Chapter One

The fixed point formula

1.1 SHIMURA VARIETIES

The reference for this section is [P2] §3.

Let $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{R}}$. Identify $\mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^{\times}$ and $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ using the morphism $a \otimes 1 + b \otimes i \longmapsto (a + ib, a - ib)$, and write $\mu_0 : \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbb{S}_{\mathbb{C}}$ for the morphism $z \longmapsto (z, 1)$.

The definition of (pure) Shimura data that will be used here is that of [P2] (3.1), up to condition (3.1.4). So a pure Shimura datum is a triple $(\mathbf{G}, \mathcal{X}, h)$ (that will often be written simply $(\mathbf{G}, \mathcal{X})$), where $\mathbf{G}$ is a connected reductive linear algebraic group over $\mathbb{Q}$, $\mathcal{X}$ is a set with a transitive action of $\mathbf{G}(\mathbb{R})$ and $h : \mathcal{X} \longrightarrow \operatorname{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$ is a $\mathbf{G}(\mathbb{R})$-equivariant morphism, satisfying conditions (3.1.1), (3.1.2), (3.1.3), and (3.1.5) of [P2], but not necessarily condition (3.1.4) (i.e., the group $\mathbf{G}^{\mathrm{ad}}$ may have a simple factor of compact type defined over $\mathbb{Q}$).

Let $(\mathbf{G}, \mathcal{X}, h)$ be a Shimura datum. The field of definition F of the conjugacy class of cocharacters $h_x \circ \mu_0 : \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbf{G}_{\mathbb{C}}$, $x \in \mathcal{X}$, is called the *reflex field* of the datum. If K is an open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$, there is an associated Shimura variety $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$, which is a quasi-projective algebraic variety over F satisfying

$$M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \setminus (\mathcal{X} \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K}).$$

If moreover K is *neat* (cf. [P1] 0.6), then $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$ is smooth over F. Let $M(\mathbf{G}, \mathcal{X})$ be the inverse limit of the $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$, taken over the set of open compact subgroups K of $\mathbf{G}(\mathbb{A}_f)$.

Let $g, g' \in \mathbf{G}(\mathbb{A}_f)$, and let K, K' be open compact subgroups of $\mathbf{G}(\mathbb{A}_f)$ such that $\mathrm{K}' \subset g\mathrm{K}g^{-1}$. Then there is a finite morphism

$$T_g : M^{\mathrm{K}'}(\mathbf{G}, \mathcal{X}) \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X}),$$

which is given on complex points by

$$\begin{cases} \mathbf{G}(\mathbb{Q}) \setminus (\mathcal{X} \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K}') & \longrightarrow & \mathbf{G}(\mathbb{Q}) \setminus (\mathcal{X} \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K}), \\ \mathbf{G}(\mathbb{Q})(x, h\mathrm{K}') & \longmapsto & \mathbf{G}(\mathbb{Q})(x, hg\mathrm{K}). \end{cases}$$

If K is neat, then the morphism T_g is étale.

Fix K. The Shimura variety $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$ is not projective over F in general, but it has a compactification $j : M^{\mathrm{K}}(\mathbf{G}, \mathcal{X}) \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$, the Satake-Baily-Borel (or Baily-Borel, or minimal Satake, or minimal) compactification, such that $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$ is a normal projective variety over F and $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$ is open dense in $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$. Note that $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$ is not smooth in general (even when K is neat). The set of complex points of $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$ is

$$M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \setminus (\mathcal{X}^* \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K}),$$

where $\mathcal{X}^*$ is a topological space having $\mathcal{X}$ as an open dense subset and such that the $\mathbf{G}(\mathbb{Q})$-action on $\mathcal{X}$ extends to a continuous $\mathbf{G}(\mathbb{Q})$-action on $\mathcal{X}^*$. As a set, $\mathcal{X}^*$ is the disjoint union of $\mathcal{X}$ and of boundary components $\mathcal{X}_P$ indexed by the set of admissible parabolic subgroups of $\mathbf{G}$ (a parabolic subgroup of $\mathbf{G}$ is called *admissible* if it is not equal to $\mathbf{G}$ and if its image in every simple factor $\mathbf{G}'$ of $\mathbf{G}^{\mathrm{ad}}$ is equal to $\mathbf{G}'$ or to a maximal parabolic subgroup of $\mathbf{G}'$, cf. [P1] 4.5). If $\mathbf{P}$ is an admissible parabolic subgroup of $\mathbf{G}$, then $\mathbf{P}(\mathbb{Q}) = \mathrm{Stab}_{\mathbf{G}(\mathbb{Q})}(\mathcal{X}_P)$; the $\mathbf{P}(\mathbb{Q})$-action on $\mathcal{X}_P$ extends to a transitive $\mathbf{P}(\mathbb{R})$-action, and the unipotent radical of $\mathbf{P}$ acts trivially on $\mathcal{X}_P$.

For every g, K, K$'$ as above, there is a finite morphism $\overline{T}_g : M^{\mathrm{K}'}(\mathbf{G}, \mathcal{X})^* \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$ extending the morphism T_g.

From now on, we will assume that $\mathbf{G}$ satisfies the following condition. Let $\mathbf{P}$ be an admissible parabolic subgroup of $\mathbf{G}$, $\mathbf{N}_P$ be its unipotent radical, $\mathbf{U}_P$ the center of $\mathbf{N}_P$, and $\mathbf{M}_P = \mathbf{P}/\mathbf{N}_P$ the Levi quotient. Then there exist two connected reductive subgroups $\mathbf{L}_P$ and $\mathbf{G}_P$ of $\mathbf{M}_P$ such that

- $\mathbf{M}_P$ is the direct product of $\mathbf{L}_P$ and $\mathbf{G}_P$;
- $\mathbf{G}_P$ contains $\mathbf{G}_1$, where $\mathbf{G}_1$ is the normal subgroup of $\mathbf{M}_P$ defined by Pink in [P2] (3.6), and the quotient $\mathbf{G}_P/\mathbf{G}_1 Z(\mathbf{G}_P)$ is $\mathbb{R}$-anisotropic;
- $\mathbf{L}_P \subset \mathrm{Cent}_{\mathbf{M}_P}(\mathbf{U}_P) \subset Z(\mathbf{M}_P)\mathbf{L}_P$;
- $\mathbf{G}_P(\mathbb{R})$ acts transitively on $\mathcal{X}_P$, and $\mathbf{L}_P(\mathbb{R})$ acts trivially on $\mathcal{X}_P$;
- for every neat open compact subgroup K_M of $\mathbf{M}_P(\mathbb{A}_f)$, $\mathrm{K}_M \cap \mathbf{L}_P(\mathbb{Q}) = \mathrm{K}_M \cap \mathrm{Cent}_{\mathbf{M}_P(\mathbb{Q})}(\mathcal{X}_P)$.

Denote by $\mathbf{Q}_P$ the inverse image of $\mathbf{G}_P$ in $\mathbf{P}$.

Remark 1.1.1 If $\mathbf{G}$ satisfies this condition, then, for every admissible parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, the group $\mathbf{G}_P$ satisfies the same condition.

Example 1.1.2 Any interior form of the general symplectic group $\mathbf{GSp}_{2n}$ or of the quasi-split unitary group $\mathbf{GU}^*(n)$ defined in section 2.1 satisfies the condition.

The boundary of $\mathbf{M}^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$ has a natural stratification (this stratification exists in general, but its description is a little simpler when $\mathbf{G}$ satisfies the above condition). Let $\mathbf{P}$ be an admissible parabolic subgroup of $\mathbf{G}$. Pink has defined a morphism $\mathcal{X}_P \longrightarrow \mathrm{Hom}(\mathbb{S}, \mathbf{G}_{P,\mathbb{R}})$ ([P2] (3.6.1)) such that $(\mathbf{G}_P, \mathcal{X}_P)$ is a Shimura datum and the reflex field of $(\mathbf{G}_P, \mathcal{X}_P)$ is F. Let $g \in \mathbf{G}(\mathbb{A}_f)$. Let $\mathrm{H}_P = g\mathrm{K}g^{-1} \cap \mathbf{P}(\mathbb{Q})\mathbf{Q}_P(\mathbb{A}_f)$, $\mathrm{H}_L = g\mathrm{K}g^{-1} \cap \mathbf{L}_P(\mathbb{Q})\mathbf{N}_P(\mathbb{A}_f)$, $\mathrm{K}_Q = g\mathrm{K}g^{-1} \cap \mathbf{Q}_P(\mathbb{A}_f)$, and $\mathrm{K}_N = g\mathrm{K}g^{-1} \cap \mathbf{N}_P(\mathbb{A}_f)$. Then (cf. [P2] (3.7)) there is a morphism, finite over its image,

$$M^{\mathrm{K}_Q/\mathrm{K}_N}(\mathbf{G}_P, \mathcal{X}_P) \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^* - M^{\mathrm{K}}(\mathbf{G}, \mathcal{X}).$$

The group H_P acts on the right on $M^{\mathrm{K}_Q/\mathrm{K}_N}(\mathbf{G}_P, \mathcal{X}_P)$, and this action factors through the finite group $\mathrm{H}_P/\mathrm{H}_L\mathrm{K}_Q$. Denote by $i_{P,g}$ the locally closed immersion

$$M^{\mathrm{K}_Q/\mathrm{K}_N}(\mathbf{G}_P, \mathcal{X}_P)/\mathrm{H}_P \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*.$$

This immersion extends to a finite morphism

$$\bar{i}_{P,g} : M^{\mathrm{K}_Q/\mathrm{K}_N}(\mathbf{G}_P, \mathcal{X}_P)^*/\mathrm{H}_P \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$$

(this morphism is not a closed immersion in general). The boundary of $M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*$ is the union of the images of the morphisms $i_{P,g}$, for $\mathbf{P}$ an admissible parabolic subgroup of $\mathbf{G}$ and $g\in\mathbf{G}(\mathbb{A}_f)$. If $\mathbf{P}'$ is another admissible parabolic subgroup of $\mathbf{G}$ and $g'\in\mathbf{G}(\mathbb{A}_f)$, then the images of the immersions $i_{P,g}$ and $i_{P',g'}$ are equal if and only if there exists $\gamma\in\mathbf{G}(\mathbb{Q})$ such that $\mathbf{P}'=\gamma\mathbf{P}\gamma^{-1}$ and $\mathbf{P}(\mathbb{Q})\mathbf{Q}_P(\mathbb{A}_f)g\mathrm{K}=\mathbf{P}(\mathbb{Q})\mathbf{Q}_P(\mathbb{A}_f)\gamma^{-1}g'\mathrm{K}$; if there is no such γ, then these images are disjoint. If K is neat, then $\mathrm{K}_Q/\mathrm{K}_N$ is also neat and the action of $\mathrm{H}_P/\mathrm{H}_L\mathrm{K}_N$ on $M^{\mathrm{K}_Q/\mathrm{K}_N}(\mathbf{G}_P,\mathcal{X}_P)$ is free (so $M^{\mathrm{K}_Q/\mathrm{K}_N}(\mathbf{G}_P,\mathcal{X}_P)/\mathrm{H}_P$ is smooth).

The images of the morphisms $i_{P,g}$, $g\in\mathbf{G}(\mathbb{A}_f)$, are the *boundary strata* of $M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*$ associated to $\mathbf{P}$.

To simplify notation, assume from now on that $\mathbf{G}^{\mathrm{ad}}$ is simple. Fix a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}$. A parabolic subgroup of $\mathbf{G}$ is called *standard* if it contains $\mathbf{P}_0$. Let $\mathbf{P}_1,\dots,\mathbf{P}_n$ be the maximal standard parabolic subgroups of $\mathbf{G}$, with the numbering satisfying $r\leq s$ if and only if $\mathbf{U}_{P_r}\subset\mathbf{U}_{P_s}$ (cf. [GHM] (22.3)). Write $\mathbf{N}_r=\mathbf{N}_{P_r}$, $\mathbf{G}_r=\mathbf{G}_{P_r}$, $\mathbf{L}_r=\mathbf{L}_{P_r}$, $i_{r,g}=i_{P_r,g}$, etc.

Let $\mathbf{P}$ be a standard parabolic subgroup of $\mathbf{G}$. Write $\mathbf{P}=\mathbf{P}_{n_1}\cap\dots\cap\mathbf{P}_{n_r}$, with $n_1<\dots<n_r$. The Levi quotient $\mathbf{M}_P=\mathbf{P}/\mathbf{N}_P$ is the direct product of $\mathbf{G}_{n_r}$ and of a Levi subgroup $\mathbf{L}_P$ of $\mathbf{L}_{n_r}$. Let $\mathcal{C}_P$ be the set of n-uples $(X_1,\dots,X_r)$, where

- X_1 is a boundary stratum of $M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*$ associated to $\mathbf{P}_{n_1}$;
- for every $i\in\{1,\dots,r-1\}$, X_{i+1} is a boundary stratum of X_i associated to the maximal parabolic subgroup $(\mathbf{P}_{n_{i+1}}\cap\mathbf{Q}_{n_i})/\mathbf{N}_{n_i}$ of $\mathbf{G}_{n_i}$.

Let $\mathcal{C}_P^1$ be the quotient of $\mathbf{G}(\mathbb{A}_f)\times\mathbf{Q}_{n_1}(\mathbb{A}_f)\times\dots\times\mathbf{Q}_{n_{r-1}}(\mathbb{A}_f)$ by the following equivalence relation: $(g_1,\dots,g_r)$ is equivalent to $(g_1',\dots,g_r')$ if and only if, for every $i\in\{1,\dots,r\}$,

$$(\mathbf{P}_{n_1}\cap\dots\cap\mathbf{P}_{n_i})(\mathbb{Q})\mathbf{Q}_{n_i}(\mathbb{A}_f)g_i\dots g_1\mathrm{K}=(\mathbf{P}_{n_1}\cap\dots\cap\mathbf{P}_{n_i})(\mathbb{Q})\mathbf{Q}_{n_i}(\mathbb{A}_f)g_i'\dots g_1'\mathrm{K}.$$

Proposition 1.1.3 (i) *The map* $\mathbf{G}(\mathbb{A}_f)\longrightarrow\mathcal{C}_P^1$ *that sends* g *to the class of* $(g,1,\dots,1)$ *induces a bijection* $\mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f)\setminus\mathbf{G}(\mathbb{A}_f)/\mathrm{K}\xrightarrow{\sim}\mathcal{C}_P^1$.

(ii) *Define a map* $\varphi':\mathcal{C}_P^1\longrightarrow\mathcal{C}_P$ *in the following way. Let* $(g_1,\dots,g_r)\in\mathbf{G}(\mathbb{A}_f)\times\mathbf{Q}_{n_1}(\mathbb{A}_f)\times\dots\times\mathbf{Q}_{n_{r-1}}(\mathbb{A}_f)$. *For every* $i\in\{1,\dots,r\}$, *write*

$$\mathrm{H}_i=(g_i\dots g_1)\mathrm{K}(g_i\dots g_1)^{-1}\cap(\mathbf{P}_{n_1}\cap\dots\cap\mathbf{P}_{n_i})(\mathbb{Q})\mathbf{Q}_{n_i}(\mathbb{A}_f),$$

and let K_i *be the image of* $\mathrm{H}_i\cap\mathbf{Q}_{n_i}(\mathbb{A}_f)$ *by the obvious morphism* $\mathbf{Q}_{n_i}(\mathbb{A}_f)\longrightarrow\mathbf{G}_{n_i}(\mathbb{A}_f)$. *Then* φ' *sends the class of* $(g_1,\dots,g_r)$ *to the* n*-tuple* $(X_1,\dots,X_r)$, *where* $X_1=\mathrm{Im}(i_{n_1,g_1})=M^{\mathrm{K}_1}(\mathbf{G}_{n_1},\mathcal{X}_{n_1})/\mathrm{H}_1$ *and, for every* $i\in\{1,\dots,r-1\}$, X_{i+1} *is the boundary stratum of* $X_i=M^{\mathrm{K}_i}(\mathbf{G}_{n_i},\mathcal{X}_{n_i})/\mathrm{H}_i$ *image of the morphism* $i_{P',g}$, *with* $\mathbf{P}'=(\mathbf{P}_{n_{i+1}}\cap\mathbf{Q}_{n_i})/\mathbf{N}_{n_i}$ *(a maximal parabolic subgroup of* $\mathbf{G}_{n_i}$*) and* $g=g_{i+1}\mathbf{N}_{n_i}(\mathbb{A}_f)\in\mathbf{G}_{n_i}(\mathbb{A}_f)$.

Then this map $\mathcal{C}_P^1\longrightarrow\mathcal{C}_P$ *is well defined and bijective.*

The proposition gives a bijection $\varphi_P:\mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f)\setminus\mathbf{G}(\mathbb{A}_f)/\mathrm{K}\xrightarrow{\sim}\mathcal{C}_P$. On the other hand, there is a map from $\mathcal{C}_P$ to the set of boundary strata of $M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*$ associated to $\mathbf{P}_{n_r}$, defined by sending $(X_1,\dots,X_r)$ to the image of X_r in $M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*$. After identifying $\mathcal{C}_P$ to $\mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f)\setminus\mathbf{G}(\mathbb{A}_f)/\mathrm{K}$ using φ_P and the second set to

$\mathbf{P}_{n_r}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}$ using $g \longrightarrow \mathrm{Im}(i_{n_r,g})$, this map becomes the obvious projection $\mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K} \longrightarrow \mathbf{P}_{n_r}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}$.

Proof.

(i) As $\mathbf{Q}_{n_r} \subset \mathbf{Q}_{n_{r-1}} \subset \cdots \subset \mathbf{Q}_{n_1}$, it is easy to see that, in the definition of $\mathcal{C}^1_P$, $(g_1, \dots, g_r)$ is equivalent to $(g'_1, \dots, g'_r)$ if and only if

$$(\mathbf{P}_{n_1} \cap \cdots \cap \mathbf{P}_{n_r})(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f)g_r \dots g_1\mathrm{K} = (\mathbf{P}_{n_1} \cap \cdots \cap \mathbf{P}_{n_r})(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f)g'_r \dots g'_1\mathrm{K}.$$

The result now follows from the fact that $\mathbf{P} = \mathbf{P}_{n_1} \cap \cdots \cap \mathbf{P}_{n_r}$.

(ii) We first check that φ' is well defined. Let $i \in \{1, \dots, r-1\}$. If $X_i = M^{\mathrm{K}_i}(\mathbf{G}_{n_i}, \mathcal{X}_{n_i})/\mathrm{H}_i$ and X_{i+1} is the boundary stratum $\mathrm{Im}(i_{P',g})$ of X_i, with $\mathbf{P}'$ and g as in the proposition, then $X_{i+1} = M^{\mathrm{K}'}(\mathbf{G}_{n_{i+1}}, \mathcal{X}_{n_{i+1}})/\mathrm{H}'$, where $\mathrm{H}' = g_{i+1}\mathrm{H}_i g_{i+1}^{-1} \cap \mathbf{P}_{n_{i+1}}(\mathbb{Q})\mathbf{Q}_{n_{i+1}}(\mathbb{A}_f)$ and K' is the image of $\mathrm{H}' \cap \mathbf{Q}_{n_{i+1}}(\mathbb{A}_f)$ in $\mathbf{G}_{n_{i+1}}(\mathbb{A}_f)$. As $g_{i+1} \in \mathbf{Q}_{n_i}(\mathbb{A}_f)$,

$$\mathrm{H}' = (g_{i+1} \dots g_1)\mathrm{K}(g_{i+1} \dots g_1)^{-1} \cap (\mathbf{P}_{n_1} \cap \cdots \cap \mathbf{P}_{n_i})(\mathbb{Q})\mathbf{Q}_{n_i}(\mathbb{A}_f) \cap \mathbf{P}_{n_{i+1}}(\mathbb{Q})\mathbf{Q}_{n_{i+1}}(\mathbb{A}_f).$$

On the other hand, it is easy to see that

$$(\mathbf{P}_{n_1} \cap \cdots \cap \mathbf{P}_{n_i})(\mathbb{Q})\mathbf{Q}_{n_i}(\mathbb{A}_f) \cap \mathbf{P}_{n_{i+1}}(\mathbb{Q})\mathbf{Q}_{n_{i+1}}(\mathbb{A}_f) = (\mathbf{P}_{n_1} \cap \cdots \cap \mathbf{P}_{n_{i+1}})(\mathbb{Q})\mathbf{Q}_{n_{i+1}}(\mathbb{A}_f).$$

Hence $\mathrm{H}' = \mathrm{H}_{i+1}$, and $X_{i+1} = M^{\mathrm{K}_{i+1}}(\mathbf{G}_{n_{i+1}}, \mathcal{X}_{n_{i+1}})/\mathrm{H}_{i+1}$. It is also clear that the n-tuple $(X_1, \dots, X_r)$ defined in the proposition does not change if $(g_1, \dots, g_r)$ is replaced by an equivalent r-tuple.

It is clear that φ' is surjective. We want to show that it is injective. Let $c, c' \in \mathcal{C}^1_P$; write $(X_1, \dots, X_r) = \varphi'(c)$ and $(X'_1, \dots, X'_r) = \varphi'(c')$, and suppose that $(X_1, \dots, X_r) = (X'_1, \dots, X'_r)$. Fix representatives $(g_1, \dots, g_n)$ and $(g'_1, \dots, g'_n)$ of c and c'. As before, write, for every $i \in \{1, \dots, n\}$,

$$\mathrm{H}_i = (g_i \dots g_1)\mathrm{K}(g_i \dots g_1)^{-1} \cap (\mathbf{P}_{n_1} \cap \cdots \cap \mathbf{P}_{n_i})(\mathbb{Q})\mathbf{Q}_{n_i}(\mathbb{A}_f),$$

$$\mathrm{H}'_i = (g'_i \dots g'_1)\mathrm{K}(g'_i \dots g'_1)^{-1} \cap (\mathbf{P}_{n_1} \cap \cdots \cap \mathbf{P}_{n_i})(\mathbb{Q})\mathbf{Q}_{n_i}(\mathbb{A}_f).$$

Then the equality $X_1 = X'_1$ implies that $\mathbf{P}_{n_1}(\mathbb{Q})\mathbf{Q}_{n_1}(\mathbb{A}_f)g_1\mathrm{K} = \mathbf{P}_{n_1}(\mathbb{Q})\mathbf{Q}_{n_1}(\mathbb{A}_f)g'_1\mathrm{K}$ and, for every $i \in \{1, \dots, r-1\}$, the equality $X_{i+1} = X'_{i+1}$ implies that

$$\mathbf{P}_{n_{i+1}}(\mathbb{Q})\mathbf{Q}_{n_{i+1}}(\mathbb{A}_f)g_{i+1}\mathrm{H}_i(g_i \dots g_1) = \mathbf{P}_{n_{i+1}}(\mathbb{Q})\mathbf{Q}_{n_{i+1}}(\mathbb{A}_f)g'_{i+1}\mathrm{H}'_i(g'_i \dots g'_1).$$

So $(g_1, \dots, g_r)$ and $(g'_1, \dots, g'_r)$ are equivalent, and $c = c'$. □

1.2 LOCAL SYSTEMS AND PINK'S THEOREM

Fix a number field K. If $\mathbf{G}$ is a linear algebraic group over $\mathbb{Q}$, let $\mathrm{Rep}_{\mathbf{G}}$ be the category of algebraic representations of $\mathbf{G}$ defined over K. Fix a prime number ℓ and a place λ of K over ℓ.

Let $\mathbf{M}$ be a connected reductive group over $\mathbb{Q}$, $\mathbf{L}$ and $\mathbf{G}$ connected reductive subgroups of $\mathbf{M}$ such that $\mathbf{M}$ is the direct product of $\mathbf{L}$ and $\mathbf{G}$, and $(\mathbf{G}, \mathcal{X})$ a Shimura datum. Extend the $\mathbf{G}(\mathbb{A}_f)$-action on $M(\mathbf{G}, \mathcal{X})$ to an $\mathbf{M}(\mathbb{A}_f)$-action by the obvious map $\mathbf{M}(\mathbb{A}_f) \longrightarrow \mathbf{G}(\mathbb{A}_f)$ (so $\mathbf{L}(\mathbb{A}_f)$ acts trivially). Let K_M be a neat open compact subgroup of $\mathbf{M}(\mathbb{A}_f)$. Write $\mathrm{H} = \mathrm{K}_M \cap \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f)$, $\mathrm{H}_L = \mathrm{K}_M \cap \mathbf{L}(\mathbb{Q})$ (an arithmetic subgroup of $\mathbf{L}(\mathbb{Q})$), and $\mathrm{K} = \mathrm{K}_M \cap \mathbf{G}(\mathbb{A}_f)$. The group H acts on the Shimura variety $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$, and the quotient $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})/\mathrm{H}$ is equal to $M^{\mathrm{H}/\mathrm{H}_L}(\mathbf{G}, \mathcal{X})$ (H/H_L is a neat open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$).

Remark 1.2.1 It is possible to generalize the morphisms T_g of section 1.1: If $m \in \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f)$ and K'_M is an open compact subgroup of $\mathbf{M}(\mathbb{A}_f)$ such that $\mathrm{K}'_M \cap \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f) \subset m\mathrm{H}m^{-1}$, then there is a morphism

$$T_m : M(\mathbf{G}, \mathcal{X})/\mathrm{H}' \longrightarrow M(\mathbf{G}, \mathcal{X})/\mathrm{H},$$

where $\mathrm{H}' = \mathrm{K}'_M \cap \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f)$ and $\mathrm{H} = \mathrm{K}_M \cap \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f)$. This morphism is simply the one induced by the injection $\mathrm{H}' \longrightarrow \mathrm{H}$, $h \longrightarrow mhm^{-1}$ (equivalently, it is induced by the endomorphism $x \longrightarrow xm$ of $M(\mathbf{G}, \mathcal{X})$).

There is an additive triangulated functor $V \longrightarrow \mathcal{F}^{\mathrm{H}/\mathrm{H}_L} R\Gamma(\mathrm{H}_L, V)$ from the category $D^b(\mathrm{Rep}_{\mathbf{M}})$ to the category of λ-adic complexes on $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})/\mathrm{H}$,[1] constructed using the functors $\mu_{\Gamma,\varphi}$ of Pink (cf. [P1] (1.10)) for the profinite étale (and Galois of group H/H_L) covering $M(\mathbf{G}, \mathcal{X}) \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})/\mathrm{H}$ and the properties of the arithmetic subgroups of $\mathbf{L}(\mathbb{Q})$. This construction is explained in [M1] 2.1.4. For every $V \in \mathrm{Ob}\, D^b(\mathrm{Rep}_{\mathbf{M}})$ and $k \in \mathbb{Z}$, $\mathrm{H}^k \mathcal{F}^{\mathrm{H}/\mathrm{H}_L} R\Gamma(\mathrm{H}_L, V)$ is a lisse λ-adic sheaf on $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})/\mathrm{H}$, whose fiber is (noncanonically) isomorphic to

$$\bigoplus_{i+j=k} \mathrm{H}^i(\mathrm{H}_L, \mathrm{H}^j V).$$

Remark 1.2.2 If Γ is a neat arithmetic subgroup of $\mathbf{L}(\mathbb{Q})$ (e.g., $\Gamma = \mathrm{H}_L$), then it is possible to compute $R\Gamma(\Gamma, V)$ in the category $D^b(\mathrm{Rep}_{\mathbf{G}})$, because Γ is of type FL (cf. [BuW], theorem 3.14).

We will now state a theorem of Pink about the direct image of the complexes $\mathcal{F}^{\mathrm{H}/\mathrm{H}_L} R\Gamma(\mathrm{H}_L, V)$ by the open immersion $j : M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})/\mathrm{H} \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*/\mathrm{H}$. Let $\mathbf{P}$ be an admissible parabolic subgroup of $\mathbf{G}$ and $g \in \mathbf{G}(\mathbb{A}_f)$. Write

$$\mathrm{H}_P = g\mathrm{H}g^{-1} \cap \mathbf{L}(\mathbb{Q})\mathbf{P}(\mathbb{Q})\mathbf{Q}_P(\mathbb{A}_f),$$

$$\mathrm{H}_{P,L} = g\mathrm{H}g^{-1} \cap \mathbf{L}(\mathbb{Q})\mathbf{L}_P(\mathbb{Q})\mathbf{N}_P(\mathbb{A}_f),$$

$$\mathrm{K}_N = g\mathrm{H}g^{-1} \cap \mathbf{N}_P(\mathbb{A}_f),$$

$$\mathrm{K}_G = (g\mathrm{H}g^{-1} \cap \mathbf{Q}_P(\mathbb{A}_f))/(g\mathrm{H}g^{-1} \cap \mathbf{N}_P(\mathbb{A}_f)),$$

and $i = i_{P,g} : M^{\mathrm{K}_G}(\mathbf{G}_P, \mathcal{X}_P)/\mathrm{H}_P \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*/\mathrm{H}$.

Then theorem 4.2.1 of [P2] implies the following result (cf. [M1] 2.2).

[1] Here, and in the rest of the book, the notation $R\Gamma$ will be used to denote the right derived functor of the functor H^0.

Theorem 1.2.3 *For every $V \in \mathrm{Ob}\, D^b(\mathrm{Rep}_{\mathbf{M}})$, there are canonical isomorphisms*

$$\begin{aligned} i^* R j_* \mathcal{F}^{\mathrm{H}/\mathrm{H}_L} R\Gamma(\mathrm{H}_L, V) &\simeq \mathcal{F}^{\mathrm{H}_P/\mathrm{H}_{P,L}} R\Gamma(\mathrm{H}_{P,L}, V) \\ &\simeq \mathcal{F}^{\mathrm{H}_P/\mathrm{H}_{P,L}} R\Gamma(\mathrm{H}_{P,L}/\mathrm{K}_N, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)). \end{aligned}$$

The last isomorphism uses van Est's theorem, as stated (and proved) in [GHM] 24.

We will also use local systems on locally symmetric spaces that are not necessarily hermitian. We will need the following notation. Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$. Fix a maximal compact subgroup K_∞ of $\mathbf{G}(\mathbb{R})$. Let $\mathbf{A}_G$ be the maximal ($\mathbb{Q}$-)split torus of the center of $\mathbf{G}$, $\mathcal{X} = \mathbf{G}(\mathbb{R})/\mathrm{K}_\infty \mathbf{A}_G(\mathbb{R})^0$ and $q(\mathbf{G}) = \dim(\mathcal{X})/2 \in \frac{1}{2}\mathbb{Z}$. Write

$$M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \setminus (\mathcal{X} \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K})$$

(even though $(\mathbf{G}, \mathcal{X})$ is not a Shimura datum in general, and $\mathbf{M}^{\mathrm{K}}(\mathbf{G}, \mathcal{X})(\mathbb{C})$ is not always the set of complex points of an algebraic variety). If K is small enough (e.g., neat), this quotient is a real analytic variety. There are morphisms T_g ($g \in \mathbf{G}(\mathbb{A}_f)$) defined exactly as in section 1.1.

Let $V \in \mathrm{Ob}\,\mathrm{Rep}_{\mathbf{G}}$. Let $\mathcal{F}^{\mathrm{K}} V$ be the sheaf of local sections of the morphism

$$\mathbf{G}(\mathbb{Q}) \setminus (V \times \mathcal{X} \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K}) \longrightarrow \mathbf{G}(\mathbb{Q}) \setminus (\mathcal{X} \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K})$$

(where $\mathbf{G}(\mathbb{Q})$ acts on $V \times \mathcal{X} \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K}$ by $(\gamma, (v, x, g\mathrm{K})) \longrightarrow (\gamma.v, \gamma.x, \gamma g\mathrm{K})$). As suggested by the notation, there is a connection between this sheaf and the local systems defined above: if $(\mathbf{G}, \mathcal{X})$ is a Shimura datum, then $\mathcal{F}^{\mathrm{K}} V \otimes K_\lambda$ is the inverse image on $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})(\mathbb{C})$ of the λ-adic sheaf $\mathcal{F}^{\mathrm{K}} V$ on $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$ (cf. [L1], p. 38 or [M1] 2.1.4.1).

Let Γ be a neat arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Then the quotient $\Gamma \setminus \mathcal{X}$ is a real analytic variety. For every $V \in \mathrm{Ob}\,\mathrm{Rep}_{\mathbf{G}}$, let $\mathcal{F}^{\Gamma} V$ be the sheaf of local sections of the morphism

$$\Gamma \setminus (V \times \mathcal{X}) \longrightarrow \Gamma \setminus \mathcal{X}$$

(where Γ acts on $V \times \mathcal{X}$ by $(\gamma, (v, x)) \longrightarrow (\gamma.v, \gamma.x)$).

Let K be a neat open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$, and let $(g_i)_{i \in I}$ be a system of representatives of the double quotient $\mathbf{G}(\mathbb{Q}) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}$. For every $i \in I$, let $\Gamma_i = g_i \mathrm{K} g_i^{-1} \cap \mathbf{G}(\mathbb{Q})$. Then the Γ_i are neat arithmetic subgroups of $\mathbf{G}(\mathbb{Q})$,

$$M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})(\mathbb{C}) = \coprod_{i \in I} \Gamma_i \setminus \mathcal{X},$$

and, for every $V \in \mathrm{Ob}\,\mathrm{Rep}_{\mathbf{G}}$,

$$\mathcal{F}^{\mathrm{K}} V = \underset{i \in I}{\quad} \mathcal{F}^{\Gamma_i} V.$$

1.3 INTEGRAL MODELS

Notation is as in section 1.1. Let $(\mathbf{G}, \mathcal{X})$ be a Shimura datum such that $\mathbf{G}^{\mathrm{ad}}$ is simple and that the maximal parabolic subgroups of $\mathbf{G}$ satisfy the condition of section 1.1.

The goal here is to show that there exist integral models (i.e., models over a suitable localization of $\mathcal{O}_F$) of the varieties and sheaves of sections 1.1 and 1.2 such that Pink's theorem is still true. The exact conditions that we want these models to satisfy are given more precisely below (conditions (1)–(7)).

Fix a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}$, and let $(\mathbf{G}_1, \mathcal{X}_1), \dots, (\mathbf{G}_n, \mathcal{X}_n)$ be the Shimura data associated to the standard maximal parabolic subgroups of $\mathbf{G}$. We will also write $(\mathbf{G}_0, \mathcal{X}_0) = (\mathbf{G}, \mathcal{X})$. Note that, for every $i \in \{0, \dots, n\}$, $\mathbf{P}_0$ determines a minimal parabolic subgroup of $\mathbf{G}_i$. It is clear that, for every $i \in \{0, \dots, n\}$, the Shimura data associated to the standard maximal parabolic subgroups of $\mathbf{G}_i$ are the $(\mathbf{G}_j, \mathcal{X}_j)$, with $i + 1 \leq j \leq n$.

Remember that F is the reflex field of $(\mathbf{G}, \mathcal{X})$. It is also the reflex field of all the $(\mathbf{G}_i, \mathcal{X}_i)$ ([P1] 12.1 and 11.2(c)). Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$; as F is by definition a subfield of $\mathbb{C}$, it is included in $\overline{\mathbb{Q}}$. For every prime number p, fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and an injection $F \subset \overline{\mathbb{Q}}_p$.

Fix a point x_0 of $\mathcal{X}$, and let $h_0 : \mathbb{S} \longrightarrow \mathbf{G}_{\mathbb{R}}$ be the morphism corresponding to x_0. Let w be the composition of h_0 and of the injection $\mathbb{G}_{m,\mathbb{R}} \subset \mathbb{S}$. Then w is independent of the choice of h_0, and it is defined over $\mathbb{Q}$ (cf. [P2] 5.4). An algebraic representation $\rho : \mathbf{G} \longrightarrow \mathbf{GL}(V)$ of $\mathbf{G}$ is said to be *pure of weight m* if $\rho \circ w$ is the multiplication by the character $\lambda \longrightarrow \lambda^m$ of $\mathbb{G}_m$ (note that the sign convention here is not the same as in [P2] 5.4).

Consider the following data:

- for every $i \in \{0, \dots, n\}$, a set $\mathcal{K}_i$ of neat open compact subgroups of $\mathbf{G}_i(\mathbb{A}_f)$, stable by $\mathbf{G}(\mathbb{A}_f)$-conjugacy;
- for every $i \in \{0, \dots, n\}$, a subset A_i of $\mathbf{G}_i(\mathbb{A}_f)$ such that $1 \in A_i$;
- for every $i \in \{0, \dots, n\}$, a full abelian subcategory $\mathcal{R}_i$ of $\mathrm{Rep}_{\mathbf{G}_i}$, stable by taking direct factors.

These data should satisfy the following conditions. Let $i, j \in \{0, \dots, n\}$ be such that $j > i$ and $\mathrm{K} \in \mathcal{K}_i$. Let $\mathbf{P}$ be the standard maximal parabolic subgroup of $\mathbf{G}_i$ associated to $(\mathbf{G}_j, \mathcal{X}_j)$.

Then

(a) For every $g \in \mathbf{G}_i(\mathbb{A}_f)$,

$$(g\mathrm{K}g^{-1} \cap \mathbf{Q}_P(\mathbb{A}_f))/(g\mathrm{K}g^{-1} \cap \mathbf{N}_P(\mathbb{A}_f)) \in \mathcal{K}_j,$$

and, for every $g \in \mathbf{G}_i(\mathbb{A}_f)$ and every standard parabolic subgroup $\mathbf{P}'$ of $\mathbf{G}_i$ such that $\mathbf{Q}_P \subset \mathbf{P}' \subset \mathbf{P}$,

$$(g\mathrm{K}g^{-1} \cap \mathbf{P}'(\mathbb{Q})\mathbf{N}_{P'}(\mathbb{A}_f)\mathbf{Q}_P(\mathbb{A}_f))/(g\mathrm{K}g^{-1} \cap \mathbf{L}_{P'}(\mathbb{Q})\mathbf{N}_{P'}(\mathbb{A}_f)) \in \mathcal{K}_j,$$

$$(g\mathrm{K}g^{-1} \cap \mathbf{P}'(\mathbb{A}_f))/(g\mathrm{K}g^{-1} \cap \mathbf{L}_{P'}(\mathbb{A}_f)\mathbf{N}_{P'}(\mathbb{A}_f)) \in \mathcal{K}_j.$$

(b) Let $g \in A_i$ and $\mathrm{K}' \in \mathcal{K}_i$ be such that $\mathrm{K}' \subset g\mathrm{K}g^{-1}$. Let $h \in \mathbf{P}(\mathbb{Q})\mathbf{Q}_P(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f)/\mathrm{K}$ and $h' \in \mathbf{P}(\mathbb{Q})\mathbf{Q}_P(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f)/\mathrm{K}'$ be such that $\mathbf{P}(\mathbb{Q})\mathbf{Q}_P(\mathbb{A}_f)h\mathrm{K} = \mathbf{P}(\mathbb{Q})\mathbf{Q}_P(\mathbb{A}_f)h'g\mathrm{K}$. Then there exist $p \in \mathbf{L}_P(\mathbb{Q})$ and $q \in \mathbf{Q}_P(\mathbb{A}_f)$ such that $pqh\mathrm{K} = h'g\mathrm{K}$ and the image of q in $\mathbf{G}_j(\mathbb{A}_f) = \mathbf{Q}_P(\mathbb{A}_f)/\mathbf{N}_P(\mathbb{A}_f)$ is in A_j.

(c) For every $g \in \mathbf{G}_i(\mathbb{A}_f)$ and $V \in \operatorname{Ob} \mathcal{R}_i$,

$$R\Gamma(\Gamma_L, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)) \in \operatorname{Ob} D^b(\mathcal{R}_j),$$

where

$$\Gamma_L = (g\mathrm{K}g^{-1} \cap \mathbf{P}(\mathbb{Q})\mathbf{Q}_P(\mathbb{A}_f))/(g\mathrm{K}g^{-1} \cap \mathbf{Q}_P(\mathbb{A}_f)).$$

Let Σ be a finite set of prime numbers such that the groups $\mathbf{G}_0, \ldots, \mathbf{G}_n$ are unramified outside Σ. For every $p \notin \Sigma$, fix $\mathbb{Z}_p$-models of these groups such that the group of $\mathbb{Z}_p$-points is hyperspecial. Let

$$\mathbb{A}_\Sigma = \prod_{p \in \Sigma} \mathbb{Q}_p.$$

Fix $\ell \in \Sigma$ and a place λ of K above ℓ, and consider the following conditions on Σ:

(1) For every $i \in \{0, \ldots, n\}$, $A_i \subset \mathbf{G}_i(\mathbb{A}_\Sigma)$ and every $\mathbf{G}(\mathbb{A}_f)$-conjugacy class in $\mathcal{K}_i$ has a representative of the form $\mathrm{K}_\Sigma \mathrm{K}^\Sigma$, with $\mathrm{K}_\Sigma \subset \mathbf{G}_i(\mathbb{A}_\Sigma)$ and $\mathrm{K}^\Sigma = \prod_{p \notin \Sigma} \mathbf{G}_i(\mathbb{Z}_p)$.

(2) For every $i \in \{0, \ldots, n\}$ and $\mathrm{K} \in \mathcal{K}_i$, there exists a smooth quasi-projective scheme $\mathcal{M}^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)$ over $\operatorname{Spec}(\mathcal{O}_F[1/\Sigma])$ whose generic fiber is $M^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)$.

(3) For every $i \in \{0, \ldots, n\}$ and $\mathrm{K} \in \mathcal{K}_i$, there exists a normal scheme $\mathcal{M}^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)^*$, projective over $\operatorname{Spec}(\mathcal{O}_F[1/\Sigma])$, containing $\mathcal{M}^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)$ as a dense open subscheme and with generic fiber $M^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)^*$. Moreover, the morphisms $i_{P,g}$ (resp., $\bar{i}_{P,g}$) of section 1.1 extend to locally closed immersions (resp., finite morphisms) between the models over $\operatorname{Spec}(\mathcal{O}_F[1/\Sigma])$, and the boundary of $\mathcal{M}^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)^* - \mathcal{M}^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)$ is still the disjoint union of the images of the immersions $i_{P,g}$.

(4) For every $i \in \{0, \ldots, n\}$, $g \in A_i$ and $\mathrm{K}, \mathrm{K}' \in \mathcal{K}_i$ such that $\mathrm{K}' \subset g\mathrm{K}g^{-1}$, the morphism $\overline{T}_g : M^{\mathrm{K}'}(\mathbf{G}_i, \mathcal{X}_i)^* \longrightarrow M^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)^*$ extends to a finite morphism $\mathcal{M}^{\mathrm{K}'}(\mathbf{G}_i, \mathcal{X}_i)^* \longrightarrow \mathcal{M}^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)^*$, which will still be denoted by $\overline{T}_g$, whose restriction to the strata of $\mathcal{M}^{\mathrm{K}'}(\mathbf{G}_i, \mathcal{X}_i)^*$ (including the open stratum $\mathcal{M}^{\mathrm{K}'}(\mathbf{G}_i, \mathcal{X}_i)$) is étale.

(5) For every $i \in \{0, \ldots, n\}$ and $\mathrm{K} \in \mathcal{K}_i$, there exists a functor $\mathcal{F}^\mathrm{K}$ from $\mathcal{R}_i$ to the category of lisse λ-adic sheaves on $\mathcal{M}^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)$ that, after passing to the special fiber, is isomorphic to the functor $\mathcal{F}^\mathrm{K}$ of section 1.2.

(6) For every $i \in \{0, \ldots, n\}$, $\mathrm{K} \in \mathcal{K}_i$, and $V \in \operatorname{Ob} \mathcal{R}_i$, the isomorphisms of Pink's theorem (1.2.3) extend to isomorphisms between λ-adic complexes on the $\operatorname{Spec}(\mathcal{O}_F[1/\Sigma])$-models.

(7) For every $i \in \{0, \ldots, n\}$, $\mathrm{K} \in \mathcal{K}_i$ and $V \in \operatorname{Ob} \mathcal{R}_i$, the sheaf $\mathcal{F}^\mathrm{K} V$ on $\mathcal{M}^\mathrm{K}(\mathbf{G}_i, \mathcal{X}_i)$ is mixed ([D2] 1.2.2). If moreover V is pure of weight m, then $\mathcal{F}^\mathrm{K} V$ is pure of weight $-m$.

The fact that suitable integral models exist for PEL Shimura varieties has been proved by Kai-Wen Lan, who constructed the toroidal and minimal compactifications of the integral models.

Proposition 1.3.1 *Suppose that the Shimura datum $(\mathbf{G}, \mathcal{X})$ is of the type considered in [K11] §5; more precisely, we suppose fixed data as in [Lan] 1.2. Let be a finite set of prime numbers that contains all bad primes (in the sense of [Lan] 1.4.1.1). For every $i \in \{0, \dots, n\}$, let $A_i = \mathbf{G}_i(\mathbb{A})$, let $\mathcal{K}_i$ be the union of the $\mathbf{G}_i(\mathbb{A}_f)$-conjugacy classes of neat open compact subgroups of the form* K K *with* $\mathrm{K} \subset \ _{p\in} \mathbf{G}_i(\mathbb{Z}_p)$ *and* $\mathrm{K} = \ _{p\in} \mathbf{G}_i(\mathbb{Z}_p)$*, and let* $\mathcal{R}_i = Rep_{\mathbf{G}_i}$.

Then the set satisfies conditions (1)–(7), and moreover the schemes $\mathcal{M}^{\mathrm{K}}(\mathbf{G}_i, \mathcal{X}_i)$ *of (2) are the schemes representing the moduli problem of [Lan] 1.4.*

Proof. This is just putting together Lan's and Pink's results. Condition (1) is automatic. Condition (2) (in the more precise form given in the proposition) is a consequence of theorem 1.4.1.12 of [Lan]. Conditions (3) and (4) are implied by theorem 7.2.4.1 and proposition 7.2.5.1 of [Lan]. The construction of the sheaves in condition (5) is the same as in [P2] 5.1, once the integral models of condition (2) are known to exist. In [P2] 4.9, Pink observed that the proof of his theorem extends to integral models if toroidal compactifications and a minimal compactification of the integral model satisfying the properties of section 3 of [P2] have been constructed. This has been done by Lan (see, in addition to the results cited above, theorem 6.4.1.1 and propositions 6.4.2.3, 6.4.2.9 and 6.4.3.4 of [Lan]), so condition (6) is also satisfied. In the PEL case, $\mathbf{G}^{\mathrm{ad}}$ is automatically of abelian type in the sense of [P2] 5.6.2 (cf. [K11] §5). So $\mathbf{G}_i^{\mathrm{ad}}$ is of abelian type for all i, and condition (7) is implied by proposition 5.6.2 in [P2]. □

Remark 1.3.2 Let $(\mathbf{G}, \mathcal{X})$ be one of the Shimura data defined in 2.1, and let K be a neat open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$. Then there exists a finite set S of primes such that $\mathrm{K} = \mathrm{K}_S \ _{p\in S} \mathbf{G}(\mathbb{Z}_p)$, with $\mathrm{K}_S \subset \ _{v\in S} \mathbf{G}(\mathbb{Q}_v)$ (and with the $\mathbb{Z}$-structure on $\mathbf{G}$ defined in remark 2.1.1). Let be the union of S and of all prime numbers that are ramified in E. Then contains all bad primes, so proposition 1.3.1 above applies to .

Remark 1.3.3 The convention we use here for the action of the Galois group on the canonical model is that of Pink ([P2] 5.5), which is different from the convention of Deligne (in [D1]) and hence also from the convention of Kottwitz (in [K11]); so what Kottwitz calls canonical model of the Shimura variety associated to the Shimura datum $(\mathbf{G}, \mathcal{X}, h^{-1})$ is here the canonical model of the Shimura variety associated to the Shimura datum $(\mathbf{G}, \mathcal{X}, h)$.

Let us indicate another way to find integral models when the Shimura datum is not necessarily PEL. The problem with this approach is that the set of "bad" primes is unknown.

Proposition 1.3.4 *Let $\mathcal{K}_i$ and A_i be as above (and satisfying conditions (a) and (b)). Suppose that, for every $i \in \{0, \dots, n\}$, $\mathcal{K}_i$ is finite modulo $\mathbf{G}_i(\mathbb{A}_f)$-conjugacy and A_i is finite. If $\mathbf{G}^{\mathrm{ad}}$ is of abelian type (in the sense of [P2] 5.6.2), then there exists a finite set of prime numbers satisfying conditions (1)–(7), with $\mathcal{R}_i = Rep_{\mathbf{G}_i}$ for every $i \in \{0, \dots, n\}$.*

In general, there exists a finite set of prime numbers satisfying conditions (1)–(6), with $\mathcal{R}_i = \mathrm{Rep}_{\mathbf{G}_i}$ *for every* $i \in \{0, \ldots, n\}$. *Let* $\mathcal{R}'_i$, $0 \le i \le n$, *be full subcategories of* $\mathrm{Rep}_{\mathbf{G}_i}$, *stable by taking direct factors and by isomorphism, containing the trivial representation, satisfying condition (c) and minimal for all these properties (this determines the* $\mathcal{R}'_i$*). Then there exists* $' \supset$ *finite such that* $'$ *and the* $\mathcal{R}'_i$ *satisfy condition (7).*

This proposition will typically be applied to the following situation: $g \in \mathbf{G}(\mathbb{A}_f)$ and K, K′ are neat open compact subgroups of $\mathbf{G}(\mathbb{A}_f)$ such that $\mathrm{K}' \subset \mathrm{K} \cap g\mathrm{K}g^{-1}$, and we want to study the Hecke correspondence $(T_g, T_1) : M^{\mathrm{K}'}(\mathbf{G}, \mathcal{X})^* \longrightarrow (M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*)^2$. In order to reduce this situation modulo p, choose sets $\mathcal{K}_i$ such that $\mathrm{K}, \mathrm{K}' \in \mathcal{K}_0$ and condition (a) is satisfied, and minimal for these properties, sets A_i such that $1, g \in A_0$ and condition (b) is satisfied, and minimal for these properties; take $\mathcal{R}_i = \mathrm{Rep}_{\mathbf{G}_i}$ if $\mathbf{G}^{\mathrm{ad}}$ is of abelian type and $\mathcal{R}_i$ equal to the $\mathcal{R}'_i$ defined in the proposition in the other cases; fix such that conditions (1)–(7) are satisfied, and reduce modulo $p \in$.

Proof. First we show that, in the general case, there is a finite set of prime numbers satisfying conditions (1)–(6), with $\mathcal{R}_i = \mathrm{Rep}_{\mathbf{G}_i}$. It is obviously possible to find satisfying conditions (1)–(4). Proposition 3.6 of [W] implies that we can find satisfying conditions (1)–(5). To show that there exists satisfying conditions (1)–(6), reason as in the proof of proposition 3.7 of [W], using the generic base change theorem of Deligne (cf. SGA 4 1/2 [Th. finitude] théorème 1.9). As in the proof of proposition 1.3.1, if $\mathbf{G}^{\mathrm{ad}}$ is of abelian type, then condition (7) is true by proposition 5.6.2 of [P2]. In the general case, let $\mathcal{R}'_i$ be defined as in the statement of the proposition. Condition (7) for these subcategories is a consequence of proposition 5.6.1 of [P2] (reason as in the second proof of [P2] 5.6.6). □

Remark 1.3.5 Note that it is clear from the proof that, after replacing by a bigger finite set, we can choose the integral models $\mathcal{M}^{\mathrm{K}}(\mathbf{G}_i, \mathcal{X}_i)$ to be any integral models specified before (as long as they satisfy the conditions of (2)).

When we later talk about reducing Shimura varieties modulo p, we will always implicitly fix as in proposition 1.3.1 (or proposition 1.3.4) and take $p \in$. The prime number ℓ will be chosen among elements of (or added to).

1.4 WEIGHTED COHOMOLOGY COMPLEXES AND INTERSECTION COMPLEX

Let $(\mathbf{G}, \mathcal{X})$ be a Shimura datum and K be a neat open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$. Assume that $\mathbf{G}$ satisfies the conditions of section 1.1 and that $\mathbf{G}^{\mathrm{ad}}$ is simple. Fix a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}$ and maximal standard parabolic subgroups $\mathbf{P}_1, \ldots, \mathbf{P}_n$ as before proposition 1.1.3. Fix prime numbers p and ℓ as at the end of section 1.3, and a place λ of K above ℓ. In this section, we will write $\mathbf{M}^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$, etc. for the reduction modulo p of the varieties of 1.1.

Write $M_0 = M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$ and $d = \dim M_0$, and, for every $r \in \{1, \ldots, n\}$, denote by M_r the union of the boundary strata of $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$ associated to $\mathbf{P}_r$, by d_r the dimension of M_r and by i_r the inclusion of M_r in $M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*$. Then $(M_0,\dots,M_n)$ is a stratification of $M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*$ in the sense of [M2] 3.3.1. Hence, for every $\underline{a}=(a_0,\dots,a_n)\in(\mathbb{Z}\cup\{\pm\infty\})^{n+1}$, the functors $w_{\leq\underline{a}}$ and $w_{>\underline{a}}$ of [M2] 3.3.2 are defined (on the category $D_m^b(M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*,K_\lambda)$ of mixed λ-adic complexes on $M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*$). We will recall the definition of the intersection complex and of the weighted cohomology complexes. Remember that j is the open immersion $M^{\mathrm{K}}(\mathbf{G},\mathcal{X})\longrightarrow M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*$.

Remark 1.4.1 We will need to use the fact that the sheaves $\mathcal{F}^{\mathrm{K}}V$ are mixed with known weights. So we fix categories $\mathcal{R}_0,\dots,\mathcal{R}_n$ as in section 1.3, satisfying conditions (c) and (7) of 1.3. If $\mathbf{G}^{\mathrm{ad}}$ is of abelian type, we can simply take $\mathcal{R}_0=\mathrm{Rep}_{\mathbf{G}}$.

Definition 1.4.2 (i) Let $V\in\mathrm{Ob}\,\mathrm{Rep}_{\mathbf{G}}$. The intersection complex on $M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*$ with coefficients in V is the complex

$$IC^{\mathrm{K}}V=(j_{!*}(\mathcal{F}^{\mathrm{K}}V[d]))[-d].$$

(ii) (cf. [M2] 4.1.3) Let $t_1,\dots,t_n\in\mathbb{Z}\cup\{\pm\infty\}$. For every $r\in\{1,\dots,n\}$, write $a_r=-t_r+d_r$. Define an additive triangulated functor

$$W^{\geq t_1,\dots,\geq t_n}:D^b(\mathcal{R}_0)\longrightarrow D_m^b(M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*,K_\lambda)$$

in the following way: for every $m\in\mathbb{Z}$, if $V\in\mathrm{Ob}\,D^b(\mathcal{R}_0)$ is such that all $\mathrm{H}^i\,V$, $i\in\mathbb{Z}$, are pure of weight m, then

$$W^{\geq t_1,\dots,\geq t_n}V=w_{\leq(-m+d,-m+a_1,\dots,-m+a_n)}Rj_*\mathcal{F}^{\mathrm{K}}V.$$

The definition of the weighted cohomology complex in (ii) was inspired by the work of Goresky, Harder and MacPherson ([GHM]). Proposition 4.1.5 of [M2] admits the following obvious generalization.

Proposition 1.4.3 *Let $t_1,\dots,t_n\in\mathbb{Z}$ be such that, for every $r\in\{1,\dots,n\}$, $d_r-d\leq t_r\leq 1+d_r-d$. Then, for every $V\in\mathrm{Ob}\,\mathcal{R}_0$, there is a canonical isomorphism*

$$IC^{\mathrm{K}}V\qquad W^{\geq t_1,\dots,\geq t_n}V.$$

We now want to calculate the restriction to boundary strata of the weighted cohomology complexes. The following theorem is a consequence of propositions 3.3.4 and 3.4.2 of [M2].

Theorem 1.4.4 *Let $\underline{a}=(a_0,\dots,a_n)\in(\mathbb{Z}\cup\{\pm\infty\})^{n+1}$. Then, for every $L\in\mathrm{Ob}\,D_m^b(M^{\mathrm{K}}(\mathbf{G},\mathcal{X}),K_\lambda)$ such that all perverse cohomology sheaves of L are pure of weight a_0, there is an equality of classes in the Grothendieck group of $D_m^b(M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*,K_\lambda)$:*

$$[w_{\leq\underline{a}}Rj_*L]=\sum_{1\leq n_1<\cdots<n_r\leq n}(-1)^r[i_{n_r!}w_{\leq a_{n_r}}i_{n_r}^!\dots i_{n_1!}w_{\leq a_{n_1}}i_{n_1}^!j_!L].$$

Therefore it is enough to calculate the restriction to boundary strata of the complexes $i_{n_r!}w_{\leq a_{n_r}}i_{n_r}^!\dots i_{n_1!}w_{\leq a_{n_1}}i_{n_1}^!j_!\mathcal{F}^{\mathrm{K}}V$, $1\leq n_1<\cdots<n_r\leq n$. The following proposition generalizes proposition 4.2.3 of [M2] and proposition 5.2.3 of [M1].

Proposition 1.4.5 *Let* $n_1, \dots, n_r \in \{1, \dots, n\}$ *be such that* $n_1 < \dots < n_r$, $a_1, \dots, a_r \in \mathbb{Z} \cup \{\pm\infty\}$, $V \in \mathrm{Ob}\, D^b(\mathcal{R}_0)$ *and* $g \in \mathbf{G}(\mathbb{A}_f)$. *Write* $\mathbf{P} = \mathbf{P}_{n_1} \cap \dots \cap \mathbf{P}_{n_r}$; *remember that, in section 1.1, before proposition 1.1.3, we constructed a set* $\mathcal{C}_P \quad \mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f)/\mathrm{K}$ *and a map from this set to the set of boundary strata of* $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$ *associated to* $\mathbf{P}_{n_r}$. *For every* $i \in \{1, \dots, r\}$, *let* $w_i : \mathbb{G}_m \longrightarrow \mathbf{G}_{n_i}$ *be the cocharacter associated to the Shimura datum* $(\mathbf{G}_{n_i}, \mathcal{X}_{n_i})$ *as in section 1.3; the image of* w_i *is contained in the center of* $\mathbf{G}_{n_i}$, *and* w_i *can be seen as a cocharacter of* $\mathbf{M}_P$. *For every* $i \in \{1, \dots, r\}$, *write* $t_i = -a_i + d_{n_i}$. *Let*

$$L = i^*_{n_r,g} R i_{n_r *} w_{>a_r} i^*_{n_r} \dots R i_{n_1 *} w_{>a_1} i^*_{n_1} R j_* \mathcal{F}^{\mathrm{K}} V.$$

Then there is a canonical isomorphism

$$L \qquad \underset{C}{} T_{C*} L_C,$$

where the direct sum is over the set of $C = (X_1, \dots, X_r) \in \mathcal{C}_P$ *that are sent to the stratum* $\mathrm{Im}(i_{n_r,g})$, T_C *is the obvious morphism* $X_r \longrightarrow \mathrm{Im}(i_{n_r,g})$ *(a finite étale morphism), and* L_C *is an* λ*-adic complex on* X_r *such that, if* $h \in \mathbf{G}(\mathbb{A}_f)$ *is a representative of* C, *there is an isomorphism*

$$L_C \qquad \mathcal{F}^{\mathrm{H}/\mathrm{H}_L} R\Gamma(\mathrm{H}_L/\mathrm{K}_N, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)_{<t_1,\dots,<t_r}),$$

where $\mathrm{H} = h\mathrm{K}h^{-1} \cap \mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f)$, $\mathrm{H}_L = h\mathrm{K}h^{-1} \cap \mathbf{P}(\mathbb{Q})\mathbf{N}_{n_r}(\mathbb{A}_f) \cap \mathbf{L}_{n_r}(\mathbb{Q})\mathbf{N}_{n_r}(\mathbb{A}_f)$, $\mathrm{K}_N = h\mathrm{K}h^{-1} \cap \mathbf{N}_P(\mathbb{Q})\mathbf{N}_{n_r}(\mathbb{A}_f)$ *and, for every* $i \in \{1, \dots, r\}$, *the subscript "*$< t_i$*" means that the complex* $R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)$ *of representations of* $\mathbf{M}_P$ *is truncated by the weights of* $w_i(\mathbb{G}_m)$ *(cf. [M2] 4.1.1).*

Remember that the Levi quotient $\mathbf{M}_P$ is the direct product of $\mathbf{G}_{n_r}$ and a Levi subgroup $\mathbf{L}_P$ of $\mathbf{L}_{n_r}$. Write $\Gamma_L = \mathrm{H}_L/\mathrm{K}_N$ and $X_L = \mathbf{L}_P(\mathbb{R})/\mathrm{K}_{L,\infty}\mathbf{A}_{L_P}(\mathbb{R})^0$, where $\mathrm{K}_{L,\infty}$ is a maximal compact subgroup of $\mathbf{L}_P(\mathbb{R})$ and $\mathbf{A}_{L_P}$ is, as in section 1.2, the maximal split subtorus of the center of $\mathbf{L}_P$; also remember that $q(\mathbf{L}_P) = \dim(X_L)/2$. Then Γ_L is a neat arithmetic subgroup of $\mathbf{L}_P(\mathbb{Q})$, and, for every $W \in \mathrm{Ob}\, D^b(\mathrm{Rep}_{\mathbf{L}_P})$,

$$R\Gamma(\Gamma_L, W) = R\Gamma(\Gamma_L \backslash X_L, \mathcal{F}^{\Gamma_L} W).$$

Write

$$R\Gamma_c(\Gamma_L, W) = R\Gamma_c(\Gamma_L \backslash X_L, \mathcal{F}^{\Gamma_L} W).$$

If $W \in \mathrm{Ob}\, D^b(\mathrm{Rep}_{\mathbf{M}_P})$, then this complex can be seen as an object of $D^b(\mathrm{Rep}_{\mathbf{G}_{n_r}})$, because it is the dual of $R\Gamma(\Gamma_L, W^*)[\dim(X_L)]$ (where W^* is the contragredient of W). Define in the same way a complex $R\Gamma_c(\mathrm{K}_L, W)$ for K_L a neat open compact subgroup of $\mathbf{L}_P(\mathbb{A}_f)$ and $W \in \mathrm{Ob}\, D^b(\mathrm{Rep}_{\mathbf{L}_P})$.

Corollary 1.4.6 *Write*

$$M = i^*_{n_r,g} i_{n_r !} w_{\leq a_r} i^!_{n_r} \dots i_{n_1 !} w_{\leq a_1} i^!_{n_1} j_! \mathcal{F}^{\mathrm{K}} V.$$

Then there is a canonical isomorphism

$$M \qquad \underset{C}{} T_{C*} M_C,$$

where the sum is as in the proposition above and, for every $C = (X_1, \ldots, X_r) \in \mathcal{C}_P$ that is sent to the stratum $\mathrm{Im}(i_{n_r,g})$, M_C is an λ-adic complex on X_r such that, if h is a representative of C, then there is an isomorphism (with the notation of the proposition)

$$M_C \qquad \mathcal{F}^{\mathrm{H}/\mathrm{H}_L} R\Gamma_c(\mathrm{H}_L/\mathrm{K}_N, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)_{\geq t_1,\ldots,\geq t_r})[-\dim(\mathbf{A}_{M_P}/\mathbf{A}_G)].$$

Proof. Let V^* be the contragredient of V. The complex dual to M is

$$\begin{aligned} D(M) &= i^*_{n_r,g} R i_{n_r*} w_{\geq -a_r} i^*_{n_r} \ldots R i_{n_1*} w_{\geq -a_1} i^*_{n_1} R j_* D(\mathcal{F}^{\mathrm{K}} V) \\ &= i^*_{n_r,g} R i_{n_r*} w_{\geq -a_r} i^*_{n_r} \ldots R i_{n_1*} w_{\geq -a_1} i^*_{n_1} R j_* (\mathcal{F}^{\mathrm{K}} V^*[2d](d)) \\ &= (i^*_{n_r,g} R i_{n_r*} w_{\geq 2d-a_r} i^*_{n_r} \ldots R i_{n_1*} w_{\geq 2d-a_1} i^*_{n_1} R j_* \mathcal{F}^{\mathrm{K}} V^*)[2d](d). \end{aligned}$$

For every $i \in \{1, \ldots, r\}$, let $s_i = -(2d - a_i - 1) + d_{n_i} = 1 - t_i - 2(d - d_{n_i})$. By proposition 1.4.5,

$$D(M) \qquad {}_{C} \; T_{C*} M'_C,$$

with

$$M'_C \qquad \mathcal{F}^{\mathrm{H}/\mathrm{H}_L} R\Gamma(\mathrm{H}_L/\mathrm{K}_N, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V^*)_{<s_1,\ldots,<s_r})[2d](d).$$

Take $M_C = D(M'_C)$. It remains to prove the formula for M_C.

Let $m = \dim(\mathbf{N}_P)$. By lemma (10.9) of [GHM],

$$\begin{aligned} &R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)_{\geq t_1,\ldots,\geq t_r} \\ &\qquad R\,\mathrm{Hom}(R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V^*)_{<s_1,\ldots,<s_r}, \mathrm{H}^m(\mathrm{Lie}(\mathbf{N}_P), \mathbb{Q}))[-m], \end{aligned}$$

and $\mathrm{H}^m(\mathrm{Lie}(\mathbf{N}_P), \mathbb{Q})$ is the character $\gamma \longrightarrow \det(\mathrm{Ad}(\gamma), \mathrm{Lie}(\mathbf{N}_P))^{-1}$ of $\mathbf{M}_P$ (only the case of groups $\mathbf{G}$ with anisotropic center is treated in [GHM], but the general case is similar). In particular, $\mathrm{H}_L/\mathrm{K}_N$ acts trivially on $\mathrm{H}^m(\mathrm{Lie}(\mathbf{N}_P), \mathbb{Q})$, and the group $w_r(\mathbb{G}_m)$ acts by the character $\lambda \longrightarrow \lambda^{2(d-d_{n_r})}$ (w_r is defined as in proposition 1.4.5). Hence

$$M_C \qquad \mathcal{F}^{\mathrm{H}/\mathrm{H}_L} R\Gamma_c(\mathrm{H}_L/\mathrm{K}_N, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)_{\geq t_1,\ldots,\geq t_r})[a],$$

with

$$\begin{aligned} a &= 2d_{n_r} + m + 2q(\mathbf{L}_P) - 2d = 2q(\mathbf{G}_{n_r}) + 2q(\mathbf{L}_P) + \dim(\mathbf{N}_P) - 2q(\mathbf{G}) \\ &= -\dim(\mathbf{A}_{M_P}/\mathbf{A}_G). \end{aligned}$$

□

Proof of proposition 1.4.5. Let $C = (X_1, \ldots, X_r) \in \mathcal{C}_P$. Let I_1 be the locally closed immersion $X_1 \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$ and, for every $m \in \{1, \ldots, r-1\}$, denote by j_m the open immersion $X_m \longrightarrow X^*_m$ and by I_{m+1} the locally closed immersion $X_{m+1} \longrightarrow X^*_m$ (where X^*_m is the Baily-Borel compactification of X_m). Define a complex L_C on X_r by

$$L_C = w_{>a_r} I^*_r R j_{r-1*} w_{>a_{r-1}} I^*_{r-1} \ldots w_{>a_1} I^*_1 R j_* \mathcal{F}^{\mathrm{K}} V.$$

Let us show by induction on r that L is isomorphic to the direct sum of the $T_{C*} L_C$, for $C \in \mathcal{C}_P$ that is sent to the stratum $Y := \mathrm{Im}(i_{n_r,g})$. The statement is obvious if $r = 1$. Suppose that $r \geq 2$ and that the statement is true for $r - 1$. Let $Y_1, \dots, Y_m$ be the boundary strata of $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$ associated to $\mathbf{P}_{n_{r-1}}$ whose adherence contains Y. For every $i \in \{1, \dots, m\}$, let $u_i : Y_i \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$ be the inclusion, and let

$$L_i = u_i^* Ri_{n_{r-1}*} w_{>a_{r-1}} i_{n_{r-1}}^* \dots Ri_{n_1 *} w_{>a_1} i_{n_1}^* Rj_* \mathcal{F}^{\mathrm{K}} V.$$

It is obvious that

$$L = \bigoplus_{i=1}^{m} i_{n_r,g}^* Ru_{i*} w_{>a_r} L_i.$$

Write $\mathbf{P}' = \mathbf{P}_{n_1} \cap \dots \cap \mathbf{P}_{n_{r-1}}$. Let $i \in \{1, \dots, m\}$. By the induction hypothesis, L_i is isomorphic to the direct sum of the $T_{C'*} L_{C'}$ over the set of $C' \in \mathcal{C}_{P'}$ that are sent to Y_i, where $L_{C'}$ is defined in the same way as L_C. Fix $C' = (X_1, \dots, X_{r-1})$ that is sent to Y_i; let us calculate $i_{n_r,g}^* Ru_{i*} w_{>a_r} T_{C'*} L_{C'}$. There is a commutative diagram, with squares cartesian up to nilpotent elements:

$$\begin{array}{ccccc} Y' & \xrightarrow{I'} & X_{r-1}^* & \xleftarrow{j_{r-1}} & X_{r-1} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \overline{T}_{C'}} & & \downarrow{\scriptstyle T_{C'}} \\ Y & \longrightarrow & \overline{Y}_i & \longleftarrow & Y_i \end{array},$$

where Y' is a disjoint union of boundary strata of X_{r-1}^* associated to the parabolic subgroup $(\mathbf{P}_{n_r} \cap \mathbf{Q}_{n_{r-1}})/\mathbf{N}_{r_{n-1}}$. Moreover, the vertical arrows are finite maps, and the maps T and $T_{C'}$ are étale. By the proper base change isomorphism and the fact that the functors $w_{>a}$ commute with taking the direct image by a finite étale morphism, there is an isomorphism

$$i_{n_r,g}^* Ru_{i*} w_{>a_r} T_{C'*} L_{C'} = T_* w_{>a_r} I'^* Rj_{r-1*} L_{C'}.$$

The right-hand side is the direct sum of the complexes

$$(T \circ I_r)_* w_{>a_r} I_r^* Rj_{r-1*} L_{C'} = T_{C*} L_C,$$

for $I_r : X_r \longrightarrow X_{r-1}^*$ in the set of boundary strata of X_{r-1}^* included in Y' and $C = (X_1, \dots, X_r)$. These calculations clearly imply the statement that we were trying to prove.

It remains to prove the formula for L_C given in the proposition. Again, use induction on r. If $r = 1$, the formula for L_C is a direct consequence of Pink's theorem (1.2.3) and of lemma 4.1.2 of [M2]. Supppose that $r \geq 2$ and that the result is known for $r - 1$. Let $C = (X_1, \dots, X_r) \in \mathcal{C}_P$, and let $h \in \mathbf{G}(\mathbb{A}_f)$ be a representative of C. Write $\mathbf{P}' = \mathbf{P}_{r_1} \cap \dots \cap \mathbf{P}_{r_{n-1}}$, $C' = (X_1, \dots, X_{r-1})$,

$$\mathrm{H} = h\mathrm{K}h^{-1} \cap \mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f),$$

$$\mathrm{H}_L = h\mathrm{K}h^{-1} \cap \mathbf{P}(\mathbb{Q})\mathbf{N}_{n_r}(\mathbb{A}_f) \cap \mathbf{L}_{n_r}(\mathbb{Q})\mathbf{N}_{n_r}(\mathbb{A}_f) = \mathrm{H} \cap \mathbf{L}_{n_r}(\mathbb{Q})\mathbf{N}_{n_r}(\mathbb{A}_f),$$

$$\mathrm{K}_N = h\mathrm{K}h^{-1} \cap \mathbf{N}_P(\mathbb{Q})\mathbf{N}_{n_r}(\mathbb{A}_f),$$

$$\mathrm{H}' = h\mathrm{K}h^{-1} \cap \mathbf{P}'(\mathbb{Q})\mathbf{Q}_{n_{r-1}}(\mathbb{A}_f),$$

$$\mathrm{H}'_L = h\mathrm{K}h^{-1} \cap \mathbf{P}'(\mathbb{Q})\mathbf{N}_{n_{r-1}}(\mathbb{A}_f) \cap \mathbf{L}_{n_{r-1}}(\mathbb{Q})\mathbf{N}_{n_{r-1}}(\mathbb{A}_f) = \mathrm{H}' \cap \mathbf{L}_{n_{r-1}}(\mathbb{Q})\mathbf{N}_{n_{r-1}}(\mathbb{A}_f),$$

$$\mathrm{K}'_N = h\mathrm{K}h^{-1} \cap \mathbf{N}_{P'}(\mathbb{Q})\mathbf{N}_{n_{r-1}}(\mathbb{A}_f).$$

By the induction hypothesis, there is a canonical isomorphism

$$L_{C'} \quad \mathcal{F}^{\mathrm{H}'/\mathrm{H}'_L} R\Gamma(\mathrm{H}'_L/\mathrm{K}'_N, R\Gamma(\mathrm{Lie}(\mathbf{N}_{P'}), V)_{<t_1,\dots,<t_{r-1}}).$$

Applying Pink's theorem, we get a canonical isomorphism

$$L_C \quad w_{>a_r}\mathcal{F}^{\mathrm{H}/\mathrm{H}_L} R\Gamma(\mathrm{H}_L/\mathrm{H}'_L, R\Gamma(\mathrm{H}'_L/\mathrm{K}'_N, R\Gamma(\mathrm{Lie}(\mathbf{N}_{n_{r-1}}), V)_{<t_1,\dots,<t_{r-1}})).$$

There are canonical isomorphisms

$$\begin{aligned} R\Gamma(\mathrm{H}_L/\mathrm{H}'_L, R\Gamma(\mathrm{H}'_L/\mathrm{K}'_N, -)) \quad & R\Gamma(\mathrm{H}_L/\mathrm{K}_N, R\Gamma(\mathrm{K}_N/\mathrm{K}'_N, -)) \\ & R\Gamma(\mathrm{H}_L/\mathrm{K}_N, R\Gamma(\mathrm{Lie}(\mathbf{N}_{n_r}/\mathbf{N}_{n_{r-1}}), -)) \end{aligned}$$

(the last isomorphism comes from van Est's theorem, cf. [GKM] §24). On the other hand, for every $i \in \{1, \dots, r-1\}$, the image of the cocharacter $w_i : \mathbb{G}_m \longrightarrow \mathbf{G}_{n_i}$ is contained in the center of $\mathbf{G}_{n_i}$; hence it commutes with $\mathbf{G}_{n_{r-1}}$. This implies that

$$\begin{aligned} & R\Gamma(\mathrm{Lie}(\mathbf{N}_{n_r}/\mathbf{N}_{n_{r-1}}), R\Gamma(\mathrm{Lie}(\mathbf{N}_{n_{r-1}}), V)_{<t_1,\dots,t_{r-1}}) \\ & \quad = R\Gamma(\mathrm{Lie}(\mathbf{N}_{n_r}), V)_{<t_1,\dots,<t_{r-1}}, \end{aligned}$$

so that

$$L_C \quad w_{>a_r}\mathcal{F}^{\mathrm{H}/\mathrm{H}_L} R\Gamma(\mathrm{H}_L/\mathrm{K}_N, R\Gamma(\mathrm{Lie}(\mathbf{N}_{n_r}), V)_{<t_1,\dots,<t_{r-1}}).$$

To finish the proof, it suffices to apply lemma 4.1.2 of [M2] and to notice that the image of $w_r : \mathbb{G}_m \longrightarrow \mathbf{G}_{n_r}$ commutes with $\mathbf{L}_{n_r}(\mathbb{Q})$, hence also with its subgroup $\mathrm{H}_L/\mathrm{K}_N$. □

1.5 COHOMOLOGICAL CORRESPONDENCES

Notation 1.5.1 Let $(T_1, T_2) : X' \longrightarrow X_1 \times X_2$ be a correspondence of separated schemes of finite type over a finite field, and let $c : T_1^* L_1 \longrightarrow T_2^! L_2$ be a cohomological correspondence with support in (T_1, T_2). Denote by Φ the absolute Frobenius morphism of X_1. For every $j \in \mathbb{N}$, we write $\Phi^j c$ for the cohomological correspondence with support in $(\Phi^j \circ T_1, T_2)$ defined as the following composition of maps:

$$(\Phi^j \circ T_1)^* L_1 = T_1^* \Phi^{j*} L_1 \quad T_1^* L_1 \xrightarrow{c} T_2^! L_2.$$

First we will define Hecke correspondences on the complexes of 1.2. Fix $\mathbf{M}$, $\mathbf{L}$ and $(\mathbf{G}, \mathcal{X})$ as in 1.2. Let $m_1, m_2 \in \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f)$ and $\mathrm{K}'_M, \mathrm{K}_M^{(1)}, \mathrm{K}_M^{(2)}$ be neat open compact subgroups of $\mathbf{M}(\mathbb{A}_f)$ such that $\mathrm{H}' \subset m_1 \mathrm{H}^{(1)} m_1^{-1} \cap m_2 \mathrm{H}^{(2)} m_2^{-1}$, where $\mathrm{H}' = \mathrm{K}'_M \cap \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f)$ and $\mathrm{H}^{(i)} = \mathrm{K}_M^{(i)} \cap \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f)$. This gives two finite étale morphisms $T_{m_i} : M(\mathbf{G}, \mathcal{X})/\mathrm{H}' \longrightarrow M(\mathbf{G}, \mathcal{X})/\mathrm{H}^{(i)}$, $i = 1, 2$. Write $\mathrm{H}_L^{(i)} = \mathrm{H}^{(i)} \cap \mathbf{L}(\mathbb{Q})$ and $\mathrm{H}'_L = \mathrm{H}' \cap \mathbf{L}(\mathbb{Q})$. Let $V \in \mathrm{Ob}\,\mathrm{Rep}_{\mathbf{M}}$. For $i = 1, 2$, write

$$L_i = \mathcal{F}^{\mathrm{H}^{(i)}/\mathrm{H}_L^{(i)}} R\Gamma(\mathrm{H}_L^{(i)}, V).$$

By [P2] 1.11.5, there are canonical isomorphisms

$$T_{m_i}^* L_i \quad \mathcal{F}^{\mathrm{H}'/\mathrm{H}'_L} \theta_i^* R\Gamma(\mathrm{H}_L^{(i)}, V),$$

where $\theta_i^* R\Gamma(\mathrm{H}_L^{(i)}, V)$ is the inverse image by the morphism $\theta_i : \mathrm{H}'/\mathrm{H}'_L \longrightarrow \mathrm{H}^{(i)}/\mathrm{H}_L^{(i)}$, $h \longrightarrow m_i^{-1} h m_i$, of the complex of $\mathrm{H}^{(i)}/\mathrm{H}_L^{(i)}$-modules $R\Gamma(\mathrm{H}_L^{(i)}, V)$. Using the injections $\mathrm{H}'_L \longrightarrow \mathrm{H}_L^{(i)}$, $h \longrightarrow m_i^{-1} h m_i$, we get an adjunction morphism $\theta_1^* R\Gamma(\mathrm{H}_L^{(1)}, V) \xrightarrow{\mathrm{adj}} R\Gamma(\mathrm{H}'_L, V)$ and a trace morphism $R\Gamma(\mathrm{H}'_L, V) \xrightarrow{\mathrm{Tr}} \theta_2^* R\Gamma(\mathrm{H}_L^{(2)}, V)$ (this last morphism exists because the index of H'_L in $\mathrm{H}_L^{(2)}$ is finite); these morphisms are $\mathrm{H}'/\mathrm{H}'_L$-equivariant. The Hecke correspondence

$$c_{m_1,m_2} : T_{m_1}^* L_1 \longrightarrow T_{m_2}^! L_2 = T_{m_2}^* L_2$$

is the map

$$T_{m_1}^* L_1 \quad \mathcal{F}^{\mathrm{H}'/\mathrm{H}'_L} \theta_1^* R\Gamma(\mathrm{H}_L^{(1)}, V)$$
$$\xrightarrow{\mathrm{adj}} \mathcal{F}^{\mathrm{H}'/\mathrm{H}'_L} R\Gamma(\mathrm{H}'_L, V) \xrightarrow{\mathrm{Tr}} \mathcal{F}^{\mathrm{H}'/\mathrm{H}'_L} \theta_2^* R\Gamma(\mathrm{H}_L^{(2)}, V) \quad T_{m_2}^* L_2.$$

Note that, if $\mathbf{L} = \{1\}$, then this correspondence is an isomorphism.

Remarks 1.5.2 (1) Assume that $\mathrm{K}'_M \subset m_1 \mathrm{K}_M^{(1)} m_1^{-1} \cap m_2 \mathrm{K}_M^{(2)} m_2^{-1}$, and write $\mathrm{K}'_L = \mathrm{K}'_M \cap \mathbf{L}(\mathbb{A}_f)$ and $\mathrm{K}_L^{(i)} = \mathrm{K}_M^{(i)} \cap \mathbf{L}(\mathbb{A}_f)$. Using the methods of [M1] 2.1.4 (and the fact that, for every open compact subgroup K_L of $\mathbf{L}(\mathbb{A}_f)$, $R\Gamma(\mathrm{K}_L, V) = \bigoplus_{i \in I} R\Gamma(g_i \mathrm{K}_L g_i^{-1} \cap \mathbf{L}(\mathbb{Q}), V)$, where $(g_i)_{i \in I}$ is a system of representatives of $\mathbf{L}(\mathbb{Q}) \setminus \mathbf{L}(\mathbb{A}_f)/\mathrm{K}_L$), it is possible to construct complexes $M_i = \mathcal{F}^{\mathrm{K}_M^{(i)}/\mathrm{K}_L^{(i)}} R\Gamma(\mathrm{K}_L^{(i)}, V)$ and $\mathcal{F}^{\mathrm{K}'_M/\mathrm{K}'_L} R\Gamma(\mathrm{K}'_L, V)$. There are a correspondence

$$(T_{m_1}, T_{m_2}) : M^{\mathrm{K}'_M/\mathrm{K}'_L}(\mathbf{G}, \mathcal{X}) \longrightarrow M^{\mathrm{K}_M^{(1)}/\mathrm{K}_L^{(1)}}(\mathbf{G}, \mathcal{X}) \times M^{\mathrm{K}_M^{(2)}/\mathrm{K}_L^{(2)}}(\mathbf{G}, \mathcal{X}),$$

and a cohomological correspondence, constructed as above,

$$c_{m_1,m_2} : T_{m_1}^* M_1 \longrightarrow T_{m_2}^! M_2.$$

(2) There are analogous correspondences, constructed by replacing $R\Gamma(\mathrm{H}_L^{(i)}, V)$ and $R\Gamma(\mathrm{H}'_L, V)$ (resp. $R\Gamma(\mathrm{K}_L^{(i)}, V)$ and $R\Gamma(\mathrm{K}'_L, V)$) with $R\Gamma_c(\mathrm{H}_L^{(i)}, V)$ and $R\Gamma_c(\mathrm{H}'_L, V)$ (resp. $R\Gamma_c(\mathrm{K}_L^{(i)}, V)$ and $R\Gamma_c(\mathrm{K}'_L, V)$). We will still use the notation c_{m_1,m_2} for these correspondences.

Use the notation of section 1.4, and fix $g \in \mathbf{G}(\mathbb{A}_f)$ and a second open compact subgroup K' of $\mathbf{G}(\mathbb{A}_f)$, such that $\mathrm{K}' \subset \mathrm{K} \cap g\mathrm{K}g^{-1}$. Fix prime numbers p and ℓ as at the end of 1.3. In particular, it is assumed that $g \in \mathbf{G}(\mathbb{A}_f^p)$ and that K (resp., K′) is of the form $\mathrm{K}^p \mathbf{G}(\mathbb{Z}_p)$ (resp., $\mathrm{K}'^p \mathbf{G}(\mathbb{Z}_p)$), with $\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$ (resp., $\mathrm{K}'^p \subset \mathbf{G}(\mathbb{A}_f^p)$) and $\mathbf{G}(\mathbb{Z}_p)$ a hyperspecial maximal compact subgroup of $\mathbf{G}(\mathbb{Q}_p)$. As in section 1.4, we will use the notations $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$, etc, for the reductions modulo p of the varieties of section 1.1.

Let Φ be the absolute Frobenius morphism of $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$. For every $V \in \mathrm{Ob}\, D^b(\mathrm{Rep}_{\mathbf{G}})$ and $j \in \mathbb{Z}$, let $u_j : (\Phi^j T_g)^* \mathcal{F}^{\mathrm{K}} V \longrightarrow T_1^! \mathcal{F}^{\mathrm{K}} V$ be the cohomological correspondence $\Phi^j c_{g,1}$ on $\mathcal{F}^{\mathrm{K}} V$ (with support in $(\Phi^j T_g, T_1)$).

Let $V \in \operatorname{Ob} D^b(\mathcal{R}_0)$. By [M2] 5.1.2 and 5.1.3:

- for every $t_1, \dots, t_n \in \mathbb{Z} \cup \{\pm\infty\}$, the correspondence u_j extends in a unique way to a correspondence
$$\overline{u}_j : (\Phi^j \overline{T}_g)^* W^{\geq t_1, \dots, \geq t_n} V \longrightarrow \overline{T}_1^! W^{\geq t_1, \dots, \geq t_n} V;$$
- for every $n_1, \dots, n_r \in \{1, \dots, n\}$ such that $n_1 < \dots < n_r$ and every $a_1, \dots, a_r \in \mathbb{Z} \cup \{\pm\infty\}$, the correspondence u_j gives in a natural way a cohomological correspondence on $i_{n_r!} w_{\leq a_r} i_{n_r}^! \dots i_{n_1!} w_{\leq a_1} i_{n_1}^! j_! \mathcal{F}^{\mathrm{K}} V$ with support in $(\Phi^j \overline{T}_g, \overline{T}_1)$; write $i_{n_r!} w_{\leq a_r} i_{n_r}^! \dots i_{n_1!} w_{\leq a_1} i_{n_1}^! j_! u_j$ for this correspondence.

Moreover, there is an analog of theorem 1.4.4 for cohomological correspondences (cf. [M2] 5.1.5). The goal of this section is to calculate the correspondences $i_{n_r!} w_{\leq a_r} i_{n_r}^! \dots i_{n_1!} w_{\leq a_1} i_{n_1}^! j_! u_j$.

Fix $n_1, \dots, n_r \in \{1, \dots, n\}$ such that $n_1 < \dots < n_r$ and $a_1, \dots, a_r \in \mathbb{Z} \cup \{\pm\infty\}$, and write
$$L = i_{n_r!} w_{\leq a_r} i_{n_r}^! \dots i_{n_1!} w_{\leq a_1} i_{n_1}^! j_! \mathcal{F}^{\mathrm{K}} V,$$
$$u = i_{n_r!} w_{\leq a_r} i_{n_r}^! \dots i_{n_1!} w_{\leq a_1} i_{n_1}^! j_! u_j.$$

Use the notation of corollary 1.4.6. By this corollary, there is an isomorphism
$$L \qquad \underset{C \in \mathcal{C}_P}{} (i_C T_C)_! L_C,$$
where, for every $C = (X_1, \dots, X_r) \in \mathcal{C}_P$, i_C is the inclusion in $M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^*$ of the boundary stratum image of X_r (i.e., of the stratum $\operatorname{Im}(i_{n_r,h})$, if $h \in \mathbf{G}(\mathbb{A}_f)$ is a representative of C). Hence the correspondence u can be seen as a matrix $(u_{C_1,C_2})_{C_1,C_2 \in \mathcal{C}_P}$, and we want to calculate the entries of this matrix.

Let $\mathcal{C}'_P$ be the analog of the set $\mathcal{C}_P$ obtained when K is replaced with K$'$. The morphisms $\overline{T}_g, \overline{T}_1$ define maps $T_g, T_1 : \mathcal{C}'_P \longrightarrow \mathcal{C}_P$, and these maps correspond via the bijections $\mathcal{C}_P \quad \mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f)/\mathrm{K}$ and $\mathcal{C}'_P \quad \mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f)/\mathrm{K}'$ of proposition 1.1.3 to the maps induced by $h \longrightarrow hg$ and $h \longrightarrow h$.

Let $C_1 = (X_1^{(1)}, \dots, X_r^{(1)})$, $C_2 = (X_1^{(2)}, \dots, X_r^{(2)}) \in \mathcal{C}_P$, and choose representatives $h_1, h_2 \in \mathbf{G}(\mathbb{A}_f)$ of C_1 and C_2. Let $C' = (X'_1, \dots, X'_r) \in \mathcal{C}'_P$ be such that $T_g(C') = C_1$ and $T_1(C') = C_2$. Fix a representative $h' \in \mathbf{G}(\mathbb{A}_f)$ of C'. There exist $q_1, q_2 \in \mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f)$ such that $q_1 h' \in h_1 g\mathrm{K}$ and $q_2 h' \in h_2 \mathrm{K}$. Let $\overline{q}_1, \overline{q}_2$ be the images of q_1, q_2 in $\mathbf{L}_{n_r}(\mathbb{Q})\mathbf{G}_{n_r}(\mathbb{A}_f)$. The following diagrams are commutative:

$$\begin{array}{ccc} X'_r & \xrightarrow{i_{C'} T_{C'}} & M^{\mathrm{K}'}(\mathbf{G}, \mathcal{X})^* \\ {\scriptstyle T_{\overline{q}_1}} \downarrow & & \downarrow {\scriptstyle \overline{T}_g} \\ X_r^{(1)} & \xrightarrow{i_{C_1} T_{C^{(1)}}} & M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^* \end{array} \qquad \begin{array}{ccc} X'_r & \xrightarrow{i_{C'} T_{C'}} & M^{\mathrm{K}'}(\mathbf{G}, \mathcal{X})^* \\ {\scriptstyle T_{\overline{q}_2}} \downarrow & & \downarrow {\scriptstyle \overline{T}_1} \\ X_r^{(2)} & \xrightarrow{i_{C_2} T_{C^{(2)}}} & M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})^* \end{array}$$

By corollary 1.4.6, there are isomorphisms
$$L_{C_1} \quad \mathcal{F}^{\mathrm{H}^{(1)}/\mathrm{H}_L^{(1)}} R\Gamma_c(\mathrm{H}_L^{(1)}/\mathrm{K}_N^{(1)}, R\Gamma(\operatorname{Lie}(\mathbf{N}_{n_r}), V)_{\geq t_1, \dots, \geq t_r})[a]$$

and

$$L_{C_2} \quad \mathcal{F}^{\mathrm{H}^{(2)}/\mathrm{H}_L^{(2)}} R\Gamma_c(\mathrm{H}_L^{(2)}/\mathrm{K}_N^{(2)}, R\Gamma(\mathrm{Lie}(\mathbf{N}_{n_r}), V)_{\geq t_1,\dots,\geq t_r})[a],$$

where $t_1, \dots, t_r$ are defined as in proposition 1.4.5, $a = -\dim(\mathbf{A}_{M_P}/\mathbf{A}_G)$, $\mathrm{H}^{(i)} = h_i \mathrm{K} h_i^{-1} \cap \mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f)$, $\mathrm{H}_L^{(i)} = \mathrm{H}^{(i)} \cap \mathbf{L}_{n_r}(\mathbb{Q})\mathbf{N}_{n_r}(\mathbb{A}_f)$ and $\mathrm{K}_N^{(i)} = \mathrm{H}^{(i)} \cap \mathbf{N}_{n_r}(\mathbb{A}_f)$. We get a cohomological correspondence

$$\Phi^j c_{\overline{q}_1,\overline{q}_2} : (\Phi^j T_{\overline{q}_1})^* L_{C_1} \longrightarrow T_{\overline{q}_2}^! L_{C_2}.$$

Define a cohomological correspondence

$$u_{C'} : (\Phi^j \overline{T}_g)^*(i_{C_1} T_{C_1})_! L_{C_1} \longrightarrow \overline{T}_1^!(i_{C_2} T_{C_2})_! L_{C_2}$$

by taking the direct image with compact support of the previous correspondence by $(i_{C_1} T_{C_1}, i_{C_2} T_{C_2})$ (the direct image of a correspondence by a proper morphism is defined in [SGA 5] III 3.3; the direct image by a locally closed immersion is defined in [M2] 5.1.1 (following [F] 1.3.1), and the direct image with compact support is defined by duality). Finally, write

$$N_{C'} = [\mathrm{K}_N^{(2)} : h_2 \mathrm{K}' h_2^{-1} \cap \mathbf{N}_{n_r}(\mathbb{A}_f)].$$

Proposition 1.5.3 *The coefficient u_{C_1,C_2} in the above matrix is equal to*

$$\sum_{C'} N_{C'} u_{C'},$$

where the sum is taken over the set of $C' \in \mathcal{C}'_P$ such that $T_g(C') = C_1$ and $T_1(C') = C_2$.

This proposition generalizes (the dual version of) theorem 5.2.2 of [M2] and can be proved in exactly the same way (by induction on r, as in the proof of proposition 1.4.5). The proof of theorem 5.2.2 of [M2] uses proposition 2.2.3 of [M2] (via the proof of corollary 5.2.4), but this proposition is simply a reformulation of proposition 4.8.5 of [P2], and it is true as well for the Shimura varieties considered here.

1.6 THE FIXED POINT FORMULAS OF KOTTWITZ AND GORESKY-KOTTWITZ-MACPHERSON

In this section, we recall two results about the fixed points of Hecke correspondences, which will be used in 1.7.

Theorem 1.6.1 *([K11] 19.6) Notation is as in 1.5. Assume that the Shimura datum $(\mathbf{G}, \mathcal{X})$ is of the type considered in [K11] §5, and that we are not in case (D) of that article (i.e., that $\mathbf{G}$ is not an orthogonal group). Fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Let $V \in \mathrm{Ob}\,\mathrm{Rep}_{\mathbf{G}}$. For every $j \geq 1$, denote by $T(j, g)$ the sum over the set of fixed points in $M^{\mathrm{K}'}(\mathbf{G}, \mathcal{X})(\mathbb{F})$ of the correspondence $(\Phi^j \circ T_g, T_1)$ of the naive local terms (cf. [P3] 1.5) of the cohomological correspondence u_j on $\mathcal{F}^{\mathrm{K}} V$ defined in section 1.5. Then*

$$T(j, g) = \sum_{(\gamma_0;\gamma,\delta) \in C_{\mathbf{G},j}} c(\gamma_0; \gamma, \delta) O_\gamma(f^p) T O_\delta(\phi_j^{\mathbf{G}}) \,\mathrm{Tr}(\gamma_0, V).$$

Let us explain briefly the notation (see [K9] §§2 and 3 for more detailed explanations).

The function $f^p \in C_c^\infty(\mathbf{G}(\mathbb{A}_f^p))$ is defined by the formula

$$f^p = \frac{\mathbb{1}_{g\mathrm{K}^p}}{\mathrm{vol}(\mathrm{K}'^p)}.$$

For every $\gamma \in \mathbf{G}(\mathbb{A}_f^p)$, write

$$O_\gamma(f^p) = \int_{\mathbf{G}(\mathbb{A}_f^p)_\gamma \backslash \mathbf{G}(\mathbb{A}_f^p)} f^p(x^{-1}\gamma x)d\overline{x},$$

where $\mathbf{G}(\mathbb{A}_f^p)_\gamma$ is the centralizer of γ in $\mathbf{G}(\mathbb{A}_f^p)$.

Remember that we fixed an injection $F \subset \overline{\mathbb{Q}}_p$; this determines a place $\wp$ of F over p. Let $\mathbb{Q}_p^{nr}$ be the maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$, L be the unramified extension of degree j of $F_\wp$ in $\overline{\mathbb{Q}}_p$, $r = [L : \mathbb{Q}_p]$, ϖ_L be a uniformizer of L and $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{nr}/\mathbb{Q}_p)$ be the element lifting the arithmetic Frobenius morphism of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$. Let $\delta \in \mathbf{G}(L)$. Define the norm $N\delta$ of δ by

$$N\delta = \delta\sigma(\delta)\ldots\sigma^{r-1}(\delta) \in \mathbf{G}(L).$$

The σ-centralizer of δ in $\mathbf{G}(L)$ is by definition

$$\mathbf{G}(L)_\delta^\sigma = \{x \in \mathbf{G}(L) | x\delta = \delta\sigma(x)\}.$$

We say that $\delta' \in \mathbf{G}(L)$ is σ-conjugate to δ in $\mathbf{G}(L)$ if there exists $x \in \mathbf{G}(L)$ such that $\delta' = x^{-1}\delta\sigma(x)$.

By definition of the reflex field F, the conjugacy class of cocharacters $h_x \circ \mu_0 : \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbf{G}_\mathbb{C}$, $x \in \mathcal{X}$, of section 1.1 is defined over F. Choose an element μ in this conjugacy class that factors through a maximal split torus of $\mathbf{G}$ over $\mathcal{O}_L$ (cf. [K9] §3 p. 173), and write

$$\phi_j^\mathbf{G} = \mathbb{1}_{\mathbf{G}(\mathcal{O}_L)\mu(\varpi_L^{-1})\mathbf{G}(\mathcal{O}_L)} \in \mathcal{H}(\mathbf{G}(L), \mathbf{G}(\mathcal{O}_L)).$$

($\mathcal{H}(\mathbf{G}(L), \mathbf{G}(\mathcal{O}_L))$ is the Hecke algebra of functions with compact support on $\mathbf{G}(L)$ that are bi-invariant by $\mathbf{G}(\mathcal{O}_L)$.) For every $\delta \in \mathbf{G}(L)$ and $\phi \in C_c^\infty(\mathbf{G}(L))$, write

$$TO_\delta(\phi) = \int_{\mathbf{G}(L)_\delta^\sigma \backslash \mathbf{G}(L)} \phi(y^{-1}\delta\sigma(y))d\overline{y}.$$

Let $\widehat{\mathbf{T}}$ be a maximal torus of $\widehat{\mathbf{G}}$. The conjugacy class of cocharacters $h_x \circ \mu_0$, $x \in \mathcal{X}$, corresponds to a Weyl group orbit of characters of $\widehat{\mathbf{T}}$; denote by μ_1 the restriction to $Z(\widehat{\mathbf{G}})$ of any of these characters (this does not depend on the choices).

It remains to define the set $C_{\mathbf{G},j}$ indexing the sum of the theorem and the coefficients $c(\gamma_0; \gamma, \delta)$. Consider the set of triples $(\gamma_0; \gamma, \delta) \in \mathbf{G}(\mathbb{Q}) \times \mathbf{G}(\mathbb{A}_f^p) \times \mathbf{G}(L)$ satisfying the following conditions (we will later write (C) for the list of these conditions):

- γ_0 is semisimple and elliptic in $\mathbf{G}(\mathbb{R})$ (i.e., there exists an elliptic maximal torus $\mathbf{T}$ of $\mathbf{G}_\mathbb{R}$ such that $\gamma_0 \in \mathbf{T}(\mathbb{R})$).
- For every place $v = p, \infty$ of $\mathbb{Q}$, γ_v (the local component of γ at v) is $\mathbf{G}(\overline{\mathbb{Q}}_v)$-conjugate to γ_0.

- $N\delta$ and γ_0 are $\mathbf{G}(\overline{\mathbb{Q}}_p)$-conjugate.
- The image of the σ-conjugacy class of δ by the map $B(\mathbf{G}_{\mathbb{Q}_p}) \longrightarrow X^*(Z(\widehat{\mathbf{G}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)})$ of [K9] 6.1 is the restriction of $-\mu_1$ to $Z(\widehat{\mathbf{G}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$.

Two triples $(\gamma_0; \gamma, \delta)$ and $(\gamma_0'; \gamma', \delta')$ are called equivalent if γ_0 and γ_0' are $\mathbf{G}(\overline{\mathbb{Q}})$-conjugate, γ and γ' are $\mathbf{G}(\mathbb{A}_f^p)$-conjugate, and δ and δ' are σ-conjugate in $\mathbf{G}(L)$.

Let $(\gamma_0; \gamma, \delta)$ be a triple satisfying conditions (C). Let I_0 be the centralizer of γ_0 in $\mathbf{G}$. There is a canonical morphism $Z(\widehat{\mathbf{G}}) \longrightarrow Z(\widehat{I_0})$, and the exact sequence

$$1 \longrightarrow Z(\widehat{\mathbf{G}}) \longrightarrow Z(\widehat{I_0}) \longrightarrow Z(\widehat{I_0})/Z(\widehat{\mathbf{G}}) \longrightarrow 1$$

induces a morphism

$$\pi_0((Z(\widehat{I_0})/Z(\widehat{\mathbf{G}}))^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}) \longrightarrow \mathrm{H}^1(\mathbb{Q}, Z(\widehat{\mathbf{G}})).$$

Denote by $\mathfrak{K}(I_0/\mathbb{Q})$ the inverse image by this morphism of the subgroup

$$\mathrm{Ker}^1(\mathbb{Q}, Z(\widehat{\mathbf{G}})) := \mathrm{Ker}\left(\mathrm{H}^1(\mathbb{Q}, Z(\widehat{\mathbf{G}})) \longrightarrow \prod_{v \text{ place of } \mathbb{Q}} \mathrm{H}^1(\mathbb{Q}_v, Z(\widehat{\mathbf{G}}))\right).$$

In [K9] §2, Kottwitz defines an element $\alpha(\gamma_0; \gamma, \delta) \in \mathfrak{K}(I_0/\mathbb{Q})^D$ (where, for every group A, $A^D = \mathrm{Hom}(A, \mathbb{C}^\times)$); this element depends only on the equivalence class of $(\gamma_0; \gamma, \delta)$. For every place $v = p, \infty$ of $\mathbb{Q}$, denote by $I(v)$ the centralizer of γ_v in $\mathbf{G}_{\mathbb{Q}_v}$; as γ_0 and γ_v are $\mathbf{G}(\overline{\mathbb{Q}}_v)$-conjugate, the group $I(v)$ is an inner form of I_0 over $\mathbb{Q}_v$. On the other hand, there exists a $\mathbb{Q}_p$-group $I(p)$ such that $I(p)(\mathbb{Q}_p) = \mathbf{G}(L)_\delta^\sigma$, and this group is an inner form of I_0 over $\mathbb{Q}_p$. There is a similar object for the infinite place: in the beginning of [K9] §3, Kottwitz defines an inner form $I(\infty)$ of I_0; $I(\infty)$ is an algebraic group over $\mathbb{R}$, anisotropic modulo $\mathbf{A}_G$. Kottwitz shows that, if $\alpha(\gamma_0; \gamma, \delta) = 1$, then there exists an inner form I of I_0 over $\mathbb{Q}$ such that, for every place v of $\mathbb{Q}$, $I_{\mathbb{Q}_v}$ and $I(v)$ are isomorphic (Kottwitz's statement is more precise, cf. [K9] pp. 171–172).

The set $C_{\mathbf{G},j}$ indexing the sum of the theorem is the set of equivalence classes of triples $(\gamma_0; \gamma, \delta)$ satisfying conditions (C) and such that $\alpha(\gamma_0; \gamma, \delta) = 1$. For every $(\gamma_0; \gamma, \delta)$ in $C_{\mathbf{G},j}$, let

$$c(\gamma_0; \gamma, \delta) = \mathrm{vol}(I(\mathbb{Q}) \setminus I(\mathbb{A}_f)) |\mathrm{Ker}(\mathrm{Ker}^1(\mathbb{Q}, I_0) \longrightarrow \mathrm{Ker}^1(\mathbb{Q}, \mathbf{G}))|.$$

Finally, the Haar measures are normalized as in [K9] §3. Take on $\mathbf{G}(\mathbb{A}_f^p)$ (resp., $\mathbf{G}(\mathbb{Q}_p)$, resp., $\mathbf{G}(L)$) the Haar measure such that the volume of K^p (resp., $\mathbf{G}(\mathbb{Z}_p)$, resp., $\mathbf{G}(\mathcal{O}_L)$) is equal to 1. Take on $I(\mathbb{A}_f^p)$ (resp., $I(\mathbb{Q}_p)$) a Haar measure such that the volume of every open compact subgroup is a rational number, and use inner twistings to transport these measures to $\mathbf{G}(\mathbb{A}_f^p)_\gamma$ and $\mathbf{G}(L)_\delta^\sigma$.

Remark 1.6.2 If $\mathrm{K}' = \mathrm{K} \cap g\mathrm{K}g^{-1}$, we may replace f^p with the function

$$\frac{\mathbb{1}_{\mathrm{K}^p g \mathrm{K}^p}}{\mathrm{vol}(\mathrm{K}^p)} \in \mathcal{H}(\mathbf{G}(\mathbb{A}_f^p), \mathrm{K}^p) := C_c^\infty(\mathrm{K}^p \setminus \mathbf{G}(\mathbb{A}_f^p)/\mathrm{K}^p)$$

(cf. [K11] §16 p. 432).

Remark 1.6.3 There are two differences between the formula given here and formula (19.6) of [K11]:

(1) Kottwitz considers the correspondence $(T_g, \Phi^j \circ T_1)$ (and not $(\Phi^j \circ T_g, T_1)$) and does not define the naive local term in the same way as Pink (cf. [K11] §16 p. 433). But is is easy to see (by comparing the definitions of the naive local terms and composing Kottwitz's correspondence by $T_{g^{-1}}$) that the number $T(j, f)$ of [K11] (19.6) is equal to $T(j, g^{-1})$. This explains that the function of $C_c^\infty(\mathbf{G}(\mathbb{A}_f^p))$ appearing in theorem 1.6.1 is $\mathrm{vol}(\mathrm{K}'^p)^{-1}\mathbb{1}_{g\mathrm{K}^p}$, instead of the function $\widetilde{f}^p = \mathrm{vol}(\mathrm{K}'^p)^{-1}\mathbb{1}_{\mathrm{K}^p g^{-1}}$ of [K11] §16 p. 432. (Kottwitz also takes systematically $\mathrm{K}' = \mathrm{K} \cap g\mathrm{K}g^{-1}$, but his result generalizes immediately to the case where K' is of finite index in $\mathrm{K} \cap g\mathrm{K}g^{-1}$.)

(2) Below formula (19.6) of [K11], Kottwitz notes that this formula is true for the canonical model of a Shimura variety associated to the datum $(\mathbf{G}, \mathcal{X}, h^{-1})$ (and not $(\mathbf{G}, \mathcal{X}, h)$). The normalization of the global class field isomorphism used in [K9], [K11], and here are the same (it is also the normalization of [D1] 0.8 and [P2] 5.5). However, the convention for the action of the Galois group on the special points of the canonical model that is used here is the convention of [P2] 5.5, and it differs (by a sign) from the convention of [D1] 2.2.4 (because the reciprocity morphism of [P2] 5.5 is the inverse of the reciprocity morphism of [D1] 2.2.3). As Kottwitz uses Deligne's conventions, what he calls canonical model of a Shimura variety associated to the datum $(\mathbf{G}, \mathcal{X}, h^{-1})$ is what is called here canonical model of a Shimura variety associated to the datum $(\mathbf{G}, \mathcal{X}, h)$.

Remark 1.6.4 Actually, Kottwitz proves a stronger result in [K11] §19: For every $\gamma \in \mathbf{G}(\mathbb{A}_f^p)$, let $N(\gamma)$ be the number of fixed points x' in $M^{\mathrm{K}'}(\mathbf{G}, \mathcal{X})(\mathbb{F})$ that can be represented by an element $\widetilde{x}$ of $M(\mathbf{G}, \mathcal{X})(\mathbb{F})$ such that there exist $k \in \mathrm{K}$ and $g \in \mathbf{G}(\mathbb{A}_f)$ with $\Phi^j(\widetilde{x})g = \widetilde{x}k$ and gk^{-1} $\mathbf{G}(\mathbb{A}_f^p)$-conjugate to γ (this condition depends only on x', and not on the choice of $\widetilde{x}$). Then

$$N(\gamma) = \sum_\delta c(\gamma_0; \gamma, \delta) O_\gamma(f^p) TO_\delta(\phi_j^{\mathbf{G}}),$$

where the sum is taken over the set of σ-conjugacy classes of $\delta \in \mathbf{G}(L)$ such that there exists $\gamma_0 \in \mathbf{G}(\mathbb{Q})$ such that the triple $(\gamma_0; \gamma, \delta)$ is in $C_{\mathbf{G},j}$ (if such a γ_0 exists, it is unique up to $\mathbf{G}(\overline{\mathbb{Q}})$-conjugacy, because, for every place $v = p, \infty$ of $\mathbb{Q}$, it is conjugate under $\mathbf{G}(\overline{\mathbb{Q}}_v)$ to the component at v of γ). Moreover, if x' is a fixed point contributing to $N(\gamma)$, then the naive local term at x' is $\mathrm{Tr}(\gamma_\ell, V)$ (where γ_ℓ is the ℓ-adic component of γ).

Remark 1.6.5 Some of the Shimura varieties that will be used later are not of the type considered in [K11] §5, so we will need another generalization of Kottwitz's result, in a very particular (and easy) case. Let $(\mathbf{G}, \mathcal{X}, h)$ be a Shimura datum (in the sense of section 1.1) such that $\mathbf{G}$ is a torus. Let $\mathcal{Y}$ be the image of $\mathcal{X}$ by the morphism $h : \mathcal{X} \longrightarrow \mathrm{Hom}(\mathbb{S}, \mathbf{G})$ ($\mathcal{Y}$ is a point because $\mathbf{G}$ is commutative, but the cardinality of $\mathcal{X}$ can be greater than 1 in general; remember that the morphism

h is assumed to have finite fibers, but that it is not assumed to be injective). Let $\mathbf{G}(\mathbb{R})^+$ be the subgroup of $\mathbf{G}(\mathbb{R})$ stabilizing a connected component of $\mathcal{X}$ (this group does not depend on the choice of the connected component) and $\mathbf{G}(\mathbb{Q})^+ = \mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})^+$. The results of theorem 1.6.1 and of remark 1.6.4 are true for the Shimura datum $(\mathbf{G}, \mathcal{Y})$ (in this case, they are a consequence of the description of the action of the Galois group on the special points of the canonical model, cf. [P2] 5.5). For the Shimura datum $(\mathbf{G}, \mathcal{X})$, these results are also true if the following changes are made:

- multiply the formula giving the trace in theorem 1.6.1 and the formula giving the number of fixed points in remark 1.6.4 by $|\mathcal{X}|$;
- replace $C_{\mathbf{G},j}$ with the subset of triples $(\gamma_0; \gamma, \delta) \in C_{\mathbf{G},j}$ such that $\gamma_0 \in \mathbf{G}(\mathbb{Q})^+$.

This fact is also an easy consequence of [P2] 5.5.

The fixed point formula of Goresky, Kottwitz and MacPherson applies to a different situation, that of the end of 1.2. Use the notation introduced there. Let $V \in \operatorname{Ob}\operatorname{Rep}_{\mathbf{G}}$, $g \in \mathbf{G}(\mathbb{A}_f)$, and let K, K' be neat open compact subgroups of $\mathbf{G}(\mathbb{A}_f)$ such that $\mathrm{K}' \subset \mathrm{K} \cap g\mathrm{K}g^{-1}$. This gives two finite étale morphisms T_g, $T_1 : M^{\mathrm{K}'}(\mathbf{G}, \mathcal{X})(\mathbb{C}) \longrightarrow M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})(\mathbb{C})$. Define a cohomological correspondence

$$u_g : T_g^* \mathcal{F}^{\mathrm{K}} V \xrightarrow{\sim} T_1^! \mathcal{F}^{\mathrm{K}} V$$

as at the beginning of section 1.5. The following theorem is a particular case of theorem 7.14.B of [GKM] (cf. [GKM] (7.17)).

Theorem 1.6.6 *The trace of the correspondence u_g on the cohomology with compact support $R\Gamma_c(M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})(\mathbb{C}), \mathcal{F}^{\mathrm{K}} V)$ is equal to*

$$\sum_{\mathbf{M}} (-1)^{\dim(\mathbf{A}_M/\mathbf{A}_G)} (n_M^G)^{-1} \sum_{\gamma} \iota^M(\gamma)^{-1} \chi(\mathbf{M}_\gamma) O_\gamma(f_{\mathbf{M}}^\infty) |D_M^G(\gamma)|^{1/2} \mathrm{Tr}(\gamma, V),$$

where the first sum is taken over the set of $\mathbf{G}(\mathbb{Q})$-conjugacy classes of cuspidal Levi subgroups $\mathbf{M}$ of $\mathbf{G}$ and, for every $\mathbf{M}$, the second sum is taken over the set γ of semisimple $\mathbf{M}(\mathbb{Q})$-conjugacy classes that are elliptic in $\mathbf{M}(\mathbb{R})$.

Let us explain the notation.

- $f^\infty = \dfrac{\mathbb{1}_{g\mathrm{K}}}{\mathrm{vol}(\mathrm{K}')} \in C_c^\infty(\mathbf{G}(\mathbb{A}_f))$, and $f_{\mathbf{M}}^\infty$ is the constant term of f^∞ at $\mathbf{M}$ (cf. [GKM] (7.13.2)).
- Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{G}$. Let $\mathbf{A}_M$ be the maximal ($\mathbb{Q}$-)split subtorus of the center of $\mathbf{M}$ and
$$n_M^G = |\operatorname{Nor}_{\mathbf{G}}(\mathbf{M})(\mathbb{Q})/\mathbf{M}(\mathbb{Q})|.$$
$\mathbf{M}$ is called *cuspidal* if the group $\mathbf{M}_{\mathbb{R}}$ has a maximal ($\mathbb{R}$-)torus $\mathbf{T}$ such that $\mathbf{T}/\mathbf{A}_{M,\mathbb{R}}$ is anisotropic.
- Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{G}$ and $\gamma \in \mathbf{M}(\mathbb{Q})$. Let $\mathbf{M}^\gamma$ be the centralizer of γ in $\mathbf{M}$, $\mathbf{M}_\gamma = (\mathbf{M}^\gamma)^0$,
$$\iota^M(\gamma) = |\mathbf{M}^\gamma(\mathbb{Q})/\mathbf{M}_\gamma(\mathbb{Q})|$$

and

$$D_M^G(\gamma) = \det(1 - \mathrm{Ad}(\gamma), \mathrm{Lie}(\mathbf{G})/\mathrm{Lie}(\mathbf{M})).$$

- $\chi(\mathbf{M}_\gamma)$ is the Euler characteristic of $\mathbf{M}_\gamma$, cf. [GKM] (7.10).

Remark 1.6.7 According to [GKM] 7.14.B, the formula of the theorem should use $\mathrm{Tr}(\gamma, V^*)$ (or $\mathrm{Tr}(\gamma^{-1}, V)$) and not $\mathrm{Tr}(\gamma, V)$. The difference between the formula given here and that of [GKM] comes from the fact that [GKM] uses a different convention to define the trace of u_g (cf. [GKM] (7.7)); the convention used here is that of [SGA 5] III and of [P1].

1.7 THE FIXED POINT FORMULA

Use the notation introduced before proposition 1.5.3 and in section 1.6. Assume that the Shimura data $(\mathbf{G}, \mathcal{X})$ and $(\mathbf{G}_i, \mathcal{X}_i)$, $1 \leq i \leq n-1$, are of the type considered [K11] §5, with case (D) excluded. (In particular, $\mathbf{G}^{\mathrm{ad}}$ is of abelian type, so we can take $\mathcal{R}_0 = \mathrm{Rep}_{\mathbf{G}}$, i.e., choose any $V \in \mathrm{Ob}\, D^b(\mathrm{Rep}_{\mathbf{G}})$.) Assume moreover that $(\mathbf{G}_n, \mathcal{X}_n)$ is of the type considered in [K11] §5 (case (D) excluded) or that $\mathbf{G}_n$ is a torus.

We want to calculate the trace of the cohomological correspondence

$$\overline{u}_j : (\Phi^j \overline{T}_g)^* W^{\geq t_1, \dots, \geq t_n} V \longrightarrow \overline{T}_1^! W^{\geq t_1, \dots, \geq t_n} V.$$

Assume that $w(\mathbb{G}_m)$ acts on the $\mathrm{H}^i V$, $i \in \mathbb{Z}$, by $t \longrightarrow t^m$, for a certain $m \in \mathbb{Z}$ (where $w : \mathbb{G}_m \longrightarrow \mathbf{G}$ is the cocharacter of 1.3).

Let

$$f^{\infty,p} = \mathrm{vol}(\mathrm{K}'^p)^{-1} \mathbb{1}_{g\mathrm{K}^p}.$$

Let $\mathbf{P}$ be a standard parabolic subgroup of $\mathbf{G}$. Write $\mathbf{P} = \mathbf{P}_{n_1} \cap \dots \cap \mathbf{P}_{n_r}$, with $n_1 < \dots < n_r$. Let

$$T_P = m_P \sum_{\mathbf{L}} (-1)^{\dim(\mathbf{A}_L/\mathbf{A}_{L_P})} (n_L^{L_P})^{-1} \sum_{\gamma_L} \iota^L(\gamma_L)^{-1} \chi(\mathbf{L}_{\gamma_L}) |D_L^{L_P}(\gamma_L)|^{1/2}$$

$$\sum_{(\gamma_0;\gamma,\delta) \in C'_{\mathbf{G}_{n_r},j}} c(\gamma_0;\gamma,\delta) O_{\gamma_L \gamma}(f^{\infty,p}_{\mathbf{LG}_{n_r}}) O_{\gamma_L}(\mathbb{1}_{\mathbf{L}(\mathbb{Z}_p)}) \delta^{1/2}_{P(\mathbb{Q}_p)}(\gamma_0) TO_\delta(\phi_j^{\mathbf{G}_{n_r}})$$

$$\times \delta^{1/2}_{P(\mathbb{R})}(\gamma_L \gamma_0) \,\mathrm{Tr}(\gamma_L \gamma_0, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)_{\geq t_{n_1}+m, \dots, \geq t_{n_r}+m}),$$

where the first sum is taken over the set of $\mathbf{L}_P(\mathbb{Q})$-conjugacy classes of cuspidal Levi subgroups $\mathbf{L}$ of $\mathbf{L}_P$, the second sum is taken over the set of semisimple conjugacy classes $\gamma_L \in \mathbf{L}(\mathbb{Q})$ that are elliptic in $\mathbf{L}(\mathbb{R})$, and

- $\mathbf{L}(\mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup of $\mathbf{L}(\mathbb{Q}_p)$;
- $m_P = 1$ if $n_r < n$ or if $(\mathbf{G}_n, \mathcal{X}_n)$ is of the type considered in [K11] §5, and $m_P = |\mathcal{X}_{n_r}|$ if $n_r = n$ and $\mathbf{G}_{n_r}$ is a torus;
- $C'_{\mathbf{G}_{n_r},j} = C_{\mathbf{G}_{n_r},j}$ if $n_r < n$ or if $(\mathbf{G}_n, \mathcal{X}_n)$ is of the type considered in [K11] §5, and, if $\mathbf{G}_n$ is a torus, $C'_{\mathbf{G}_n,j}$ is the subset of $C_{\mathbf{G}_n,j}$ defined in remark 1.6.5.

Write also

$$T_G = \sum_{(\gamma_0;\gamma,\delta)\in C_{\mathbf{G},j}} c(\gamma_0;\gamma,\delta) O_\gamma(f^{\infty,p}) TO_\delta(\phi_j^{\mathbf{G}}) \operatorname{Tr}(\gamma_0, V).$$

Theorem 1.7.1 *If j is positive and big enough, then*

$$\operatorname{Tr}(\overline{u}_j, R\Gamma(M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*_{\mathbb{F}}, (W^{\geq t_1,\dots,\geq t_n}V)_{\mathbb{F}})) = T_G + \sum_{\mathbf{P}} T_P,$$

where the sum is taken over the set of standard parabolic subgroups of **G**. *Moreover, if $g = 1$ and $\mathrm{K} = \mathrm{K}'$, then this formula is true for every $j \in \mathbb{N}^*$.*

Proof. For every $i \in \{1,\dots,n\}$, let $a_i = -t_i - m + \dim(M_i)$. For every standard parabolic subgroup $\mathbf{P} = \mathbf{P}_{n_1} \cap \dots \cap \mathbf{P}_{n_r}$, with $n_1 < \dots < n_r$, let

$$T'_P = (-1)^r \operatorname{Tr}(i_{n_r!} w_{\leq a_{n_r}} i^!_{n_r} \dots i_{n_1!} w_{\leq a_{n_1}} i^!_{n_1} \overline{u}_j).$$

Let

$$T'_G = \operatorname{Tr}(\overline{u}_j, R\Gamma(M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*_{\mathbb{F}}, (j_! \mathcal{F}^{\mathrm{K}} V)_{\mathbb{F}})).$$

Then, by the dual of proposition 5.1.5 of [M2] and the definition of $W^{\geq t_1,\dots,\geq t_n}V$,

$$\operatorname{Tr}(\overline{u}_j, R\Gamma(M^{\mathrm{K}}(\mathbf{G},\mathcal{X})^*_{\mathbb{F}}, (W^{\geq t_1,\dots,\geq t_n}V)_{\mathbb{F}})) = T'_G + \sum_{\mathbf{P}} T'_P,$$

where the sum is taken over the set of standard parabolic subgroups of **G**. So we want to show that $T'_G = T_G$ and $T'_P = T_P$. Fix $\mathbf{P} = \mathbf{G}$ (and $n_1,\dots,n_r$). It is easy to see that

$$\dim(\mathbf{A}_{M_P}/\mathbf{A}_G) = r.$$

Let $h \in \mathbf{G}(\mathbb{A}_f^p)$. Write

$$\mathrm{K}_{N,h} = h\mathrm{K}h^{-1} \cap \mathbf{N}(\mathbb{A}_f),$$

$$\mathrm{K}_{P,h} = h\mathrm{K}h^{-1} \cap \mathbf{P}(\mathbb{A}_f),$$

$$\mathrm{K}_{M,h} = \mathrm{K}_{P,h}/\mathrm{K}_{N,h},$$

$$\mathrm{K}_{L,h} = (h\mathrm{K}h^{-1} \cap \mathbf{L}_P(\mathbb{A}_f)\mathbf{N}_P(\mathbb{A}_f))/\mathrm{K}_{N,h},$$

$$\mathrm{H}_h = h\mathrm{K}h^{-1} \cap \mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f),$$

$$\mathrm{H}_{L,h} = h\mathrm{K}h^{-1} \cap \mathbf{L}_P(\mathbb{Q})\mathbf{N}_P(\mathbb{A}_f).$$

Define in the same way groups $\mathrm{K}'_{N,h}$, etc., by replacing K with K′. If there exists $q \in \mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f)$ such that $qh\mathrm{K} = hg\mathrm{K}$, let $\overline{q}$ be the image of q in $\mathbf{M}_P(\mathbb{A}_f)$, and let u_h be the cohomological correspondence on $\mathcal{F}^{\mathrm{H}_h/\mathrm{H}_{L,h}} R\Gamma_c(\mathrm{H}_{L,h}, R\Gamma(\mathrm{Lie}(\mathbf{N}_{n_r}), V)_{\geq t_{n_1},\dots,\geq t_{n_r}})[a]$ with support in $(\Phi^j T_{\overline{q}}, T_1)$ equal to $\Phi^j c_{\overline{q},1}$ (we may assume that $q \in \mathbf{P}(\mathbb{A}_f^p)$, hence that $\overline{q} \in \mathbf{M}_P(\mathbb{A}_f^p)$). This correspondence is called $u_{C'}$ in section 1.5, where C' is the image of h in $\mathcal{C}'_P$. If there is no such $q \in \mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f)$, take $u_h = 0$. Similarly, if there exists $q \in \mathbf{P}(\mathbb{A}_f)$ such that

$qh\mathrm{K} = hg\mathrm{K}$, let $\overline{q}$ be the image of q in $\mathbf{M}_P(\mathbb{A}_f)$, and let v_h be the cohomological correspondence on $\mathcal{F}^{\mathrm{K}_{M,h}/\mathrm{K}_{L,h}}R\Gamma_c(\mathrm{K}_{L,h}, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)_{\geq t_{n_1},\dots,\geq t_{n_r}})[a]$ with support in $(\Phi^j T_{\overline{q}}, T_1)$ equal to $\Phi^j c_{\overline{q},1}$ (we may assume that $q \in \mathbf{P}(\mathbb{A}_f^p)$). If there is no such $q \in \mathbf{P}(\mathbb{A}_f)$, take $v_h = 0$. Finally, let $N_h = [\mathrm{K}_{N,h} : \mathrm{K}'_{N,h}]$.

Let $h \in \mathbf{G}(\mathbb{A}_f^p)$ be such that there exists $q \in \mathbf{P}(\mathbb{A}_f)$ with $qh\mathrm{K} = hg\mathrm{K}$. By proposition 1.7.2 below,

$$\mathrm{Tr}(v_h) = \sum_{h'} \mathrm{Tr}(u_{h'}),$$

where the sum is taken over a system of representatives $h' \in \mathbf{G}(\mathbb{A}_f^p)$ of the double classes in $\mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}'$ that are sent to the class of h in $\mathbf{P}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/K'$ (apply proposition 1.7.2 with $\mathbf{M} = \mathbf{M}_P$, $\mathrm{K}_M = \mathrm{K}_{M,h}$, m equal to the image of q in $\mathbf{M}_P(\mathbb{A}_f)$). On the other hand, by proposition 1.5.3,

$$T'_P = (-1)^r \sum_h N_h \,\mathrm{Tr}(u_h),$$

where the sum is taken over a system of representatives $h \in \mathbf{G}(\mathbb{A}_f^p)$ of the double classes in $\mathbf{P}(\mathbb{Q})\mathbf{Q}_{n_r}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}'$. Hence

$$T'_P = (-1)^r \sum_h N_h \,\mathrm{Tr}(v_h),$$

where the sum is taken over a system of representatives $h \in \mathbf{G}(\mathbb{A}_f^p)$ of the double classes in $\mathbf{P}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}'$.

Let $h \in \mathbf{G}(\mathbb{A}_f^p)$. Assume that there exists $q \in \mathbf{P}(\mathbb{A}_f^p)$ such that $qh\mathrm{K} = hg\mathrm{K}$. Let $\overline{q}$ be the image of q in $\mathbf{M}_P(\mathbb{A}_f^p)$. Write $\overline{q} = q_L q_H$, with $q_L \in \mathbf{L}_P(\mathbb{A}_f^p)$ and $q_H \in \mathbf{G}_{n_r}(\mathbb{A}_f^p)$. Let

$$f_{G,h}^{\infty,p} = \mathrm{vol}(\mathrm{K}'_{M,h}/\mathrm{K}'_{L,h})^{-1}\mathbb{1}_{q_H(\mathrm{K}_{M,h}/\mathrm{K}_{L,h})}.$$

Notice that $\mathrm{K}'_{L,h} \subset \mathrm{K}_{L,h} \cap q_L \mathrm{K}_{L,h} q_L^{-1}$. Let u_{q_L} be the endomorphism of $R\Gamma_c(\mathrm{K}_{L,h}, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)_{\geq t_{n_1},\dots,\geq t_{n_r}})$ induced by the cohomological correspondence $c_{q_L,1}$.

To calculate the trace of v_h, we will use Deligne's conjecture, which has been proved by Pink (cf. [P3]) assuming some hypotheses (that are satisfied here), and in general by Fujiwara ([F]) and Varshavsky ([V]). This conjecture (which should now be called a theorem) says that, if j is big enough, then the fixed points of the correspondence between schemes underlying v_h are all isolated, and that the trace of v_h is the sum over these fixed points of the naive local terms. By theorem 1.6.1 and remarks 1.6.4 and 1.6.5, if j is big enough, then

$$\mathrm{Tr}(v_h) = (-1)^r m_P \sum_{(\gamma_0;\gamma,\delta)\in C'_{\mathbf{G}_{n_r},j}} c(\gamma_0;\gamma,\delta) O_\gamma(f_{G,h}^{\infty,p}) TO_\delta(\phi_j^{\mathbf{G}_{n_r}}) \\ \mathrm{Tr}(u_{q_L}\gamma_0, R\Gamma_c(\mathrm{K}_{L,h}, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)_{\geq t_{n_1},\dots,\geq t_{n_r}})).$$

Let

$$f_{L_P,h}^{\infty} = \mathrm{vol}(\mathrm{K}'_{L,h})^{-1}\mathbb{1}_{q_L \mathrm{K}_{L,h}}.$$

Then

$$f^{\infty}_{L_P,h} = \mathbb{1}_{\mathbf{L}_P(\mathbb{Z}_p)} f^{\infty,p}_{L_P,h},$$

with $f^{\infty,p}_{L_P,h} \in C_c^\infty(\mathbf{L}_P(\mathbb{A}_f^p))$. By theorem 1.6.6, for every $\gamma_0 \in \mathbf{G}_{n_r}(\mathbb{Q})$,

$$\begin{aligned}
&\mathrm{Tr}(u_{q_L}\gamma_0, R\Gamma_c(\mathrm{K}_{L,h}, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)_{\geq t_{n_1},\dots,\geq t_{n_r}}))\\
&= \sum_{\mathbf{L}} (-1)^{\dim(\mathbf{A}_L/\mathbf{A}_{L_P})}(n_L^{L_P})^{-1}\\
&\sum_{\gamma_L} \iota^L(\gamma_L)^{-1}\chi(\mathbf{L}_{\gamma_L})|D_L^{L_P}(\gamma_L)|^{1/2} O_{\gamma_L}((f^\infty_{L_P,h})_L)\\
&\quad\times \mathrm{Tr}(\gamma_L\gamma_0, R\Gamma(\mathrm{Lie}(\mathbf{N}_P), V)_{\geq t_{n_1},\dots,\geq t_{n_r}}),
\end{aligned}$$

where the first sum is taken over the set of conjugacy classes of cuspidal Levi subgroups $\mathbf{L}$ of $\mathbf{L}_P$ and the second sum is taken over the set of semisimple conjugacy classes γ_L of $\mathbf{L}(\mathbb{Q})$ that are elliptic in $\mathbf{L}(\mathbb{R})$. To show that $T'_P = T_P$, it is enough to show that, for every Levi subgroup $\mathbf{L}$ of $\mathbf{L}_P$, for every $\gamma_L \in \mathbf{L}(\mathbb{Q})$ and every $(\gamma_0; \gamma, \delta) \in C_{\mathbf{G}_{n_r},j}$,

$$\sum_h N_h O_{\gamma_L}((f^\infty_{L_P,h})_L) O_\gamma(f^{\infty,p}_{G,h}) = O_{\gamma_L\gamma}(f^{\infty,p}_{\mathbf{LG}_{n_r}})\delta^{1/2}_{P(\mathbb{Q}_p)}(\gamma_L\gamma_0) O_{\gamma_L}(\mathbb{1}_{\mathbf{L}(\mathbb{Z}_p)})\delta^{1/2}_{P(\mathbb{R})}(\gamma_0),$$

where the sum is taken over a system of representatives $h \in \mathbf{G}(\mathbb{A}_f^p)$ of the double classes in $\mathbf{P}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}'$ (with $f^\infty_{L_P,h} = 0$ and $f^{\infty,p}_{G,h} = 0$ if there is no $q \in \mathbf{P}(\mathbb{A}_f)$ such that $qh\mathrm{K} = hg\mathrm{K}$).

Fix a parabolic subgroup $\mathbf{R}$ of $\mathbf{L}_P$ with Levi subgroup $\mathbf{L}$, and let $\mathbf{P}' = \mathbf{R}\mathbf{G}_{n_r}\mathbf{N}_P$ (a parabolic subgroup of $\mathbf{G}$ with Levi subgroup $\mathbf{LG}_{n_r}$). Fix a system of representatives $(h_i)_{i\in I}$ in $\mathbf{G}(\mathbb{A}_f^p)$ of $\mathbf{P}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}'$. For every $i \in I$, fix a system of representatives $(m_{ij})_{j\in J_i}$ in $\mathbf{L}_P(\mathbb{A}_f^p)$ of $\mathbf{R}(\mathbb{A}_f) \setminus \mathbf{L}_P(\mathbb{A}_f)/\mathrm{K}'_{L,h_i}$. Then $(m_{ij}h_i)_{i,j}$ is a system of representatives of $\mathbf{P}'(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}'$. By lemma 1.7.4 below,

$$O_{\gamma_L\gamma}(f^{\infty,p}_{\mathbf{LG}_{n_r}}) = \delta^{1/2}_{P'(\mathbb{A}_f^p)}(\gamma_L\gamma)\sum_{i,j} r(m_{ij}h_i) O_{\gamma_L\gamma}(f_{P',m_{ij}h_i}),$$

where

$$r(m_{ij}h_i) = [(m_{ij}h_i)\mathrm{K}(m_{ij}h_i)^{-1} \cap \mathbf{N}_{P'}(\mathbb{A}_f) : (m_{ij}h_i)\mathrm{K}'(m_{ij}h_i)^{-1} \cap \mathbf{N}_{P'}(\mathbb{A}_f)]$$

and $f_{P',m_{ij}h_i}$ is equal to the product of

$$\mathrm{vol}\left(((m_{ij}h_i)\mathrm{K}'(m_{ij}h_i)^{-1} \cap \mathbf{P}'(\mathbb{A}_f))/((m_{ij}h_i)\mathrm{K}'(m_{ij}h_i)^{-1} \cap \mathbf{N}_{P'}(\mathbb{A}_f))\right)^{-1}$$

and of the characteristic function of the image in $(\mathbf{LG}_{n_r})(\mathbb{A}_f^p) = \mathbf{M}_{P'}(\mathbb{A}_f^p)$ of $(m_{ij}h_i)g\mathrm{K}(m_{ij}h_i)^{-1} \cap \mathbf{P}'(\mathbb{A}_f^p)$. Note that

$$r(m_{ij}h_i) = N_{h_i} r'(m_{ij}),$$

where

$$r'(m_{ij}) = [m_{ij}\mathrm{K}_{L,h_i}m_{ij}^{-1} \cap \mathbf{N}_R(\mathbb{A}_f) : m_{ij}\mathrm{K}'_{L,h_i}m_{ij}^{-1} \cap \mathbf{N}_R(\mathbb{A}_f)],$$

that

$$\delta_{P'(\mathbb{A}_f^p)}(\gamma_L\gamma) = \delta_{R(\mathbb{A}_f^p)}(\gamma_L)\delta_{P(\mathbb{A}_f^p)}(\gamma_L\gamma),$$

and that

$$f_{P',m_{ij}h_i} = f_{R,m_{ij}} f^{\infty,p}_{G,h_i},$$

where $f_{R,m_{ij}}$ is the product of

$$\mathrm{vol}\left((m_{ij}\mathrm{K}'_{L,h_i}m_{ij}^{-1} \cap \mathbf{R}(\mathbb{A}_f))/(m_{ij}\mathrm{K}'_{L,h_i}m_{ij}^{-1} \cap \mathbf{N}_R(\mathbb{A}_f))\right)^{-1}$$

and of the characteristic function of the image in $\mathbf{L}(\mathbb{A}^p_f) = \mathbf{M}_R(\mathbb{A}^p_f)$ of $(m_{ij}h_i)g\mathrm{K}(m_{ij}h_i)^{-1} \cap \mathbf{R}(\mathbb{A}_f)\mathbf{N}_P(\mathbb{A}_f)$. By applying lemma 1.7.4 again, we find, for every $i \in I$,

$$\sum_{j\in J_i} r'(m_{ij})O_{\gamma_L}(f_{R,m_{ij}}) = \delta^{-1/2}_{R(\mathbb{A}^p_f)}(\gamma_L)O_{\gamma_L}((f^{\infty,p}_{L_P,h_i})_{\mathbf{L}}).$$

Finally,

$$\begin{aligned}
&\sum_{i\in I} N_{h_i} O_{\gamma_L}((f^{\infty}_{L_P,h_i})_{\mathbf{L}})O_\gamma(f^{\infty,p}_{G,h_i}) \\
&= O_{\gamma_L}((\mathbb{1}_{\mathbf{L}_P(\mathbb{Z}_p)})_{\mathbf{L}})\sum_{i\in I} N_{h_i} O_{\gamma_L}((f^{\infty,p}_{L_P,h_i})_{\mathbf{L}})O_\gamma(f^{\infty,p}_{G,h_i}) \\
&= O_{\gamma_L}((\mathbb{1}_{\mathbf{L}_P(\mathbb{Z}_p)})_{\mathbf{L}})\sum_{i\in I} N_{h_i} O_\gamma(f^{\infty,p}_{G,h_i})\delta^{1/2}_{R(\mathbb{A}^p_f)}(\gamma_L)\sum_{j\in J_i} r'(m_{ij})O_{\gamma_L}(f_{R,m_{ij}}) \\
&= O_{\gamma_L}((\mathbb{1}_{\mathbf{L}_P(\mathbb{Z}_p)})_{\mathbf{L}})\delta^{1/2}_{R(\mathbb{A}^p_f)}(\gamma_L)\sum_{i\in I}\sum_{j\in J_i} r(m_{ij}h_i)O_{\gamma_L\gamma}(f_{P',m_{ij}h_i}) \\
&= O_{\gamma_L}((\mathbb{1}_{\mathbf{L}_P(\mathbb{Z}_p)})_{\mathbf{L}})\delta^{1/2}_{R(\mathbb{A}^p_f)}(\gamma_L)\delta^{-1/2}_{P'(\mathbb{A}^p_f)}(\gamma_L\gamma)O_{\gamma_L\gamma}(f^{\infty,p}_{\mathbf{LG}_{n_r}}) \\
&= O_{\gamma_L}((\mathbb{1}_{\mathbf{L}_P(\mathbb{Z}_p)})_{\mathbf{L}})\delta^{-1/2}_{P(\mathbb{A}^p_f)}(\gamma_L\gamma)O_{\gamma_L\gamma}(f^{\infty,p}_{\mathbf{LG}_{n_r}}).
\end{aligned}$$

To finish the proof, it suffices to notice that $(\mathbb{1}_{\mathbf{L}_P(\mathbb{Z}_p)})_{\mathbf{L}} = \mathbb{1}_{\mathbf{L}(\mathbb{Z}_p)}$, that $\delta^{-1/2}_{P(\mathbb{A}^p_f)}(\gamma_L\gamma) = \delta^{-1/2}_{P(\mathbb{A}^p_f)}(\gamma_L\gamma_0)$, that, as $\gamma_L\gamma_0 \in \mathbf{M}_P(\mathbb{Q})$, the product formula gives

$$\delta^{-1/2}_{P(\mathbb{A}^p_f)}(\gamma_L\gamma_0) = \delta^{1/2}_{P(\mathbb{Q}_p)}(\gamma_L\gamma_0)\delta^{1/2}_{P(\mathbb{R})}(\gamma_L\gamma_0)$$

and that

$$\delta_{P(\mathbb{Q}_p)}(\gamma_L\gamma_0) = \delta_{P(\mathbb{Q}_p)}(\gamma_L)\delta_{P(\mathbb{Q}_p)}(\gamma_0) = \delta_{P(\mathbb{Q}_p)}(\gamma_0)$$

if $O_{\gamma_L}(\mathbb{1}_{\mathbf{L}(\mathbb{Z}_p)}) = 0$ (because this implies that γ_L is conjugate in $\mathbf{L}(\mathbb{Q}_p)$ to an element of $\mathbf{L}(\mathbb{Z}_p)$).

If j is big enough, we can calculate T'_G using theorem 1.6.1 and Deligne's conjecture. It is obvious $T'_G = T_G$.

If $g = 1$ and $\mathrm{K} = \mathrm{K}'$, then $\overline{u}_j$ is simply the cohomological correspondence induced by Φ^j. In this case, we can calculate the trace of $\overline{u}_j$, for every $j \in \mathbb{N}^*$, using Grothendieck's trace formula (cf. [SGA 4 1/2] Rapport 3.2). □

Proposition 1.7.2 *Let* $\mathbf{M}$, $\mathbf{L}$ *and* $(\mathbf{G}, \mathcal{X})$ *be as in section 1.2. Let* $m \in \mathbf{M}(\mathbb{A}_f)$, *and let* K'_M, K_M *be neat open compact subgroups of* $\mathbf{M}(\mathbb{A}_f)$ *such that* $\mathrm{K}'_M \subset \mathrm{K}_M \cap m\mathrm{K}_M m^{-1}$. *Let* $\mathrm{K}_L = \mathrm{K}_M \cap \mathbf{L}(\mathbb{A}_f)$ *and* $\mathrm{K} = \mathrm{K}_M/\mathrm{K}_L$. *Consider a system of* *representatives* $(m_i)_{i\in I}$ *of the set of double classes* $c \in \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f) \setminus \mathbf{M}(\mathbb{A}_f)/\mathrm{K}'_M$ *such that* $cm\mathrm{K}_M = c\mathrm{K}_M$. *For every* $i \in I$, *fix* $l_i \in \mathbf{L}(\mathbb{Q})$ *and* $g_i \in \mathbf{G}(\mathbb{A}_f)$ *such that* $l_i g_i m_i \in m_i m \mathrm{K}_M$. *Assume that the Shimura varieties and the morphisms that we get from the above data have good reduction modulo* p *as in section 1.3 (in particular,* K_M *and* K'_M *are hyperspecial at* p, *and* $m, m_i \in \mathbf{M}(\mathbb{A}_f^p)$, $g_i \in \mathbf{G}(\mathbb{A}_f^p)$*). Let* $\mathbb{F}_q$ *be the field of definition of these varieties and* $\mathbb{F}$ *be an algebraic closure of* $\mathbb{F}_q$.

For every $i \in I$, *let* $\mathrm{H}_i = m_i \mathrm{K}_M m_i^{-1} \cap \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f)$, $\mathrm{H}_{i,L} = \mathrm{H}_i \cap \mathbf{L}(\mathbb{Q})$ *and* $\mathrm{K}_i = \mathrm{H}_i/\mathrm{H}_{i,L}$. *Fix* $V \in \operatorname{Ob}\operatorname{Rep}_{\mathbf{G}}$. *Let*

$$L = \mathcal{F}^{\mathrm{K}} R\Gamma(\mathrm{K}_L, V),$$

$$L_i = \mathcal{F}^{\mathrm{K}_i} R\Gamma(\mathrm{H}_{i,L}, V),$$

$$M = \mathcal{F}^{\mathrm{K}} R\Gamma_c(\mathrm{K}_L, V),$$

$$M_i = \mathcal{F}^{\mathrm{K}} R\Gamma_c(\mathrm{H}_{i,L}, V).$$

Then, for every $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$,

(1) $\sum_{i\in I} \mathrm{Tr}(\sigma c_{l_i g_i,1}, R\Gamma(M^{\mathrm{K}_i}(\mathbf{G},\mathcal{X})_{\mathbb{F}}, L_{i,\mathbb{F}})) = \mathrm{Tr}(\sigma c_{m,1}, R\Gamma(M^{\mathrm{K}}(\mathbf{G},\mathcal{X})_{\mathbb{F}}, L_{\mathbb{F}}))$.

(2) $\sum_{i\in I} \mathrm{Tr}(\sigma c_{l_i g_i,1}, R\Gamma_c(M^{\mathrm{K}_i}(\mathbf{G},\mathcal{X})_{\mathbb{F}}, M_{i,\mathbb{F}})) = \mathrm{Tr}(\sigma c_{m,1}, R\Gamma_c(M^{\mathrm{K}}(\mathbf{G},\mathcal{X})_{\mathbb{F}}, M_{\mathbb{F}}))$.

Proof. Write $m = lg$, with $l \in \mathbf{L}(\mathbb{A}_f)$ and $g \in \mathbf{G}(\mathbb{A}_f)$. We may assume that $m_i \in \mathbf{L}(\mathbb{A}_f)$, hence $g_i = g$, for every $i \in I$. Let $\mathrm{K}^0 = \mathrm{H}_i \cap \mathbf{G}(\mathbb{A}_f) = m\mathrm{K}_M m^{-1} \cap \mathbf{G}(\mathbb{A}_f)$.

Point (1) implies point (2) by duality.

Let us prove (1). Let c_m be the endomorphism of $R\Gamma(\mathrm{K}_M, V)$ equal to

$$R\Gamma(\mathrm{K}_M, V) \longrightarrow R\Gamma(\mathrm{K}'_M, V) \xrightarrow{\mathrm{Tr}} R\Gamma(\mathrm{K}_M, V),$$

where the first map is induced by the injection $\mathrm{K}'_M \longrightarrow \mathrm{K}_M$, $k \longrightarrow m^{-1}km$, and the second map is the trace morphism associated to the injection $\mathrm{K}'_M \subset \mathrm{K}_M$. Define in the same way, for every $i \in I$, an endomorphism $c_{l_i g_i}$ of $R\Gamma(\mathrm{H}_i, V)$. Then

$$R\Gamma(\mathrm{K}_M, V) \qquad \underset{i\in I}{} R\Gamma(\mathrm{H}_i, V)$$

and $c_m = {}_{i\in I}\, c_{l_i g_i}$, so it is enough to show that this decomposition is $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$-equivariant. Let $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. Then σ induces an endomorphism of $R\Gamma(\mathrm{K}^0, V) = R\Gamma(M^{\mathrm{K}^0}(\mathbf{G},\mathcal{X})_{\mathbb{F}}, \mathcal{F}^{\mathrm{K}^0} V_{\mathbb{F}})$, that will still be denoted by σ, and, by the lemma below, the endomorphism of $R\Gamma(\mathrm{K}_M, V)$ (resp., $R\Gamma(\mathrm{H}_i, V)$) induced by σ is

$$R\Gamma(\mathrm{K}_M/(\mathrm{K}_M \cap \mathbf{L}(\mathbb{A}_f)), \sigma)$$

$$(\text{resp.,} \quad R\Gamma(\mathrm{H}_i/(\mathrm{H}_i \cap \mathbf{L}(\mathbb{Q})), \sigma)).$$

This finishes the proof. □

Lemma 1.7.3 *Let* $\mathbf{M}$, $\mathbf{L}$ *and* $(\mathbf{G}, \mathcal{X})$ *be as in the proposition above. Let* K_M *be a neat open compact subgroup of* $\mathbf{M}(\mathbb{A}_f)$. *Let* $\mathrm{K}_L = \mathrm{K}_M \cap \mathbf{M}(\mathbb{A}_f)$, $\mathrm{K}_G = \mathrm{K}_M \cap \mathbf{G}(\mathbb{A}_f)$, $\mathrm{H} = \mathrm{K}_M \cap \mathbf{L}(\mathbb{Q})\mathbf{G}(\mathbb{A}_f)$, $\mathrm{H}_L = \mathrm{K}_M \cap \mathbf{L}(\mathbb{Q})$, $\mathrm{K} = \mathrm{K}_M/\mathrm{K}_L$ *and* $\mathrm{K}' = \mathrm{H}/\mathrm{H}_L$. *Let* $V \in \mathrm{Ob}\,\mathrm{Rep}_{\mathbf{M}}$ *and* $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. *The element* σ *induces an endomorphism of* $R\Gamma(\mathrm{K}_G, V) = R\Gamma(M^{\mathrm{K}_G}(\mathbf{G}, \mathcal{X})_{\mathbb{F}}, \mathcal{F}^{\mathrm{K}_G} V_{\mathbb{F}})$ *(resp.,* $R\Gamma(\mathrm{K}_M, V) = R\Gamma(M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})_{\mathbb{F}}, \mathcal{F}^{\mathrm{K}} R\Gamma(\mathrm{K}_L, V)_{\mathbb{F}})$, *resp.,* $R\Gamma(\mathrm{H}, V) = R\Gamma(M^{\mathrm{K}'}(\mathbf{G}, \mathcal{X})_{\mathbb{F}}, \mathcal{F}^{\mathrm{K}'} R\Gamma(\mathrm{H}_L, V)_{\mathbb{F}})$*), that will be denoted by* φ_0 *(resp.* φ*, resp.* φ'*). Then*

$$\varphi = R\Gamma(\mathrm{K}_M/\mathrm{K}_G, \varphi_0)$$

and

$$\varphi' = R\Gamma(\mathrm{H}/\mathrm{K}_G, \varphi_0).$$

Proof. The two equalities are proved in the same way. Let us prove the first one. Let $Y = M^{\mathrm{K}_G}(\mathbf{G}, \mathcal{X})$, $X = M^{\mathrm{K}}(\mathbf{G}, \mathcal{X})$, let $f : Y \longrightarrow X$ be the (finite étale) morphism T_1 and $L = \mathcal{F}^{\mathrm{K}} R\Gamma(\mathrm{K}_L, V)$. Then, $f^*L = \mathcal{F}^{\mathrm{K}_G} R\Gamma(\mathrm{K}_L, V)$ by [P1] (1.11.5), and L is canonically a direct factor of $f_* f^* L$ because f is finite étale, so it is enough to show that the endomorphism of

$$R\Gamma(Y_{\mathbb{F}}, f^*L) = R\Gamma(\mathrm{K}_G, R\Gamma(\mathrm{K}_L, V)) = R\Gamma(\mathrm{K}_L, R\Gamma(\mathrm{K}_G, V))$$

induced by σ is equal to $R\Gamma(\mathrm{K}_L, \varphi_0)$. The complex $M = \mathcal{F}^{\mathrm{K}_G} V$ on Y is a complex of K_L-sheaves in the sense of [P2] (1.2), and $R\Gamma(\mathrm{K}_L, M) = f^*L$ by [P2] (1.9.3). To conclude, apply [P2] (1.6.4). □

The following lemma of [GKM] is used in the proof of theorem 1.7.1. Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$, $\mathbf{M}$ a Levi subgroup of $\mathbf{G}$ and $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ with Levi subgroup $\mathbf{M}$. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{P}$. If $f \in C_c^\infty(\mathbf{G}(\mathbb{A}_f))$, the constant term $f_M \in C_c^\infty(\mathbf{M}(\mathbb{A}_f))$ of f at $\mathbf{M}$ is defined in [GKM] (7.13) (the function f_M depends on the choice of $\mathbf{P}$, but its orbital integrals do not depend on that choice). For every $g \in \mathbf{M}(\mathbb{A}_f)$, let

$$\delta_{P(\mathbb{A}_f)}(g) = |\det(\mathrm{Ad}(g), \mathrm{Lie}(\mathbf{N}) \otimes \mathbb{A}_f)|_{\mathbb{A}_f}.$$

Let $g \in \mathbf{G}(\mathbb{A}_f)$ and let K', K be open compact subgroups of $\mathbf{G}(\mathbb{A}_f)$ such that $\mathrm{K}' \subset g\mathrm{K}g^{-1}$. For every $h \in \mathbf{G}(\mathbb{A}_f)$, let $\mathrm{K}_M(h)$ be the image in $\mathbf{M}(\mathbb{A}_f)$ of $hg\mathrm{K}h^{-1} \cap \mathbf{P}(\mathbb{A}_f)$,

$$f_{P,h} = \mathrm{vol}((h\mathrm{K}'h^{-1} \cap \mathbf{P}(\mathbb{A}_f))/(h\mathrm{K}'h^{-1} \cap \mathbf{N}(\mathbb{A}_f)))^{-1} \mathbb{1}_{\mathrm{K}_M(h)} \in C_c^\infty(\mathbf{M}(\mathbb{A}_f)),$$

and

$$r(h) = [h\mathrm{K}h^{-1} \cap \mathbf{N}(\mathbb{A}_f) : h\mathrm{K}'h^{-1} \cap \mathbf{N}(\mathbb{A}_f)].$$

(Note that, if there is no element $q \in \mathbf{P}(\mathbb{A}_f)$ such that $qh\mathrm{K} = hg\mathrm{K}$, then $\mathrm{K}_M(h)$ is empty, hence $f_{P,h} = 0$.) Let

$$f = \mathrm{vol}(\mathrm{K}')^{-1} \mathbb{1}_{g\mathrm{K}}$$

and

$$f_P = \sum_h r(h) f_{P,h},$$

where the sum is taken over a system of representatives of the double quotient $\mathbf{P}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}'$.

Lemma 1.7.4 *([GKM] 7.13.A) The functions* f_M *and* $\delta_{P(\mathbb{A}_f)}^{1/2} f_P$ *have the same orbital integrals.*

In [GKM], the g is on the right of the K (and not on the left), and $\delta_{P(\mathbb{A}_f)}^{-1/2}$ appears in the formula instead of $\delta_{P(\mathbb{A}_f)}^{1/2}$, but it is easy to see that their proof adapts to the case considered here. There are obvious variants of this lemma obtained by replacing $\mathbb{A}_f$ with $\mathbb{A}_f^p$ or $\mathbb{Q}_p$, where p is a prime number.

Remark 1.7.5 The above lemma implies in particular that the function $\gamma \longrightarrow O_\gamma(f_{\mathbf{M}})$ on $\mathbf{M}(\mathbb{A}_f)$ has its support contained in a set of the form $\bigcup_{m\in\mathbf{M}(\mathbb{A}_f)} mXm^{-1}$, where X is a compact subset of $\mathbf{M}(\mathbb{A}_f)$, because the support of $\gamma \longrightarrow O_\gamma(f_{\mathbf{M}})$ is contained in the union of the conjugates of $\mathrm{K}_M(h)$, for h in a system of representatives of the finite set $\mathbf{P}(\mathbb{A}_f) \setminus \mathbf{G}(\mathbb{A}_f)/\mathrm{K}'$. Moreover, if $g = 1$, then we may assume that X is a finite union of compact subgroups of $\mathbf{M}(\mathbb{A}_f)$, that are neat if K is neat (because the $\mathrm{K}_M(h)$ are subgroups of $\mathbf{M}(\mathbb{A}_f)$ in that case).