1

The Simplest Model of Financial Markets

The main goal of the first chapter is to introduce the one-period finite state model of financial markets with elementary financial concepts such as *basis assets*, *focus assets*, *portfolio*, *Arrow–Debreu securities*, *hedging* and *replication*. Alongside the financial topics we will encounter mathematical tools—linear algebra and matrices—essential for formulating and solving basic investment problems. The chapter explains vector and matrix notation and important concepts such as linear independence.

After reading the first two chapters you should understand the meaning of and be able to solve questions of the following type.

Example 1.1 (replication of securities). Suppose that there is a risky security (call it stock) with tomorrow's value $S = 3$, 2 or 1 depending on the state of the market tomorrow. The first state (first scenario) happens with probability $\frac{1}{2}$, the second with probability $\frac{1}{6}$ and the third with probability $\frac{1}{3}$. There is also a risk-free security (bond) which pays 1 no matter what happens tomorrow. We are interested in replicating two call options written on the stock, one with strike 1.5 and the second with strike 1.

1. Find a portfolio of the stock, bond and the first call option that replicates the second call option (so-called gamma hedging).
2. If the initial stock price is 2 and the risk-free rate of return is 5%, what is the no-arbitrage price of the second option?
3. Find the portfolio of the bond and stock which is the best hedge to the first option in terms of the expected squared replication error (so-called delta hedging).

This chapter is important for two reasons. Firstly, the one-period model of financial markets is the main building block of a dynamic multi-period model which will be discussed later and which represents the main tool of any financial analyst. Secondly, matrices provide an effective way of describing the relationships among several variables, random or deterministic, and as such they are used with great advantage throughout the book.

1.1 One-Period Finite State Model

It is a statement of the obvious that the returns in financial markets are uncertain. The question is how to model this uncertainty. The simplest model assumes that

Table 1.1. Hypothetical scenarios.

Event	Scenario #1 probability $\frac{1}{4}$	Scenario #2 probability $\frac{1}{6}$	Scenario #3 probability $\frac{1}{3}$	Scenario #4 probability $\frac{1}{4}$
Value of FTSE100	5000	4500	4200	4100
LIBOR rate	6.25	6.5	6.75	7.00
Weather	Rain	Rain	Rain	Rain and fog
Chelsea–Wimbledon	5:0	4:0	2:3	0:9
etc.				

there are only two dates, which we will call today and tomorrow, but which could equally well be called this week and next week, this year and next year, or now and in 10 min. The essential feature of our two-date, one-period model is that no investment decisions are taken between the two dates. One should be thinking of a world which is at a standstill apart from at 12 noon each day when all economic activity (work, consumption, trading, etc.) is carried out in a split second.

It is assumed that we do not know today what the market prices will be tomorrow, in other words the state of tomorrow's world is uncertain. However, we assume that there is only a finite number of scenarios that can take place, each of which is known today down to the smallest detail. One of these scenarios is drawn at random, using a controlled experiment whereby the probability of each scenario being drawn is known today. The result of the draw is made public at noon tomorrow and all events take place as prescribed by the chosen scenario (see Table 1.1 for illustration).

Let us stop for a moment and reflect how realistic the finite state model is. First of all, how many scenarios are necessary? In the above table we have four random variables: the value of the FTSE index, the level of UK base interest rate, UK weather and the result of the Chelsea–Wimbledon football game. Assuming that each of these variables has five different outcomes and that any combination of individual outcomes is possible we would require $5^4 = 625$ different scenarios. Given that in finance one usually works with two or three scenarios, 625 seems more than sufficient. And yet if you realize that this only allows five values for each random variable (only five different results of the football match!), then 625 scenarios do not appear overly exigent.

Next, do we know the probability of each of the 625 scenarios? Well, we might have a *subjective* opinion on how much these probabilities are but since the weather, football match or development in financial markets can hardly be thought of as controlled random experiments we do not know what the *objective* probabilities of those scenarios are. There is even a school of thought stating that objective probabilities do not exist; see the notes at the end of the chapter.

Hence the finite state model departs from reality in two ways: firstly, with a small number of scenarios (states of the world) it provides only a patchy coverage of the actual outcomes, and secondly we do not know the objective probabilities of each scenario, we only have our subjective opinion of how much they might be.

1.2 Securities and Their Payoffs

Security is a legal entitlement to receive (or an obligation to pay) an amount of money. A security is characterized by its known price today and its generally uncertain payoff tomorrow. What constitutes the payoff depends to some extent on the given security. For example, consider a model with just two scenarios and one security, a share in publicly traded company TRADEWELL Inc. Let us assume that the initial price of the share is 1, and tomorrow it can either increase to 1.2 or drop to 0.9. Assume further that the shareholders will receive a dividend of 0.1 per share tomorrow, no matter what happens to the share price.

The *security payoff* is the amount of money one receives after selling the security tomorrow plus any additional payment such as the dividend, coupon or rebate one is entitled to by virtue of holding the security. In our case the payoff of one TRADEWELL share is 1.3 or 1 depending on the state of the world tomorrow.

Security price plays a dual role. The stock price today is just that—a price. The stock price tomorrow is part of the stock's uncertain payoff.

Throughout this chapter and for a large part of the next chapter we will ignore today's prices and will only talk about the security payoffs. We will come back to pricing in Chapter 2, Section 2.5. Throughout this book we assume *frictionless trading*, meaning that one can buy or sell any amount of any security at the market price without transaction costs. This assumption is justified in liquid markets.

Example 1.2. Suppose S is the stock price at maturity. A *call option* with strike K is a derivative security paying

$$\begin{array}{rl} S-K & \text{if } S>K, \\ 0 & \text{if } S\leqslant K. \end{array}$$

The payoffs of options in Example 1.1 are in Table 1.2.

1.3 Securities as Vectors

An n-tuple of real numbers is called an n-dimensional vector. For

$$x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad y=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

we write $x, y \in \mathbb{R}^n$. Each n-dimensional vector refers to a point in n-dimensional space. The above is a representation of such a point as a *column vector*, which is nothing other than an $n \times 1$ matrix, since it has n rows and 1 column. Of course, the same point can be written as a row vector instead. Whether to use columns or rows is a matter of personal taste, but it is important to be consistent.

Table 1.2. Call option payoffs.

Probability	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$
Stock	3	2	1
Call option #1 ($K = 1.5$)	1.5	0.5	0
Call option #2 ($K = 1$)	2	1	0

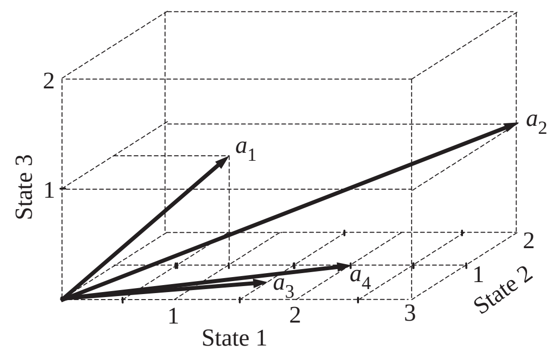

Figure 1.1. Graphical representation of security payoffs.

Example 1.3. Consider the four securities from the introductory example. Let us write the payoffs of each security in the three states (scenarios) as a three-dimensional column vector:

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \qquad a_3 = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix}, \qquad a_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

These securities are depicted graphically in Figure 1.1.

In MATLAB one would write

```
a1 = [1;1;1];
a2 = [3;2;1];
a3 = [1.5;0.5;0];
a4 = [2;1;0];
```

1.4 Operations on Securities

We can multiply vectors by a scalar. For any $\alpha \in \mathbb{R}$ we define

$$\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}.$$

This operation represents α units of security x.

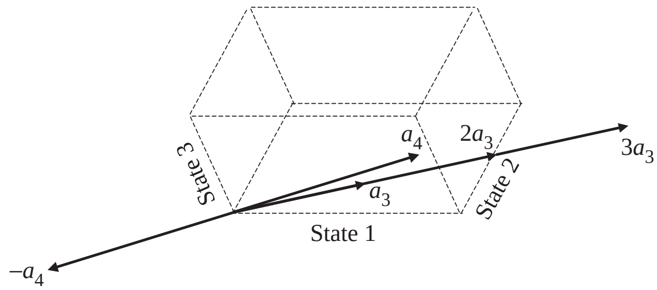

Figure 1.2. Different amounts of the same security have payoffs that lie along a common direction.

Example 1.4. Two units of the third security will have the payoff

$$2a_3 = 2\begin{bmatrix}1.5\\0.5\\0\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix}.$$

If we buy two units of the third security today, tomorrow we will collect 3 pounds (dollars, euros) in the first scenario, 1 in the second scenario and nothing in the third scenario. In MATLAB one would type

```
2*a3;
```

If we issued (wrote, sold) 1 unit of the fourth security, then our payoff tomorrow would be

$$-a_4 = -1\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\-1\\0\end{bmatrix}.$$

In other words, we would have to pay the holder of this security 2 in the first scenario, 1 in the second scenario and nothing in the third scenario. In MATLAB one types

```
-a4;
```

Various amounts of securities a_3 and a_4 are represented graphically in Figure 1.2.

One can also add vectors together:

$$x + y = \begin{bmatrix}x_1 + y_1\\x_2 + y_2\\\vdots\\x_n + y_n\end{bmatrix}.$$

With this operation we can calculate portfolio payoffs. A *portfolio* is a combination of existing securities, which tells us how many units of each security have to be bought or sold to create the portfolio. Naturally, *portfolio payoff* is what the name suggests: the payoff of the combination of securities. The word ‘portfolio’ is sometimes used as an abbreviation of ‘portfolio payoff’, creating a degree of ambiguity in the terminology.

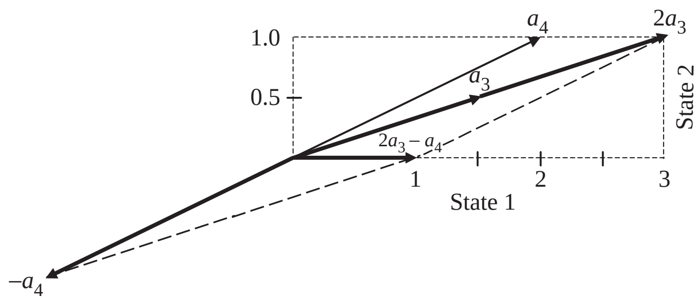

Figure 1.3. Payoff of the portfolio containing two units of security a_3 and minus one unit of security a_4.

Example 1.5. A portfolio in which we hold two units of the first option and issue one unit of the second option will have the payoff

$$2a_3 - a_4 = \begin{bmatrix} 2 \times 1.5 - 2 \\ 2 \times 0.5 - 1 \\ 2 \times 0 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Graphically, this situation is depicted in Figure 1.3. In MATLAB the portfolio payoff is

```
2*a3-a4;
```

1.5 The Matrix as a Collection of Securities

Often we need to work with a collection of securities (vectors). It is then convenient to stack the column vectors next to each other to form a matrix.

Example 1.6. The vectors a_1, a_2, a_3, a_4 from Example 1.3 form a 3×4 payoff matrix, which we denote A,

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

The market scenarios (states of the world) are in rows, securities are in columns. In MATLAB

```
A = [a1 a2 a3 a4];
```

1.6 Transposition

Sometimes we need a *row vector* rather than a column vector. This is achieved by *transposition* of a column vector:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad x^* = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

Note that x^* (transpose of x) is a $1 \times n$ matrix. Conversely, transposition of a row vector gives a column vector. Should we perform the transposition twice, we will end up with the original vector:

$$(x^*)^* = x.$$

Example 1.7.

$$\begin{aligned} a_1^* &= \begin{matrix} 1 & 1 & 1 \end{matrix} \;, \\ a_2^* &= \begin{matrix} 3 & 2 & 1 \end{matrix} \;, \\ a_3^* &= \begin{matrix} 1.5 & 0.5 & 0 \end{matrix} \;, \\ a_4^* &= \begin{matrix} 2 & 1 & 0 \end{matrix} \;. \end{aligned}$$

In MATLAB transposition is achieved by attaching a prime to the matrix name. For example, a_1^* would be written as

```
a1';
```

The vectors $a_1^*, a_2^*, a_3^*, a_4^*$ stacked under each other form a 4×3 matrix B

$$B = \begin{bmatrix} a_1^* \\ a_2^* \\ a_3^* \\ a_4^* \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1.5 & 0.5 & 0 \\ 2 & 1 & 0 \end{bmatrix}, \tag{1.1}$$

in MATLAB

```
B = [a1';a2';a3';a4']
```
(1.2)

Matrix B from equation (1.1) is in fact the *transpose of matrix* A

$$B = A^*,$$

thus instead of (1.2) in MATLAB one would simply write

```
B = A';
```

In general, we can have an $m \times n$ matrix M (denoted $M \in \mathbb{R}^{m \times n}$), where m is the number of rows and n is the number of columns. The element in the ith row and jth column is denoted M_{ij}. The entire jth column is denoted $M_{\bullet j}$ while the entire ith row is denoted $M_{i\bullet}$. According to our needs we can think of the matrix M as if it were composed of m row vectors or n column vectors:

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix} = \begin{bmatrix} M_{1\bullet} \\ M_{2\bullet} \\ \vdots \\ M_{m\bullet} \end{bmatrix} = \begin{matrix} M_{\bullet 1} & M_{\bullet 2} & \cdots & M_{\bullet n} \end{matrix} \;.$$

The transpose of a matrix is obtained by changing the columns of the original matrix into the rows of the transposed matrix:

$$M^* = \begin{bmatrix} M_{11} & M_{21} & \cdots & M_{m1} \\ M_{12} & M_{22} & \cdots & M_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1n} & M_{2n} & \cdots & M_{mn} \end{bmatrix} = \begin{bmatrix} (M_{\bullet 1})^* \\ (M_{\bullet 2})^* \\ \vdots \\ (M_{\bullet m})^* \end{bmatrix}$$
$$= \begin{bmatrix} (M_{1\bullet})^* & (M_{2\bullet})^* & \cdots & (M_{n\bullet})^* \end{bmatrix}.$$

Hence, for example, $M^*_{1\bullet} = (M_{\bullet 1})^*$ and $M^*_{\bullet 1} = (M_{1\bullet})^*$, which in words says that the first row of the transposed matrix is the transpose of the first column of the original matrix.

Example 1.8. Suppose a 3×4 payoff matrix A is given. To extract the payoff of the third security in all states, in MATLAB one would simply write

```
A(:,3);
```

On the other hand, if one wanted to know the payoff of all four securities in the first market scenario, one would look at the row

```
A(1,:);
```

1.7 Matrix Multiplication and Portfolios

The basic building block of matrix multiplication is the multiplication of a row vector by a column vector. Let $A \in \mathbb{R}^{1\times k}$ and $B \in \mathbb{R}^{k\times 1}$:

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}.$$

In this simple case the matrix multiplication AB is defined as follows:

$$AB = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_k b_k. \tag{1.3}$$

Note that A is a $1 \times k$ matrix, B is $k \times 1$ matrix and the result is a 1×1 matrix. One often thinks of a 1×1 matrix as a number.

Example 1.9. Suppose that we have a portfolio of the four securities from the introductory example which consists of x_1, x_2, x_3, x_4 units of the first, second, third and fourth security, respectively. In the third state the individual securities pay 1, 1, 0, 0 in turn. The payoff of the portfolio in the third state will be

$$x_1 \times 1 + x_2 \times 1 + x_3 \times 0 + x_4 \times 0.$$

If we take

$$A_{3\bullet} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

then the portfolio payoff can be written in matrix notation as $A_{3\bullet}x$.

In general one can multiply a matrix U $(m \times k)$ with a matrix V $(k \times n)$, regarding the former as m row vectors in $\mathbb{R}^k$ and the latter as n column vectors in $\mathbb{R}^k$. One multiplies each of the m row vectors in U with each of the n column vectors in V using the simple multiplication rule (1.3):

$$UV = \begin{bmatrix} U_{1\bullet} \\ U_{2\bullet} \\ \vdots \\ U_{m\bullet} \end{bmatrix} \begin{bmatrix} V_{\bullet 1} & V_{\bullet 2} & \cdots & V_{\bullet n} \end{bmatrix} = \begin{bmatrix} U_{1\bullet}V_{\bullet 1} & U_{1\bullet}V_{\bullet 2} & \cdots & U_{1\bullet}V_{\bullet n} \\ U_{2\bullet}V_{\bullet 1} & U_{2\bullet}V_{\bullet 2} & \cdots & U_{2\bullet}V_{\bullet n} \\ \vdots & \vdots & \ddots & \vdots \\ U_{m\bullet}V_{\bullet 1} & U_{m\bullet}V_{\bullet 2} & \cdots & U_{m\bullet}V_{\bullet n} \end{bmatrix}.$$

Facts.

- Matrix multiplication is not, in general, commutative:
$$UV \neq VU.$$
- The result of matrix multiplication does not depend on the order in which the multiplication is carried out (associativity property):
$$(UV)W = U(VW).$$
- Transposition reverses the order of multiplication!
$$(UV)^* = V^*U^*.$$

Example 1.10. Suppose we issue 2 units of call option #1 and 1 unit of call option #2. To balance this position we will buy 2 units of the stock and borrow 1 unit of the bond. What is the total exposure of this portfolio in the three scenarios?

Solution. The portfolio payoff in the first scenario is

$$\begin{bmatrix} 1 & 3 & 1.5 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix} = 1 \times (-1) + 3 \times 2 + 1.5 \times (-2) + 2 \times (-1) = 0.$$

The payoff in the second state is

$$\begin{bmatrix} 1 & 2 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix} = 1 \times (-1) + 2 \times 2 + 0.5 \times (-2) + 1 \times (-1) = 1$$

	A	B	C	D	E	F	G	H
1	pay-off matrix A					portfolio weights x		portfolio pay-off
2	1	3	1.5	2		−1		0
3	1	2	0.5	1		2		1
4	1	1	0	0		−2		1
5						−1		
6								
7								
8								
9								
10								
11								

To multiply matrix A with vector x
select the whole area H2:H4,
then type in the formula
=MMULT(A2:D4;F2:F5)
and press CTRL+SHIFT+ENTER

Figure 1.4. Matrix multiplication in Excel.

and the payoff in the third state will be

$$\begin{matrix} 1 & 1 & 0 & 0 \end{matrix} \begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix} = 1 \times (-1) + 1 \times 2 + 0 \times (-2) + 0 \times (-1) = 1.$$

The payoff in *all three states together* is now

$$\begin{bmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus the portfolio payoff can be expressed using the payoff matrix A and the portfolio vector

$$x^* = \begin{matrix} -1 & 2 & -2 & -1 \end{matrix}$$

as Ax. In MATLAB this reads `A*x`.

Example 1.11. You can perform the same matrix multiplication in Excel, using the instructions in Figure 1.4.

1.8 Systems of Equations and Hedging

A system of m equations for n unknowns $x_1, \dots, x_n$,

$$\left.\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1, \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2, \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m, \end{aligned}\right\} \tag{1.4}$$

can be written in matrix form as

$$\begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{bmatrix} x_1 + \begin{bmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$A_{\bullet 1}x_1 + A_{\bullet 2}x_2 + \cdots + A_{\bullet n}x_n = b$$

or

$$Ax = b, \tag{1.5}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

One can think of the columns of A as being n securities in m states, x being a portfolio of the n securities and b another security that we want to hedge. In such a situation the securities in A are called *basis assets* and the security b is called a *focus asset*. We know that Ax gives the payoff of the portfolio x of basis assets. To solve a system of equations $Ax = b$ therefore means finding a portfolio $\boldsymbol{x}$ of basis assets that replicates (perfectly hedges) the focus asset b.

Typically, the basis assets are liquid securities with known prices, whereas the focus asset b is an over-the-counter (OTC) security issued by an investment bank. Such securities are issued between two parties and do not have a liquid secondary market. The question is, what is a fair price of the OTC security?

By issuing the focus asset b the bank commits itself to pay different amounts of money in different states of the world and thus it enters into a risky position. *Hedging* is a simultaneous purchase of another portfolio that reduces this risk, and a *perfect hedge* is a portfolio that eliminates the risk completely. Suppose that portfolio x is a perfect hedge to the focus asset b. The bank will issue asset b (promise to pay b_i in state i tomorrow) and simultaneously purchase the replicating portfolio x of basis assets.

How much will the bank charge for issuing the OTC security? To break even, it will charge exactly the cost of the replicating portfolio (plus a fee to cover its overheads). Tomorrow, when the payment of b becomes due it will liquidate the hedging portfolio x. Since x was a perfect hedge, the payoff of the hedging portfolio Ax will exactly match the liability b in each state of the world. Hence the bank will not have incurred any risk in this operation.

Example 1.12. Let us answer parts (1) and (2) of the introductory Example 1.1. To replicate the fourth security we need to find a portfolio

$$x^* = \quad x_1 \quad x_2 \quad x_3$$

such that

$$A_{\bullet 1} \quad A_{\bullet 2} \quad A_{\bullet 3} \;\; x = A_{\bullet 4}.$$

Thus we are solving

$$\begin{array}{ccccccccc} 1 \times x_1 & + & 3 \times x_2 & + & 1.5 \times x_3 & = & 2, \\ 1 \times x_1 & + & 2 \times x_2 & + & 0.5 \times x_3 & = & 1, \\ 1 \times x_1 & + & 1 \times x_2 & + & 0 \times x_3 & = & 0. \end{array}$$

After a short manipulation we find that $x_1 = -1$, $x_2 = 1$, $x_3 = 0$ is a unique solution. In MATLAB one can obtain the replicating portfolio by typing

```
x = inv(A(:,1:3))*A(:,4);
```

Part (2) assumes that the risk-free security costs $1/1.05$ today, whereas the stock costs 2. The value of the replicating portfolio is therefore

$$\frac{x_1}{1.05} + 2x_2 = \frac{-1}{1.05} + 2 = 1.048.$$

This is how much the bank would charge for the second call option.

1.8.1 Complications

In the preceding example the hedging portfolio x

$$\begin{bmatrix} \underset{\text{bond}}{A_{\bullet 1}} & \underset{\text{stock}}{A_{\bullet 2}} & \underset{\text{option \#1}}{A_{\bullet 3}} \end{bmatrix} x = \underset{\text{option \#2}}{A_{\bullet 4}} \tag{1.6}$$

is unique and it can be expressed using an inverse matrix

$$x = \quad A_{\bullet 1} \quad A_{\bullet 2} \quad A_{\bullet 3} \quad ^{-1} A_{\bullet 4}.$$

However, if we swap the two call options around,

$$\begin{bmatrix} \underset{\text{bond}}{A_{\bullet 1}} & \underset{\text{stock}}{A_{\bullet 2}} & \underset{\text{option \#2}}{A_{\bullet 4}} \end{bmatrix} x = \underset{\text{option \#1}}{A_{\bullet 3}}, \tag{1.7}$$

we will find that (1.7) suddenly does not have a solution, and, what is more, the matrix

$$A_{\bullet 1} \quad A_{\bullet 2} \quad A_{\bullet 4}$$

is not invertible; this can be seen by typing `inv(A(:,[1 2 4]))`.

To add to the confusion, the system

$$A_{\bullet 1} \quad A_{\bullet 2} \quad x = A_{\bullet 4} \tag{1.8}$$

has a unique solution ($x_1 = -1$, $x_2 = 1$) even though the inverse of $A_{\bullet 1} \quad A_{\bullet 2}$ does not exist; try `inv(A(:,1:2))`. At the same time the system

$$A_{\bullet 1} \quad A_{\bullet 2} \quad x = A_{\bullet 3} \tag{1.9}$$

does not have a solution.

It should be stressed that the hedging problems (1.6)–(1.9) arise naturally; these are *not* special cases that you will never see in practice. Clearly, $m = n$ is neither necessary nor sufficient to find a solution and the same holds for the existence or non-existence of the inverse matrix. The next few sections explain how one solves the hedging problem in full generality. Sections 1.9 and 1.10 provide the terminology, Sections 1.11–1.14 discuss the special case when A has an inverse, and Section 2.1 solves the general case.

1.9 Linear Independence and Redundant Securities

Let the column vectors $A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet n} \in \mathbb{R}^m$ represent n securities in m scenarios, in the sense discussed above.

Definition 1.13. We say that vectors (securities) $A_{\bullet 1}, A_{\bullet 2}, \dots, A_{\bullet n}$ are *linearly independent* if the only solution to

$$A_{\bullet 1}x_1 + A_{\bullet 2}x_2 + \cdots + A_{\bullet n}x_n = 0$$

is the trivial portfolio

$$x_1 = 0, \quad x_2 = 0, \quad \dots, \quad x_n = 0.$$

Mathematicians call the sum $A_{\bullet 1}x_1 + A_{\bullet 2}x_2 + \cdots + A_{\bullet n}x_n$ a *linear combination* of vectors $A_{\bullet 1}, A_{\bullet 2}, \dots, A_{\bullet n}$ and the numbers $x_1, \dots, x_n$ are coefficients of the linear combination. To us $x_1, \dots, x_n$ represent numbers of units of each security in a portfolio and the linear combination represents the portfolio payoff.

The meaning of linear independence is best understood if we look at a situation where $A_{\bullet 1}, A_{\bullet 2}, \dots, A_{\bullet n}$ are *not* linearly independent. From the definition it means that there is a linear combination where at least one of the coefficients $x_1, \dots, x_n$ is non-zero and

$$A_{\bullet 1}x_1 + A_{\bullet 2}x_2 + \cdots + A_{\bullet n}x_n = 0. \tag{1.10}$$

Without loss of generality we can assume that $x_1 \neq 0$. One can then solve (1.10) for $A_{\bullet 1}$:

$$A_{\bullet 1} = -\left(A_{\bullet 2}\frac{x_2}{x_1} + \cdots + A_{\bullet n}\frac{x_n}{x_1}\right).$$

The last equality means that $A_{\bullet 1}$ is a linear combination of vectors $A_{\bullet 2}, \dots, A_{\bullet n}$ with coefficients $-x_2/x_1, \dots, -x_n/x_1$. In conclusion, if the vectors $A_{\bullet 1}, \dots, A_{\bullet n}$ are not linearly independent, then at least one of them can be expressed as a linear combination of the remaining $n-1$ vectors. And vice versa, if vectors $A_{\bullet 1}, \dots, A_{\bullet n}$ *are* linearly independent, then none of them can be expressed as a linear combination of the remaining $n-1$ vectors.

Securities that are linear combinations of other securities are called *redundant* and the portfolio which achieves the same payoff as that of a redundant security is called a *replicating portfolio*. Redundant securities do not add anything new to the market because their payoff can be synthesized from the payoff of the remaining securities; instead of trading a redundant security we might equally well trade the replicating portfolio with the same result.

The practical significance of linearly independent securities, on the other hand, is that each additional linearly independent security has a payoff previously unavailable in the market. The *marketed subspace* is formed by payoffs of all possible portfolios (linear combinations) of basis assets and is denoted $\mathrm{Span}(A_{\bullet 1}, A_{\bullet 2}, \dots, A_{\bullet n})$. As was mentioned above each linearly independent security adds something new to the market—it adds one extra dimension to the marketed subspace. Consequently, the maximum number of linearly independent securities in the marketed subspace is called the *dimension* of the marketed subspace. The definition of dimension is made meaningful by the following theorem.

Theorem 1.14 (Dimensionality Theorem). *Suppose $A_{\bullet 1}, A_{\bullet 2}, \dots, A_{\bullet n}$ are n linearly independent vectors. Suppose*

$$B_{\bullet 1}, B_{\bullet 2}, \dots, B_{\bullet k} \in \mathrm{Span}(A_{\bullet 1}, A_{\bullet 2}, \dots, A_{\bullet n})$$

are linearly independent. Then

$$\mathrm{Span}(B_{\bullet 1}, B_{\bullet 2}, \ldots, B_{\bullet k}) = \mathrm{Span}(A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet n})$$

if and only if $k = n$.

Proof. See website. □

We say that the *market is complete* if the marketed subspace

$$\mathrm{Span}(A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet n})$$

includes all possible payoffs over the m states, that is, if it contains all possible m-dimensional vectors. A complete market means that whatever distribution of wealth in the m market scenarios one may think of, it can always be achieved as a payoff from a portfolio of marketed securities. Since the dimension of $\mathbb{R}^m$ is m, another way of saying that the market is complete is to claim that there are m linearly independent basis securities or that the dimension of the marketed subspace is m.

1.10 The Structure of the Marketed Subspace

There is a simple procedure for finding out the dimension of the marketed subspace, based on the following two facts, which are a direct consequence of the Dimensionality Theorem.

- Suppose that $A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet k}$ are linearly independent. For the next security $A_{\bullet k+1}$ there are only two possibilities. Either $A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet k+1}$ are linearly independent, or $A_{\bullet k+1}$ is *redundant*, that is, there is a replicating portfolio
$$x^* = x_1 \quad x_2 \quad \cdots \quad x_k$$
such that
$$A_{\bullet k+1} = A_{\bullet 1}x_1 + A_{\bullet 2}x_2 + \cdots + A_{\bullet k}x_k.$$
- With m states there cannot be more than m linearly independent securities.

This allows us to sort basis assets into two groups: in one group we have linearly independent securities that span the whole marketed subspace and in the other group we have redundant securities. There is more than one way of splitting the basis assets into these two groups, and the same security may appear once as linearly independent and another time as redundant—there is no contradiction in this. However, the *number* of linearly independent securities in the first group is always the same, and we know that it is equal to the dimension of the marketed subspace.

Example 1.15. Let us split the four securities from the introductory example into linearly independent and redundant securities.

1. We will start with the first security
$$A_{\bullet 1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
and place it among the linearly independent securities.

2. For

$$A_{\bullet 2} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

there are now two possibilities: either

(a) it is redundant, which means there is x_1 such that $A_{\bullet 2} = x_1 A_{\bullet 1}$, or

(b) $A_{\bullet 1}$, $A_{\bullet 2}$ are linearly independent.

Let us examine (a), that is, try to find x_1 so that $A_{\bullet 2} = x_1 A_{\bullet 1}$ holds

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix}.$$

This implies that $x_1 = 3$ *and* $x_1 = 2$ *and* $x_1 = 1$, which is impossible. Since (a) is impossible (b) must hold, therefore we add the second security to the basket of linearly independent securities, already containing the first security.

3. Let us examine the third security:

$$A_{\bullet 3} = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix}.$$

Either

(a) $A_{\bullet 3}$ is redundant, $A_{\bullet 3} = x_1 A_{\bullet 1} + x_2 A_{\bullet 2}$, or

(b) $A_{\bullet 1}$, $A_{\bullet 2}$, $A_{\bullet 3}$ are linearly independent.

Possibility (a) would imply

$$\begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ x_1 + 2x_2 \\ x_1 + x_2 \end{bmatrix}.$$

Subtracting the third equation from the second equation we have $0.5 = x_2$, whereas the first equation minus the second equation gives $1 = x_2$ and these two statements are contradictory. Since (a) is not possible the securities $A_{\bullet 1}$, $A_{\bullet 2}$, $A_{\bullet 3}$ are linearly independent and therefore $A_{\bullet 3}$ goes into the basket with securities one and two.

4. Finally, we examine the fourth security. We could go through the process outlined above, but there is a faster way. We have three states, hence we know that there cannot be *more* than three linearly independent securities. And we already have three linearly independent securities, namely $A_{\bullet 1}$, $A_{\bullet 2}$ and $A_{\bullet 3}$. Since $A_{\bullet 4}$ cannot be independent it has to be redundant.

Note. Had we started with $A_{\bullet 4}$ and then continued with $A_{\bullet 3}$, $A_{\bullet 2}$ and $A_{\bullet 1}$, we would have found that $A_{\bullet 4}$, $A_{\bullet 3}$, $A_{\bullet 2}$ are linearly independent and that $A_{\bullet 1}$ is then redundant.

We can conclude that the market containing securities $A_{\bullet 1}$, $A_{\bullet 2}$, $A_{\bullet 3}$ and $A_{\bullet 4}$ is complete, since with three states three linearly independent securities are (necessary and) sufficient to span the whole market.

Recall that we can stack the securities into a matrix $A = \begin{bmatrix} A_{\bullet 1} & A_{\bullet 2} & \cdots & A_{\bullet n} \end{bmatrix}$ and that the portfolio payoff can be written as $A_{\bullet 1}x_1 + A_{\bullet 2}x_2 + \cdots + A_{\bullet n}x_n = Ax$. Mathematicians call the maximum number of linearly independent columns of a matrix its *rank* and denote it $r(A)$. For us $r(A)$ is nothing other than the dimension of the marketed subspace.

Facts.

- The rank of A^*A is the same as the rank of A.
- $r(AB) \leqslant \min(r(A), r(B))$.
- The ranks of A and A^* are the same—it does not matter whether we look at columns or rows.
- For the $m \times n$ matrix A it is always true that $r(A) \leqslant \min(m, n)$.

Proof. Readers with a particular interest in linear algebra can find the proofs on the website. □

When $r(A) = \min(m, n)$ we say that A has *full rank*. Square matrices with full rank are called *regular* (*non-singular, invertible*).

1.11 The Identity Matrix and Arrow–Debreu Securities

A square matrix of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

is called the identity matrix and is denoted I (or sometimes I_n to denote the dimension). The identity matrix is closely linked to *Arrow–Debreu securities*.

There are as many Arrow–Debreu securities (also called pure securities or elementary state securities) as there are states of the world. The Arrow–Debreu security for state j (denoted e_j) pays 1 in state j and 0 in all other states. Ordering all Arrow–Debreu securities into a matrix $\begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix}$ gives

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

an $m \times m$ identity matrix.

1.12 Matrix Inverse

Recall that a square matrix with full rank is called invertible (regular, non-singular).

- For every square matrix A with full rank (and only for such matrices!) there is a unique matrix B such that
$$AB = BA = I.$$
The matrix B is called the *inverse* to matrix A and it is more commonly denoted A^{-1}. Thus
$$AA^{-1} = A^{-1}A = I.$$
- When C and D are invertible, then CD is also invertible and $(CD)^{-1} = D^{-1}C^{-1}$.
- Trivially, $(A^{-1})^{-1} = A$.

1.13 Inverse Matrix and Replicating Portfolios

Remember that a matrix A must be square with linearly independent columns to have an inverse. Throughout this book we will assume that an efficient procedure for computation of A^{-1} is available. In MATLAB this procedure is called `inv()`. In this section we are interested in the *interpretation* of the inverse matrix. Let us begin with the definition:

$$AA^{-1} = I. \tag{1.11}$$

If we divide the matrices A^{-1} and I into n columns, the matrix equality (1.11) is split into n systems of the form

$$AA^{-1}_{\bullet j} = e_j,$$

where $A^{-1}_{\bullet j}$ is the jth column of the inverse matrix and e_j is the jth column of the identity matrix (see also Section 1.11), $j = 1, 2, \ldots, n$.

Thus, for example, the solution x of the system

$$Ax = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

gives us the first column of the inverse matrix.

Again, if we think of A as containing payoffs of n basis assets in n states, then solving

$$Ax = e_j$$

means finding a portfolio x that replicates the Arrow–Debreu security for state j. Existence of the inverse matrix therefore requires existence of the replicating portfolio for *each* Arrow–Debreu security and this explains why $r(A)$ must equal n for the inverse to exist.

The argument goes as follows. For the inverse to exist each elementary state security must lie in the marketed subspace formed by the basis assets (columns of matrix A). But the elementary state securities are linearly independent and if they all belong to the marketed subspace, that means that the dimension of the marketed subspace is n. We know from Section 1.9 that the dimension of the marketed subspace is equal to $r(A)$. Thus for an inverse to exist we must have $r(A) = n$.

Example 1.16. Find the inverse of

$$A = \begin{bmatrix} 1 & 3 & 1.5 \\ 1 & 2 & 0.5 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution. In MATLAB we would type

```
inv(A(:,1:3));
```

which gives

$$A^{-1} = \begin{bmatrix} 1 & -3 & 3 \\ -1 & 3 & -2 \\ 2 & -4 & 2 \end{bmatrix}.$$

To find the inverse by hand one must solve n systems of the type $Ax = \boldsymbol{I}_{\bullet i}$ for $i = 1, 2, \ldots, n$. This is best performed by Gaussian elimination, but there are other possibilities, for example, the Cramer rule applied to $Ax = \boldsymbol{I}_{\bullet i}$ will lead to the computation of the adjoint matrix ($A^{-1} = \operatorname{adj} A / \det A$). This book does not teach how to solve systems of linear equations by hand; the reader should consult the references at the end of the chapter for a detailed exposition of Gaussian elimination, the Cramer rule and related topics.

Just for illustration let us solve $Ax = \boldsymbol{I}_{\bullet 1}$, that is,

$$\begin{aligned} x_1 &+ 3x_2 &+ 1.5x_3 &= 1, & \text{(1a)} \\ x_1 &+ 2x_2 &+ 0.5x_3 &= 0, & \text{(2a)} \\ x_1 &+ x_2 &+ &= 0, & \text{(3a)} \end{aligned}$$

by Gaussian elimination. In the first instance we subtract Equation (1a) from both Equation (2a) and Equation (3a),

$$\begin{aligned} x_1 &+ 3x_2 &+ 1.5x_3 &= 1, & \text{(1b)} \\ &- x_2 &- x_3 &= -1, & \text{(2b)} \\ &- 2x_2 &- 1.5x_3 &= -1. & \text{(3b)} \end{aligned}$$

Now subtract $2 \times$ Equation (2b) from Equation (3b),

$$\begin{aligned} x_1 &+ 3x_2 &+ 1.5x_3 &= 1, & \text{(1c)} \\ &- x_2 &- x_3 &= -1, & \text{(2c)} \\ & & 0.5x_3 &= 1. & \text{(3c)} \end{aligned}$$

Equation (3c) gives $x_3 = 2$, from Equation (2c) we then have $x_2 = -1$ and finally Equation (1c) gives $x_1 = 1$. Note that x represents the first column of A^{-1} as expected.

Excel commands for computing an inverse matrix are described in Figure 1.5.

	A	B	C	D	E	F	G	H	I	J
1	Matrix A				A inverse					
2	1	3	1.5		1	−3	3			
3	1	2	0.5		−1	3	−2			
4	1	1	0		2	−4	2			
5										
6										
7										
8										
9										

To calculate the inverse of matrix A
select the whole area E2:G4,
then type in the formula
=MINVERSE(A2:D4)
and press CTRL+SHIFT+ENTER

Figure 1.5. Computation of A^{-1} in Excel.

1.14 Complete Market Hedging Formula

The inverse of the payoff matrix can be used to compute replicating portfolios. Recall that the hedging equation reads

$$Ax = b.$$

If A^{-1} exists, we can apply it on both sides to obtain x:

$$A^{-1}Ax = x = A^{-1}b.$$

Complete market without redundant basis assets. Suppose that $A \in \mathbb{R}^{m\times n}$ represents the payoff of n securities in m states. If A represents a complete market without redundant assets, then $r(A) = m = n$, which means that A is a square matrix with full rank and therefore has an inverse A^{-1}. In this case any focus asset b can be hedged perfectly; there is x such that $Ax = b$. The hedging portfolio x is unique and is given by formula

$$x = A^{-1}b. \tag{1.12}$$

Hedging formula (1.12) has a simple financial interpretation. Recall that the columns of A^{-1} represent portfolio weights that perfectly replicate Arrow–Debreu state securities. The focus asset b is a combination of Arrow–Debreu securities with exactly b_i units of the ith state security. Therefore, the hedging portfolio x is a linear combination of columns in A^{-1}; $x = A^{-1}b$.

Example 1.17. Let us take part (1) of the introductory Example 1.1. We have

$$A = \begin{bmatrix} 1 & 3 & 1.5 \\ 1 & 2 & 0.5 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

We have calculated A^{-1} in Example 1.16:

$$A^{-1} = \begin{bmatrix} 1 & -3 & 3 \\ -1 & 3 & -2 \\ 2 & -4 & 2 \end{bmatrix}.$$

	A	B	C	D	E	F	G	H	I	J
1	Matrix A				b		x			
2	1	3	1.5		2		−1			
3	1	2	0.5		1		1			
4	1	1	0		0		0			
5										
6										
7										
8										
9										

To calculate the replicating portfolio
select the whole area G2:G4,
then type in the formula
=MMULT(MINVERSE(A2:C4), E2:E4)
and press CTRL+SHIFT+ENTER

Figure 1.6. Solution of the hedging problem using A^{-1}.

The replicating portfolio is therefore

$$x = A^{-1}b = \begin{bmatrix} 1 & -3 & 3 \\ -1 & 3 & -2 \\ 2 & -4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. \tag{1.13}$$

Excel commands for computing expression (1.13) are given in Figure 1.6.

1.14.1 To Invert or not to Invert?

Note that we have already found the same x in Example 1.12, that time without computing A^{-1}. Which of the two computations would we use in practice?

The main difference between Example 1.12 and Example 1.17 is that the former solves $Ax = b$ for one specific focus asset b; if we changed b, we would have to redo the whole calculation from scratch. In contrast, once we know A^{-1} in Example 1.17 it is easy to recalculate the perfect hedge for *any* focus asset b; we just perform one matrix multiplication $A^{-1}b$. It is also true that solving $Ax = b$ for one fixed value of b (which is what we have done in Example 1.12) is about three times faster than computing the entire inverse matrix A^{-1}. Thus the conclusion is clear. If we are required to solve the hedging problem just once, it is quicker *not* to use the inverse matrix: a MATLAB command to achieve this is `x = A\b`. However, if we have to solve many hedging problems with the same set of basis assets, then it will be far more economical to compute A^{-1} once at the beginning and then recycle it using the formula

$$x = A^{-1}b. \tag{1.14}$$

MATLAB code to perform this task reads `Ainv = inv(A), x = Ainv * b`. This will be particularly useful in dynamic option replication of Chapter 5, where the number of one-period hedging problems is large.

1.15 Summary

- The simplest model of financial markets has two periods and a finite number of states. While today's prices of all securities are known, tomorrow's security payoffs are uncertain. Nevertheless, this uncertainty is rather organized. The security payoffs must follow one of the finite number of scenarios and the contents of each of these scenarios is known today together with the probability of each scenario.

- If m is the number of scenarios (states of the world), then the payoff of each security can be represented as an m-dimensional vector.
- The payoff of n securities is captured in an $m \times n$ payoff matrix A.
- A portfolio is a combination of existing securities. If we write down the number of units of each security in the portfolio into an n-dimensional portfolio vector x, then the portfolio payoff can be calculated from the matrix multiplication Ax.
- An asset whose payoff can be obtained as a combination of payoffs of other securities is called redundant. The portfolio which has the same payoff as a redundant asset is called a replicating portfolio.
- Any system of linear equations can be written down as a matrix equality and vice versa; see equations (1.4) and (1.5).
- A hedging problem with m states of the world, n basis assets and a focus asset b can be expressed as a system of m linear equations for n unknowns x, with right-hand side b:

$$Ax = b.$$

  The $m \times n$ system matrix A contains payoffs of the basis assets as its columns. The solution x of the system, if it exists, represents a portfolio of basis assets which replicates the focus asset b.
- A matrix A has an inverse if and only if it is square with full rank. The inverse, if it exists, is denoted A^{-1} and has the property,

$$AA^{-1} = A^{-1}A = I.$$

- If A is a payoff matrix of basis assets, then the individual columns of A^{-1} represent replicating portfolios to individual Arrow–Debreu securities.
- In a complete market one can hedge perfectly any focus asset b, and when there are no redundant basis assets one can express the perfect hedge as

$$x = A^{-1}b.$$

  Here one can interpret x as a linear combination of portfolios that perfectly replicate Arrow–Debreu securities.

1.16 Notes

Anton (2000) and Grossman (1994) are comprehensive guides to matrix calculations and to the underlying theory.

It is important to bear in mind that objective probabilities are in fact our *subjective* guess of how likely the different states are; in reality, we cannot hope that someone behind the scenes is flipping a coin or rolling dice to generate states according to a particular (random) formula. The classic statement of this is by de Finetti (1974a): '[objective] probability does not exist'. One can use probabilistic models with great advantage but every user has to supply his or her own ‘objective’ probabilities and each user is solely responsible for the actions he or she takes based on such models.

1.17 Exercises

Exercise 1.1. Which of the following is true of matrix multiplication of matrices A and B?

(a) It can be performed only if A and B are square matrices.
(b) Each entry of the result c_{ij} is the product of a_{ij} and b_{ij}.
(c) $AB = BA$.
(d) It can be performed only if the number of columns of A is equal to the number of rows B.

Exercise 1.2. The result of the matrix multiplication $\begin{bmatrix}1\\1\end{bmatrix} \begin{matrix}1 & 2 & 3\end{matrix}$ is

(a) not defined;
(b) 6 ;
(c) $\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\end{bmatrix}$;
(d) none of the above.

Exercise 1.3. Which of the following is true of matrices A and B if AB is a column vector?

(a) B is a column vector.
(b) A is a row vector.
(c) A and B are square matrices.
(d) The number of rows in A must equal the number of columns in B.

Exercise 1.4. The rank of the $n \times n$ identity matrix is

(a) 0;
(b) 1;
(c) n^2;
(d) none of the above.

Exercise 1.5. The rank of the $m \times n$ matrix is

(a) equal to $\max(m, n)$;
(b) only defined when $m = n$, in which case it is equal to m;
(c) not greater than $\min(m, n)$;
(d) none of the above.

Exercise 1.6. The last column of a transposed matrix is the same as

(a) the first column of the original matrix;
(b) the last row of the original matrix, but transposed;
(c) the first row of the original matrix, but transposed;
(d) none of the above.

Exercise 1.7. Let A be an $m \times n$ matrix representing the payoff of n securities in m states of the world. The assertion 'market is complete' means that

(a) $m \geqslant n$;
(b) $n \geqslant m$;
(c) $r(A) = m$;
(d) $r(A) = n$.

Exercise 1.8. When there are more securities than states of the world, then

(a) some securities are redundant;
(b) markets are complete;
(c) markets are incomplete;
(d) none of the above.

Exercise 1.9. The number of redundant securities is equal to

(a) $m - \min(m, n)$;
(b) $m - r(A)$;
(c) $n - r(A)$;
(d) none of the above.

Exercise 1.10. If A has full rank, this means that

(a) markets are complete;
(b) there are no redundant securities;
(c) sometimes (a), sometimes (b) and sometimes both;
(d) none of the above.

Exercise 1.11 (terminal wealth). An investor with initial wealth £10 000 chooses between a risk-free rate of return of 2% and a risky security with rate of return −20%, −10%, −5%, 0%, 5%, 10%, 20%, 30% with probability 0.05, 0.10, 0.15, 0.20, 0.20, 0.15, 0.10, 0.05, respectively. If α denotes the proportion of initial wealth invested in the risky asset, explain how one can express in matrix notation

(a) terminal wealth;
(b) expected terminal wealth.

Exercise 1.12 (redundant securities). In this question an $m \times n$ matrix A represents the payoff of n securities in m states. In each of the markets below divide securities into linearly independent and redundant:

(a) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$;

(b) $A = \begin{bmatrix} 2 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}$;

(c) $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$.

Exercise 1.13 (quadratic forms). Define a symmetric 2×2 matrix

$$H = \begin{matrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{matrix}$$

and a 2×1 vector

$$x = \begin{matrix} x_1 \\ x_2 \end{matrix} .$$

(a) Perform the matrix multiplication $x^* H x$. The result of the multiplication is a quadratic form in x.

(b) Consider a quadratic form $x_1^2 - 6x_1x_2 + 2x_2^2$. Find a symmetric matrix H such that

$$x_1^2 - 6x_1x_2 + 2x_2^2 = x^* H x.$$

(c) Write the expression

$$\frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(y - y_0)^2$$

in matrix form.

Exercise 1.14 (probability matrices). A probability matrix is a square matrix having two properties: (i) every component is non-negative and (ii) the sum of elements in each row is 1. The following are probability matrices:

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ 0 & 1 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} \end{bmatrix}.$$

(a) Show that PQ is a probability matrix.

(b) Show that for any pair of probability matrices P and Q the product PQ is a probability matrix.