Chapter One

Main results: Semisimple Lie groups case

The present chapter is devoted to describing the main results in the case of connected semisimple Lie groups, which is fundamental in what follows.

1.1 ADMISSIBLE SETS

We start by introducing the notion of admissibility, which describes the families of averaging sets G_t that will be the subject of our analysis.

Let G be a connected semisimple Lie group with finite center and no nontrivial compact factors. Fix a left-invariant Riemannian metric on G and denote the associated invariant distance by d and the Haar invariant measure by m_G. Let

$$\mathcal{O}_\varepsilon = \{g \in G : d(g, e) < \varepsilon\}.$$

Definition 1.1. An increasing family of bounded Borel subsets G_t, $t > 0$, of G is called *admissible* if there exist $c > 0$ and $t_0 > 0$, $\varepsilon_0 > 0$, such that for all $t \geq t_0$ and $0 < \varepsilon \leq \varepsilon_0$,

$$\mathcal{O}_\varepsilon \cdot G_t \cdot \mathcal{O}_\varepsilon \subset G_{t+c\varepsilon}, \tag{1.1}$$

$$m_G(G_{t+\varepsilon}) \leq (1 + c\varepsilon) \cdot m_G(G_t). \tag{1.2}$$

Let us briefly note the following facts (see Proposition 3.14, Chapter 7, and Proposition 5.24 for the respective proofs).

1. Admissibility is independent of the Riemannian metric chosen to define it.

2. Many of the natural families of sets in G are admissible. In particular, the radial sets G_t projecting to the Cartan-Killing Riemannian balls on the symmetric space are admissible. Furthermore, the sets $\{g\,;\ \log \|\tau(g)\| < t\}$, where τ is a faithful linear representation, are also admissible, for any choice of linear norm $\|\cdot\|$.

3. Admissibility is invariant under translations; namely, if G_t is admissible, so is gG_th, for any fixed $g, h \in G$.

Later on we will consider the corresponding *Hölder admissibility* condition, which will also play an important role.

1.2 ERGODIC THEOREMS ON SEMISIMPLE LIE GROUPS

We define β_t to be the probability measures on G obtained as the restriction of the Haar measure to G_t, normalized by $m_G(G_t)$.

The averaging operators associated to the measures β_t when G acts by measure-preserving transformations of a standard Borel probability space (X, μ) are given by

$$\pi(\beta_t)f(x) = \frac{1}{m_G(G_t)} \int_{G_t} f(g^{-1}x)dm_G(g).$$

To state our main results, we introduce the following notation (see Chapter 3 for a detailed discussion).

1. The family β_t (and G_t) will be called *(left-) radial* if it is invariant under (left-) multiplication by some fixed maximal compact subgroup K, for all sufficiently large t. *Standard averages* are those defined in Definition 3.19.

2. The action is called *irreducible* if every noncompact simple factor acts ergodically.

3. The action is said to have a strong spectral gap if each simple factor has a spectral gap, namely, admits no asymptotically invariant sequence of unit vectors (see §3.6 for a full discussion).

4. The sets G_t (and the averages β_t) will be called *balanced* if for every nontrivial direct product decomposition $G = G(1)G(2)$ and every compact subset $Q \subset G(1)$, $\beta_t(QG(2)) \to 0$. G_t will be called *well balanced* if the convergence is at a specific rate (see §3.5 for a full discussion).

Our first main result is the following pointwise ergodic theorem for admissible averages on semisimple Lie groups.

Theorem 1.2. Pointwise ergodic theorems for admissible averages. *Let G be a connected semisimple Lie group with finite center and no nontrivial compact factors. Let (X, μ) be a standard probability Borel space with a measure-preserving ergodic action of G. Assume that G_t is an admissible family.*

1. *Assume that β_t is left-radial. If the action is irreducible, then β_t satisfies the pointwise ergodic theorem in $L^p(X)$, $1 < p < \infty$; namely, for every $f \in L^p(X)$ and for almost every $x \in X$,*

$$\lim_{t\to\infty} \pi(\beta_t)f(x) = \int_X f d\mu.$$

The conclusion also holds in reducible actions of G, provided the averages are standard, well balanced, and boundary-regular (see §3.4 and §3.5 for definitions).

2. *If the action has a strong spectral gap, then β_t converges to the ergodic mean almost surely exponentially fast; namely, for every $f \in L^p(X)$, $1 < p < \infty$, and almost all $x \in X$,*

$$\left| \pi(\beta_t) f(x) - \int_X f \, d\mu \right| \leq C_p(f, x) e^{-\theta_p t},$$

where $\theta_p > 0$ depends explicitly on the spectral gap (and the family G_t).

The conclusion also holds in any action of G with a spectral gap, provided the averages satisfy the additional necessary condition of being well balanced (see §3.5 and §3.7 for definitions).

We note that there are many natural examples of averages for which the conclusions of Theorem 1.2 hold. Previously, it has been established for the Haar-uniform averages on Riemannian balls in [N3], [N4], [NS2], and [MNS]. The fact that the conditions required in Theorem 1.2 are satisfied by much more general families is demonstrated in Theorems 3.15 and 3.18 below.

Regarding Theorem 1.2(1), we remark that the proof of pointwise convergence in the case of reducible actions without a spectral gap is quite involved, and we have thus assumed in that case that the averages are standard, well balanced, and boundary-regular to make the analysis tractable. However, the reducible case will be absolutely indispensable for us below since we will induce actions of a lattice subgroup to actions of G, and these may be reducible.

Regarding Theorem 1.2(2), we note that θ_p depends explicitly on the spectral gap of the action and on natural geometric parameters of G_t, and we refer the reader to §5.2 for a full discussion including a formula for a lower bound. Furthermore, Hölder admissibility is sufficient for this part, as we shall see below.

Let us now formulate the following invariance principle for ergodic actions of G, which will play an important role below in the derivation of pointwise ergodic theorems for lattices.

Theorem 1.3. Invariance principle. *Let G, (X, μ) be as in Theorem 1.2 and let G_t be an admissible family. Then for any given measurable function f on X with $f \in L^p(X)$, the set where pointwise convergence to the ergodic mean holds; namely,*

$$\left\{ x \in X \, ; \, \lim_{t \to \infty} \frac{1}{m_G(G_t)} \int_{G_t} f(g^{-1}x) \, dm_G(g) = \int_X f \, d\mu \right\}$$

contains a G-invariant set of full measure.

We note that G is a nonamenable group and that the sets G_t are not asymptotically invariant under translations (namely, do not have the Følner property). Thus the conclusion of Theorem 1.3 is not obvious, even in the case where X is a homogeneous G-action. The special case where $G = \mathrm{SO}^0(n, 1)$ and β_t are the bi-K-invariant averages lifted from ball averages on hyperbolic space $\mathbb{H}^n$ was considered earlier in [BR].

One of our applications of ergodic theorems on G is an equidistribution theorem for isometric actions of a lattice subgroup. The proof of the latter result actually depends only on the *mean* ergodic theorem for β_t, which requires less stringent conditions than the pointwise theorem. Because of its significance later on, we formulate separately the following.

Theorem 1.4. Mean ergodic theorems for admissible averages. *Let G and (X, μ) be as in Theorem 1.2 and let G_t be an admissible family.*

1. *If the action is irreducible, or the family G_t is balanced, then*
$$\lim_{t\to\infty} \left\| \pi(\beta_t) f - \int_X f\, d\mu \right\|_{L^p(X)} = 0, \quad f \in L^p,\ 1 \le p < \infty.$$
2. *If the action has a strong spectral gap, or a spectral gap and the averages are well balanced, then*
$$\left\| \pi(\beta_t) f - \int_X f\, d\mu \right\|_{L^p(X)} \le B_p e^{-\theta_p t}, \quad f \in L^p,\ 1 < p < \infty,$$
for the same $\theta_p > 0$ as in Theorem 1.2(2).

We remark that the mean ergodic theorem actually holds under much more general conditions, and we refer the reader to [GN] for further discussion and applications of this fact.

1.3 THE LATTICE POINT–COUNTING PROBLEM IN ADMISSIBLE DOMAINS

Now let $\Gamma \subset G$ be any lattice subgroup; the lattice point–counting problem is to determine the number of lattice points in the domains G_t. Its ideal solution calls for evaluating the main term in the asymptotic expansion, establishing the existence of the limit, and estimating explicitly the error term. Our second main result gives a complete solution to this problem for all lattices and all families of admissible domains in connected semisimple Lie groups. The proof we give below will establish the general principle that a mean ergodic theorem in $L^2(G/\Gamma)$ for the averages β_t (with an explicit rate of convergence) implies a solution to the Γ-lattice point–counting problem in the admissible domains G_t (with an explicit estimate of the error term). In fact, this principle applies to lattices in general lcsc groups, and we will discuss this further below.

Assuming that G is connected semisimple with finite center and no nontrivial compact factors, we note that in this case the main term in the lattice count (namely, part 1 of the following theorem) was established in [Ba] (for uniform lattices), in [DRS] (for balls w.r.t. a norm), and in [EM] (in general). Error terms were considered for rotation-invariant norms in [DRS] and for more general norms recently in [Ma]. For a comparison of part 2 of the following theorem with these results, see Chapter 2.

Theorem 1.5. Counting lattice points in admissible domains. *Let G be a connected semisimple Lie group with finite center and no nontrivial compact factors. Let G_t be an admissible family of sets and let Γ be any lattice subgroup. Normalize the Haar measure m_G to assign measure 1 to a fundamental domain of Γ in G.*

1. *If Γ is an irreducible lattice, or the sets G_t are balanced, then*

$$\lim_{t\to\infty} \frac{|\Gamma \cap G_t|}{m_G(G_t)} = 1.$$

2. *If $(G/\Gamma, m_{G/\Gamma})$ has a strong spectral gap, or the sets G_t are well balanced, then, for all $\varepsilon > 0$,*

$$\frac{|\Gamma \cap G_t|}{m_G(G_t)} = 1 + O_\varepsilon\left(\exp\left(\frac{-t(\theta-\varepsilon)}{\dim G + 1}\right)\right),$$

where $\theta > 0$ depends on G_t and the spectral gap in G/Γ via

$$\theta = \liminf_{t\to\infty} -\frac{1}{t} \log \quad \pi_{G/\Gamma}(\beta_t) \quad {}_{L_0^2(G/\Gamma)}.$$

Remark 1.6.

1. Recall that the G-action on $(G/\Gamma, m_{G/\Gamma})$ is irreducible if and only if Γ is an irreducible lattice in G, namely, the projection of Γ to every nontrivial factor of G is a dense subgroup. If the lattice is reducible, a strong spectral gap will certainly not hold.

2. The G-action on G/Γ always has a spectral gap. If G has no nontrivial compact factors and the lattice Γ is irreducible, then it has a strong spectral gap (see §3.7 for more details). Whether this is true when G has compact factors seems to be an open problem.

3. When the action has a strong spectral gap, the parameter θ can be given explicitly in terms of the rate of volume growth of the sets G_t and the size of the gap—see §5.2.2.2 and Corollary 8.1.

4. Note that under the normalization of m_G given in Theorem 1.5, if $\Delta \subset \Gamma$ is a subgroup of finite index, then

$$\lim_{t\to\infty} \frac{|\Delta \cap G_t|}{m_G(G_t)} = \frac{1}{[\Gamma : \Delta]}.$$

Finally, we remark that the condition of admissibility is absolutely crucial in obtaining pointwise ergodic theorems for G, and thus also for Γ, when the action does not have a spectral gap. When the action has a spectral gap, Hölder admissibility is sufficient. However, lattice point–counting results, quantitative or not, hold in much greater generality. Namely, they hold for families that satisfy the weaker condition $m_G(\mathcal{O}_\varepsilon G_t \mathcal{O}_\varepsilon) \le (1 + c\varepsilon) m_G(G_t)$, which amounts to a quantitative version of the well-roundedness condition in [DRS] and [EM]. This generalization is discussed systematically in [GN], where several applications, including those to quantitative counting of lattice points in sectors, on symmetric varieties, and on Adele groups are given.

1.4 ERGODIC THEOREMS FOR LATTICE SUBGROUPS

We now turn to our third main result, namely, the solution to the problem of establishing ergodic theorems for a general action of a lattice subgroup on a probability space (X, μ). This result uses Theorem 1.2 as a basic tool; namely, it is applied to the action of G induced by the action of Γ on (X, μ). The argument generalizes the one used in the proof of Theorem 1.5, where we consider the action of G induced from the trivial action of Γ on a point. However, the increased generality requires considerable further effort and additional arguments.

To formulate the result, consider the set of lattice points $\Gamma_t = \Gamma \cap G_t$. Let λ_t denote the probability measure on Γ uniformly distributed on Γ_t.

We begin with the following fundamental mean ergodic theorem for arbitrary lattice actions.

Theorem 1.7. Mean ergodic theorem for lattice actions. *Let G, G_t, and Γ be as in Theorem 1.5. Let (X, μ) be an ergodic measure-preserving action of Γ.*

1. *Assume that the action of G induced from the Γ-action on (X, μ) is irreducible or that the sets G_t are balanced. Then for every $f \in L^p(X)$, $1 \leq p < \infty$,*

$$\lim_{t\to\infty} \left\| \frac{1}{|\Gamma_t|} \sum_{\gamma\in\Gamma_t} f(\gamma^{-1}x) - \int_X f\, d\mu \right\|_{L^p(X)} = 0.$$

2. *Assume that the action of G induced from the Γ-action on (X, μ) has a strong spectral gap or that it has a spectral gap and the sets G_t are well balanced. Then for every $f \in L^p(X)$, $1 < p < \infty$,*

$$\left\| \frac{1}{|\Gamma_t|} \sum_{\gamma\in\Gamma_t} f(\gamma^{-1}x) - \int_X f\, d\mu \right\|_{L^p(X)} \leq C e^{-\delta_p t} \|f\|_{L^p(X)},$$

where $\delta_p > 0$ is determined explicitly by the spectral gap for the induced G-action (and also depends on the family G_t).

One immediate application of Theorem 1.7 arises when we take X to be a transitive action on a finite space, namely, $X = \Gamma/\Delta$, Δ a finite index subgroup.

Corollary 1.8. Equidistribution in finite actions. *Let G, Γ, and G_t be as in Theorem 1.5. Let $\Delta \subset \Gamma$ be a subgroup of finite index and γ_0 any element in Γ. Under the assumptions of Theorem 1.7(2),*

$$\frac{1}{|\Gamma_t|} \cdot |\{\gamma \in \Gamma \cap G_t : \gamma \cong \gamma_0 \bmod \Delta\} = \frac{1}{[\Gamma : \Delta]} + O(e^{-\delta t}),$$

where $\delta > 0$ and is determined explicitly by the spectral gap in G/Δ.

We remark that a weaker conclusion than Corollary 1.8, namely, equidistribution of the lattice points in $\Gamma \cap G_t$ among the cosets of Δ in Γ, can also be obtained using the method in [GW], which employs Ratner's theory of unipotent flow. It is also possible to derive the main term from considerations related to the mixing property of flows on G/Γ.

Another application of the mean ergodic theorem is in the proof of an equidistribution theorem for the corresponding averages in isometric actions of the lattice. The result is as follows.

Theorem 1.9. Equidistribution in isometric actions of lattices. *Let G, G_t, and Γ be as in Theorem 1.7. Let (S, d) be a compact metric space on which Γ acts by isometries and assume that the action is ergodic with respect to an invariant probability measure μ whose support coincides with S. Then under the assumptions of Theorem 1.7(1), for every continuous function f on S and every point $s \in S$,*

$$\lim_{t\to\infty} \frac{1}{|\Gamma_t|} \mathop{}_{\gamma\in\Gamma_t} f(\gamma^{-1}s) = \mathop{}_{S} f\, d\mu,$$

and the convergence is uniform in $s \in S$ (i.e., in the supremum norm on $C(S)$).

Let us now formulate pointwise ergodic theorems for general actions of lattices.

Theorem 1.10. Pointwise ergodic theorems for general lattice actions. *Let G, G_t, Γ, and (X, μ) be as in Theorem 1.7.*

1. *Assume that the action induced to G is irreducible and the averages β_t are left-radial. Then the averages λ_t satisfy the pointwise ergodic theorem in $L^p(X)$, $1 < p < \infty$; namely, for $f \in L^p(X)$ and almost every $x \in X$,*

$$\lim_{t\to\infty} \frac{1}{|\Gamma_t|} \mathop{}_{\gamma\in\Gamma_t} f(\gamma^{-1}x) = \mathop{}_{X} f\, d\mu.$$

   The same conclusion also holds when the induced action is reducible, provided β_t are standard, well balanced, and boundary-regular.

2. *Retain the assumption of Theorem 1.7(2). Then the convergence of λ_t to the ergodic mean is almost surely exponentially fast; namely, for $f \in L^p(X)$, $1 < p < \infty$, and almost every $x \in X$,*

$$\frac{1}{|\Gamma_t|} \mathop{}_{\gamma\in\Gamma_t} f(\gamma^{-1}x) - \mathop{}_{X} f\, d\mu \le C_p(x, f)e^{-\zeta_p t},$$

   where $\zeta_p > 0$ is determined explicitly by the spectral gap for the induced G-action (and the family G_t).

Remark 1.11.

1. Note that if G is simple, then of course any action of G induced from an ergodic action of a lattice subgroup is irreducible. However, if G is not simple, then the induced action can be reducible and then the assumption that the averages are balanced is necessary in Theorem 1.10(1). We assume in fact that they are standard, well balanced, and boundary-regular, as we will apply Theorem 1.2(1) to the induced action.

2. Note further that if G is simple and has property T, then the assumption of a strong spectral gap stated in Theorem 1.10(2) is satisfied for every ergodic action of every lattice subgroup. Furthermore, in that case ζ_p has an explicit positive lower bound depending on G and G_t only and independent of Γ and X.

3. It may be the case that whenever G/Γ has a strong spectral gap, so does every action of G induced from an ergodic action of the irreducible lattice Γ which has a spectral gap, but this problem also seems to be open.

4. As we shall see in §6.1, the possibility of utilizing the induced G-action to deduce information on *pointwise convergence* in the inducing Γ-action depends on the invariance principle stated in Theorem 1.3 for admissible averages on G.

Further, below we will give a complete analysis valid for S-algebraic groups and their lattices in all cases, but let us here demonstrate our results in a more concrete fashion which shows, in particular, that sets Γ_t satisfying all the assumptions required do exist. Indeed, let G be a connected semisimple Lie group with finite center and no nontrivial compact factors. Let G/K be its symmetric space, let d be the Riemannian distance associated with the Cartan-Killing form, and let β_t be the Haar-uniform averages on the sets

$$G_t = \{g \in G\,;\ d(gK, K) \leq t\}\,. \tag{1.3}$$

Then G_t are admissible and well balanced, and it has been established in [N2], [N3], [NS2], and [MNS] that in every ergodic probability measure–preserving action of G, the family β_t satisfies the pointwise ergodic theorem in L^p, $1 < p < \infty$, as in Theorem 1.2(1). Furthermore, if the action has a spectral gap, then the convergence to the ergodic mean is exponentially fast, as in Theorem 1.2(2).

Now let $\Gamma \subset G$ be any lattice subgroup. Then the following result, announced in [N6, Thm. 14.4], holds.

Theorem 1.12. Ergodic theorems for lattice points in Riemannian balls. *Let G, G_t, and Γ be as in the preceding paragraph and λ_t the uniform averages on $\Gamma \cap G_t$. Then in every probability measure–preserving action of Γ, λ_t satisfy the mean ergodic theorem in L^p, $1 \leq p < \infty$, and the pointwise ergodic theorem in L^p, $1 < p \leq \infty$. If the Γ-action has a spectral gap, then λ_t satisfy the exponential mean and pointwise ergodic theorem as in Theorems 1.7(2) and 1.10(2). Finally, λ_t satisfy the equidistribution theorem w.r.t. an ergodic invariant probability measure of full support in every isometric action of Γ.*

1.5 SCOPE OF THE METHOD

Motivated by concrete lattice point–counting problems, by equidistribution problems, and by other applications of ergodic theorems, we have attempted to give a unified and comprehensive treatment for a large class of averages on all S-algebraic groups and their lattice subgroups, which applies to every ergodic action of the group or the lattice. This level of generality inevitably brings up several issues that must be addressed, including the following:

1. Natural examples such as $SL_2(\mathbb{Z}[\sqrt{2}])$, which is an irreducible lattice in the group $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, necessitate the consideration of general semisimple Lie groups.

2. Natural examples such as $SL_2(\mathbb{Z}[\frac{1}{p}])$, which is an irreducible lattice in the group $SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_p)$, necessitate the consideration of products of algebraic groups over different fields.

3. Natural lattice point–counting problems, such as the problem of integral equivalence of homogeneous forms that we will discuss below, require the consideration of a wide variety of distances, going beyond norms or invariant distances on symmetric spaces and beyond radial averages.

4. Consideration of all ergodic actions of product groups necessitates the analysis of reducible actions, namely, actions where some component of the product group does not act ergodically. This raises the issue of whether the mass distribution of the balls G_t among the simple factors is balanced, or well balanced, in a sense to be made precise below.

5. Consideration of product groups necessitates the analysis of actions with a spectral gap, but where some component group acts without a spectral gap, namely, with an asymptotically invariant sequence.

6. The fact that totally disconnected linear algebraic groups such as $PGL_2(\mathbb{Q}_p)$ can admit finite-dimensional irreducible nontrivial permutation representations where the ergodic theorems fail necessitates restricting our attention to actions where each component of the group is mixing in the orthogonal complement of the space of its invariants.

We note that resolution of all the issues listed above is essential in the proof of the ergodic theorems for lattice subgroups. Indeed, the basic underlying principle of our method of proof is to induce an action of Γ to an action of G and reduce the ergodic theorems for λ_t to those of β_t. However, it may perhaps be the case that the resulting action of G is reducible, or that it may perhaps have a spectral gap but not a strong spectral gap. In these cases, whether the averages are balanced or well balanced becomes a necessary consideration. We will introduce the tools necessary for a systematic development of the general theory in Chapter 3 and employ them in our subsequent proofs of the ergodic theorems.

As is already clear from the statements of the results above, the distinction between actions with and without a spectral gap is fundamental in determining which ergodic theorems apply, and the two cases call for rather different methods of proof. Thus the results will be established according to the following scheme:

1. Ergodic theorems for averages on semisimple S-algebraic groups in the presence of a spectral gap.

2. Ergodic theorems for averages on semisimple S-algebraic groups in the absence of a spectral gap.

3. Stability of admissible averages on semisimple S-algebraic groups and an invariance principle for their ergodic actions.

4. Mean, maximal, and pointwise ergodic theorems for lattice subgroups in the absence of a spectral gap.

5. Exponential pointwise ergodic theorem for lattice actions in the presence of a spectral gap.

6. Equidistribution for isometric lattice actions.

As we shall see below, this scheme applies in a much wider context than that of semisimple S-algebraic groups. We will formulate in Chapter 6 a general recipe to derive ergodic theorems for actions of a lattice subgroup Γ, given that the underlying averages on the enveloping lcsc group G satisfy the corresponding ergodic theorems. We will also demonstrate in Chapter 5 that the ergodic theorems for G-actions hold, provided only that certain natural spectral, geometric, and regularity conditions are satisfied by the group G and the sets G_t.