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Charles R. MacCluer: Honors Calculus

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8

Applications of the Integral

We now learn many of the important uses of the Riemann integral in mathematics, science, and engineering. The results depend in most cases on physical/geometric intuition or observed facts, not on an axiomatic foundation. Thus these results have no proofs, only persuasive verifications.

8.1 Work

By definition, *work done while moving a distance d against a constant force f* is $W = f \cdot d$. But what if the force f is not constant?

Result A. Suppose a particle at location x experiences a force¹ $f(x)$, where f is continuous on $[a, b]$. Then the total work W done by this particle moving from a to b against the force f is

$$W = - \int_a^b f(x) dx. \quad (8.1)$$

Verification. Partition $[a, b]$ into small trips, $\mathcal{P} : a = x_0 < x_1 < \dots < x_n = b$. The actual work W_i done moving the small distance $\Delta x_i = x_i - x_{i-1}$ from x_{i-1} to x_i is certainly trapped:

$$\inf_{[x_{i-1}, x_i]} f(x) \Delta x_i \leq -W_i \leq \sup_{[x_{i-1}, x_i]} f(x) \Delta x_i. \quad (8.2)$$

Thus the total actual work

$$W = \sum_{i=1}^n W_i$$

is caught between the upper and lower Darboux sums for any partition, giving (8.1).

¹By convention, positive force is to the right, negative to the left. Thus moving to the right with a force to the right is negative work.

Remark A. Many of our verifications below will use this same idea—that the actual physical or geometric quantities are Riemann sums or at least caught between all upper and lower Darboux sums. However, this is not how most scientists and engineers verify these results. Their intuition harkens back to the early viewpoint of infinitesimals. For example, they would argue result A as follows:

During the movement from x through an infinitesimal distance dx , the force f changes little and so the contribution to the total work is $dW = -f(x) dx$. Summing over all these infinitesimal motions yields the total work

$$W = \int_a^b dW = - \int_a^b f(x) dx.$$

I recommend that you embrace this intuition. It is quick and effective and rarely leads you astray.

Example 1. A mass m is free to slide along a horizontal frictionless rail (see figure 8.1) restrained by an ideal spring of spring constant k . Hooke's law states that *the restoring force F of a spring is proportional to the displacement x from its natural length*. In symbols, $F = -kx$. Hence if the mass is displaced x from rest, the work done must be

$$W = - \int F dx = \int_0^x ky dy = kx^2/2. \quad (8.3)$$

While we are here, note that as another consequence of Hooke's law, since the mass m is experiencing only the restoring force of the spring (and inertial force must be balanced by external forces), its acceleration a obeys $ma = F = -kx$. Therefore the mass is undergoing *harmonic motion*; that is, displacement x satisfies the ODE

$$\ddot{x} + \omega^2 x = 0, \quad (8.4)$$

where $\omega^2 = k/m$.

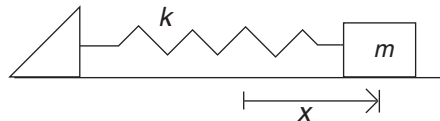


Figure 8.1 A spring-mass system.

8.2 Area

Result B. Suppose f is integrable and nonnegative on $[a, b]$. The area A of the region R bounded by the graph of $y = f(x)$, the x -axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x) \, dx. \quad (8.5)$$

See figure 8.2.

Verification. Whatever one's notion of area, certainly the area A under the curve is caught between the upper and lower Darboux sums of any partition. See figure 8.2. But there is exactly one such real number, namely the value of the integral (8.5).

Remark B. Those same scientists or engineers would instead argue that the total area is clearly the sum of all the infinitesimal areas $dA = f(x) \, dx$ with base dx and height $f(x)$. See figure 8.3. After all, they would point out, the elongated "S" integral symbol indicates it is a "smear" or "continuous" sum. See exercise 8.69.

Alert. The integral (8.5) gives the geometric (absolute) area provided $f \geq 0$. When the graph of f dips below the x -axis, the absolute area bounded by $y = f(x)$, $x = a$, $x = b$, and the x -axis is given by

$$A = \int_a^b |f(x)| \, dx. \quad (8.6)$$

To evaluate integrals of absolute values $|f|$, divide the integration into intervals where the sign of f is constant, then integrate $\text{sgn}(f(x))f(x) = |f(x)|$.

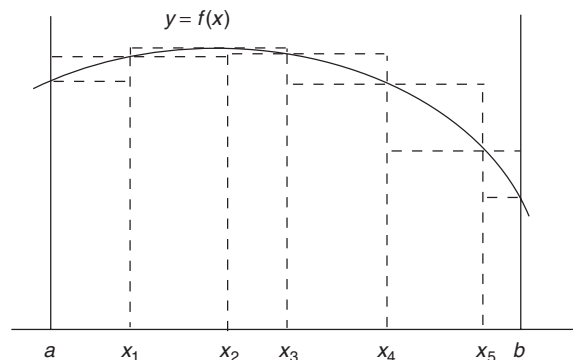


Figure 8.2 Whatever is the area A under curve $y = f(x)$, it must be trapped between the upper and lower sum.

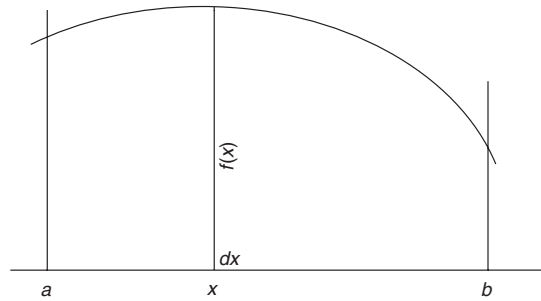


Figure 8.3 The area is the “continuous” or “smear” sum of the infinitesimal rectangles of base dx and altitude $f(x)$.

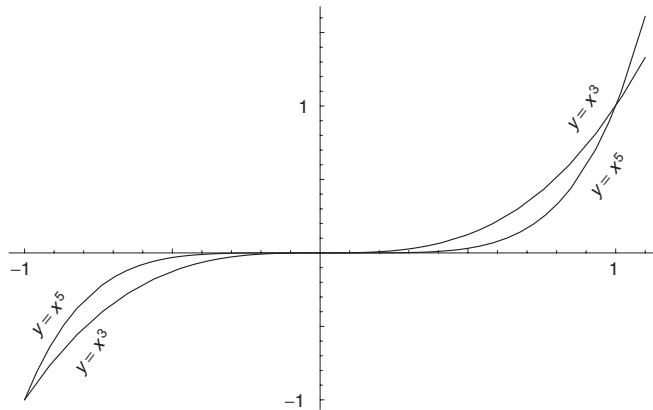


Figure 8.4 The graphs of $y = x^3$ and $y = x^5$ superimposed.

Likewise, the absolute area bounded by two curves $y = f(x)$ and $y = g(x)$ is given by

$$A = \int_a^b |f(x) - g(x)| dx. \quad (8.7)$$

Example 2. What is the area A of the region bounded by the curves $y = x^3$, $y = x^5$, $x = -1$, and $x = 2$?

Solution. A good graph is essential—see figure 8.4. The (bounded) region caught by the given curves has absolute area

$$\begin{aligned} A &= \int_{-1}^2 |x^3 - x^5| dx \\ &= \int_{-1}^0 x^5 - x^3 dx + \int_0^1 x^3 - x^5 dx + \int_1^2 x^5 - x^3 dx. \end{aligned}$$

8.3 Average Value

Result C. The average value $\text{av}(f)$ of an integrable function f on $[a, b]$ is given by

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) \, dx. \quad (8.8)$$

Verification. Partition $[a, b]$ into n equal subdivisions $\mathcal{P} : a = x_0 < x_1 < \cdots < x_n = b$, each of length $\Delta x = (b-a)/n$. Then the average of the values of f sampled at the right endpoint of each of these subintervals is

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{1}{b-a} [f(x_1) + f(x_2) + \cdots + f(x_n)] \Delta x, \quad (8.9)$$

a Riemann sum of $f/(b-a)$. Thus it is natural to define (8.8) as the average value of f . As an epigram, *the average value is the (signed) area over the base*.

Corollary. (mean value theorem for integrals) *A continuous function takes on its average value.* That is, if f is continuous on $[a, b]$, then for some $a < c < b$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx. \quad (8.10)$$

Proof. Using a partition of one,

$$\inf_{[a,b]} f(x)(b-a) \leq \int_a^b f(x) \, dx \leq \sup_{[a,b]} f(x)(b-a).$$

But because f is continuous, it takes on its infimum and supremum as well as all values in between.

Remark C. Note that this mean value theorem (8.10) for integrals is our original mean value theorem (5.22) in disguise. For by setting

$$F(x) = \int_a^x f(t) \, dt,$$

we see that

$$F'(c) = \frac{F(b) - F(a)}{b-a}$$

is exactly (8.10).

Example 3. The average value of a linear function over an interval is its height at the midpoint:

$$\begin{aligned} \frac{1}{b-a} \int_a^b (mx+c) dx &= \frac{1}{b-a} \left(\frac{mx^2}{2} + cx \right) \Big|_a^b \\ &= \frac{1}{b-a} \left[\frac{mb^2 - ma^2}{2} + c(b-a) \right] = m \frac{a+b}{2} + c. \end{aligned}$$

Example 4. The average value of $\sin x$ over a half-cycle is

$$\text{av} = \frac{1}{\pi} \int_0^\pi \sin x dx = -\frac{\cos x}{\pi} \Big|_0^\pi = \frac{2}{\pi} \approx 0.637.$$

In contrast, the average over full cycles is of course 0.

8.4 Volumes

Result D. Suppose f is continuous on $[a, b]$. Let R be the region bounded by the graph of $y = f(x)$ and the x -axis from $x = a$ to $x = b$.

- (a) When the region R is rotated about the x -axis, the resulting solid has volume

$$V = \pi \int_a^b f(x)^2 dx. \quad (8.11)$$

- (b) Assume $0 < a < b$. When this same region R is instead rotated about the y -axis, the resulting volume is

$$V = 2\pi \int_a^b x|f(x)| dx. \quad (8.12)$$

Verification. A Darboux sum verification is left as exercise 8.3. A verification by infinitesimals goes like this: Slice the solid of (a) with a plane perpendicular to the x -axis at x . The result is an infinitesimal cylinder of volume $dV = \pi r^2 dx$ of radius $r = f(x)$ and height dx . Sum up these infinitesimal volumes.

In (b), this same vertical slice is rotated about the y -axis to produce a cylindrical shell of radius x , height $|y| = |f(x)|$, and infinitesimal thickness dx with volume $dV = 2\pi x|y| dx$.

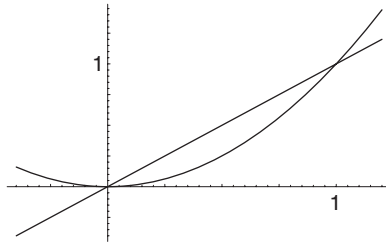


Figure 8.5 The region bounded by the curves $y = x$ and $y = x^2$.

Example 5. Consider the (bounded) region R enclosed by $y = x$ and $y = x^2$, as seen in figure 8.5. When this region is rotated about the x -axis, the resulting volume is

$$V_a = \pi \int_0^1 x^2 - x^4 dx = \dots = \frac{2\pi}{15}. \quad (8.13a)$$

When this same region is rotated about the y -axis, the resulting volume is

$$V_b = 2\pi \int_0^1 x(x - x^2) dx = \dots = \pi/6. \quad (8.13b)$$

Note that this second result can also be obtained by slicing horizontally:

$$V_b = \pi \int_0^1 (\sqrt{y})^2 - y^2 dy = \dots = \pi/6. \quad (8.13c)$$

8.5 Moments

Moments of a region or body are of central importance to mechanics and statistics. Visualize a collection of masses m_1, m_2, \dots, m_n at locations $p_1 = (x_1, y_1, z_1)$, $p_2 = (x_2, y_2, z_2)$, \dots , $p_n = (x_n, y_n, z_n)$, respectively. The *first moment* of these masses is the triple

$$M = \sum_{k=1}^n p_k m_k = \left(\sum_{k=1}^n x_k m_k, \sum_{k=1}^n y_k m_k, \sum_{k=1}^n z_k m_k \right). \quad (8.14)$$

Result E. The center of mass O is the first moment divided by the total mass, that is,

$$O = \frac{M}{m_1 + m_2 + \dots + m_n}. \quad (8.15)$$

Verification. By definition, the center of mass is the origin of a coordinate system with respect to which the masses have first moment zero. Let $O = M/m = (\tilde{x}, \tilde{y}, \tilde{z})$ and $m = m_1 + \cdots + m_n$. Then

$$\begin{aligned} 0 &= M - mO = \sum_{k=1}^n m_k \cdot (x_k, y_k, z_k) - m_k \cdot (\tilde{x}, \tilde{y}, \tilde{z}) \\ &= \sum_{k=1}^n m_k \cdot [(x_k, y_k, z_k) - (\tilde{x}, \tilde{y}, \tilde{z})]. \end{aligned}$$

Thinking of a solid V as composed of infinitesimal masses dm at location $p = (x, y, z)$, one then leaps to the continuous version of (8.14) and (8.15): *The first moment of the solid V is the triple*

$$M = \int_V p \, dm \quad (8.16a)$$

and its center of mass is at

$$O = (\tilde{x}, \tilde{y}, \tilde{z}) = \frac{M}{m}, \quad (8.16b)$$

where m is the total mass

$$m = \int_V dm.$$

Example 6. Consider a uniform thin plate in the shape of the region in the plane bounded by $y = x^2$ and $y = 1$. See figure 8.6. Where does this plate balance?

Solution: We assume constant mass density per unit area. Clearly by symmetry, the geometric center (*centroid*) lies on the y -axis and so $\tilde{x} = 0$. As for its y -coordinate, consider the mass $dm = 2\sqrt{y} \, dy$

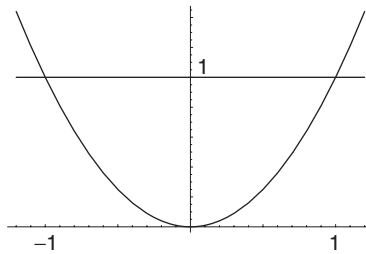


Figure 8.6 Where is the centroid of the region bounded by $y = 1$ and $y = x^2$?

of a infinitesimally thin horizontal slice with lever arm y and hence moment $dM = 2y\sqrt{y} dy$. Totalling all infinitesimal moments yields

$$\tilde{y} = \frac{\int_0^1 2y\sqrt{y} dy}{\int_0^1 2\sqrt{y} dy} = \frac{3}{5}. \quad (8.17a)$$

Alternatively, we may slice vertically, then concentrate all mass $dm = (1-x^2) dx$ at the center of this infinitesimally thin strip with lever arm $y = (1+x^2)/2$ to obtain

$$\tilde{y} = \frac{\int_{-1}^1 (1+x^2)(1-x^2) dx/2}{\int_{-1}^1 (1-x^2) dx} = \frac{\int_0^1 (1-x^4) dx}{2 \int_0^1 (1-x^2) dx} = \frac{3}{5}. \quad (8.17b)$$

A quantity central to modeling rotational motion is the *moment of inertia* of a collection of masses m_1, m_2, \dots, m_n about an axis of rotation L . Let r_k be the perpendicular distance of the mass m_k to the axis L . Then the moment of inertia about L is

$$I = \sum_{k=1}^n r_k^2 m_k, \quad (8.18a)$$

with its continuous version

$$I = \int_V r^2 dm. \quad (8.18b)$$

Example 7. What is the moment of inertia of a solid uniform cylinder about its axis?

Solution: Let the height of the cylinder be h and its radius a . Let us assume a mass density of δ per unit volume. We may collapse the mass of the cylinder onto one circular face to obtain a planar problem with mass density of $\sigma = h\delta$ per unit area. The mass at an infinitesimally thin annular ring of radius r from the axis is then $dm = 2\pi\sigma r dr$. Hence the moment of inertia is

$$I = \int_0^a r^2 dm = 2\pi\sigma \int_0^a r^3 dr = \frac{\pi\delta ha^4}{2}. \quad (8.19)$$

Result F. The *rotational kinetic energy* of a body rotating about a line L at angular velocity ω is

$$T = \frac{I\omega^2}{2}. \quad (8.20)$$

Thus *in rotational motion, the role of mass is played by the moment of inertia, velocity by angular velocity.*

Verification. The total kinetic energy T is the sum of all the kinetic energies of all the infinitesimal masses making up the solid V , giving

$$T = \int_V \frac{v^2 dm}{2} = \int_V \frac{(r\omega)^2 dm}{2} = \frac{\omega^2}{2} \int_V r^2 dm = \frac{\omega^2 I}{2}.$$

8.6 Arclength

Result G. Suppose f is continuously differentiable on $[a, b]$. Then the arclength of the graph of $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is

$$\Gamma = \int_a^b \sqrt{1 + f'(x)^2} dx. \quad (8.21)$$

Verification. Partition $[a, b]$ as usual: $a = x_0 < x_1 < \cdots < x_n = b$. The arclength Γ must be the supremum of the sum of all the straight-line distances

$$\begin{aligned} & \sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sum_{i=1}^n \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1}))}{x_i - x_{i-1}}\right)^2} \Delta x_i \\ &= (\text{by MVT}) = \sum_{i=1}^n \sqrt{1 + f'(c_i)^2} \Delta x_i, \end{aligned}$$

which are Riemann sums of the integral of (8.21).

Arguing instead with infinitesimals, the increment of arclength $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2} dx$.

Example 8. Let us rederive the formula $C = 2\pi r$ for the circumference of a circle. Consider the half-circle of radius a given by $y = \sqrt{a^2 - x^2}$, $-a \leq x \leq a$. By (8.21), the arclength of the entire circle is (exercise 8.13)

$$\begin{aligned} \Gamma &= 2 \int_{-a}^a ds = 4 \int_0^a ds = 4 \int_0^a \sqrt{1 + y'^2} dx \\ &= 4 \int_0^a \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2} dx = 4a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} \\ &= 4a \text{Arcsin}(x/a) \Big|_0^a = 4a \text{Arcsin} 1 = 2\pi a. \end{aligned} \quad (8.22)$$

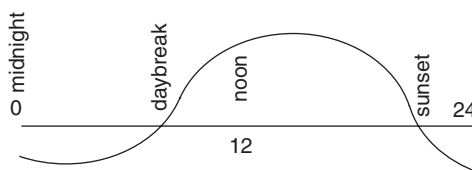


Figure 8.7 The rate of uptake of carbon dioxide is greater during photosynthesis than the rate of discharge at night. The net is the (signed) area under the curve, a positive uptake.

8.7 Accumulating Processes

The integral of a rate of production (or loss) of a quantity recovers the net accumulation of the quantity.

Example 9. Suppose $f(t)$ is the rate of CO_2 mass uptake of a given tree. During the day, sugars are formed from the ambient CO_2 , light, and water. At night, some of these sugars are metabolized, releasing CO_2 . The rate of intake and release of CO_2 follows a curve like that shown in figure 8.7. Thus,

$$U = \int_0^{24} f(t) dt$$

is the net mass of CO_2 locked away in the tissues of the tree per day.

8.8 Logarithms

Definition A. The *natural logarithm* is the function $\ln : (0, \infty) \rightarrow \mathbf{R}$ given by the rule

$$\ln x = \int_1^x \frac{dt}{t}. \quad (8.23a)$$

Of course, because $1/x$ is continuous on $(0, \infty)$,

$$D \ln x = \frac{1}{x}. \quad (8.23b)$$

Note that $\ln 1 = 0$. Moreover, for any positive constant a ,

$$D (\ln ax - \ln a - \ln x) = \frac{a}{ax} - \frac{1}{x} = 0,$$

and so $\ln ax = \ln a + \ln x + c$. But setting $x = 1$ reveals that $c = 0$. Thus we have the first of the laws of the logarithm:

$$\ln xy = \ln x + \ln y \quad (8.24)$$

for all positive x, y . In particular, for a natural number n ,

$$\begin{aligned}\ln x^n &= \ln(x \cdot x \cdots x) \quad (n \text{ times}) \\ &= \ln x + \ln x + \cdots + \ln x \quad (n \text{ times}) = n \ln x.\end{aligned}\quad (8.25)$$

Replacing x in (8.25) by $x^{1/n}$ yields that $(1/n) \ln x = \ln x^{1/n}$ and hence (exercise 8.15)

$$\ln x^r = r \ln x, \quad r \in \mathbf{Q}.\quad (8.26)$$

Note that the logarithm is an increasing bijective continuous function from $(0, \infty)$ onto $(-\infty, \infty)$ (exercise 8.14). Therefore its inverse function $\ln^{-1} x = \exp x$ has by (5.26) the derivative $D \exp x = 1/(1/\exp x)$; that is,

$$D \exp x = \exp x.\quad (8.27)$$

Note that $\exp 0 = 1$, since $\ln 1 = 0$ and that by (8.24) and (8.26),

$$\exp(x + y) = (\exp x) \cdot (\exp y), \quad x, y \in \mathbf{R}\quad (8.28a)$$

and

$$\exp(rx) = (\exp x)^r, \quad x \in \mathbf{R}, r \in \mathbf{Q}.\quad (8.28b)$$

Definition B. The real number

$$e = \exp(1)\quad (8.29)$$

is called *Napier's natural base*. Its value is approximately 2.718281828 (exercise 8.21).

Note that for rational exponents, $e^r = (\exp 1)^r = \exp r$. But how can we make sense of e^π ? Or $e^{\sqrt{2}}$?

Definition C. For any real number x , the meaning of the symbol e^x is given by

$$e^x = \exp x,\quad (8.30a)$$

which is in complete agreement with the value e^r for r rational. For any positive b , we define

$$b^x = e^{x \ln b} = \exp(x \ln b).\quad (8.30b)$$

A list of familiar results now follows easily (exercise 8.16).

Corollary A. (rules of exponents) For any positive b and real x, y ,

$$b^{x+y} = b^x b^y, \quad (b^x)^y = b^{xy}, \quad b^{-x} = \frac{1}{b^x}, \quad b^0 = 1. \quad (8.31)$$

Corollary B. For $b > 0$, the function $f(x) = b^x$ is an everywhere differentiable bijective function from the reals to the positive reals with derivative

$$D b^x = b^x \ln b. \quad (8.32)$$

In particular,

$$D e^x = e^x. \quad (8.33)$$

8.9 Methods of Integration

There are two methods for integrating complicated integrals: *by substitution* and *by parts*.

Result H. (substitution) Suppose that $u = u(x)$ is continuously differentiable on $[a, b]$ and that f is continuous on at least the interval $u([a, b])$. Then

$$\int_a^b f(u(x))u'(x) dx = \int_{u(a)}^{u(b)} f(u) du. \quad (8.34)$$

Proof. By (7.25), the derivatives with respect to b of both sides of (8.34) agree, thus they differ by a constant. But setting $b = a$ reveals the constant to be zero.

Example 10. Make the substitution $u = 1 + x^2$ so that

$$\int_0^1 x(1+x^2)^5 dx = \int_1^2 u^5 du/2 = \frac{u^6}{12} \Big|_{u=1}^{u=2}.$$

Example 11. Often the formula of substitution (8.34) is run from right to left. For example, let us rederive the formula $A = \pi r^2$ for the area of a disk. We employ the substitution $u = r \sin \theta$ whereupon $du = r \cos \theta d\theta$. Hence

$$\begin{aligned} A &= 4 \int_0^r \sqrt{r^2 - u^2} du = 4 \int_0^{\pi/2} \sqrt{r^2 - r^2 \sin^2 \theta} r \cos \theta d\theta \\ &= 4r^2 \int_0^{\pi/2} \cos^2 \theta d\theta = 4r^2 \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta = \pi r^2. \end{aligned}$$

Result I. (integration by parts²) Suppose u and v are both continuously differentiable on $[a, b]$. Then

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du,$$

that is,

$$\int_a^b u(x)v'(x) \, dx = u(x)v(x) \Big|_a^b - \int_a^b u'(x)v(x) \, dx. \quad (8.35)$$

Proof. Merely integrate the product rule $(uv)' = uv' + u'v$.

Example 12. By setting $u = x$ and $dv = e^x \, dx$ we obtain

$$\int_0^1 xe^x \, dx = xe^x \Big|_0^1 - \int_0^1 e^x \, dx = e - (e - 1) = 1.$$

Example 13. The method of *partial fractions* is the frequency-domain equivalent of integration by parts—see [MacCluer, 2000]. It is used to integrate rational functions. For instance,

$$\begin{aligned} \int_0^1 \frac{x}{x^2 + 3x + 2} \, dx &= \int_0^1 \left(\frac{A}{x+1} + \frac{B}{x+2} \right) \, dx \\ &= \int_0^1 \left(\frac{-1}{x+1} + \frac{2}{x+2} \right) \, dx = \ln \frac{(x+2)^2}{x+1} \Big|_0^1. \end{aligned}$$

8.10 Improper Integrals

Let us now extend the concept of the integral to unbounded integrands and intervals.

Example 14. (unbounded integrand) Consider the integral

$$I = \int_0^1 \frac{dx}{\sqrt{x}}. \quad (8.36)$$

Note that although the interval of integration is bounded, the integrand $f(x) = 1/\sqrt{x}$ is unbounded at $x = 0$ and so the hypotheses for the Riemann integral are not satisfied.

²Not only is integration by parts a useful computational trick, but it crops up in important mathematics—for example, it is crucial in prime number estimates and in the modern Sobolev school of partial differential equations.

However, also note that f is bounded on every subinterval $[\epsilon, b]$, $\epsilon > 0$, and in fact

$$I_\epsilon = \int_\epsilon^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big|_\epsilon^1 = 2 - 2\sqrt{\epsilon},$$

and so in the limit, as $\epsilon \rightarrow 0$, it would be natural to claim $I = 2$. Thus (8.36) is an example of a *convergent improper integral*.

Definition D. Suppose f is Riemann integrable on every closed subinterval of (a, b) . Then the (possibly improper) integral

$$\int_a^b f(x) dx$$

is said to *converge to the value I* when

$$\lim_{\delta \rightarrow 0^+} \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^{b-\delta} f(x) dx = \lim_{\epsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \int_{a+\epsilon}^{b-\delta} f(x) dx = I, \quad (8.37)$$

and we extend the meaning of equality by writing

$$\int_a^b f(x) dx = I.$$

Example 15. (unbounded interval) Consider the improper integral

$$I = \int_1^\infty \frac{dx}{x^2}. \quad (8.38)$$

Note that

$$\lim_{N \rightarrow \infty} \int_1^N \frac{dx}{x^2} = - \lim_{N \rightarrow \infty} \frac{1}{x} \Big|_1^N = 1$$

and so (8.38) is *convergent to the value 1*.

Definition E. Suppose f is Riemann integrable on every closed and bounded subinterval of $[a, \infty)$. Then the improper integral

$$\int_a^\infty f(x) dx$$

is said to *converge to the value I* when

$$\lim_{N \rightarrow \infty} \int_a^N f(x) dx = I \quad (8.39)$$

and we extend the meaning of equality by writing

$$\int_a^\infty f(x) dx = I.$$

(There is the obvious similar notion when the lower limit of integration is $-\infty$.)

Occasionally integrals are improper for several reasons, for example

$$I = \int_0^\infty f(x) dx = \int_0^\infty \frac{dx}{\sqrt{x}\sqrt{|1-x^2|}}.$$

In such cases, merely divide the interval of integration and deal with each impropriety separately. In this example, rewrite the integral as

$$I = \int_0^{1/2} f(x) dx + \int_{1/2}^1 f(x) dx + \int_1^2 f(x) dx + \int_2^\infty f(x) dx. \quad (8.40)$$

If each summand converges (exercise 8.26), one says the the original improper integral *converges*. If any one summand diverges, so does the original integral.

8.11 Statistics

Statistics is a profound example of the “unreasonable efficacy of mathematics.” Although a given phenomenon may be fundamentally random, it may nevertheless exhibit great regularity on average. The probability (frequency) that a certain measurement X of the phenomenon has value between a and b can often be predicted as an integral of the form

$$\text{prob}(a \leq X \leq b) = \int_a^b f(x) dx, \quad (8.41a)$$

where f is called the *probability density function* of the measurement X . Of course,

$$\text{prob}(-\infty < X < \infty) = \int_{-\infty}^\infty f(x) dx = 1. \quad (8.41b)$$

The expected value of the outcome (*mean*) is the first moment

$$\mu = \int_{-\infty}^\infty xf(x) dx, \quad (8.42a)$$

which, because the total mass is 1, is the the x -coordinate of the center of mass of the distribution. The *variance* v is the moment of inertia about the mean:

$$v = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx, \quad (8.42b)$$

where $\sigma > 0$ is called the *standard deviation*. The smaller the standard deviation, the more likely the outcomes cluster about the mean.

Example 16. Take a stopwatch to a supermarket and record the time between arrivals at a checkout lane. You will find close agreement with *Poisson flow*: The probability that the time T before the next arrival is at most t_0 is given by

$$\text{prob}(T \leq t_0) = \frac{1}{a} \int_0^{t_0} e^{-t/a} dt,$$

where a is the mean time between arrivals. So the statistical outcomes for this experiment (of measuring time between arrivals) are modeled by the probability density function

$$f(t) = \begin{cases} (1/a)e^{-t/a} & \text{if } 0 \leq t \\ 0 & \text{if } t < 0. \end{cases}$$

Example 17. A certain scholarship aptitude test is designed under the belief that scores should be *normally distributed* with mean $\mu = 500$ and standard deviation $\sigma = 100$. So the probability that a randomly chosen student's score X will fall between (say) $a = 550$ and $b = 675$ is given by

$$\text{prob}(a \leq X \leq b) = \frac{1}{\sqrt{2\pi} \sigma} \int_a^b e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx. \quad (8.43)$$

This famous integral has no elementary antiderivative and must be calculated numerically. See §8.13 and exercise 8.76.

Any experimental outcome that is the superposition of many unrelated phenomena will have this famous *Gaussian* (normal) distribution (8.43).³

³This is an informal statement of the *central limit theorem* of statistics to be found in any text on statistics.

8.12 Quantum Mechanics

The rules of mechanics change at the atomic scale—they become statistical rather than deterministic. One can obtain only expected (average) measurements. Mechanical rules are no longer relationships between measured quantities; they become instead the analogous relationships between the instruments taking the measurements.

The most that can be known about a particle is its *wave function* ψ , where $f(x) = |\psi(x)|^2$ is a probability density function. The wave function ψ is a solution of *Schrödinger's equation*

$$i\hbar\dot{\psi} = H\psi, \quad (8.44)$$

where H is the mathematical analog of the instrument measuring total energy. When any one measurement is taken, the wave function ψ “collapses” to one of its *stationary states*, that is, a time-independent solution of

$$H\psi = E\psi, \quad (8.45)$$

where E is the energy of that state. The mathematics necessary to model all this is called *functional analysis*, a sort of infinite dimensional matrix theory. To learn more about quantum mechanics, see the delightful book by [Davies] and my own view [MacCluer, 2004]. But let us sample the flavor of the subject by working through one problem to see how the integral comes into play.

Example 18. A particle of mass m is trapped in the interval $[0, \pi]$ by an infinite external potential. There is 0 potential within the interval. Thus all the particle's energy is kinetic. Schrödinger's equation (8.45) for the stationary states then becomes

$$-\frac{\hbar^2}{2m}\psi'' = E\psi; \quad (8.46)$$

that is, the wave function ψ satisfies the ODE of harmonic motion $\psi'' + \omega^2\psi = 0$, where in this case $\omega = \sqrt{2mE}/\hbar$. The solutions to the harmonic motion ODE are the sinusoidals $\psi(x) = a \sin(\omega x - \theta)$. But because the particle is trapped, $\psi(x)$ vanishes outside $(0, \pi)$. Thus $\theta = 0$ and $\omega = n$, $n \in \mathbf{N}$. Because $|\psi|^2$ is a probability distribution,

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_0^{\pi} |\psi(x)|^2 dx \\ &= a^2 \int_0^{\pi} \sin^2 nx dx = \frac{a^2}{2} \int_0^{\pi} (1 - \cos 2nx) dx = \pi a^2/2, \end{aligned}$$

hence $a = \sqrt{2/\pi}$. Thus the wave function of the n -th stationary state is

$$\psi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx, \quad (8.47a)$$

that must because of (8.46) have (kinetic) energy

$$E_n = \frac{\hbar^2 n^2}{2m}, \quad n \in \mathbf{N}. \quad (8.47b)$$

It so happens that position (displacement) is measured by multiplication by x . Thus the expected location of the particle is

$$\begin{aligned} \langle Q\psi, \psi \rangle &= \int_0^\pi x |\psi(x)|^2 dx \\ &= \frac{2}{\pi} \int_0^\pi x \sin^2 nx dx = (\text{exercise 8.28}) = \pi/2. \end{aligned} \quad (8.48)$$

Thus on average, the particle is to be found at the midpoint of the interval.

8.13 Numerical Integration

Although extensive tables of antiderivatives exist [CRC], many important integrals cannot be given in terms of elementary functions. One way to think about the history of mathematics is that when mathematicians encounter an equation with a solution inexpressible in terms of known quantities, the solution is named, tabulated, and added to the list of primitive notions.

For example, an integral important to statistics and heat transfer is the *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2} d\beta, \quad (8.49)$$

an integral with no antiderivative in terms of elementary functions. Before the availability of cheap computing, such integrals were often tabulated by analytic techniques, such as Taylor expansions, as follows: Since

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \\ \int_0^x e^{-\beta^2} d\beta &= \int_0^x \left(1 - \beta^2 + \frac{\beta^4}{2!} - \frac{\beta^6}{3!} + \cdots \right) d\beta. \end{aligned}$$

Nowadays we employ the digital computer to compute cleverly chosen Riemann sums.

Objective. To estimate the value of the Riemann integral

$$I = \int_a^b f(x) dx. \quad (8.50)$$

Approach. Partition the interval $[a, b]$ into n equal parts of length $\Delta x = (b - a)/n$: $a = x_0 < x_1 < x_2 < \dots < x_n = b$, where $x_i = a + i \cdot \Delta x$.

The five standard algorithms. The Riemann sums

$$L_n = \Delta x \cdot (f(x_0) + f(x_1) + \dots + f(x_{n-1})), \quad (8.51a)$$

$$R_n = \Delta x \cdot (f(x_1) + f(x_2) + \dots + f(x_n)), \quad (8.51b)$$

$$M_n = \Delta x \cdot \left(f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \dots + f\left(\frac{x_{n-1} + x_n}{2}\right) \right) \quad (8.51c)$$

are called the *left-hand*, *right-hand*, and *midpoint rule* estimates for (8.50), respectively.

The estimates

$$T_n = \frac{L_n + R_n}{2} = \Delta x \cdot \left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \dots + f(x_{n-1}) + \frac{f(x_n)}{2} \right) \quad (8.52)$$

and (when n is even),

$$S_n = \frac{\Delta x}{3} \cdot (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 4f(x_{n-1}) + f(x_n)) \quad (8.53)$$

are called the *trapezoid* and *Simpson rule* estimates for (8.50), respectively.

Example 19. Let us estimate π using each of these five rules and the integral

$$\frac{\pi}{4} = \text{Arctan } 1 = \int_0^1 \frac{dx}{1 + x^2}. \quad (8.54)$$

Partition the interval $[0, 1]$ into $n = 4$ equal parts: $0 < 1/4 < 1/2 < 3/4 < 1$. Then the five estimates to $\pi/4 \approx 0.785398164$ are

$$L_4 = \frac{1}{4} \left(\frac{1}{1+0^2} + \frac{1}{1+(1/4)^2} + \frac{1}{1+(1/2)^2} + \frac{1}{1+(3/4)^2} \right) \approx 0.84529,$$

$$R_4 = \frac{1}{4} \left(\frac{1}{1+(1/4)^2} + \frac{1}{1+(1/2)^2} + \frac{1}{1+(3/4)^2} + \frac{1}{1+1^2} \right) \approx 0.72029,$$

$$M_4 = \frac{1}{4} \left(\frac{1}{1+(1/8)^2} + \frac{1}{1+(3/8)^2} + \frac{1}{1+(5/8)^2} + \frac{1}{1+(7/8)^2} \right) \approx 0.78670,$$

$$T_4 = \frac{1}{4} \left(\frac{1}{1+0^2} + \frac{2}{1+(1/4)^2} + \frac{2}{1+(1/2)^2} + \frac{2}{1+(3/4)^2} + \frac{1}{1+1^2} \right) \approx 0.78229,$$

$$S_4 = \frac{1}{12} \left(\frac{1}{1+0^2} + \frac{4}{1+(1/4)^2} + \frac{2}{1+(1/2)^2} + \frac{4}{1+(3/4)^2} + \frac{1}{1+1^2} \right) \approx 0.785392.$$

Notice the stunning accuracy of Simpson's rule.

It is easy to instruct machines to perform these numerical integrations:

Pseudocode. (for approximating (8.50) using n subdivisions)

```

n = 4
x = a
dx = (b-a)/n
L = 0, R = 0, M = 0
T = f(x), S = f(x)
c = 1
loop (n times)
  x = x + dx
  L = L + f(x-dx)
  R = R + f(x)
  M = M + f(x-dx/2)
  T = T + 2f(x)
  S = (3 + c)f(x)
  c = -c
return to loop
L = L dx, R = R dx, M = M dx, T = (T - f(x)) dx/2
S = (S - f(x)) dx/3

```

Remark D. It is clear why the lefthand, righthand, midpoint, and trapezoid rules estimate the integral, since they are either Riemann sums or averages of Riemann sums. But why does Simpson's rule work and why so well? It works because (exercise 8.29)

$$S_n = \frac{L_n + R_n + M_{n/2}}{3} \quad (8.55)$$

and thus S_n is also an average of Riemann sums. But why is it so accurate? It is because Simpson's rule replaces the integrand with piecewise quadratics rather than piecewise linear elements. This yields an error that decreases with the *fourth* power of the number of subdivisions.

Result J. Assume the integrand f of (8.50) has at least its first four derivatives continuous on $[a, b]$. Let

$$m = \sup_{[a,b]} |f^{(4)}(x)|.$$

Then for a partition of $[a, b]$ into n (an even number) equal subintervals of length $\Delta x = (b - a)/n$, the absolute Simpson error

$$\left| S_n - \int_a^b f(x) dx \right| \leq \frac{(b - a)^5 m}{180n^4}. \quad (8.56)$$

Proof. Exercise 8.31.

Example 20. How many subdivisions n are necessary to achieve eight-place accuracy in the Simpson estimate of

$$\ln 2 = \int_1^2 \frac{dx}{x} ? \quad (8.57)$$

Solution: By (8.56) we must choose n even and large so that

$$\frac{(2 - 1)^5 m}{180n^4} < 5 \times 10^{-9},$$

where m is the maximum value on $[1, 2]$ of $|(1/x)^{(4)}| = |4!/x^5|$, i.e., $m = 24$. Thus it is sufficient that

$$n > \left(\frac{24 \cdot 10^9}{5 \cdot 180} \right)^{1/4} \approx 71.9.$$

Hence seventy-two subdivisions suffice.

Exercises

- 8.1 Compute the work needed to move a particle from $x = 1$ to $x = 4$ against the force field $f(x) = -1/x^2$.

Answer: 3/4.

- 8.2 Compute the work needed to compress a spring with spring constant $k = 3$ N/m and natural length $L_N = 2$ m from length $L = 1$ m to $L = 4$ m.
Answer: $9/2$ N-m.
- 8.3 Using Darboux sums, provide a verification of result D.
- 8.4 Deduce from example 8 the formula $A = \pi r^2$ for the area of a disc of radius r . Deduce in turn the formula $V = 4\pi r^3/3$ for the volume V of a ball of radius r from its surface area $S = 4\pi r^2$.
- 8.5 Compute the average value of \sqrt{x} over $[0, 2]$.
Answer: $2\sqrt{2}/3$.
- 8.6 A sinusoidal voltage $v = a \sin \omega t$ is applied to a resistor of resistance R . Show that the energy (in Joules) consumed by the resistor in heat every complete period T is $a^2 T/2R$. Deduce that an alternating voltage of peak value a can only deliver the energy of the lower DC voltage $a/\sqrt{2}$. (For example, the nominal "120 volt AC" North American house voltage actually peaks at approximately 170 volts.)
Outline: Instantaneous power P (in Watts) through the resistor is by Ohm's law, $P = vi = v^2/R = (a^2/R) \sin^2 \omega t$. Energy is the time-integral of power.
- 8.7 Verify the formulas (8.13).
- 8.8 Suppose we have n masses m_i moving in empty space, respectively located at $(x_i(t), y_i(t), z_i(t))$ at time t . Let $O(t)$ be the location of the center of mass of these n masses at time t . Prove that O is moving along a straight line at constant speed.
Hint: Differentiate the coordinates of $O(t)$ and use that inertial forces sum to 0.
- 8.9 Verify (8.17).
- 8.10 Calculate the moment of inertia of a solid uniform ball of density δ and radius a .
Answer: $I = 8\pi\delta a^5/15$.
Outline: Consider cylindrical shells about the axis of rotation,
- $$I = \int_0^a r^2 dm = 2\delta \int_0^a r^2 2\pi r \sqrt{a^2 - r^2} dr.$$
- Employ the substitution $r = a \sin \theta$.

- 8.11 Verify *Pappus's First Law*: The volume V obtained by revolving a planar region on one side of a line L about L is the area A of the region multiplied by the circumference traveled by the centroid of the region: $V = 2\pi\bar{r}A$. Think of a donut.
- 8.12 Consider a curve in the plane given by $y = f(x)$ for $a \leq x \leq b$. Argue that when this curve is rotated about the x -axis, the resulting surface area generated is

$$S = 2\pi \int_a^b y \, ds = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} \, dx.$$

Deduce *Pappus's Second Law*: The surface area S obtained by rotating a plane curve on one side of a line about this line is the product of the distance $2\pi r$ traveled by the geometric centroid⁴ of the curve with the arclength Γ of the curve: $S = 2\pi r\Gamma$.

For example, the (slanted) surface area of a right circular cone of radius r is its slant height times πr .

- 8.13 Verify all steps leading to (8.22).
Outline: This is more delicate than it first appears. Note that the integrand is unbounded as $x \rightarrow a$. It is an *improper integral* that nevertheless converges. Integrate up to $a - \epsilon$, then let ϵ go to 0.
- 8.14 Show from the definition (8.23) that logarithm is a bijective continuous function from $(0, \infty)$ onto $(-\infty, \infty)$.
- 8.15 Verify (8.26).
- 8.16 Prove the rules of exponents (9.31).
- 8.17 Prove the differentiation formula $D b^x = b^x \ln b$ from (8.30b).
- 8.18 Estimate $\ln 2$ and $\ln 3$ using lower and upper sums of (8.23a), respectively. Deduce that $2 < e < 3$.
- 8.19 Deduce from (8.30b) that $\alpha \ln x = \ln x^\alpha$ for any real α and $x > 0$.

⁴The centroid of a curve is the center of mass were the curve to be made from a uniform wire.

- 8.20 For $b > 0$, the *logarithm to base b* of x , in symbols $g(x) = \log_b x$, is defined as the inverse function of $f(x) = b^x$. Prove that for $x > 0$,

$$\log_b x = \frac{\ln x}{\ln b}.$$

Deduce that $\log_b(xy) = \log_b x + \log_b y$, $\alpha \log_b x = \log_b x^\alpha$, and

$$D \log_b x = \frac{1}{x \ln b}.$$

- 8.21 Estimate e to six places.

Outline: Employing Taylor's theorem (exercise 5.20),

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{e^c}{(n+1)!}.$$

But $e^c < e^1 < 3$. Thus choose n large enough so that

$$\frac{3}{(n+1)!} < 5 \times 10^{-7}.$$

- 8.22 Let $b > 0$. Prove that there is at most one extension of $f(r) = b^r$ on \mathbf{Q} to a continuous function on \mathbf{R} .

Outline: Show that if f is uniformly continuous on the rationals in $[-n, n]$, then there is but one continuous extension to $[-n, n]$.

- 8.23 Compute

$$\int_0^1 \frac{x^2 dx}{1+x^6}.$$

- 8.24 For what p does the improper integral

$$\int_0^1 \frac{dx}{x^p}$$

converge?

Answer: $p < 1$.

- 8.25 For what p does the improper integral

$$\int_1^\infty \frac{dx}{x^p}$$

converge?

Answer: $p > 1$.

- 8.26 Show that each improper summand of (8.40) converges.

Hint: Use upper estimates.

8.27 Prove that

$$\int_0^{\infty} \frac{\sin x}{x} dx$$

converges (to in fact $\pi/2$).

Outline:

$$\int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{\pi} \frac{\sin x}{x} dx + \int_{\pi}^{\infty} \frac{\sin x}{x} dx.$$

Integrate the second integral on the right by parts and apply the comparison theorem (exercise 8.49).

8.28 Verify (8.48).

8.29 Verify (8.55). That is, prove that the Simpson's rule approximation is the average of three other approximations:

$$S_n = \frac{L_n + R_n + M_{n/2}}{3}.$$

8.30 **(Project)** Use the inverse function of

$$I(y) = \int_0^y \frac{dx}{\sqrt{1-x^2}},$$

(extended periodically) as an analytic definition of $\sin x$. Derive all trigonometric results of §5.5 without dubious appeals to geometry.

8.31* Prove the error formula (8.56) for Simpson's rule.

Outline: Reduce to the case of two subdivisions then set

$$E(h) = \int_{-h}^h f(x) dx - \frac{h}{3} [f(-h) + 4f(0) + f(h)].$$

Note that $0 = E(0) = E'(0) = E''(0) = E'''(0)$, and so by the Taylor formula with integral remainder,

$$E(h) = \frac{1}{3!} \int_0^h E^{(4)}(x)(h-x)^3 dx = \frac{E(c)}{3!} = \dots$$

8.32 Estimate the average value of the function f on $[0, 4]$ knowing only the sampled data

$$\begin{array}{c|cccccc} x & 0 & 1 & 2 & 3 & 4 \\ \hline f(x) & 1.2 & 2.3 & 2.9 & 3.1 & 2.8 \end{array}.$$

Recommended engineering practice: Use Simpson's rule.

- 8.33 Verify *Cavalieri's First Rule*: The volume of a generalized cylinder is the product of the area of its base with its altitude: $V = Ah$.
(A generalized cylinder is formed by vertically lifting a horizontal planar region parallel with itself.)
Hint: Slice parallel to the base.

- 8.34 Verify *Cavalieri's Second Rule*: The volume of a generalized cone is one third of the product of the area of its base with its altitude: $V = Ah/3$.
(A generalized cone is formed by all line segments connecting a point P to all points of a planar (base) region.)
Hint: Slice parallel to the base and use that

$$\int_0^h x^2 dx = \frac{h^3}{3}.$$

- 8.35 Use Pappus's rules to deduce the surface area and volume of a ball from the circumference and area of a disk. What is the surface area and volume of a donut?
- 8.36 The *Gamma function* is given by the improper integral

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx, \quad s > 0.$$

Show that $\Gamma(1) = 1$ and that $\Gamma(s+1) = s\Gamma(s)$. Deduce that $\Gamma(n+1) = n!$.

Remark: Many calculators have the Gamma function built in disguised as $\Gamma(s) = (s-1)!$. Experiment with yours. Does entering “(-0.5)!” yield $\sqrt{\pi} = \Gamma(1/2) \approx 1.77245$? If so, graph $y = \Gamma(x)$.

- 8.37 The *Beta function* B is defined by the integral

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx \quad p, q > 0.$$

Prove by a clever change of variables that for $\alpha, \beta > -1$,

$$\int_0^{\pi/2} \cos^\alpha \theta \sin^\beta \theta d\theta = \frac{1}{2} B\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right).$$

- 8.38 Calculate the volume $V_n(r)$ of the first several n -dimensional balls $x_1^2 + x_2^2 + \cdots + x_n^2 \leq r^2$.
Answer: $V_1(r) = 2r$, $V_2(r) = \pi r^2$, $V_3(r) = 4\pi r^3/3$, $V_4(r) = \pi^2 r^4/2$, $V_5(r) = 8\pi^2 r^5/15$.

Hint: By slicing, the n -ball of radius r has volume

$$V_n(r) = \int_{-r}^r V_{n-1}(\sqrt{r^2 - x^2}) dx.$$

8.39 Calculate the work done against the force $f(x) = -x^3e^{-x}$ while moving from $x = 1$ to $x = 2$.

8.40 Derive the integration formula

$$\int_a^b e^{\beta x} \sin x dx = \frac{\beta \sin x - \cos x}{1 + \beta^2} e^{\beta x} \Big|_a^b.$$

Hint: Integrate twice by parts.

8.41 Integrate

$$\int_a^b \operatorname{Arctan} x dx.$$

Hint: Integrate by parts with $u = \operatorname{Arctan} x$.

8.42 Integrate

$$\int_a^b \ln x dx.$$

8.43 Integrate

$$\int_a^b \frac{x^2 - 3x - 1}{(x - 1)(x - 2)(x^2 + x + 1)} dx.$$

Suggestion: Employ the partial fraction expansion

$$\frac{x^2 - 3x - 1}{(x - 1)(x - 2)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{Cx + D}{x^2 + x + 1}.$$

8.44 Find the centroid of the planar region bounded above by $y = \sin x$ and below by the x -axis, $0 \leq x \leq \pi/2$.

8.45 Suppose the ground state (lowest energy stationary state) of a particle trapped in $[0, \infty)$ has wave function $\psi(x)$, where $|\psi(x)|^2 = xe^{-x}$. Find the expected location of this particle.

Outline: Since the instrument Q observing displacement is given by the rule $Q\psi = x\psi$,

$$\langle Q\psi, \psi \rangle = \langle x\psi, \psi \rangle = \int_0^\infty x^2 e^{-x} dx = \dots = 2.$$

- 8.46 The *Rayleigh* probability density function

$$f(r) = \frac{re^{-r^2/2\sigma^2}}{\sigma^2}$$

is a good model (for instance) of the distance error r from the bullseye in target shooting (since horizontal and vertical error are normally distributed). What is the mean error?

Answer:

$$\mu = \sqrt{2}\sigma\Gamma\left(\frac{3}{2}\right) = \sigma\sqrt{\frac{\pi}{2}}.$$

- 8.47 Estimate the arclength of $y = \sin x$ for $0 \leq x \leq \pi$. (The integral for arclength in this case is *elliptic* and hence has no antiderivative in terms of elementary functions. Numerical methods are necessary.)
- 8.48 What is the resulting surface area S when the curve $y = 1 - x^2$, $0 \leq x \leq 2$, is rotated about the y -axis?

Outline: By exercise 8.12,

$$S = 2\pi \int_0^2 x ds = \dots = \frac{\pi}{6}[(17)^{3/2} - 1].$$

- 8.49 Prove the *comparison test*: Suppose $0 \leq f(x) \leq g(x)$ on $[a, \infty)$. Then

$$\int_a^\infty g(x) dx \text{ convergent implies } \int_a^\infty f(x) dx \text{ is convergent.}$$

- 8.50 Using the comparison test of exercise 8.49, prove the convergence of

$$\int_0^\infty \frac{dx}{3x^2 + 2x + 2 + \sin x + \sqrt{x}}.$$

- 8.51 Calculate

$$\int_2^\infty \frac{dx}{x \ln^2 x}.$$

- 8.52 Using Simpson's rule with four subdivisions, estimate

$$I = \int_0^1 \frac{dx}{1 + x^3}.$$

- 8.53 How many subdivisions are sufficient to approximate

$$I = \int_0^1 e^{x^2} dx$$

to three places via Simpson's rule?

8.54 Develop an integration formula for integrals of type

$$\int x^n \sin \alpha x \, dx.$$

8.55 Find the following antiderivatives:

a. $\int \sqrt{1 + \cos 2x} \, dx$

b. $\int \frac{dx}{\sqrt{e^{2x} - 1}}$

c. $\int \frac{dx}{1 + \sqrt{x}}$

d. $\int \frac{x^3 \, dx}{x^2 - 2x + 1}$

e. $\int \frac{x^2 \, dx}{x^4 - 16}$

f. $\int \frac{4x \, dx}{x^4 + 1}$

g. $\int \tan^2 x \, dx$

h. $\int \cos^3 x \sin^2 x \, dx$

8.56 The *hyperbolic trigonometric* functions are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Show that every point on the hyperbola $x^2 - y^2 = 1$ is of the form $(\pm \cosh t, \pm \sinh t)$.

8.57 Prove the differentiation formulas

$$D \cosh x = \sinh x,$$

$$D \sinh x = \cosh x,$$

and derive the Taylor expansions

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots,$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

8.58 Argue that the complex exponential $e^{i\theta}$, if definable at all, must satisfy $e^{i\theta} = \cos \theta + i \sin \theta$.

Hint: Think Taylor series.

- 8.59 Galileo estimated the acceleration g of gravity by rolling solid balls down an inclined plane at extreme inclinations. Show that because of rotational inertia, his estimates of g were doomed to be $5g/7$.
Hint: The kinetic energy gained is the potential energy lost. Total kinetic energy is the sum of translational and rotational kinetic energies. Apply exercise 8.10.

- 8.60 A hoop, a solid cylinder, and a solid ball are simultaneously allowed to roll down an inclined ramp. In what order do they finish at the bottom?
Answer: The ball is first, then cylinder, then hoop.

- 8.61 The *convolution* of two continuous functions f, g on $[0, \infty)$ is the (continuous) function $h = f * g$ given by the rule

$$h(t) = \int_0^t f(\alpha)g(t - \alpha) d\alpha.$$

Calculate the convolutions $t * t$ and $t * e^{-t}$.

Answers: $t^3/6$ and $-1 + t + e^{-t}$.

- 8.62 The *Laplace transform* F of a continuous function f on $[0, \infty)$ is (where convergent) given by the improper integral

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt.$$

Show that the Laplace transform of $f(t) = t^n$ is $F(s) = n!/s^{n+1}$ for $s > 0$ and that the transform of $g(t) = e^{-at}$ is $G(s) = 1/(s + a)$ for $s > -a$.

- 8.63 Ignoring technical convergence details, substantiate that the Laplace transform of a convolution is the product of their transforms. Using this result, redo exercise 8.61. Notice how integration by parts in the time domain becomes partial fraction expansion in the Laplace s -domain.
- 8.64 Show that the work V required to move a satellite of mass m at a distance r from a planet of mass M to infinity is $V = GMm/r$.
- 8.65 Compute the energy (work) W necessary to lift a satellite of mass m from the surface vertically to a distance r from the center of the Earth.
Answer: $W = gmR(r - R)/r$, where R is the radius of the Earth and g is the acceleration at its surface.

- 8.66 (Coulomb's Law) It is observed that two charges Q_1 and Q_2 of the same polarity will repel one another with the force (in Newtons)

$$F = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2},$$

where r is the distance (in meters) separating the two charges and where ϵ_0 is the *permittivity* of free space (in the appropriate units). Glue a $Q_1 = 2$ Coulomb charge to the origin $x = 0$. Calculate the work required to move a $Q_2 = 6$ Coulomb charge from $x = 4$ to $x = 1$.
Answer: $9/4\pi\epsilon_0$ N-m.

- 8.67 Suppose the z -axis is uniformly charged at $\rho > 0$ Coulomb per meter. Show that the electrostatic force F on a unit charge of $Q = 1$ Coulomb located at (x, y, z) is directed perpendicular to the z -axis and is of magnitude

$$F = \frac{\rho Q}{2\pi\epsilon_0\sqrt{x^2 + y^2}}.$$

- 8.68 (*Cauchy principal value*) How far should we push the enlargement of equality? Consider the following ridiculous result when the limits at infinity are taken simultaneously:

$$P.V. \int_{-\infty}^{\infty} x^2 \sin x \, dx = \lim_{N \rightarrow \infty} \int_{-N}^N x^2 \sin x \, dx = 0.$$

Should we allow this even more forgiving improper integral?

- 8.69 (**Project**) Newton did not use the modern integral sign. What was his notation? Did in fact Leibniz choose the elongated "S" symbol to indicate that it is a "smear sum" of infinitesimals?
- 8.70 Using sectors of infinitesimal angle width $d\theta$, argue that the area of a region in the plane given by a polar equation $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 \, d\theta.$$

Also, formulate hypotheses and provide a formal Darboux sum proof of this same result.

Hint: See exercise 5.19.

- 8.71 Find the area of one leaf of $r = \sin 4\theta$.
Answer: $A = \pi/16$.

- 8.72 The E -field voltage gain pattern of a certain Yagi antenna is given by $r = \cos 2\theta$, $-\pi/4 \leq \theta \leq \pi/4$. Its *half-power beamwidth* is $\theta = \pm\pi/8$ since power r^2 has dropped off by half by this angle. What percentage of all the power transmitted is contained within this half-power beamwidth?

Answer: $1/2 + 1/\pi \approx 82\%$.

- 8.73 Show that the period p of the planar pendulum $\ddot{\theta} = -(g/a) \sin \theta$ of Exercise 6.58 is given by the rule

$$p = \sqrt{\frac{a}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}},$$

where θ_0 is the angle of maximal deflection.

Outline: Time is the accumulation of infinitesimal displacement over rate:

$$\frac{p}{4} = \int_0^{\theta_0} \frac{ds}{v} = \int_0^{\theta_0} \frac{d\theta}{d\theta/dt}.$$

Using conservation of energy, rewrite $d\theta/dt$ as a function of θ .

- 8.74 Investigate the convergence of the improper integral

$$I = \int_0^{\infty} \cos x^2 dx.$$

- 8.75 (H. C. Urey) Why are the first several pages of a table of logarithms more worn than the last several?

- 8.76 Prove that if the experimental outcome X is normally distributed with mean μ and standard deviation σ , then the outcomes cluster about the mean as follows:

$$\begin{aligned} \text{prob}(-\sigma < X - \mu < \sigma) &\approx 0.6827, \\ \text{prob}(-2\sigma < X - \mu < 2\sigma) &\approx 0.9545, \\ \text{prob}(-3\sigma < X - \mu < 3\sigma) &\approx 0.9973, \\ \text{prob}(-4\sigma < X - \mu < 4\sigma) &\approx 0.9999. \end{aligned}$$

Outline: Reduce to the standard case $\mu = 0$ and $\sigma = 1$ by means of the change of variable $Y = (X - \mu)/\sigma$, then proceed numerically.

- 8.77 Continuing with example 17, what percentage of the students received a score between 600 and 700 on a normally distributed test outcome with mean $\mu = 500$ and standard deviation $\sigma = 100$?

Answer: 13.6%.

- 8.78 Repeatedly perform sixteen tosses of a coin (either manually or via simulation). Assign $X = -1$ to tails and $X = 1$ to heads. Add up the outcomes X_i and divide by 4 to obtain a new experimental outcome Y ; that is,

$$Y = \frac{X_1 + X_2 + \cdots + X_{16}}{4},$$

where X_i is the outcome of the i -th toss, $i = 1, 2, \dots, 16$. Construct a histogram of the many outcomes of Y . Assemble graphical evidence that the experimental outcome Y appears normally distributed with $\mu = 0$ and $\sigma = 1$.