Chapter One

Introduction

In this short introductory chapter, we introduce the main problem of stability and stabilization of equilibria, and indicate briefly the central role it plays in mathematical control theory. The presentation here is mostly informal. Precise definitions are given later. The chapter serves to give some perspective while stating the primary theme of the text.

We start with a discussion of simple equations from an elementary differential equations course in order to contrast open loop control and feedback control. These examples lead us to a statement of the main problem considered in the book, followed by an indication of the central importance of stability and stabilization in mathematical control theory. We then note a few important omissions. A separate section gives a complete chapter-by-chapter description of the book. The final section of the chapter is a list of suggested collateral reading.

1.1 OPEN LOOP CONTROL

Students of elementary differential equations already have experience with open loop controls. These controls appear as a given time-dependent forcing term in the second order linear equations that are covered in the first course on the subject. A couple of simple examples will serve to illustrate the notion of open loop control and allow us to set the stage for a discussion of feedback control in the next section.

The Forced Harmonic Oscillator. Consider the nonhomogeneous linear mass-spring equation with unit mass and unit spring constant,

$$\ddot{y} + y = u(t).$$

We use $\dot{y}$ and $\ddot{y}$ to denote the first and second derivatives of $y(t)$ with respect to time. The equation involves a known right-hand side, which can be viewed as a preprogrammed, or *open loop*, control defined by $u(t)$. The general real-valued solution for such equations is considered in differential equations courses, and it takes the form

$$y(t) = y_h(t) + y_p(t),$$

where $y_p(t)$ is any particular solution of the nonhomogeneous equation and $y_h(t)$ denotes the general solution of the homogeneous equation, $\ddot{y} + y = 0$.

For this mass-spring equation, we have

$$y_h(t) = c_1 \cos t + c_2 \sin t,$$

where the constants c_1 and c_2 are uniquely determined by initial conditions for $y(0)$ and $\dot{y}(0)$.

Suppose the input signal is $u(t) = \sin t$. This would not be an effective control, for example, if our purpose is to damp out the motion asymptotically or to regulate the motion to track a specified position or velocity trajectory. Since the frequency of the input signal equals the natural frequency of the unforced harmonic oscillator, $\ddot{y} + y = 0$, the sine input creates a resonance that produces unbounded motion of the mass.

On the other hand, the decaying input $u(t) = e^{-t}$ yields a particular solution given by $y_p(t) = \frac{1}{2}e^{-t}$. In this case, every solution approaches a periodic response as $t \to \infty$, given by $y_h(t)$, which depends on the initial conditions $y(0)$ and $\dot{y}(0)$, but not on the input signal.

Suppose we wanted to apply a continuous input signal which would guarantee that all solutions approach the origin defined by zero position and zero velocity. It is not difficult to see that we cannot do this with a continuous open loop control. The theory for second-order linear equations implies that there is no continuous open loop control $u(t)$ such that each solution of $\ddot{y} + y = u(t)$ approaches the origin as $t \to \infty$, independently of initial conditions.

The Double Integrator. An even simpler equation is $\ddot{y} = u(t)$. The general solution has the form $y(t) = c_1 + c_2 t + y_p(t)$, where $y_p(t)$ is a particular solution that depends on $u(t)$. Again, there is no continuous control $u(t)$ that will guarantee that the solutions will approach the origin defined by zero position and zero velocity, independently of initial conditions.

Open loop, or preprogrammed, control does not respond to the state of the system it controls during operation. A standard feature of engineering design involves the idea of injecting a signal into a system to determine the response to an impulse, step, or ramp input signal. Recent work on the active approach to the design of signals for failure detection uses open loop controls as test signals to detect abnormal behavior [22]; an understanding of such open loop controls may enable more autonomous operation of equipment and condition-based maintenance, resulting in less costly or safer operation.

The main focus of this book is on principles of stability and feedback stabilization of an equilibrium of a dynamical system. The next section explains this terminology and gives a general statement of this core problem of dynamics and control.

1.2 THE FEEDBACK STABILIZATION PROBLEM

The main theme of stability and stabilization is focused by an emphasis on time invariant (autonomous) systems of the form

$$\dot{x} = f(x),$$

where $f : \mathcal{D} \subset \mathbf{R}^n \to \mathbf{R}^n$ is a continuously differentiable mapping (a smooth vector field on an open set $\mathcal{D} \subset \mathbf{R}^n$) and $\dot{x} := \frac{dx}{dt}$. If f is continuously differentiable, then f satisfies a local Lipschitz continuity condition in a neighborhood of each point in its domain. From the theory of ordinary differential equations, the condition of local Lipschitz continuity of f guarantees the existence and uniqueness of solutions of initial value problems

$$\dot{x} = f(x), \quad x(0) = x_0,$$

where x_0 is a given point of $\mathcal{D}$.

The state of the system at time t is described by the vector x. Assuming that $f(0) = 0$, so that the origin is an equilibrium (constant) solution of the system, the core problem is to determine the stability properties of the equilibrium. The main emphasis is on conditions for asymptotic stability of the equilibrium. A precise definition of the term *asymptotic stability* of $x = 0$ is given later. For the moment, we simply state its intuitive meaning: Solutions $x(t)$ with initial condition close to the origin are defined for all forward time $t \geq 0$ and remain close to $x = 0$ for all $t \geq 0$; moreover, initial conditions sufficiently close to the equilibrium yield solutions that approach the equilibrium asymptotically as $t \to \infty$.

We can now discuss the meaning of *feedback stabilization* of an equilibrium. Let $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ be a continuously differentiable function of $(x, u) \in \mathbf{R}^n \times \mathbf{R}^m$. The introduction of a feedback control models the more complicated process of actually measuring the system state and employing some mechanism to feed the measured state back into the system as a real time control on system operation. The feedback stabilization problems in this book involve autonomous systems with control u, given by

$$\dot{x} = f(x, u).$$

In this framework, the introduction of a smooth (continuously differentiable) state feedback control $u = k(x)$ results in the *closed loop system*

$$\dot{x} = f(x, k(x)),$$

which is autonomous as well. If $f(0, 0) = 0$, then the origin $x_0 = 0$ is an equilibrium of the unforced system, $\dot{x} = f(x, 0)$. If the feedback satisfies $k(0) = 0$, then it preserves the equilibrium; that is, the closed loop system also has an equilibrium at the origin.

We apply stability theory in several different settings to study the question of existence of a continuously differentiable feedback $u = k(x)$ such that the origin $x_0 = 0$ is an asymptotically stable equilibrium of the closed loop system. For certain system classes and conditions, explicit stabilizing feedback controls are constructed. The system classes we consider are not chosen arbitrarily; they are motivated by (i) their relevance in the research activity on stabilization of recent decades, and (ii) their accessibility in an introductory text.

Feedback in the Harmonic Oscillator and Double Integrator Systems. The system corresponding to the undamped and unforced harmonic oscillator, obtained by writing $x_1 = y$ and $x_2 = \dot{y}$, and setting $u = 0$, is given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1. \end{aligned}$$

This system does not have an asymptotically stable equilibrium at the origin $(x_1, x_2) = (0, 0)$. If we had both state components available for feedback, we could define a feedback control of the form $u = k_1 x_1 + k_2 x_2$, producing the closed loop system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= (k_1 - 1)x_1 + k_2 x_2. \end{aligned}$$

If we can measure only the position variable x_1 and use it for feedback, say $u = k_1 x_1$, then we are not able to make the origin $(0, 0)$ asymptotically stable, no matter what the value of the real coefficient k_1 may be. However, using only feedback from the velocity, if available, say $u = k_2 x_2$, it is possible to make the origin an asymptotically stable equilibrium of the closed loop system. Verification of these facts is straightforward, and to accomplish it, we can even use the second order form for the closed loop system; for position feedback only, $\ddot{y} + (1 - k_1)y = 0$; for velocity feedback only, $\ddot{y} - k_2\dot{y} + y = 0$. For position feedback, the characteristic equation is $r^2 + (1 - k_1) = 0$, and the general real-valued solution for $t \geq 0$ is (i) periodic for $k_1 < 1$, (ii) the sum of an increasing exponential term and a decreasing exponential term for $k_1 > 1$, and (iii) a constant plus an unbounded linear term for $k_1 = 1$. For velocity feedback, choosing $k_2 < 0$ ensures that all solutions that start close to the origin at time $t = 0$ remain close to the origin for all $t \geq 0$, and also satisfy $(x_1(t), x_2(t)) \to (0, 0)$ as $t \to \infty$.

For the simpler double integrator equation, $\ddot{y} = u(t)$, or its equivalent system,

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u, \end{aligned}$$

one can check that neither position feedback, $u = k_1 x_1$, nor velocity feedback, $u = k_2 x_2$, can make all solutions approach the origin as $t \to \infty$. However, feedback using both position and velocity, $u = k_1 x_1 + k_2 x_2$, will accomplish this if $k_1 > 0$ and $k_2 > 0$.

The study of stability and stabilization of equilibria for ordinary differential equations (ODEs) is a vast area of applications-oriented mathematics. The restriction to smooth feedback still leaves a huge area of results. This area will be explored in selected directions in the pages of this introductory text.

The restriction to smooth feedback avoids some technical issues that arise with discontinuous feedback, or even with merely continuous feedback.

Discontinuous feedback is mathematically interesting and relevant in many applications. For example, the solutions of many optimal control problems (not discussed in this book) involve discontinuous feedback. However, a systematic study of such feedback requires a reconsideration of the type of system under study and the meaning of solution. These questions fall essentially outside the scope of the present book.

Although we consider primarily smooth feedback controls, at several places in the text the admissible open loop controls are piecewise continuous, or at least integrable on any finite interval, that is, *locally integrable*.

The Importance of the Subject

Stability theory provides core techniques for the analysis of dynamical systems, and it has done so for well over a hundred years, at least since the 1892 work of A. M. Lyapunov; see [73]. An earlier feedback control study of a steam engine governor, by J. Clerk Maxwell, was probably the first modern analysis of a control system and its stability. Stability concepts have always been a central concern in the study of dynamical control systems and their applications. The problem of feedback stabilization of equilibria is a core problem of mathematical control theory. Possibly the most important point to make here is that many other issues and problems of control theory depend on concepts and techniques of stability and stabilization for their mathematical foundation and expression. Some of these areas are indicated in the end-of-chapter Notes and References sections.

Some Important Omissions

There are many important topics of stability, stabilization, and, more generally, mathematical control theory which are not addressed in this book. In particular, as mentioned in the Preface, there is no discussion of transfer function analysis for linear time invariant systems, and transfer functions are not used in the text. Also, there is no systematic coverage of optimal control beyond the single section on the algebraic Riccati equation. Since there is no coverage of numerical computation issues in this text, readers interested specifically in numerical methods should be aware of the text by B. N. Datta, *Numerical Methods for Linear Control Systems*, Elsevier Academic Press, London, 2004.

The end-of-chapter Notes and References sections have resources for a few other areas not covered in the text.

1.3 CHAPTER AND APPENDIX DESCRIPTIONS

In general, the chapters follow a natural progression. It may be helpful to mention that readers with a background in the state space framework of linear system theory and a primary interest in nonlinear systems might proceed with Chapter 8 (Stability Theory) after the introductory material of Chapter 2 (Mathematical Background) and Chapter 3 (Linear Systems and Stability). Definitions and examples of stability and instability appear in Chapter 3. For such readers, Chapters 4–7 could be used for reference as needed.

Chapter 2. The Mathematical Background chapter includes material mainly from linear algebra and differential equations. For basic analysis we reference Appendix B or the text [7]. The section on linear and matrix algebra includes some basic notation, linear independence and rank, similarity of matrices, invariant subspaces, and the primary decomposition theorem. The section on matrix analysis surveys ifferentiation and integration of matrix functions, inner products and norms, sequences and series of functions, and quadratic forms. A section on ordinary differential equations states the existence and uniqueness theorem for locally Lipschitz vector fields and defines the Jacobian linearization of a system at a point. The final section has examples of linear and nonlinear mass-spring systems, pendulum systems, circuits, and population dynamics in the phase plane. These examples are familiar from a first course in differential equations. The intention of the chapter is to present only enough to push ahead to the first chapter on linear systems.

Chapter 3. This chapter develops the basic facts for linear systems of ordinary differential equations. It includes existence and uniqueness of solutions for linear systems, the stability definitions that apply throughout the book, stability results for linear systems, and some theory of Lyapunov equations. Jordan forms are introduced as a source of examples and insight into the structure of linear systems. The chapter also includes the Cayley-Hamilton theorem. A few basic facts on linear time varying systems are included as well.

The next four chapters, Chapters 4–7, provide an introduction to the four fundamental structural concepts of linear system theory: controllability, observability, stabilizability, and detectability. We discuss the invariance (or preservation) of these properties under linear coordinate change and certain feedback transformations. All four properties are related to the study of stability and stabilization throughout these four chapters. While there is some focus on single-input single-output (SISO) systems in the examples, we include basic results for multi-input multi-output (MIMO) systems as well. Throughout Chapters 4–7, Jordan form systems are used as examples to help develop insight into each of the four fundamental concepts.

Chapter 4. Controllability deals with the input-to-state interaction of the system. This chapter covers controllability for linear time invariant systems. Single-input controllable systems are equivalent to systems in a special companion form (controller form). Controllability is a strong sufficient condition for stabilization by linear feedback, and it ensures the solvability of transfer-of-state control problems. The chapter includes the eigenvalue placement theorem for both SISO and MIMO systems, a controllability normal form (for uncontrollable systems) and the PBH controllability test. (The PBH controllability test and related tests for observability, stabilizability, and detectability are so designated in recognition of the work of V. M. Popov, V. Belevitch, and M. L. J. Hautus.)

Chapter 5. Observability deals with the state-to-output interaction of the system. The chapter covers the standard rank criteria for observability and the fundamental duality between observability and controllability. Lyapunov equations are considered under some special hypotheses. The chapter includes an observability normal form (for unobservable systems), and a brief discussion of output feedback versus full-state feedback.

Chapter 6. This chapter on stabilizability begins with a couple of standard stabilizing feedback constructions for controllable systems, namely, linear feedback stabilization using the controllability Gramian and Ackermann's formula. We characterize stabilizability with the help of the controllability normal form and note the general limitations on eigenvalue placement by feedback when the system is not controllable. The chapter also includes the PBH stabilizability test and some discussion on the construction of the controllability and observability normal forms.

Chapter 7. Detectability is a weaker condition than observability, but it guarantees that the system output is effective in distinguishing trajectories asymptotically, and this makes the property useful, in particular, in stabilization studies. The chapter begins with an example of an observer system for asymptotic state estimation. We define the detectability property, and establish the PBH detectability test and the duality of detectability and stabilizability. We discuss the role of detectability and stabilizability in defining observer systems, the role of observer systems in observer-based dynamic stabilization, and general linear dynamic controllers and stabilization. The final section provides a brief look at the algebraic Riccati equation, its connection with the linear quadratic regulator problem, and its role in generating stabilizing linear feedback controls.

Chapter 8. Chapter 8 presents the basic concepts and most important Lyapunov theorems on stability in the context of nonlinear systems. We discuss the use of linearization for determining asymptotic stability and instability of equilibria, and we define critical problems of stability and smooth stabilization. We state Brockett's necessary condition for smooth stabilization. This chapter also develops basic properties of limit sets and includes the invariance theorem. There is a discussion of scalar equations which is useful for examples, a section on the basin of attraction for asymptotically stable equilibria, and a statement of converse Lyapunov theorems.

CHAPTER 9. Chapter 9 develops the stability properties of equilibria for cascade systems. The assumptions are strengthened gradually through the chapter, yielding results on Lyapunov stability, local asymptotic stability, and global asymptotic stability. Two foundational results lead to the main stability results: first, the theorem on total stability of an asymptotically stable equilibrium under a class of perturbations; second, a theorem establishing that the boundedness of certain driven trajectories in a cascade implies the convergence of those trajectories to equilibrium. Cascade systems play a central role in control studies; they arise in control problems directly by design or as a result of attempts to decompose, or to transform, a system for purposes of analysis. The final section shows that cascade forms may also be obtained by appropriate aggregation of state components.

CHAPTER 10. Center manifold theory provides tools for the study of critical problems of stability, the problems for which Jacobian linearization cannot decide the issue. Many critical problems can be addressed by the theorems of Chapter 8 or Chapter 9. However, center manifold theory is an effective general approach. The chapter begins with examples to show the value of the center manifold concept and the significance of dynamic behavior on a center manifold. Then we state the main results of the theory: (i) the existence of a center manifold; (ii) the reduction of stability analysis to the behavior on a center manifold; and (iii) the approximation of a center manifold to an order sufficient to accomplish the analysis in (ii). Two applications of these ideas are given in this chapter: the preservation of smooth stabilizability when a stabilizable system is augmented by an integrator, and a center manifold proof of a result on asymptotic stability in cascades with a linear driving system. Another application, on the design of a center manifold, appears in Chapter 11.

CHAPTER 11. In this chapter we consider single-input single-output systems and the zero dynamics concept. We define the relative degree at a point, the normal form, the zero dynamics manifold, and the zero dynamics subsystem on that manifold. Next, we consider asymptotic stabilization by an analysis of the zero dynamics subsystem, including critical cases. A simple model problem of aircraft control helps in contrasting linear and nonlinear problems and their stability analysis. The concept of vector relative degree for multi-input multi-output systems is defined, although it is used within the text only for the discussion of passive systems with uniform relative degree one. (Further developments on MIMO systems with vector relative degree, or on systems without a well-defined relative degree, are available through resources in the Notes and References.) The chapter ends with two applications: the design of a center manifold for the airplane example, and the computation of zero dynamics for low-dimensional controllable linear systems which is useful in Chapter 15.

Chapter 12. We consider feedback linearization only for single-input single-output systems. A single-input control-affine nonlinear system is locally equivalent, under coordinate change and regular feedback transformation in a neighborhood of the origin, to a linear controllable system, if and only if the relative degree at the origin is equal to the dimension of the state space. Feedback linearizable systems are characterized by geometric conditions that involve the defining vector fields of the system. The proof of the main theorem involves a special case of the Frobenius theorem, which appears in an Appendix. Despite the lack of robustness of feedback linearization, there are important areas, for example mechanical systems, where feedback linearization has achieved successes. Most important, the ideas of feedback linearization have played an important role in the development of nonlinear geometric control.

Chapter 13. In the first section of this chapter, we present a theorem on the global stabilization of a special class of nonlinear systems using the feedback construction known as damping control (also known as *L_gV control*, or *Jurdjevic-Quinn feedback*). This theorem provides an opportunity to contrast the strong connections among Lie brackets, controllability, and stabilization for linear systems, with the very different situation of nonlinear systems. Thus, the second section shows that the Lie bracket-based generalization of the controllability rank condition does not imply a local controllability property of the nonlinear system, and even global controllability does not imply stabilizability by smooth feedback. (The definition of controllability used here is the same one used for linear systems: any point can be reached from any other point in finite time along a trajectory corresponding to some admissible open loop control.) We give references for more information on controllability ideas and their application.

Chapter 14. The passivity concept has roots in the study of passive circuit elements and circuit networks. Passivity is defined as an input-output property, but passive systems can be characterized in state-space terms by the KYP property. (The KYP property is so designated in recognition of the work of R. E. Kalman, V. A. Yakubovich, and V. M. Popov.) This chapter develops the stability and stabilization properties of passive systems. It is an exploration of systems having relative degree at the opposite extreme from the feedback linearizable systems: passive systems having a smooth storage function have uniform relative degree one. Moreover, systems that are feedback passive, that is, passive with a smooth positive definite storage function after application of smooth feedback, are characterized by two conditions: they have uniform relative degree one and Lyapunov stable zero dynamics in a neighborhood of the origin. Passivity plays an important role in the feedback stabilization of cascades in Chapter 15.

Chapter 15. Chapter 15 returns to cascade systems. Partial-state feedback, which uses only the states of the driving system, is sufficient for local asymptotic stabilization of a cascade. In general, however, partial-state feedback cannot guarantee global asymptotic stabilization without restrictive growth assumptions on the interconnection term in the driven system of the cascade. This chapter considers an important situation in which global stabilization is assured using full-state feedback. We assume that the driving system is feedback passive with an output function that appears in the interconnection term in an appropriate factored form; global asymptotic stabilization is then achieved with a constructible feedback control.

Chapter 16. This chapter motivates the input-to-state stability concept based on earlier considerations in the text. In particular, input-to-state stability (ISS) addresses the need for a condition on the driven system of a cascade that guarantees not only (i) bounded solutions in response to bounded inputs, but also (ii) converging solutions from converging inputs. This material requires an introduction to the properties of comparison functions from Appendix E. The comparison functions are used in a proof of the basic Lyapunov theorems on stability and asymptotic stability. We give the definition of ISS Lyapunov function and present the main result concerning them: a system is ISS if and only if an ISS Lyapunov function exists for it. This result is applied to establish the ISS property for several examples. We state a result on the use of input-to-state stability in cascade systems and provide some further references.

Chapter 17. This brief chapter collects some additional notes on further reading.

Appendix A. This brief key to notation provides a convenient reference.

Appendix B. This appendix provides a quick reference for essential facts from basic analysis in $\mathbf{R}$ and $\mathbf{R}^n$.

Appendix C. This material on ordinary differential equations is self-contained and includes proofs of basic results on existence and uniqueness of solutions, continuation of solutions, and continuous dependence of solutions on initial conditions and on right-hand sides.

Appendix D. This material on manifolds and the preimage theorem is useful background for the center manifold chapter (Chapter 10) as well as Chapters 11 and 12, which deal with some aspects of geometric nonlinear control. The material on distributions and the Frobenius theorem supports Chapters 11–12 specifically.

Appendix E. The comparison functions are standard tools for the study of ordinary differential equations; they provide a convenient language for expressing basic inequality estimates in stability theory. This material is used explicitly in the text only in Chapter 16 (and in a brief appearance in the proof of Theorem 10.2 (d) on center manifold reduction).

Appendix F. Some hints and answers to selected exercises are included here.

1.4 NOTES AND REFERENCES

For some review of a first course in differential equations, see [21] and [84]. For additional recommended reading in differential equations to accompany this text, see [15] and [40]. In addition, see the text by V. I. Arnold, *Ordinary Differential Equations*, MIT Press, Cambridge, MA, 1973, for its geometric and qualitative emphasis.

The texts [9] and [72] have many examples of control systems described by ordinary differential equations. The material in these books is accessible to an audience having a strong background in upper level undergraduate mathematics. The same is true of the texts [53] and [80]. A senior level course in control engineering is contained in K. Ogata, *Modern Control Engineering*, Prentice-Hall, Upper Saddle River, NJ, third edition, 1997.

For the mathematical foundations of control theory for linear and nonlinear systems, see [91], which is a comprehensive mathematical control theory text on deterministic finite-dimensional systems. It includes material on a variety of model types: continuous and discrete, time invariant and time varying, linear and nonlinear. The presentation in [91] also includes many bibliographic notes and references on stabilization and its development, as well as three chapters on optimal control.

For an interesting and mostly informal article on feedback, see [60].