : \Omega \to \mathbb{R}^{2\times 2}$. This leads us to the quasiregular mapping $f = u + iv$ in much the same way as an analytic function is composed of a harmonic function and its harmonic conjugate.

There are also very interesting questions concerning the convergence of *sequences* of operators that we shall address in the book. Here we will meet the notion of G-convergence, which emerges in quite a natural way and exploits the normal family (equicontinuity) properties of quasiregular mappings.

While we give evidence of substantial progress in the theory of elliptic second-order equations in the complex plane, we are sure there remains many interesting phenomena to be discovered and interesting connections to other areas of mathematics to be found.