CHAPTER 1

The First Labor: The Nemean Lion

From Apollodorus:

First, Eurystheus ordered him to bring the skin of the Nemean lion; now that was an invulnerable beast begotten by Typhon. On his way to attack the lion he came to Cleonae and lodged at the house of a day-laborer, Molorchus; and when his host would have offered a victim in sacrifice, Hercules told him to wait for thirty days, and then, if he had returned safe from the hunt, to sacrifice to Saviour Zeus, but if he were dead, to sacrifice to him as to a hero. And having come to Nemea and tracked the lion, he first shot an arrow at him, but when he perceived that the beast was invulnerable, he heaved up his club and made after him. And when the lion took refuge in a cave with two mouths, Hercules built up the one entrance and came in upon the beast through the other, and putting his arm round its neck held it tight till he had choked it; so laying it on his shoulders he carried it to Cleonae. And finding Molorchus on the last of the thirty days about to sacrifice the victim to him as to a dead man, he sacrificed to Saviour Zeus and brought the lion to Mycenae. Amazed at his manhood, Eurystheus forbade him thenceforth to enter the city, but ordered him to exhibit the fruits of his labours before the gates. They say, too, that in his fear he had a bronze jar made for himself to hide in under the earth, and that he sent his commands for the labours through a herald, Copreus, son of Pelops the Elean. This Copreus had killed Iphitus and fled to Mycenae, where he was purified by Eurystheus and took up his abode.

1.1. The Tasks

The Nemean lion was ravaging the countryside near the town of Nemea, which is northwest of Mycenae. Hercules has three tasks to complete. After he tracks the lion, he shoots an arrow at the beast. First, he determines the speed at which his arrow strikes the invulnerable lion, given an angle of elevation and a distance to the lion (the **Shooting an Arrow** problem). In the second task, Hercules determines which set of regular polygons will allow him to tile the area of the cave mouth and trap the lion inside the cave (the **Closing the Cave Mouth** problem). As an associated exercise, Hercules chooses the mouth of the cave which will give him the greatest chance of finding the lion (the **Zeus Makes a Deal** problem).

1.1.1. Shooting an Arrow

TASK: Calculate the speed at which an arrow strikes the lion at a distance of 200 meters given a launch angle of 20°. Assume Hercules aims for the lion's head and shoulder area, which is the same distance off the ground as the arrow when it leaves Hercules' bow. Ignoring air resistance, how long does it take the arrow to travel from Hercules' bow to the lion?

1.1.2. Hercules Closes the Cave Mouth

TASK: To defeat the lion, Hercules must close up one cave entrance and attack the lion through the other. He finds several stacks of tiles nearby, each of which contains sets of regular polygons. There is one stack of equilateral triangles, one stack of squares, one stack of regular pentagons, one stack of regular hexagons, and one stack of regular octagons. Which stack(s) of polygons will allow Hercules to construct an edge-to-edge tiling in order to close up the mouth of the cave with no two tiles overlapping?

1.1.3. Exercise: Zeus Makes a Deal

TASK: Suppose that the cave has three, rather than two, mouths and that the lion is hiding just inside one of the mouths. Hercules selects one of the three cave mouths at random and is about to enter when Zeus, the king of the gods, suddenly tells him that the lion is not in a second cave mouth (not the one Hercules has choosen). Should Hercules change his mind and enter the remaining third mouth to the cave?

1.2. The Solutions

1.2.1. Shooting an Arrow

From Apollodorus:

> First, Eurystheus ordered him to bring the skin of the Nemean lion; now that was an invulnerable beast begotten by Typhon. On his way to attack the lion he came to Cleonae and lodged at the house of a day-laborer, Molorchus; and when his host would have offered a victim in sacrifice, Hercules told him to wait for thirty days, and then, if he had returned safe from the hunt, to sacrifice to Saviour Zeus, but if he were dead, to sacrifice to him as to a hero. And having come to Nemea and tracked the lion, he first shot an arrow at him.

TASK: Calculate the speed at which an arrow strikes the lion at a distance of 200 meters given a launch angle of 20°. Assume Hercules aims for the lion's head and shoulder area, which is the same distance off the ground as the arrow when it leaves Hercules' bow. Ignoring air resistance, how long does it take the arrow to travel from Hercules' bow to the lion?

SOLUTION: Let's place a coordinate axis system so that the point where the arrow leaves Hercules' bow is at the origin when time $t = 0$. It is natural to assume constant acceleration and let the positive y-axis be vertically upward. The constant acceleration, directed only downward,

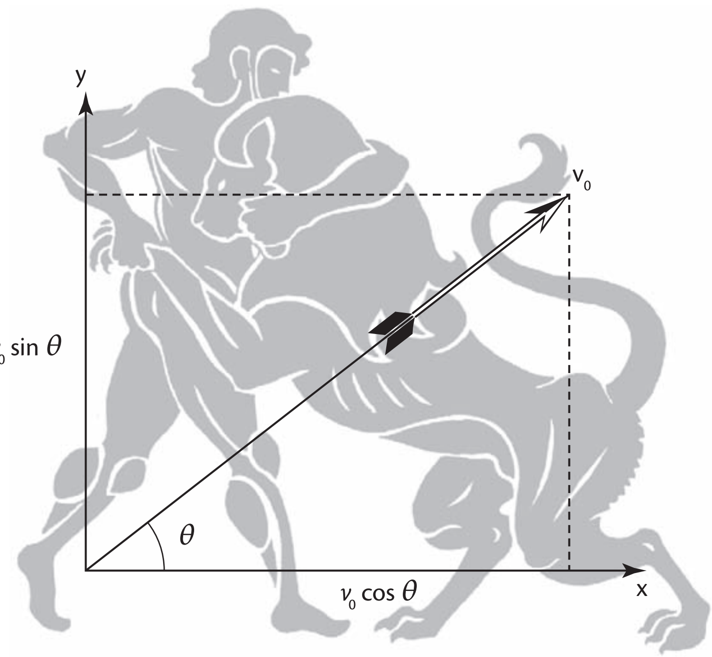

Figure 1.1. Determining the velocity components of Hercules' arrow.

is due to gravity and is denoted by $-g$. Further, think of this as a two-dimensional motion of the arrow through the air. We will neglect any effects the air might have on the arrow (in a simplified model, Hercules can neglect any effects of wind on the arrow's time of flight).

Because of our reference system, the initial position is given by $x(0) = y(0) = 0$. The initial velocity at time $t = 0$, which is at the exact instant the arrow begins its flight, is given by $v(0) = v_0$. With constant acceleration, we obtain the velocity by multiplying the acceleration by time t and adding the initial velocity:

$$v(t) = -gt + v_0. \tag{1.1}$$

Using trigonometry (see Figure 1.1), we can then determine the initial x- and y-components of v_0 as

$$v_{x0} = v_0 \cos\theta \quad \text{and} \quad v_{y0} = v_0 \sin\theta.$$

Since there is no acceleration in the x-direction, the horizontal component of velocity will remain constant. What does this mean? The horizontal velocity component ($v_0 \cos\theta$) will keep its initial value throughout the flight of the arrow. The vertical component, however, will change because of the downward acceleration. The x- and y-components of velocity (at any time t) become

$$v_x = v_0 \cos\theta \quad \text{and} \quad v_y = v_0 \sin\theta - gt.$$

The expression for v_y comes from replacing v_0 in Equation 1.1 with $v_0 \sin\theta$. Integrating these velocity components with respect to time, we can now determine the x- and y-components of the arrow's position at any time. These are given by

$$x = (v_0 \cos\theta)t \quad \text{and} \quad y = (v_0 \sin\theta)t - \frac{gt^2}{2}.$$

We can now calculate the horizontal distance (the range) that the arrow will travel before striking the lion. Setting the y-component to zero (recall that the vertical height where the arrow strikes the lion is the same as the height of the arrow leaving the bow), we obtain

$$(v_0 \sin\theta)t - \frac{gt^2}{2} = 0.$$

So, solving for t, we find that either $t = 0$ or

$$t = (2v_0 \sin\theta)/g. \tag{1.2}$$

Substituting this expression for t into the x-component, we find that the distance the arrow travels is

$$\begin{aligned} \text{distance} &= (v_0 \cos\theta) \times \frac{2v_0 \sin\theta}{g} \\ &= \frac{2v_0^2 \sin\theta \cos\theta}{g}. \end{aligned}$$

We can now solve for the speed at which the arrow leaves Hercules' bow, v_0. Substituting in the launch angle ($\theta = 20^\circ$), range (200 meters), and gravitational constant (9.8 meters per second squared), we find that

$$v_0^2 = \frac{200 \times 9.8}{2 \times \sin 20^\circ \times \cos 20^\circ}.$$

Evaluating and taking the square root of both sides,

$$v_0 \approx 55.22\,\text{meters/second} \approx 198.79\,\text{kilometers/hour.}$$

Therefore, Hercules must fire the arrow with an initial velocity of almost 200 kilometers per hour (125 miles per hour) from his mighty bow. This assumes that the speed of the arrow remains constant in flight (as mentioned above).

How long does it take the arrow to reach the lion after it leaves the bow? Substituting $v_0 \approx 55.22$ meters per second into Equation 1.2, we find

$$t = \frac{2v_0 \sin\theta}{g} = \frac{2 \times 55.22 \times \sin 20^\circ}{9.8} \approx 3.85\,\text{seconds.}$$

Unfortunately, Hercules discovers that “the beast [is] invulnerable,” as Apollodorus writes. Hercules’ arrow will not penetrate the lion’s hide. To capture the mighty lion, Hercules moves to another task in this labor.

1.2.2. Hercules Closes the Cave Mouth

From Apollodorus:

> And when the lion took refuge in a cave with two mouths, Hercules built up the one entrance and came in upon the beast through the other, and putting his arm round its neck held it tight till he had choked it; so laying it on his shoulders he carried it to Cleonae.

TASK: To defeat the lion, Hercules must close up one cave entrance and attack the lion through the other. He finds several stacks of tiles nearby, each of which contains sets of regular polygons. There is one stack of equilateral triangles, one stack of squares, one stack of regular pentagons, one stack of regular hexagons, and one stack of regular octagons (see Figure 1.2). Which stack(s) of polygons will allow Hercules to construct an edge-to-edge tiling in order to close up the mouth of the cave with no two tiles overlapping?

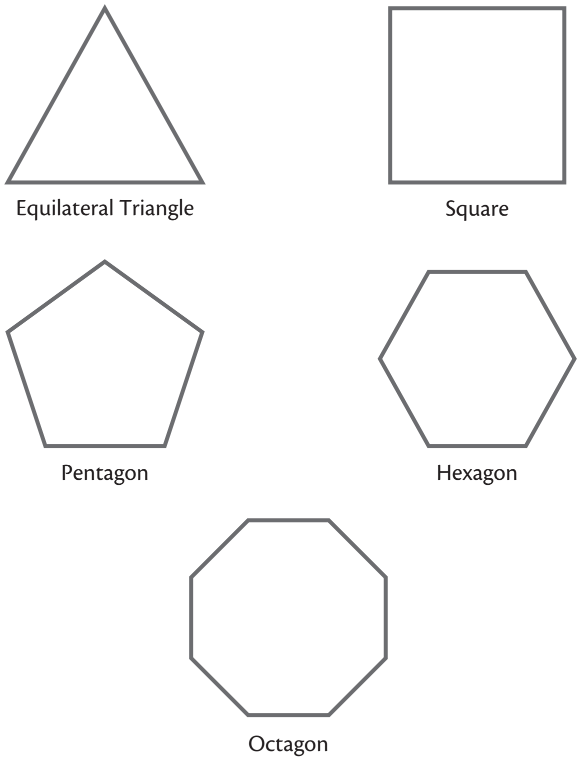

Figure 1.2. The stacks of regular polygons.

SOLUTION: Polygons are figures having many sides, usually four or more. A *tiling* is a covering of the entire plane with nonoverlapping figures. An *edge-to-edge tiling* is one in which the edge of a tile coincides completely with the edge of a bordering tile. We can assume that each entrance to the cave lies in a plane, so that the polygons can close it up by being stacked vertically edge upon edge (if possible). Hercules is not concerned with small gaps at the polygon/cave mouth interface.

Regular polygons were thought to have special meaning for the ancient Greeks, possibly beacuse of their simple yet high degree of symmetry. A *regular polygon* is a figure whose sides are equal and whose interior angles are also equal. Examples of regular polygons are equilateral triangles and squares. The ancient Greek philosopher Plato speculated that geometric solids formed from regular polygons (called *polyhedra*) were the shapes of the fundamental components

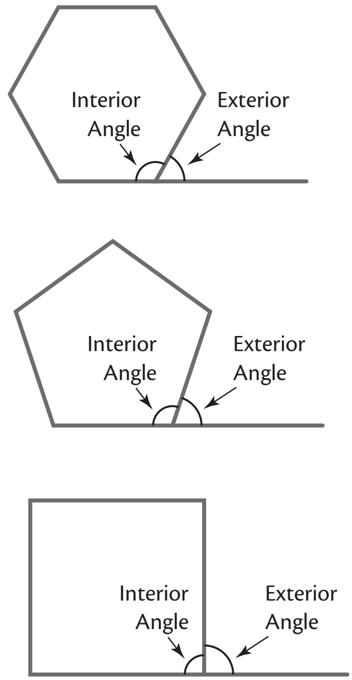

Figure 1.3. Regular polygons with interior and exterior angles.

of the physical universe. The word “polyhedra” also comes from the Greek and means “many faces.” There are five basic Platonic solids: the *tetrahedron*, formed with 4 equilateral triangles; the *cube*, formed with 6 squares; the *octahedron*, formed with 8 equilateral triangles; the *dodecahedron*, formed with 12 pentagons; and the *icosahedron*, formed with 20 equilateral triangles. For the universe to be in harmony, there should be a one-to-one correspondence between the five solids and the primordial elements—earth, fire, water, and air. Therefore, Plato dedicated the dodecahedron (with its pentagonal faces) to the heavens and the rest to the planet Earth.

When Hercules takes a particular tile from one of the stacks, he measures the angles of the polygon. An *interior* angle is an angle formed between two adjacent sides of a polygon; an *exterior* angle is formed between one side and an extension of another side of a polygon. Figure 1.3 shows a few regular polygons with both interior and exterior angles depicted.

So, back to the task at hand, determining how many of each polygon can meet at a point. In order to have an edge-to-edge tiling, Hercules must fit the polygons together in such a way that their interior angles add up to 360° without overlap. Of the five regular polygons that are available to Hercules (triangle, square, pentagon, hexagon, and octagon), which may be used as an edge-to-edge tiling to block the mouth to the cave, thus trapping the Nemean lion inside? Each exterior angle in an n-sided polygon has $360/n$ degrees. As the number of sides of the polygon increases, the total of the interior angles also increases. In fact, the total of the interior angles is equal to $(n-2) \times 180°$. The table below lists the five regular polygons and their respective interior and exterior angles.

Polygon	*Interior Angle*	*Exterior Angle*
Triangle	60°	120°
Square	90°	90°
Pentagon	108°	72°
Hexagon	120°	60°
Octagon	135°	45°

So, for any regular polygon having more than six sides, each interior angle must be greater than 120° and each exterior angle must be less than 60°. Notice that the octagon satisfies this requirement. Before we develop the solutions, photocopy Figure 1.2 and try to make a tiling with the regular polygons.

Let's start with the triangles. Six equilateral triangles come together at a single point. The sum of the interior angles is thus $6 \times 60° = 360°$. Hercules can use equilateral triangles to close the cave without overlap. Since the sum of the interior angles in a square is $4 \times 90° = 360°$, Hercules finds that squares will also suffice to close up the cave using an edge-to-edge tiling. What about the pentagon? If three pentagons come together, the sum of their interior angles will be $3 \times 108° = 324°$, so three pentagons would be too few. If four pentagons come together, the sum of their interior angles will be $4 \times 108° = 432°$, so four pentagons would be too many. Pentagons will not work.

Hercules then adds up the interior angles for hexagons and octagons. If three hexagons come together, the sum of their interior angles will be $3 \times 120^\circ = 360^\circ$, which shows that three hexagons will work as an edge-to-edge tiling. If three octagons come together, the sum of their interior angles will be $3 \times 135^\circ = 405^\circ$, so three octagons is too many. But if only two octagons are placed together, the sum of their interior angles will be $2 \times 135^\circ = 270^\circ$, which is too few.

Is the hexagon the polygon with the largest number of sides that will tile the plane? For what values of n does $180(n-2)$ divide $360n$? Unless $2n/(n-2)$ is an integer, an n-gon will not work. For $n \geq 7$,

$$2 < \frac{2n}{n-2} < 3,$$

so $n = 6$ is the largest n-gon. Hercules can use the equilateral triangles, the squares, or the regular hexagons to close up one mouth of the cave. Then he can enter through the other mouth and trap and capture the lion.

1.2.3. Exercise: Zeus Makes a Deal

From Apollodorus:

> And when the lion took refuge in a cave with two mouths, Hercules built up the one entrance and came in upon the beast through the other, and putting his arm round its neck held it tight till he had choked it.

TASK: Suppose that the cave has three, rather than two, mouths and that the lion is hiding just inside one of the mouths. Hercules selects one of the three cave mouths at random and is about to enter when Zeus, the king of the gods, suddenly tells him that the lion is not in a second cave mouth (not the one Hercules has choosen). Should Hercules change his mind and enter the remaining third mouth to the cave?

SOLUTION: First, if one of the cave mouths is eliminated, then there seems to be a 50-50 chance that the lion is in either of the two remaining cave mouths. Let's see if this is true.

As an example, assume that Hercules chooses cave mouth 1 and that Zeus has told him that the lion is *not* in cave mouth 2. Hercules must decide whether to switch his choice and enter cave mouth 3 instead of staying with cave mouth 1. There are three outcomes associated with Hercules' choice:

- Find the lion by switching cave mouths
- Don't find the lion by switching cave mouths
- Find the lion by not switching cave mouths

Is it advantageous to Hercules to switch after hearing what Zeus has to say?

We assume that no matter which cave mouth Hercules chooses, Zeus does two things: first, he reveals a mouth that Hercules has not chosen, and second, he does not reveal the actual cave mouth where the lion is (or perhaps the other gods might protest). To solve this task we model the problem as a conditional probability situation. Given two events, how does the information that event 2 has occurred affect the probability assigned to event 1?

We must determine the probability that Hercules will find the lion by switching, given that Zeus tells him the lion is not in one of the mouths. Is this an easy problem to solve? If Hercules develops a strategy to never switch, he will find the lion only one-third of the time. We could think of this in terms of the probability of failure, which is two-thirds. If Hercules does switch, then the probability of finding the lion (success) is two-thirds. In other words, if Hercules switches after receiving information from Zeus, then there are now only two possible entrances to the cave which will lead to success (finding the lion)– the entrance Hercules initially chose and the one that Zeus does not mention.

Using conditional probability, we could derive the mathematics behind the probabilities. However, it makes the discussion much more complicated and really is not necessary (I'll leave it up to you). For example, the probability that Zeus tells Hercules the lion is *not* in cave mouth 2 *given* it is actually in cave mouth 3 is exactly two-thirds. It is therefore advantageous for Hercules to switch and enter a different cave mouth.

There are many internet web sites dedicated to this problem, which is sometimes referred to as "The Let's Make a Deal Problem" or "The Monty Hall Paradox," after the game show host. Several include educational and fun Java applets as well. Give them a try. Type "Java Applet Let's Make a Deal" into your search engine. Pick any one of the applets. As you repeatedly try the applet (say 30 times), you should see that the results converge to the true probability of two-thirds success by switching and one-third success by not switching.

Here's another option: try playing a game with a friend. Your friend will be Hercules, and you will be Zeus. Select three playing cards (one of which represents the lion) and place them face down on a table. Only you (as the mighty Zeus) know which card is the lion. Have your friend select a card but not turn it over. As Zeus, turn over another card which is not the lion. Keep track of your friend's outcomes. After switching every time, the cumulative probability of success will approach two-thirds.

At long last, Hercules successfully captures and then chokes the lion. In paintings and sculptures, Hercules is often seen with the skin of the lion draped over his shoulders. Diodorus writes, "Since he could cover his whole body with it because of its great size, he had in it a protection against the perils which were to follow." Euripides corroborates this story, writing that Hercules "put its skin upon his back, hiding his yellow hair in its fearful tawny gaping jaws."