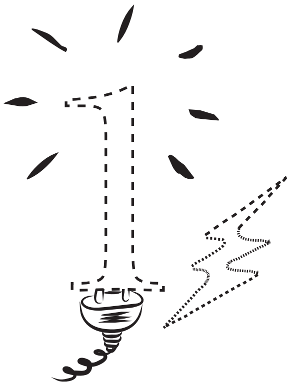


Three Examples of the Mutual Embrace

With arithmetic the Creator adjusted the World to unity, with geometry he balanced the design to give it stability. . . .
— Nicholas of Cusa, *Of Learned Ignorance* (1440)

I omit their [astronomers'] vain disputes about Eccentricks, Concentricks, Epicycles, Retrogradations, Tripidations, accessus, recessus, swift motions and circles of motion, as being the works neither of God nor Nature, but the Fiddle-Faddles and Trifles of Mathematicians.
— Henry Cornelius Agrippa, *Of the Vanity and Uncertainties of Arts and Sciences* (1531)

1.1 Unphysical Laws

As the epigraphs illustrate, the role of mathematics as an aid to understanding the world has been evaluated differently by different philosophers. (I side with Nicholas, not Henry, as you have probably guessed.) The central thesis of this book is that physics needs mathematics, but the converse is often true, too. In this opening discussion I'll show you three examples of what I'm getting at.[1] In the first example you'll see a physical problem that is not difficult to state, and for which it is only very slightly more difficult to actually calculate a solution. It all comes about so smoothly, in fact, that it is easy to convince yourself that the job is done. But it is not, because that solution makes no *physical* sense, and it will be elementary mathematical arguments that show us that. The second example illustrates the converse: a physical problem that is easy to state but that clearly has a nonsensical *mathematical* solution. And it will now be physics that shows us the way out. The final example mirrors my earlier story of C, M, and P, that is, it will show you yet another example of physics "deriving" a mathematical theorem—probably the most famous theorem in all of mathematics, in fact.

For my first example, imagine a one-kilogram mass that we are going to accelerate straight down the positive x-axis, starting from rest at $x = 1$. We'll accelerate the mass by applying the benign-looking position-dependent force $F = x^2$. So, for example, at time $t = 0$ we have the mass motionless at $x = 1$ with an initial applied force of $F = 1$ newton directed toward increasing x. We now ask what appears to be a simple question: where on the x-axis is the mass at time $t = 5$ seconds? The astonishing (I think) answer is, *there is no answer*! The mathematics seems easy enough—I'll show it to you starting right now—but the problem is simply *physical* nonsense. Here's why.

From Newton's second law of motion (net applied force equals mass times acceleration), we have

$$F = x^2 = \frac{d^2x}{dt^2}, \tag{1.1}$$

where our initial conditions are

$$\left.\frac{dx}{dt}\right|_{t=0} = 0, x(t = 0) = 1.$$

To integrate (1.1), begin by defining

$$v = \frac{dx}{dt}.$$

Then (1.1) becomes

$$x^2 = \frac{dv}{dt},$$

or, because of the chain rule from calculus,

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx},$$

and so we have

$$x^2 = v\frac{dv}{dx}.$$

That is,

$$x^2 dx = v dv,$$

which integrates by inspection to give

$$\frac{1}{3}x^3 = \frac{1}{2}v^2 + C,$$

where C is the constant of indefinite integration. From the initial conditions, $x(t = 0) = 1$ while $v(t = 0) = 0$, we see that $C = 1/3$, and so

$$\frac{1}{3}x^3 = \frac{1}{2}v^2 + \frac{1}{3},$$

or

$$x^3 = \frac{3}{2}v^2 + 1 = \frac{3}{2}\left(\frac{dx}{dt}\right)^2 + 1,$$

or

$$\frac{dx}{dt} = v = \sqrt{\frac{2}{3}\left(x^3 - 1\right)}. \tag{1.2}$$

This result tells us the speed of the accelerating mass as a function of its location. In particular, for any finite value of x the speed, too, is finite.

Now, (1.2) integrates immediately to give us our second formal answer, which provides a direct relationship connecting x and t (w, of course, is simply a dummy variable of integration):

$$t = \sqrt{\frac{3}{2}}\int_1^x \frac{dw}{\sqrt{w^3 - 1}}. \tag{1.3}$$

In (1.3) we have a formula that, for a given x, gives us the time in which the mass reaches x, and with (1.2) we can calculate how fast the mass is moving at that location. So, with these two results we are done, right? Well, some readers might wonder about the lower limit on the integral because, at $w = 1$, the denominator of the integrand is zero, and so there might be concern over the very existence of the integral.

However, as I'll show you next, the integral does indeed exist, so that is not a concern. Ironically, however, that very existence is a *problem*, because the integral exists (i.e., t has a finite value) even when the upper limit (x) is infinity! That is,

$$\int_1^\infty \frac{dw}{\sqrt{w^3-1}} < \infty. \tag{1.4}$$

This is disastrous, because that means our one-kilogram mass has moved infinitely far in finite time. Okay, you think about that for a bit while I demonstrate the above claim of existence for (1.4).

We can write the integral in (1.3), for an infinite upper limit, as

$$\int_1^\infty \frac{dw}{\sqrt{w^3-1}} = \int_1^\infty \frac{dw}{\sqrt{w-1}\sqrt{w^2+w+1}} < \int_1^\infty \frac{dw}{\sqrt{w-1}\sqrt{w^2}}.$$

That is,

$$\int_1^\infty \frac{dw}{\sqrt{w^3-1}} < \int_1^\infty \frac{dw}{w\sqrt{w-1}}.$$

Next, change variable to $q = w - 1$. Then,

$$\int_1^\infty \frac{dw}{\sqrt{w^3-1}} < \int_0^\infty \frac{dq}{(q+1)\sqrt{q}} = \int_0^1 \frac{dq}{q^{3/2}+q^{1/2}} + \int_1^\infty \frac{dq}{q^{3/2}+q^{1/2}}.$$

We make the inequality even stronger by replacing each of the two integrals on the right by even larger ones:

$$\int_1^\infty \frac{dw}{\sqrt{w^3-1}} < \int_0^1 \frac{dq}{q^{1/2}} + \int_1^\infty \frac{dq}{q^{3/2}}.$$

That is,

$$\int_1^\infty \frac{dw}{\sqrt{w^3-1}} < (2q^{1/2})\big|_0^1 - (2q^{-1/2})\big|_1^\infty = 2 + 2 = 4.$$

Thus, our one-kilogram mass has moved infinitely far away in less than $\sqrt{\frac{3}{2}} \cdot 4 = 4.899$ seconds.[2] Where the mass is at $t = 5$ seconds, therefore, simply has no answer (can anything be further away than infinity?) Before discussing just what is going on to cause this outrageous result, let me show you another seemingly faultless mathematical analysis that results in an even more outrageous conclusion (as hard as that may be to conceive at this moment).

Imagine that we now have a speed-dependent force, e.g., $F = v^3$. I'll further imagine that, as before, we have a one-kilogram mass to be accelerated by this force down the positive x-axis. But now we'll have it start from the origin $x = 0$ with an initial speed of 1 m/s. Then, $x(t = 0) = 0$ and $v(t = 0) = 1$, and

$$F = v^3 = \frac{d^2x}{dt^2} = \frac{dv}{dt}.$$

This integrates immediately to

$$t = -\frac{1}{2v^2} + C,$$

where C is an arbitrary constant. Since $v(t = 0) = 1$, we have $C = 1/2$, and so

$$\frac{1}{2v^2} = \frac{1}{2} - t,$$

or

$$v = \frac{dx}{dt} = \frac{1}{\sqrt{1-2t}}. \tag{1.5}$$

So, the mass approaches infinite speed as t approaches 1/2 second. This looks bad, alright, but what makes this result really bad is *where* the mass is when its speed becomes arbitrarily large. It is not infinitely far down the x-axis but rather pretty close to where it started from! This is easy to demonstrate.

Integrating (1.5), we have

$$\int_0^x du = \int_0^t \frac{du}{\sqrt{1-2u}} = \left\{ -(1-2u)^{\frac{1}{2}} \right\} \Big|_0^t,$$

or

$$x = 1 - \sqrt{1-2t}. \tag{1.6}$$

That is, as $t \to 1/2$, we have $x \to 1$, which means that our mass achieves infinite speed in just 1/2 second after moving just one meter! After $t = 1/2$ second and $x = 1$ meter, our solution offers no answers to either how fast the mass is moving or where it is. The mass leaves any realm of reality *right in front of our eyes*![3]

This is, of course, absolutely absurd. The problem with both of these examples is that we have assumed *unphysical* forces that have no bound, as well as used physics that is not relativistically correct as the speed of the mass approaches the speed of light. The initial physical situations initially appear to be so benign, however, that our sense of well-being is not threatened or alerted—or at least not until we work through the detailed mathematics. Then we can see the inherent absurdity of the physics and observe just how quickly things spin out of control. Now, let me reverse the situation and give you a situation where it is the mathematics of an analysis that leads us into silliness and physics that both identifies the origin of the difficulty and removes the confusion.

1.2 When Math Goes Wrong

I don't think there has been a calculus textbook written that doesn't have a problem, either as an example or as an end-of-chapter question, that goes something as follows. Imagine, as shown in Figure 1.1, that a pole of length L is leaning against a wall at angle $\theta_0 = \theta(t = 0)$. By some means the bottom end is pulled away from the wall (all the while remaining in contact with the floor) at the constant speed v_0. How fast is the other end sliding down the wall? The intent of such a question in a calculus textbook is purely mathematical, simply to provide a physical situation in which one has to calculate derivatives. From the geometry of the problem we have, from the Pythagorean theorem,

$$x^2 + y^2 = L^2. \tag{1.7}$$

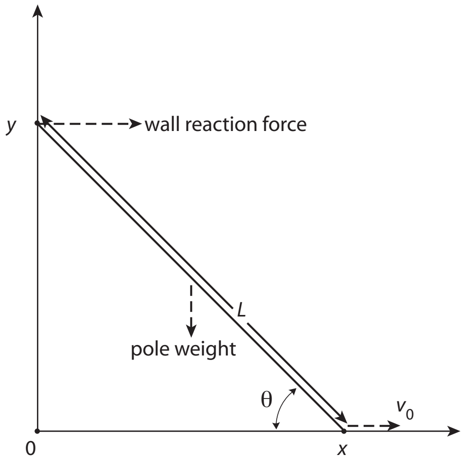


Figure 1.1. A sliding pole.

Then, differentiating (1.7) with respect to time,

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0,$$

or

$$x\cdot v_x+y\cdot v_y=0,$$

where v_x and v_y are, respectively, the speed of the floor end and the speed of the wall end of the pole. Now, from Figure 1.1 we see that $x=L\cos(\theta)$ and $y=L\sin(\theta)$, so (as $v_x=v_0$) we have the answer to the textbook question:

$$v_y=-\frac{v_0L\cos(\theta)}{L\sin(\theta)}=-v_0\cot(\theta). \tag{1.8}$$

The minus sign in (1.8) means that the wall end of the pole is sliding in the direction of decreasing y, i.e., that end is sliding *down* the wall. Notice that, as $\theta\to 0$, the magnitude of the speed of the wall end continually increases; that is, the falling end of the pole is accelerating. This says not only that θ is becoming more and more negative as the fall progresses but also that the rate of increase in the "negativeness" of θ is increasing; that is, $\frac{d^2\theta}{dt^2}<0$, or $-\frac{d^2\theta}{dt^2}>0$.

This is generally all that calculus textbook writers expect students to get from this calculation, but there is an implication in the result that bothers attentive readers. At the instant the falling end of the pole has slid all the way down to the floor, at the moment $\theta = 0$, (1.8) indicates that $v_y = -\infty$. That's right, our analysis says that the left end of the pole is moving infinitely fast at the moment it hits the floor! That can't be right. (Invoking relativistic physics isn't the answer, either: before the speed becomes infinite, the math says the speed at some finite time exceeds the speed of sound, which ought to raise an eyebrow, too: do falling ladders generate sonic booms?) But where did the mathematics go wrong? *That* is the question we have to answer. The origin of the problem lies in (1.7). That's not to say that the Pythagorean theorem is wrong but rather that it simply does not *physically* apply over the entire duration of the fall. At some point during the slide, at some intermediate angle $\theta = \theta_c$, (1.7) ceases to describe what is happening. That means, for $\theta < \theta_c$, it is no longer true that the wall end of the pole remains in contact with the wall. At the instant when θ becomes less than θ_c, the pole will "break away" from the wall. Mathematics alone cannot tell us what θ_c is, but physics *and* mathematics can. Here's how.

Figure 1.1 shows the pole and all the forces acting on it. There is the weight of the pole (mg, where m is the mass of the pole and g is the acceleration of gravity at the Earth's surface) acting at the pole's center of mass, which for a uniform pole is at the midpoint of the pole, directed perpendicular to the floor; there is the reaction force of the wall (F_w) acting on the wall end of the pole perpendicular to the wall; there is the applied force at the floor end of the pole, acting parallel to the floor, and it is the direct cause of the constant speed v_0; and there are, in general, friction wall and floor forces acting at the two ends of the pole. At this point, however, I'll make the simplifying assumption that the wall is frictionless (whether the floor has friction or not won't matter). Then, using Figure 1.1 as a guide, we can write Newton's *rotational* analogue of the second law of motion (net applied torque equals moment of inertia times angular acceleration) as

$$mg\cos(\theta)\cdot\frac{1}{2}L - F_w\sin(\theta)\cdot L = -I\frac{d^2\theta}{dt^2}, \tag{1.9}$$

where I is the moment of inertia of a uniform pole rotating about one end (the floor end). (If the minus sign on the right-hand side of (1.9) puzzles you, look back at my comments just after (1.8).) As shown in any good freshman physics textbook,

$$I = \frac{1}{3} m L^2.$$

Now, a few words about each term in (1.9). Since a *torque* in this problem is the product of a force acting perpendicular to the pole times the *moment arm* of that force (that is, the distance of the application point of the force from the axis of rotation, which is the floor end of the pole), we can see what is going on with the left-hand side of (1.9). The first term is the normal (to the pole) component of the pole's weight times half the length of the pole, and the second term is the normal (to the pole) component of the wall reaction force times the full length of the pole. It is clear that these two torques tend to rotate the pole in opposite senses, and (1.9) follows the convention that a positive torque causes a counterclockwise rotation, which a look at Figure 1.1 shows is in the sense of decreasing θ. The right-hand side of (1.9) is simply moment of inertia times angular acceleration, and as you'll soon see, during the initial part of the slide, when the pole is in contact with the wall, $-I\frac{d^2\theta}{dt^2}$ is indeed positive (because of the net positive torque), just as it should be for a pole that is rotating counterclockwise, i.e., that has its wall end sliding downward toward the floor.[4]

From Figure 1.1 we have $x = L\cos(\theta)$, and so

$$\frac{dx}{dt} = v_0 = -L\sin(\theta)\cdot\frac{d\theta}{dt}, \tag{1.10}$$

or

$$\frac{d\theta}{dt} = -\frac{v_0}{L\sin(\theta)}. \tag{1.11}$$

Differentiating (1.10) and using $\frac{dv_0}{dt} = 0$, we have

$$0 = \sin(\theta)\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}\cos(\theta)\frac{d\theta}{dt},$$

or

$$-\sin(\theta)\frac{d^2\theta}{dt^2} = \cos(\theta)\left(\frac{d\theta}{dt}\right)^2.$$

Using (1.11) in this last result, we have

$$-\frac{d^2\theta}{dt^2} = \frac{v_0^2\cos(\theta)}{L^2\sin^3(\theta)},$$

which is, as claimed, positive over the entire interval $0 \le \theta \le \theta_0 < \pi/2$. Thus, (1.9) becomes

$$\frac{1}{2}mgL\cos(\theta) - F_w L\sin(\theta) = \frac{1}{3}\frac{mv_0^2\cos(\theta)}{\sin^3(\theta)},$$

or, solving for F_w,

$$F_w = \frac{\frac{1}{2}mgL\cos(\theta)}{L\sin(\theta)} - \frac{mv_0^2\cos(\theta)}{3L\sin^4(\theta)}.$$

That is,

$$F_w = \frac{1}{2}mg\cot(\theta) - \frac{\frac{1}{2}mgv_0^2\cot(\theta)}{3\cdot\frac{1}{2}\cdot Lg\sin^3(\theta)},$$

or

$$F_w = \frac{1}{2}mg\cot(\theta)\left[1 - \frac{2v_0^2}{3Lg\sin^3(\theta)}\right]. \tag{1.12}$$

The important observation to make with (1.12) is that, as long as the wall end of the pole is leaning against the wall, we'll have $F_w \ge 0$, as it is physically meaningless to talk of $F_w < 0$. After all, the wall can't pull on the pole! The instant $F_w < 0$ is the instant the wall end of the pole loses contact with the wall.

We can normalize (1.12) by not talking of F_w or v_0^2 directly but of their ratios with the naturally occurring force and speed (squared), mg and gL, respectively. That is, let's write (1.12) as

$$\frac{F_w}{mg} = \frac{1}{2}\cot(\theta)\left[1 - \frac{\frac{2}{3}\cdot\frac{v_0^2}{gL}}{\sin^3(\theta)}\right]. \tag{1.13}$$

When the pole is simply leaning against the wall, and we have not yet begun to pull the lower end, (1.13) tells us that for $v_0^2 = 0$ the normalized reaction force of the wall is

$$\frac{F_w}{mg} = \frac{1}{2}\cot(\theta),\ v_0 = 0.$$

Once we start pulling on the lower end, however, what happens depends, as (1.13) shows, on whether v_0^2/gL is less than or greater than 3/2.

For the first case, that is, for $v_0^2/gL < 3/2$, (1.13) says the normalized wall reaction force does not drop to zero until

$$1 - \frac{\frac{2}{3} \cdot \frac{v_0^2}{gL}}{\sin^3(\theta)} = 0.$$

That is, the wall end of the pole stays in contact with the wall until it reaches the "breakaway" angle of θ_c, where

$$\theta_c = \sin^{-1}\left\{\sqrt[3]{\frac{2}{3} \cdot \frac{v_0^2}{gL}}\right\}. \tag{1.14}$$

If the initial pole angle $\theta_0 < \theta_c$, then the wall end of the pole breaks away from the wall as soon as we start pulling the floor end at speed v_0. And that is precisely what happens in the second case of $v_0^2/gL > 3/2$ no matter what v_0 may be: the pole breaks away from the wall as soon as we start pulling the floor end at *any* speed.

Suppose we concentrate from this point on the first case, of $v_0^2/gL < 3/2$, and assume $\theta_0 > \theta_c$. From (1.14) we have

$$\cot(\theta_c) = \frac{\sqrt{1 - \left(\frac{2}{3} \cdot \frac{v_o^2}{gL}\right)^{2/3}}}{\left(\frac{2}{3} \cdot \frac{v_o^2}{gL}\right)^{1/3}},$$

and so from (1.8) we see that the magnitude of the speed v_{ba} with which the wall end of the pole is sliding downward is, at the breakaway moment,

$$v_{ba} = v_0\sqrt{\left(\frac{3}{2} \cdot \frac{gL}{v_0^2}\right)^{2/3} - 1}.$$

We can write this in a more transparent form as

$$v_{ba} = \sqrt{\left(\frac{3gL}{2}\right)^{2/3} \cdot \frac{v_0^2}{v_0^{4/3}} - v_0^2} = \sqrt{\left(\frac{3gL}{2}\right)^{2/3} \cdot v_0^{2/3} - v_0^2},$$

or

$$v_{ba} = \sqrt{\alpha v_0^{2/3} - v_0^2}, \quad \alpha = \left(\frac{3gL}{2}\right)^{2/3}. \tag{1.15}$$

It is clear from (1.15) that there is some value of v_0 in the interval 0 to $\alpha^{3/4}$ for which v_{ba} reaches its maximum value. We can find that value for v_0 by setting the derivative of (1.15) to zero, which results in the equation

$$\frac{1}{2}(\alpha v_0^{2/3} - v_0^2)^{-1/2}\left\{\frac{2}{3}\alpha v_0^{-1/3} - 2v_0\right\} = 0,$$

or

$$2v_0 = \frac{\frac{2}{3}\alpha}{v_0^{1/3}},$$

or

$$v_0^{4/3} = \frac{\alpha}{3}.$$

This says that

$$v_0^{2/3} = \left(\frac{\alpha}{3}\right)^{1/2}$$

and also that

$$v_0^2 = \left(\frac{\alpha}{3}\right)^{3/2}. \tag{1.16}$$

Substituting these last two results back into (1.15), we get the maximum value for v_{ba}, what I'll call v_{bam}, to be

$$\begin{aligned} v_{bam} &= \sqrt{\left(\frac{3gL}{2}\right)^{2/3}\left(\frac{1}{3}\right)^{1/2}\left(\frac{3gL}{2}\right)^{1/3} - \left(\frac{1}{3}\right)^{3/2}\left(\frac{3gL}{2}\right)} \\ &= \sqrt{\frac{3gL}{2}}\sqrt{\frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}} = \sqrt{\frac{3gL}{2}}\sqrt{\frac{2}{3\sqrt{3}}} = \sqrt{\frac{gL}{\sqrt{3}}} \\ &= \frac{1}{\sqrt[4]{3}}\sqrt{gL} = 0.76\sqrt{gL}. \end{aligned}$$

And finally, from (1.16), we can write

$$\frac{v_0^2}{gL} = \frac{1}{gL}\cdot\left(\frac{\alpha}{3}\right)^{3/2} = \frac{1}{gL}\cdot\frac{1}{3^{3/2}}\cdot\left\{\left(\frac{3gL}{2}\right)^{2/3}\right\}^{3/2} = \frac{1}{gL}\cdot\frac{1}{3^{3/2}}\cdot\frac{3gL}{2} = \frac{1}{2\sqrt{3}}.$$

Inserting this result into (1.14), we find the pole angle θ at the instant the pole breaks away from the wall, for $v_{ba} = v_{bam}$, to be

$$\theta_c = \sin^{-1}\left\{\sqrt[3]{\frac{2}{3}\cdot\frac{1}{2\sqrt{3}}}\right\} = \sin^{-1}\left\{\frac{1}{\sqrt[3]{3^{3/2}}}\right\} = \sin^{-1}\left\{\frac{1}{\sqrt{3}}\right\} = 35.3\text{ degrees.}$$

We could do a lot more similar calculations, but this is enough to make my point: the original calculus textbook solution of (1.8) is only the beginning of what can be done if physics is brought into the analysis, along with the mathematics of calculating derivatives.

1.3 Math from Physics

In the Preface I told you a story in which physics "derived" a mathematical theorem. What I'll show you next is another example of this amazing entanglement of mathematics and physics. When I was a freshman in Physics 51 at Stanford more than fifty years ago (1958), I took a lot of examinations, but one in particular sticks in my memory. One of the questions on that test described a physical situation in which, at the end, the problem was to calculate how far up a glass tube capillary action would draw a fluid. It was a gift question, one the professor had put on the exam to get everybody off to a good start; to answer it all you had to do was remember a formula that had been derived in lecture and in the course text and that we had used on at least a couple of homework assignments. All the exam required was plugging numbers into the formula. The professor had kindly provided all the numbers, too. Unfortunately, I couldn't remember the formula, and so no gift points for me.

Later, back in the dorm, I was talking with a friend in the class, who was most grateful for that gift question; he wasn't doing well in the course, and the "free" points were nice. "So, you remembered the formula, right?" I asked. "Nope, but you didn't have to. I nailed that one, anyway," he replied. "What do you mean, you didn't have to remember the formula?" I asked, a sinking feeling in my stomach. "All you had to do," my friend grinned back, "was just take all the different

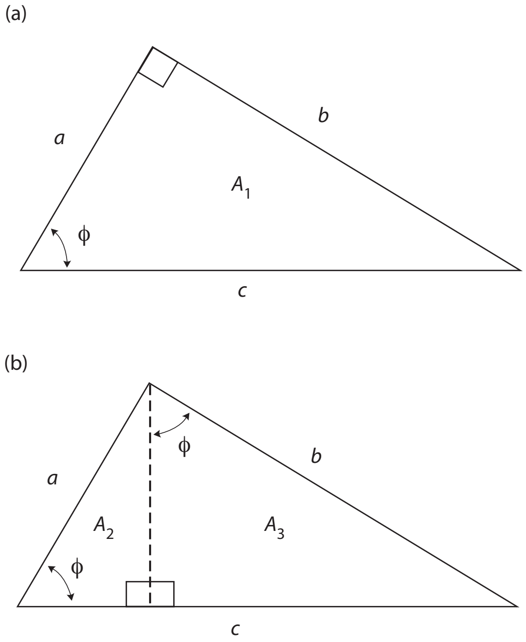


Figure 1.2. Deriving the Pythagorean theorem.

numbers the prof gave us and try them in different ways until the units worked out as a *length*, the unit of *distance up the tube*." "But, but," I sputtered, "that's, that's ... *cheating*!"

But of course, it wasn't cheating. I was just angry at myself for not being sharp enough to have thought of the same idea my friend had. It was my first (painful) introduction to the honorable technique of dimensional analysis. Here, then, as another example of that idea, is how a physicist might derive the Pythagorean theorem using dimensional analysis.

In the upper half of Figure 1.2 I've drawn a right triangle with perpendicular sides of lengths a and b and a hypotenuse of length c. One of the interior acute angles of the triangle is ϕ. I think it obvious that the triangle is *determined* once we know the values of c and ϕ. That is, for a given c and a given ϕ, the other side lengths (a and b) and the remaining interior acute angle all have unique values. And certainly, then, the *area* A_1 of the triangle is also determined. Since area has units of length squared, and since ϕ is dimensionless, it must be that the area depends on the square of c. So, let's write the area of our triangle as

$$A_1 = c^2 f(\phi), \tag{1.17}$$

where $f(\phi)$ is some function of ϕ. (we do not, as you'll soon see, have to actually know the detailed nature of $f(\phi)$!)

Now, draw the perpendicular line from the right angle to the hypotenuse of the triangle, as shown in the lower half of Figure 1.2. This divides the triangle, into two smaller right triangles, one with area A_2, an acute angle ϕ, and hypotenuse a, and another with area A_3, an acute angle ϕ, and hypotenuse b. Thus, just as in (1.17), we can write

$$A_2 = a^2 f(\phi)$$

and

$$A_3 = b^2 f(\phi).$$

Since $A_1 = A_2 + A_3$, then $c^2 f(\phi) = a^2 f(\phi) + b^2 f(\phi)$, and so the unknown function $f(\phi)$ cancels away (that's why we don't have to know what it is) and we suddenly have, dramatically and seemingly out of thin air, the well-known

$$a^2 + b^2 = c^2.$$

That's it. Pretty nifty, don't you think?

CP. P1.1:

> Looking back at the sliding pole problem, assume $\theta_0 > \theta_c$. What is the acceleration of the wall end of the sliding pole at the instant of breakingaway? *Hint*: The answer is independent of v_0 and L and is simply of the form kg, where k is a particular number.

Notes and References

1. The examples discussed here are based on the papers by A. John Mallinckrodt, "The Pathological Kinematics of Unphysical Force Laws"

(*American Journal of Physics*, March 1992, pp. 238–241), and M. Freeman and P. Palffy-Muhoray, “On Mathematical and Physical Ladders” (*American Journal of Physics*, March 1985, pp. 276–277).

2. Matters are actually worse than this. A numerical evaluation of (1.3)—I simply typed the following MATLAB Symbolic Toolbox command line, `sqrt(3/2)*double(int(1/sqrt(x^3-1),1,inf))`—almost instantly produced the value $2.97447742\cdots$ seconds for how long it takes the mass to move out to infinity. (The command `double` is MATLAB’s “double precision” command.) That is, in less than three seconds it becomes meaningless to ask where the mass is.

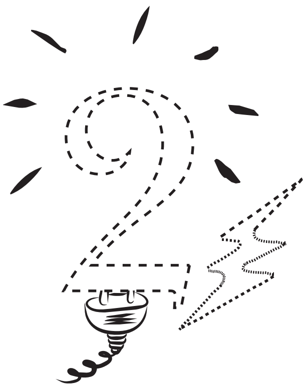
3. As I wrote the line in the text describing the disappearing mass, I couldn’t help but be reminded of a wonderful passage from H. G. Wells’s classic novella, *The Time Machine* (1895). When the story opens, the Time Traveller has invited some friends to his home for dinner, and afterward, over drinks, tells them of his time machine. Soon after he demonstrates a working model of the machine by sending it on a journey through time, in a fantastic scene described by one of the invited friends:

> We all saw the lever turn. I am absolutely certain there was no trickery. There was a breath of wind, and the lamp flame jumped. One of the candles on the mantel was blown out, and the little machine suddenly swung around, became indistinct, was seen as a ghost for a second perhaps, as an eddy of faintly glittering brass and ivory; and it was gone—vanished! Save for the lamp the table was bare. Every one was silent for a minute. Then Filby said he was damned.

4. You can see now that both the friction (if any) force and the “pulling” force at the floor end of the pole don’t appear in (1.9) because those forces have *zero-length* moment arms, and so neither of them produces a torque. If we *did* have a non-zero friction wall force, then that force, acting on the wall end of the downward-sliding pole, would produce a clockwise torque. That complication would not change the character of the results but would complicate the mathematics. Hence my assumption of frictionless surfaces. There is also a perpendicular contact force of the floor on the floor end of the pole, but it too has a zero-length moment arm and so produces no torque. As a final note, observe that all torques are taken about a *moving* point (the floor end of the pole), which is legitimate if the moving point is not accelerating. Since the floor end is moving at the constant speed v_0, this condition is satisfied. For a nice tutorial discussion of this issue, see Fredy R. Zypman, “Moments to Remember: The Conditions for Equating Torque and Rate of

Change of Angular Momentum" (*American Journal of Physics*, January 1990, pp. 41–43). One reviewer has observed that, since the floor is frictionless, to maintain a constant speed for the floor end of the pole, one would actually have to push to the left rather than pull to the right. I have put the word "pulling" in quotes, but perhaps this comment may help make the physics of what is happening more transparent.

5. The definitive modern work on the Pythagorean theorem is the book by Eli Maor, *The Pythagorean Theorem: A 4,000-Year History* (Princeton, N.J.: Princeton University Press, 2007). The dimensional analysis derivation in this discussion is, however, not in Maor's book; I came across it while browsing in a book by A. B. Migdale, *Qualitative Methods in Quantum Theory* (W. A. Benjamin, New York: 1977 [published originally in Russian in 1975], p. 2). Dimensional analysis has been around in mathematical physics for some time. For example, it is used several times in Rayleigh's classic 1877 work, *The Theory of Sound*. (Lord Rayleigh will appear again in this book, most prominently in Discussion 14.)



Measuring Gravity

“Kids, you *should* do this at home!”

2.1 First, a Little Theory

In a number of the discussions to follow in this book, we’ll find that we will need to know the numerical value of the gravitational acceleration at the Earth’s surface, usually denoted, as in the first discussion, by g (pronounced *gee*). The experimental determination of the value of g is, in fact, a classic experiment performed every year in thousands of college freshman physics labs worldwide. I remember well when I did it as a freshman in Physics 51 at Stanford (1958). I remember it as a clunky, uninspiring experiment that required watching a high-speed, pulsed spark generator burn holes through a strip of falling wax paper (I recall that even the graduate student teaching assistant looked like she would rather have been somewhere else). That was followed by the measurement of the distances between adjacent burn holes to eventually arrive, with some arcane intermediate calculations, at a value for g.

Here’s a better way—faster and pedagogically superior—to measure g with a simple two-step experiment. All you’ll need in the way of equipment is a yardstick, a bouncy rubber ball, and a stopwatch.[1] You don’t need an expensive and mysterious (to most college freshmen, anyway) spark generator. You do need to be able to follow a little elementary physics and some simple high school algebra. Then you can measure g where you live in less than 60 seconds.

Begin by imagining that you are holding a nice round bouncy rubber ball in your hand, about hip high, over a concrete floor. You let the ball fall. What happens? Well, of course, it eventually hits the floor and then bounces up, then falls again, then bounces up again, then falls. . . . With each new bounce after the first one the ball doesn't go quite as high as it did on the previous bounce, because energy is being dissipated during this process. Every time the ball hits the concrete floor, for example, both the floor and the ball get a little hotter, and of course the impact noise itself is energy traveling away from the impact point at the speed of sound. After a while the ball stops bouncing and rolls away to a stop somewhere.

Physicists model the decreasing bounce heights with the aid of what is called the *coefficient of restitution*. In our case, we look at the speed of the ball just before it hits the floor (call that speed v_b) and then just after the bounce (call that rebound speed v_a). The coefficient of restitution is then defined to be

$$c = \frac{v_a}{v_b},$$

where, because of energy dissipation on impact, we know $v_a \leq v_b$, and so $c \leq 1$. If we suppose that we initially drop our ball (with mass m) from height h_0, then conservation of energy says

$$\frac{1}{2}mv_b^2 = mgh_0,$$

which is simply equating the kinetic energy of the ball just before the first impact with the potential energy at the start of the first fall. (By doing this you should realize that we are ignoring resistive air drag on the moving ball as being an important energy dissipation mechanism.^{2}) Thus,

$$v_b = \sqrt{2gh_0},$$

and so

$$v_a = c\sqrt{2gh_0}.$$

The kinetic energy of the ball just after the start of the first bounce is therefore

$$\frac{1}{2}mv_a^2 = mc^2gh_0,$$

and so, when the ball reaches the top of the first bounce, all that energy will be potential energy. Therefore, the height of the first bounce (call it h_1) is given by

$$mgh_1 = mc^2gh_0.$$

Thus, $h_1 = c^2h_0$, or

$$c = \sqrt{h_1/h_0}, \tag{2.1}$$

and this gives us a quick way to measure c for our ball and floor. All we have to do is position a yardstick vertically upright, hold the ball in front of and at the top of the yardstick ($h_0 = 36$ inches), and then let the ball fall. If we then observe how high up the yardstick the ball rebounds (h_1), we can calculate c from (2.1), and that's the first half of our experiment to measure g. Now, what do we do with c?

Begin by noticing that if the initial drop height is h_0, then (from above) the first rebound height is $h_1 = c^2h_0$, as well as the next drop height, and so the second rebound height is $h_2 = c^4h_0$. But that says the third rebound height is $h_3 = c^6h_0$, and so on. Now, let's call the first impact of the ball with the floor the $n = 0$ impact. The second impact will be the $n = 1$ impact, and so on. Starting with the $n = 0$ impact, let's write ΔT_1 as the time interval until the $n = 1$ impact, ΔT_2 as the time interval between the $n = 1$ impact and the $n = 2$ impact, and so on. For example, ΔT_1 is the time it takes the ball to rise from the floor at speed v_a up to the height h_1 and then to fall back down to the floor, where it will again be moving at speed v_a just before the second impact. Since the rise and fall times are equal (remember, we are assuming zero air drag—see note 2 again), and since the fall time for a ball to fall through distance h_1 when starting from rest (at the height of the first rebound) is given by $\sqrt{2h_1/g}$ (remember, the distance traveled from rest with a constant acceleration g during a time interval of duration t is $\frac{1}{2}gt^2$), we see that

$$\Delta T_1 = 2\sqrt{2h_1/g}.$$

In the same way

$$\Delta T_2 = 2\sqrt{2h_2/g},$$

$$\Delta T_3 = 2\sqrt{2h_3/g},$$

and in general

$$\Delta T_n = 2\sqrt{2h_n/g}$$

where ΔT_n is the time interval between the $n-1$st and the nth impact.

If we write T_n as the total time from the $n=0$ impact (the first impact) until the nth impact (i.e., n more bounces after the first bounce), then

$$\begin{aligned} T_n &= \Delta T_1 + \Delta T_2 + \Delta T_3 + \cdots + \Delta T_n \\ &= 2\sqrt{\frac{2}{g}}\left(\sqrt{h_1} + \sqrt{h_2} + \sqrt{h_3} + \cdots + \sqrt{h_n}\right), \end{aligned}$$

or

$$\begin{aligned} T_n &= 2\sqrt{\frac{2}{g}}\left(\sqrt{c^2h_0} + \sqrt{c^4h_0} + \sqrt{c^6h_0} + \cdots + \sqrt{c^{2n}h_0}\right) \\ &= 2\sqrt{\frac{2h_0}{g}}(c + c^2 + c^3 + \cdots + c^n). \end{aligned}$$

The expression in parentheses is a geometric series, easily summed to give

$$T_n = 2\sqrt{\frac{2h_0}{g}} \cdot c \cdot \frac{1-c^n}{1-c}.$$

Solving for g gives us the result we are after:

$$g = \frac{8h_0c^2}{T_n^2} \cdot \left(\frac{1-c^n}{1-c}\right)^2. \tag{2.2}$$

2.2 Out in the Author's Garage

From the first part of the analysis we already know c for our ball, so all we need to do now to measure g is to drop the ball from a known height and, beginning with the first bounce, start a timer and measure how long it takes for n more bounces. What could be easier? And so, taking a yardstick, a stopwatch, and a bouncy rubber ball, I went out to my garage (and its concrete floor) one cold New England winter evening and measured the value of g in Lee, New Hampshire. First, with $h_0 = 36$ inches (the midpoint of the ball), I observed a rebound height of thirty inches, so (2.1) says

$$c = \sqrt{\frac{30}{36}} = 0.913.$$

Then, again using $h_0 = 36$ inches, I again dropped the ball and, at the instant of the first impact, I started my stopwatch. I counted off eight *more* bounces ($n = 8$) and, at the last impact, I stopped the watch. Observing $T_8 = 4.6$ seconds, the formula for g, (2.2), gave a value of

$$g = \frac{8 \cdot 36 \cdot (0.913)^2}{4.6^2} \cdot \left(\frac{1 - 0.913^8}{1 - 0.913}\right)^2 = 401\,\text{in./s}^2 = 33.4\,\text{ft/s}^2.$$

The typical textbook value for g is 32.2 ft/s^2 (9.81 m/s^2), and so I made a less than 4% error. Not too shabby for a sixty-six-year-old guy—my age, when I did this experiment—with poor eyesight and questionable reflexes, to say nothing about doing it while freezing in my garage. (This had to be luck! As I recall, my Physics 51 lab report was far less impressive in its estimate of g.)[3]

CP. P2.1:

> Here are a couple of little challenge calculations—they are easy!—for your amusement. What is the total distance, up *and* down, traveled by our bouncing ball as it goes through an infinity of bounces? How long does it take the ball, from the instant you release it until it has completed that infinity of bounces, to come to rest? *Hint*: Neither answer is infinity.

Notes and References

1. The analysis in this discussion is based entirely on the work of J. G. Dodd, “Determination of g by a Bouncing Ball” (*American Journal of Physics*, April 1958, p. 268). It may amuse you to know that, all the while I was writing this discussion, the words of Bobby Vee’s 1960 smash pop hit “Rubber Ball”

reverberated through my mind. Here are a few of the opening lines to get your feet tapping:

> (*Key of A major*)
>
> Rubber ball, I come bouncin' back to you,
> rubber ball, I come bouncin' back to you
> hoo-ah-ooh-ooh.
> I'm like a—rubber ball, baby that's all
> that I am to you (bouncy, bouncy) (bouncy, bouncy)
> Just a rubber ball 'cuz you think you
> can be true to two (bouncy, bouncy) (bouncy, bouncy).
> You bounce my heart around (You don't even put her down)
> And like a rubber ball, I come bouncin' back to you
> rubber ball, I come bouncin' back to you.

Ah, yes, one of the deep philosophical chants from the dawning of the Age of Aquarius. Undeniably dumb, it was, but very hard to sing correctly and very popular in its day.

2. In later discussions in this book, I have much more to say on the mathematical physics of air drag.

3. The measurement of T_n can be made more accurate using electronics (which, of course, makes the experiment more complicated): see G. Guercio and V. Zanetti, "Determination of the Gravitational Acceleration Using a Rubber Ball" (*American Journal of Physics*, January 1987, pp. 59–63). A nice discussion of the experimental errors and of the theoretical assumptions inherent in this experiment is presented by the authors. Dodd's much earlier (1958) paper—see note 1 again—is not cited, however.