CHAPTER 1

What Does It Say?

When the cube and the things together
Are equal to some discrete number,
Find two other numbers differing in this one.
Then . . . their product should always be equal
Exactly to the cube of a third of the things.
The remainder then as a general rule
Of their cube roots subtracted
Will be equal to your principal thing.

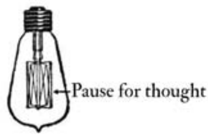
—From Niccolò Tartaglia's account of the solutions to the cubic equation (1539) in Fauvel and Gray, *The History of Mathematics: A Reader*, pp. 255–56.

That's quite a mouthful. In your study of the history of mathematics, you'll quite often come across things like this. They can be baffling at first sight. On the other hand, the same piece of mathematics might be presented like this:

To solve $x^3 + cx = d$,
find u, v such that $u - v = d$ and $uv = (c/3)^3$.
Then $x = \sqrt[3]{u} \quad \sqrt[3]{v}$.

This looks much more straightforward: it's in a mathematical language which we can understand without much difficulty, and we can easily check whether it is true or not.

But it's not really obvious that the two versions say the same thing. Let's look in detail and see if we can trace how you get from one to the other. Before we start, pause for a moment and see how much of it you can make out yourself.



How far did you get? Give yourself a pat on the back if you managed to translate all eight lines into algebra and got something that made sense. Here's how it goes.

When the cube and things together

That's pretty cryptic, for a start. But I've told you that this is about solving cubic equations, so it's fair to assume that there's an unknown quantity—call it x—involved, and that "the cube" means x^3.

What about these "things"? Well, if this is a cubic equation, they can only be (1) a multiple of x^2, (2) a multiple of x, or (3) a constant. If Tartaglia meant a multiple of x^2, he would surely say something about "squares" or "the square," so we can rule out (1). There seems to be no way to tell whether he means a multiple of x or a constant for the moment, so let's leave that and look at the next line.

Are equal to some discrete number,

That makes things a bit clearer. "Some discrete number" sounds pretty much like a constant—let's call it d. That means that "things" is most likely a multiple of x, not another constant. Let's call it cx. So putting the first two lines together gives us this: "x^3 and *cx* together are equal to *d.*" Or, to put it another way: $x^3 + cx = d$.

We're getting somewhere. The first two lines state the problem; the rest of the quote presumably tells us how to solve it.

Find two other numbers differing in this one.

Suddenly we're lost again. Find two numbers—find *u* and *v*, say—differing in "this one." This what? Tartaglia means "this number": that is, the "discrete number" from the previous line, the constant that we called *d*. So this line means "find *u* and *v* differing by *d*" or "find *u*, *v* such that $u - v = d$."

Then . . . their product should always be equal
Exactly to the cube of a third of the things.

"Their product" is the product of *u* and *v*. It's meant to be equal to "the cube of a third of the things." The last time the word "things" was mentioned it meant *cx*. Here that would give us $uv = (cx/3)^3$, right?

Wait a moment. If *x* is our unknown, we can't have it in our definition of *u* and *v*. What else can "things" mean?

Perhaps it means the coefficient: not *cx* but just *c*. That gives us $uv = (c/3)^3$, which makes a lot more sense. Now,

The remainder then as a general rule
Of their cube roots subtracted

"The remainder . . . of their cube roots subtracted"—this has to mean "the remainder when their cube roots are subtracted *from each other*." If we subtracted them from anything else, we wouldn't get one remainder, but two. So these lines mean $\sqrt[3]{u} \quad \sqrt[3]{v}$.

Will be equal to your principal thing.

There's no prize for guessing that "your principal thing" is the unknown quantity we are looking for, x. So these final lines mean $x = \sqrt[3]{u} \quad \sqrt[3]{v}$.

If you go back and look at what we've done, you'll see that the first two lines tell us that $x^3 + cx = d$; then the next six lines go on to tell us how to solve this equation: in line 3 we're told to find u and v such that $u - v = d$, and in lines 4 and 5 Tartaglia says we must also have $uv = (c/3)^3$. Then, in lines 6 through 8 he reveals that this gives us a solution: $x = \sqrt[3]{u} \quad \sqrt[3]{v}$.

In Modern Terms

If you're faced with a piece of historical mathematics that's not in modern notation, you can . . .

Work through it and translate it into modern terms.

If there are "quantities" or "numbers," make them x's and y's.

If there are squares and cubes, addition, multiplication, etc., write them out using algebra.

If there are things you can't work out, move on. Perhaps the next few lines will make it clear.

You could also . . .

Find a modern version of this particular passage and compare it with your own modern version.

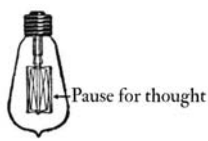
Find a modern version of the same mathematical result and compare it with your own modern version.

You've just seen how to translate sixteenth-century words into modern algebra: not a trivial task, but not an impossible one either. If you want some practice, there's another example similar to this one at the end of the chapter.

Here's another example, with some harder mathematics in it.

> Quantities . . . which in any finite time constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal.
>
> —Isaac Newton, *Philosophiæ naturalis principia mathematica*, Book 1, Lemma 1, translated by I. Bernard Cohen and Anne Whitman, 1999.

Once again, you might like to pause before reading on and have a try at translating this passage into modern notation, just as we did with the piece from Tartaglia. See how far you can get, and don't worry if you get stuck.



Here's how we might translate this passage into modern notation.

"Quantities," the extract begins. Newton is talking about two quantities: let's call them X and Y. They change over time, and we're interested in their behavior over time, so we'll consider them functions $X(t)$ and $Y(t)$ of time t. In particular, we are interested in their behavior "in a given time." Assuming that time is finite, we can call it the period from $t = 0$ to $t = t_1$.

During that time, Newton says, the two quantities "constantly tend to equality." That means the difference between them always gets smaller; in other words, $|X(t) - Y(t)|$ is decreasing during our time interval.

Next, Newton gives a second condition on the two quantities, a more demanding one. During the specified time period they "approach so close to one another that their difference is less than any given quantity." If we call the "given quantity" ε, what that means is that sooner or later the difference will be less than ε. That is to say, we can find a time t' for which $|X(t') - Y(t')| < \varepsilon$. And that's true for *any* "given quantity": any value of ε. In modern terms, for all ε there exists t' in $[0, t_1]$ such that $|X(t') - Y(t')| < \varepsilon$.

The last part of Newton's sentence tells us the consequence of these two conditions. If the conditions are met, he says, the two quantities "become ultimately equal." "Equal" obviously means that $X = Y$; "ultimately" means that this happens at the end of the time interval we're considering, at $t = t_1$. So, $X(t_1) = Y(t_1)$.

So we've found that Newton's statement can be rewritten in modern notation like this:

> Given $X(t)$ and $Y(t)$ with $|X(t) - Y(t)|$ decreasing from $t = 0$ to $t = t_1$,
> if $\;\varepsilon\;\; t' \quad [0, t_1]$ such that $|X(t') - Y(t')| < \varepsilon$,
> then $X(t_1) = Y(t_1)$.

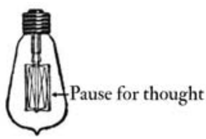
A bit harder than Tartaglia, but again, not an impossible task.

Spotting the difference

But now let's mess things up. Do you think this modern version is really—strictly—equivalent to what Newton wrote?

Be as picky as you can, and see if you can find some places where the two are not quite the same. Go back through our process of translation if you want, and check whether everything is absolutely watertight. You might find that it's not: you might have noticed even as we made our translation that we were introducing some changes that are not just matters of notation or of style.

Can you spot any differences? You might get some hints by comparing our modern version of Newton with the definition of a limit that you find in a modern textbook.



What points did you come up with? These are really picky things, and I'd like to consider just four of them, though you might have found more than that.

First: What *exactly* does Newton mean by "constantly tend"? We've said that $|X(t) - Y(t)|$ is "decreasing," but for us that could mean that $X(t) - Y(t)$ is constant in places, or even constant everywhere, making $X(t)$ and $Y(t)$ equal across the whole interval. I doubt that's what Newton means—"constantly tend" gives the impression of things changing, not staying the same. Perhaps the idea of two quantities "constantly" tending to become equal might be better expressed by saying explicity that (1) they're not equal to begin with and (2) the difference between them is strictly (i.e., always) decreasing. Should we put that into our modern version?

Second: What exactly does Newton mean by a "quantity"? A real number? Yes and no. When were the real numbers first rigorously defined? Not until a long time after Newton: the late 1800s, in fact. That doesn't stop him from intuitively using them, and it seems obvious that he's thinking about quantities which change *continuously* here. On the other hand, it's not clear that the result is still true if the "quantities" and the "given quantity" are rationals, yet it seems a bit cheeky to foist onto Newton the condition that everything in sight is a real number when he wouldn't actually have known precisely what that meant. Should we have said that X, Y, t, ε, and t' are all real numbers, or shouldn't we?

Third: Can ε be negative? No, it can't; that wouldn't make sense—and it can't be zero either. So if we want our modern version to be strictly rigorous, we would have to specify that > 0. Newton doesn't say that; he just says "any given quantity" without pointing out that it has to be a positive nonzero quantity. Maybe for him a "quantity" is necessarily positive. Maybe he just thought it was obvious. Should we leave our version as it is, or should we put in that > 0?

Fourth: These are not the only differences you might have spotted, but it's time to move on—suppose that $X(t) = 1/(t_1 - t)$ and $Y(t) = X(t) - t_1 + t$. Think about it for a moment. The difference between them behaves as it should: it's $t_1 - t$ in the interval, and it satisfies the second condition too. But neither X nor Y is defined at t_1, so the result we're interested in—$X(t_1) = Y(t_1)$—is false in this case. What can we do?

There are ways to fix this up—we could add a condition that X and Y be bounded in the interval, for example, or that they be defined at t_1—but once again we'd be putting in something that Newton didn't say. Apparently he either didn't know about badly behaved functions like these or didn't care—or maybe he just thought it was obvious that they were not what he was talking about. Is it better to fix our modern version so it excludes cases like these, or to leave it closer to what's in Newton's version?

For each of the changes I've suggested making to our modern version, you could argue it either way, and I'll leave it up to you to decide what's best to do. The point is that there *are* some differences between what Newton said and our translation of it into modern mathematical language, and it's not easy to eradicate them without straying from what Newton really wrote. We'll come back to this again and again later in this chapter and throughout this book.

You probably found this exercise a bit picky. But think what we've learned about Newton's mathematics that we might have missed otherwise. His idea of "constantly tending" says more than our idea of "decreasing." When he says "quantities," he might be talking about the real numbers, but we can't really be sure exactly what he has in mind. He assumes that "any given quantity" is greater than zero, without saying so. And he assumes that his "quantities" are finite and generally well behaved in the interval he considers.

You've now seen two ways of finding out about a piece of old mathematics. The first is to translate it into modern terms; the second is to look closely at how the translation isn't *exactly* the same as the original. You can practice them as you go through the rest of this book, and on any other bits of historical mathematics you meet—and you'll find you can learn an awful lot about a piece of old mathematics like this. The box gives a summary of this second way. It gives a selection of questions you can ask to help you spot the difference between a modern

Spotting the Difference

Once you've made—or found—a modern version of a historical piece of mathematics, you can . . .

Look closely at the terms and the concepts, and how you've translated them: Is a "quantity" quite the same thing as a "number"? Is "tending" quite the same thing as "decreasing"?

Look closely at the assumptions you've made or would like to make: Do quantities need to be real, positive, bounded, well-defined? Does the author point those things out or not?

Look closely at the original argument and how precise it is: Is it actually correct? Are there counter-examples, extra conditions, special cases, etc., that the author doesn't bother to point out? There might even be mistakes, or gaps in the argument.

version and the original; but those are not the only questions you can ask. See if you can think of some others of your own. Throughout this book we'll see that being a historian of mathematics is about learning to ask your own questions.

Translation

There's something else, too. Newton didn't say "quantity," and Tartaglia didn't say "some discrete number" or anything like it.

They wrote in foreign languages: Newton in Latin and Tartaglia in Italian. Does that matter?

In a word, yes. Very often in the history of mathematics you'll read texts that have been translated from one language to another. A good translation will turn the words into English but should leave the mathematics and its notation more or less alone. If you suspect that the mathematics itself has been tampered with, or if you're lucky enough to be able to read the original language, you can always try to find a copy of the original version and have a look at it.

We'll spend some time in chapter 2 thinking about how to find sources for the history of mathematics in libraries and on the web. For the moment I'll just say that the first (Latin) edition of Newton's *Principia* is on at least one open-access website. In chapter 3 you'll learn about some of the other things you can find out by looking at the original source, and we'll also think in depth about some of the ways that sources can be tampered with before they get to us.

If you don't, or can't, have a look at the original, it might be possible to find more than one English translation of it. There aren't a great many pieces of historical mathematics that have been translated into English more than once, but not surprisingly Newton's *Principia* is one of them. Here's the same passage we looked at above, but from an older translation. (In fact, this was the first English translation of Newton to be published—in 1729—and it's on Google Books).[1]

> Quantities . . . which in any finite time converge continually to equality, and before the end of that time approach

[1] books.google.co.uk/books?id=Tm0FAAAAQAAJ.

> nearer the one to the other than by any given difference, become ultimately equal.

I think that's similar enough to be quite reassuring. The differences seem small: "converge continually" instead of "constantly tend"; "nearer the one to the other" instead of "so close to one another"; and "any given difference" instead of "any given quantity." Do any of these affect our translation into modern terms? I don't think they do. Do they affect what we can learn about what Newton was thinking? Again, I think not—though you might feel that this older version has a slightly stronger emphasis on motion and change: "converge" and "any given difference" rather than "tend" and "any given quantity." But if there is a difference, it's quite a subtle one.

That means we can probably rely on either of these translations to give us a pretty accurate idea of what Newton was saying, and unless you are willing and able to tackle the Latin that's as much as you can hope for. (If you're keen on these things, you might like to look at the introductions to Florian Cajori's English version of the *Principia* (1934) and the one by Bernard Cohen and Anne Whitman (1999): Why did the editors feel there was a need for a new version? Cohen and Whitman have quite a lot to say about the problems of translating the *Principia*. And if you *can* tackle the Latin, do. The original text is on the web on Gallica, and you can make up your own mind about exactly how it should be translated.[2])

[2] gallica.bnf.fr/ark:/12148/bpt6k3363w. Not to get bogged down in complications, you should know that this is the first edition, whereas the two English translations were based on the third edition, which is slightly different. How it's different is interesting too, but that's another story.

How they thought

When you become accustomed to reading pieces of historical mathematics, you'll find that you don't always need to make modern versions of them in order to see what they are about. And you might also start to feel that modern versions have their limits. You'd be right. While it can be very handy to have the basic mathematical contents of Newton's text—or whoever's—set out in modern algebra, the point of reading historical mathematics is often not just *what they said* but *how they said it*. After all, if you just wanted to see a definition of a limit or a solution to the cubic equation, you could look in a modern textbook.

All the differences we've looked at—the ways our modern version didn't say exactly what Newton said, and the fact that two different English translations of the passage aren't *exactly* the same—are ways you can learn more about a piece of historical mathematics. By looking at not just *what authors said* but also *how they said it*, you can start to learn about *how they thought*.

That sounds hard, but you've started to do it already, when you noticed that Newton says "quantity" when we say "real number," and that he didn't bother to point out some of the assumptions that we want to make explicit. And so far we haven't even mentioned one of the the biggest differences between our version of Newton and the original. That difference is the notation—or rather the fact that our version *has* some mathematical notation and Newton's doesn't; it just has words. You've seen that it can be difficult just to understand a mathematical statement written down in words alone. Imagine how hard it must be to make mathematical discoveries like that, without using any algebraic notation. The solutions to the cubic and the quartic equations were first worked out like that, for example—it's very hard to imagine how they did it. There's an example at the end of this chapter where you can learn more about doing mathematics without notation.

What does the lack of notation mean for the passage from Newton? We already noticed that Newton says "quantities" when we say X, and we would like to make X a real number. More than that, in what Newton says, there are no variables and no functions at all, even though our modern version has both. When we think about this piece of mathematics, we probably imagine a graph of the functions X and Y against the variable t, with two curves that approach one another and eventually meet. But Newton was probably imagining exactly what he says—a quantity which changes over time. Maybe he was thinking of moving bodies whose positions approached one another over time; maybe their *speeds* approached one another; maybe he was thinking of a geometric construction where two *lengths* gradually became the same as he changed a third length (for example, two sides of a triangle: however they start out, if you make the third side decrease to zero, they will become equal).

The word "function" is a notoriously complicated term in historical mathematics. When Leonhard Euler wrote "function" in 1748 he meant, by his own definition, "an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities." When he wrote "function" in 1755, he meant, again by his own definition at the time, that "when quantities depend on others in such a way that [the former] undergo changes themselves when [the latter] change, then [the former] are called functions of [the latter]. . . ." When we say it, there are a few different definitions, depending on context, but none of them is really the same as either of Euler's.[3] This is a good example of how words and concepts don't stay still over time—and how looking at them can show us how historical mathematicians thought.

Something that can often help in this situation is to look look at how the author uses his words and concepts later on in his book or paper. When you see the mathematical words and concepts in use, it can become a lot clearer what they are about; looking ahead often allows you to figure out what the author meant originally. In the case of Newton, if we look ahead in the *Principia,* we find him, again and again, using his idea of a limit in geometric situations where he is interested in the relationships between quantities—lengths or areas—that change over time. For him this piece of mathematics is about changing geometric quantities and what we can say about them when they gradually become equal.

This difference—that we're thinking about functions or a graph and he's thinking about a changing quantity—is a difference we can't fix by modifying our modern version. His concepts are just different from ours. This can be confusing, but it's also helpful to us as historians because it gives us a window into how Newton was thinking—not just what he said and how he said it, but how he thought.

You'll have plenty of chances in this book to practice thinking about this. Old mathematics might use different concepts from the ones we'd use in the same situation—like Newton

[3] There are more examples in Victor Katz's *History of Mathematics* on p. 724 of the second edition.

talking about "quantities" where we'd talk about "real numbers." It might use concepts that you've never come across before—then you would have to see if you could figure out what the author meant, or if the same author gave a definition that would help you. Or it might use concepts that seem familiar, but use them in surprising ways or with surprising meanings—like Euler's different meanings of "function."

By noticing any of these things—even if they seem a bit confusing—you learn something about the historical mathematics and its author. You learn not just *what mathematics it has in it* or *how that mathematics is expressed*, but you also learn something about *how a historical mathematician thought*.

How They Thought

To learn about how a historical mathematician thought, you can ask . . .

What notation does the text use? What words? What concepts? How are these different from what you would use in the same situation?

Does it use words or concepts you don't recognize? Can you work out what they mean, or find out what they mean from the author's definitions?

Does it use familiar words or notation, but with different meanings from what you would expect? Again, can you work out or find out exactly what the author means by them?

Conclusion

You've now seen three ways to learn about a piece of historical mathematics: three kinds of questions you can ask yourself that will help you make sense of *what the author is saying*, *how the author says it*, and *how the author is thinking*. See the box for a summary of what you've learned in this chapter.

There will be times, too, when you *don't* have the original piece of mathematics in front of you: just a modern version of it. This book is about reading historical mathematics, but what you've learned so far might also help you to read the modern version in that situation—and think about what it does or doesn't tell you about the original author and the original ideas.

What Does It Say?

When you're reading a piece of historical mathematics, you can ask . . .

What does it say? What mathematics is in it? Can you translate it into modern terms?

How does it say it? What are the differences between the modern version and the original, and what do they tell you?

How is the author thinking? How are the terms and concepts, and the approach to this piece of mathematics, different from what yours would be?

I'll leave that for you to think about because now we're going to move on to consider mathematical authors themselves.

To think about

(1) Translate the following into modern algebra:

> When the squares and the things are equal to a number, first you must reduce all the equation to one square, that is if there is less than one square you must equally restore and make good. And if there is more than one square you must reduce to one square, and reducing is done by dividing the whole of the equation by the amount of the squares. And when you have done this, halve the things, and multiply one half by itself. The number is added to this product, and the root of this sum minus the half of the things is the value of the thing required.
> —Luca Pacioli, *Summa de arithmetica . . .* (1494), in Fauvel and Gray, p. 251.

(2) Find a statement of Newton's Second Law of Motion in a modern textbook. Compare it with this, Newton's original statement of the law (Cohen and Whitman's 1999 translation):

> A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.

You might like to think about the concepts Newton does and doesn't use, and you might also want to see if you can find out exactly what he means by the words "motion" and "motive force."

(3) Consider this version of Newton's Lemma 1, the piece of the *Principia* that we discussed above:

> If two quantities $X(t)$ and $Y(t)$, depending continuously on "time" t, and neither of which vanishes in the range $t_0 < t < \infty$, are such that
>
> $$\lim_{t \quad t_1} \frac{X(t)}{Y(t)} \qquad 1,$$
>
> for some assigned $t = t_1$, then
>
> $$X(t_1) = Y(t_1).$$
>
> —S. Chandrasekhar, *Newton's Principia for the Common Reader*, 1995.

In what ways does this say more than our modernized version or than Newton's own words? In what ways does it say less? Do you think it is preferable to ours? Why?

(4) Discuss the following:

> If a glass tube, 36 inches long, closed at top, be sunk perpendicularly into water, till its lower or open end be 30 inches below the surface of the water; how high will the water rise within the tube[,] the quicksilver in the common barometer at the same time standing at 29½ inches?
>
> —Mathematical question no. 1041 by Mr. John Ryley, of Leeds, *The Ladies' Diary*, 1798.

Let l = 36 inches the length of the tube, b = 30 inches the part immersed, x = the height of water in the tube, and f = 413 inches, the height of a column of water equal to the pressure of the atmosphere, when the quicksilver stands at 29½ inches. Then, since the spaces occupied by the same quantity of air, are reciprocally as the compressing forces, it will be, as $l - x : l :: f : \frac{lf}{l \quad x}$ = force of the air in $l - x$; hence $\frac{lf}{l \quad x} + x = b + f$, and x = 2.2654115 inches.

—Answer by Miss Maria Middleton, Eden,
near Durham, *The Ladies' Diary*, 1799.

(You might like to think about the notation—can you work out what :: means?—as well as the physical assumptions the two writers make, any gaps in the mathematical proof, and the level of precision of the final answer.)