Chapter One

The History of Mathematics: Alternative Perspectives

A Justification for This Book

An interest in history marks us for life. How we see ourselves and others is shaped by the history we absorb, not only in the classroom but also from the Internet, films, newspapers, television programs, novels, and even strip cartoons. From the time we first become aware of the past, it can fire our imagination and excite our curiosity: we ask questions and then seek answers from history. As our knowledge develops, differences in historical perspectives emerge. And, to the extent that different views of the past affect our perception of ourselves and of the outside world, history becomes an important point of reference in understanding the clash of cultures and of ideas. Not surprisingly, rulers throughout history have recognized that to control the past is to master the present and thereby consolidate their power.

During the last four hundred years, Europe and its cultural dependencies[1] have played a dominant role in world affairs. This is all too often reflected in the character of some of the historical writing produced by Europeans in the past. Where other people appear, they do so in a transitory fashion whenever Europe has chanced in their direction. Thus the history of the Africans or the indigenous peoples of the Americas often appears to begin only after their encounter with Europe.

An important aspect of this Eurocentric approach to history is the manner in which the history and potentialities of non-European societies were represented, particularly with respect to their creation and development of science and technology. The progress of Europe during the last four hundred years is often inextricably—or even causally—linked with the rapid growth of science and technology during the period. In the minds of some, scientific progress becomes a uniquely European phenomenon, which can be emulated by other nations only if they follow a specifically European path of scientific and social development.

Such a representation of societies outside the European cultural milieu raises a number of issues that are worth exploring, however briefly. First, these societies, many of them still in the grip of an intellectual dependence that is the legacy of European political domination, should ask themselves some questions. Was their indigenous scientific and technological base innovative and self-sufficient during their precolonial period? Case studies of India, China, and parts of Africa, contained, for example, in the work of Dharampal (1971), Needham (1954), and Van Sertima (1983) and summarized by Teresi (2002), seem to indicate the existence of scientific creativity and technological achievements long before the incursions of Europe into these areas. If this is so, we need to understand the dynamics of precolonial science and technology in these and other societies and to identify the material conditions that gave rise to these developments. This is essential if we are to see why modern science did not develop in these societies, only in Europe, and to find meaningful ways of adapting to present-day requirements the indigenous and technological forms that still remain.[2]

Second, there is the wider issue of who "makes" science and technology. In a material and nonelitist sense, each society, impelled by the pressures and demands of its environment, has found it necessary to create a scientific base to cater to its material requirements. Perceptions of what constitute the particular requirements of a society would vary according to time and place, but it would be wrong to argue that the capacity to "make" science and technology is a prerogative of one culture alone.

Third, if one attributes all significant historical developments in science and technology to Europe, then the rest of the world can impinge only marginally, either as an unchanging residual experience to be contrasted with the dynamism and creativity of Europe, or as a rationale for the creation of academic disciplines congealed in subjects such as development studies, anthropology, and oriental studies. These subjects in turn served as the basis from which more elaborate Eurocentric theories of social development and history were developed and tested.[3]

One of the more heartening aspects of academic research in the last four or five decades is that the shaky foundations of these "adjunct" disciplines are being increasingly exposed by scholars, a number of whom originate from countries that provide the subject matter of these disciplines. "Subversive" analyses aimed at nothing less than the unpackaging of prevailing Eurocentric paradigms became the major preoccupation of many of these scholars. Syed Husain Alatas (1976) studied intellectual dependence and imitative thinking among social scientists in developing countries. The growing movement toward promoting a form of indigenous anthropology that sees its primary task as questioning, redefining, and if necessary rejecting particular concepts that grew out of colonial experience in Western anthropology is thoroughly examined by Fahim (1982). Edward Said (1978) has brilliantly described the motives and methods of the so-called orientalists who set out to construct a fictitious entity called "the Orient" and then ascribe to it qualities that are a mixture of the exotic, the mysterious, and the otherworldly. The rationale for such constructs is being examined in terms of the recent history of Europe's relations with the rest of the world.

In a similar vein, and in the earlier editions of this book, it was the intention to show that the standard treatment of the history of non-European mathematics exhibited a deep-rooted historiographical bias in the selection and interpretation of facts, and that mathematical activity outside Europe has as a consequence been ignored, devalued, or distorted. It is interesting in this context that since the first edition of this book there has been a growing recognition of the mathematics outside the European and Greek traditions, especially in the mainstream teaching of the history of mathematics. The Eurocentric argument has shifted its ground and now questions both the nature of the European debt to other mathematical traditions and the existence and quality of proofs and demonstrations in traditions outside Europe. A brief discussion of the shifting ground of Eurocentrism in the history of mathematics is found in the preface to this edition.[4]

The Development of Mathematical Knowledge

A concise and meaningful definition of mathematics is difficult. In the context of this book, the following aspects of the subject are highlighted. Modern mathematics has developed into a worldwide language with a particular kind of logical structure. It contains a body of knowledge relating to number and space, and prescribes a set of methods for reaching conclusions about the physical world. And it is an intellectual activity which calls for both intuition and imagination in deriving "proofs" and reaching conclusions. Often it rewards the creator with a strong sense of aesthetic satisfaction.

The "Classical" Eurocentric Trajectory

Most histories of mathematics that have had a great influence on later work were written in the late nineteenth or early twentieth century. During that period, two contrasting developments were taking place that had an impact on both the content and the balance of these books, especially those produced in Britain and the United States. Exciting discoveries of ancient mathematics on papyri in Egypt and clay tablets in Mesopotamia pushed back the origins of written mathematical records by at least fifteen hundred years. But a far stronger and countervailing influence was the culmination of European domination in the shape of political control of vast tracts of Africa and Asia. Out of this domination arose the ideology of European superiority that permeated a wide range of social and economic activities, with traces to be found in histories of science that emphasized the unique role of Europe in providing the soil and spirit for scientific discovery. The contributions of the colonized peoples were ignored or devalued as part of the rationale for subjugation and dominance. And the development of mathematics before the Greeks—notably in Egypt and Mesopotamia—suffered a similar fate, dismissed as of little importance to the later history of the subject. In his book *Black Athena* (1987), Martin Bernal has shown how respect for ancient Egyptian science and civilization, shared by ancient Greece and pre-nineteenth-century Europe alike, was gradually eroded, leading eventually to a Eurocentric model with Greece as the source and Europe as the inheritor and guardian of the Greek heritage.

Figure 1.1 presents the "classical" Eurocentric view of how mathematics developed over the ages. This development is seen as taking place in two sections, separated by a period of stagnation lasting for over a thousand years: Greece (from about 600 BC to AD 400), and post-Renaissance Europe from the sixteenth century to the present day. The intervening period of inactivity was the "Dark Ages"—a convenient label that expressed both post-Renaissance Europe's prejudices about its immediate past and the intellectual self-confidence of those who saw themselves as the true inheritors of the "Greek miracle" of two thousand years earlier.

Two passages, one by a well-known historian of mathematics writing at the turn of the century and the other by a more recent writer whose books are still referred to on both sides of the Atlantic, show the durability of this Eurocentric view and its imperviousness to new evidence and sources:

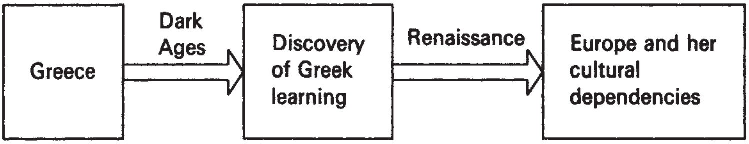

FIGURE 1.1: The "classical" Eurocentric trajectory

> The history of mathematics cannot with certainty be traced back to any school or period before that of the Ionian Greeks. (Rouse Ball 1908, p. 1)

> [Mathematics] finally secured a new grip on life in the highly congenial soil of Greece and waxed strongly for a short period. . . . With the decline of Greek civilization the plant remained dormant for a thousand years . . . when the plant was transported to Europe proper and once more imbedded in fertile soil. (Kline 1953, pp. 9–10)

The first statement is a reasonable summary of what was popularly known and accepted as the origins of mathematics at that time, except for the neglect of the early Indian mathematics contained in the *Sulbasutras* (The Rules of the Cord), belonging to the period between 800 and 500 BC, which would make it at least as old as the earliest-known Greek mathematics. Thibaut's translations of these works, made around 1875, were known to historians of mathematics at the turn of the century. The mathematics contained in the *Sulbasutras* is discussed in chapter 8.

The second statement, however, ignores a considerable body of research evidence pointing to the development of mathematics in Mesopotamia, Egypt, China, pre-Columbian America, India, and the Islamic world that had come to light in the intervening period. Subsequent chapters will bear testimony to the volume and quality of the mathematics developed in these areas. But in both these quotations mathematics is perceived as an exclusive product of European civilization. And that is the central message of the Eurocentric trajectory depicted in figure 1.1.

This comforting rationale for European dominance became increasingly untenable for a number of reasons. First, there is the full acknowledgment given by the ancient Greeks themselves of the intellectual debt they owed the Egyptians. There are scattered references from Herodotus (c. 450 BC) to Proclus (c. AD 400) of the knowledge acquired from the Egyptians in fields such as astronomy, mathematics, and surveying, while other commentators even considered the priests of Memphis to be the true founders of science.

To Aristotle (c. 350 BC), Egypt was the cradle of mathematics. His teacher, Eudoxus, one of the notable mathematicians of the time, had studied in Egypt before teaching in Greece. Even earlier, Thales (d. 546 BC), the legendary founder of Greek mathematics, and Pythagoras (c. 500 BC), one of the earliest and greatest of Greek mathematicians, were reported to have traveled widely in Egypt and Mesopotamia and learned much of their mathematics from these areas. Some sources even credit Pythagoras with having traveled as far as India in search of knowledge, which could explain some of the parallels between Indian and Pythagorean philosophy and religion.[5]

A second reason why the trajectory depicted in figure 1.1 was found to be wanting arose from the combined efforts of archaeologists, translators, and interpreters, who between them unearthed evidence of a high level of mathematics practiced in Mesopotamia and in Egypt at the beginning of the second millennium, providing further confirmation of Greek reports. In particular, the Mesopotamians had invented a place-value number system, knew different methods of solving what today would be described as quadratic equations (methods that would not be improved upon until the sixteenth century AD), and understood (but had not proved) the relationship between the sides of a right-angled triangle that came to be known as the Pythagorean theorem.[6] Indeed, as we shall see in later chapters, this theorem was stated and demonstrated in different forms all over the world.

A four-thousand-year-old clay tablet, kept in a Berlin museum, gives the value of $n^3 + n^2$ for $n = 1, 2, \ldots, 10, 20, 30, 40, 50$, from which it has been surmised that the Mesopotamians may have used these values in solving cubic equations after reducing them to the form $x^3 + x^2 = c$. A remarkable solution in Egyptian geometry found in the Moscow Papyrus from the Middle Kingdom (c. 2000–1800 BC) follows from the correct use of the formula for the volume of a truncated square pyramid. These examples and other milestones will be discussed in the relevant chapters of this book.

The neglect of the Islamic contribution to the development of European intellectual life in general and mathematics in particular is another serious drawback of the "classical" view. The course of European cultural history and the history of European thought are inseparably tied up with the activities of Islamic scholars during the Middle Ages and their seminal contributions to mathematics, the natural sciences, medicine, and philosophy.[7]

In particular, we owe to the Islamic world in the field of mathematics the bringing together of the technique of measurement, evolved from its Egyptian roots to its final form in the hands of the Alexandrians, and the remarkable instrument of computation (our number system) that originated in India. These strands were supplemented by a systematic and consistent language of calculation that came to be known by its Arabic name, algebra. An acknowledgment of this debt in more recent books contrasts sharply with the neglect of other Islamic contributions to science.[8]

Finally, in discussing the Greek contribution, there is a need to recognize the differences between the Classical period of Greek civilization (i.e., from about 600 to 300 BC) and the post-Alexandrian period (i.e., from about 300 BC to AD 400). In early European scholarship, the Greeks of the ancient world were perceived as an ethnically homogeneous group, originating from areas that were mainly within the geographical boundaries of present-day Greece. It was part of the Eurocentric mythology that from the mainland of Europe had emerged a group of people who had created, virtually out of nothing, the most impressive civilization of ancient times. And from that civilization had emerged not only the cherished institutions of present-day Western culture but also the mainspring of modern science. The reality, however, is different and more complex. The term "Greek," when applied to times before the appearance of Alexander (356–323 BC), really refers to a number of independent city-states, often at war with one another but exhibiting close ethnic or cultural affinities and, above all, sharing a common language. The conquests of Alexander changed the situation dramatically, for at his death his empire was divided among his generals, who established separate dynasties. The two notable dynasties from the point of view of mathematics were the Ptolemaic dynasty of Egypt and the Seleucid dynasty, which ruled over territories that included the earlier sites of the Mesopotamian civilization. The most famous center of learning and trade became Alexandria in Egypt, established in 332 BC and named after the conqueror. From its foundation, one of its most striking features was its cosmopolitanism—part Egyptian, part Greek, with a liberal sprinkling of Jews, Persians, Phoenicians, and Babylonians, and even attracting scholars and traders from as far away as India. A lively contact was maintained with the Seleucid dynasty. Alexandria thus became the meeting place for ideas and different traditions. The character of Greek mathematics began to change slowly, mainly as a result of continuing cross-fertilization between different mathematical traditions, notably the algebraic and empirical basis of Mesopotamian and Egyptian mathematics interacting with the geometric and antiempirical traditions of Greek mathematics. And from this mixture came some of the greatest mathematicians of antiquity, notably Archimedes and Diophantus. It is therefore important to recognize the Alexandrian dimension to Greek mathematics while noting that Greek intellectual and cultural traditions served as the main inspiration and the Greek language as the medium of instruction and writing in Alexandria. In a later chapter, based on some innovative work of Friberg (2005, 2007), we will examine the close and hitherto unexamined links between Egyptian, Mesopotamian, and Greek mathematics.

A Modified Eurocentric Trajectory

Figure 1.2 takes on board some of the objections raised about the "classical" Eurocentric trajectory. The figure acknowledges that there is growing awareness of the existence of mathematics before the Greeks, and of their debt to earlier mathematical traditions, notably those of Mesopotamia and Egypt. But this awareness was until recently tempered by a dismissive rejection of their importance in relation to Greek mathematics: the "scrawling of children just learning to write as opposed to great literature" (Kline 1962, p. 14).

The differences in character of the Greek contribution before and after Alexander are also recognized to a limited extent in figure 1.2 by the separation of Greece from the Hellenistic world (in which the Ptolemaic and Seleucid dynasties became the crucial instruments of mathematical creation). There is also some acknowledgment of the "Arabs" but mainly as custodians of Greek learning during the Dark Ages in Europe. The role of the Islamic world as transmitter and creator of knowledge is often ignored; so are the contributions of other civilizations—notably China and India—which have been perceived either as borrowers from Greek sources or as having made only minor contributions that played an insignificant role in mainstream mathematical development (i.e., the development eventually culminating in modern mathematics).

Figure 1.2 is therefore still a flawed representation of how mathematics developed: it contains a series of biases and remains quite impervious to new evidence and arguments. Until a couple of decades ago, and with minor modifications, it was the model to which a number of books on the history of mathematics conformed. But this has changed even during the twenty-odd years that this book has been in print. "Mainsteam" histories of mathematics are casting a wider net by seriously considering the contributions not only

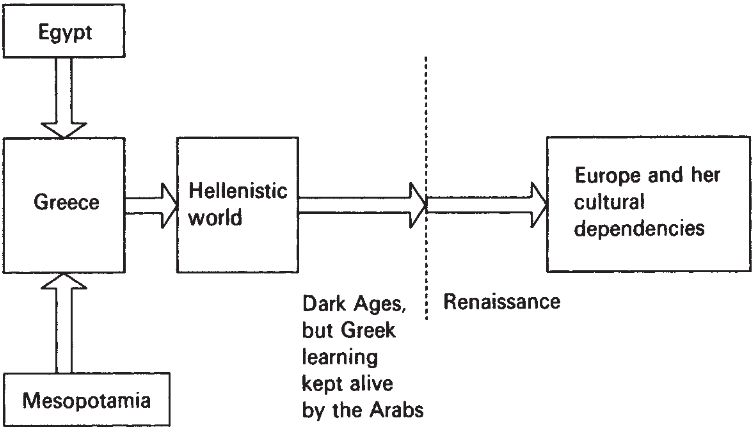

FIGURE 1.2: A modified Eurocentric trajectory

of ancient Egyptian and Mesopotamian civilizations; they are punctilious in incorporating as well the contributions of Chinese, Indian, and Islamic civilizations. The recent sourcebook for the histories of mathematics of the five civilizations edited by Katz (2007) is a testimony to this change.

It is interesting that a similar Eurocentric bias had existed in other disciplines as well: for example, diffusion theories in anthropology and social geography implied that "civilization" has spread from the center ("greater" Europe) to the periphery (the rest of the world). And the theories of globalization or evolution developed in recent years within some Marxist and neo-Marxist frameworks were characterized by a similar type of Eurocentrism. In all such conceptual schemes, the development of Europe is seen as a precedent for the way in which the rest of the world will follow—a trajectory whose spirit is not dissimilar to the one suggested by figures 1.1 and 1.2.

An Alternative Trajectory for the Dark Ages

If we are to construct an unbiased alternative to figures 1.1 and 1.2, our guiding principle should be to recognize that different cultures in different periods of history have contributed to the world's stock of mathematical knowledge. Figure 1.3 presents such a trajectory of mathematical development but confines itself to the period between the fifth and fifteenth centuries AD—the period represented by the arrow labeled in figures 1.1 and 1.2 as the "Dark Ages" in Europe. The choice of this trajectory as an illustration is deliberate: it serves to highlight the variety of mathematical

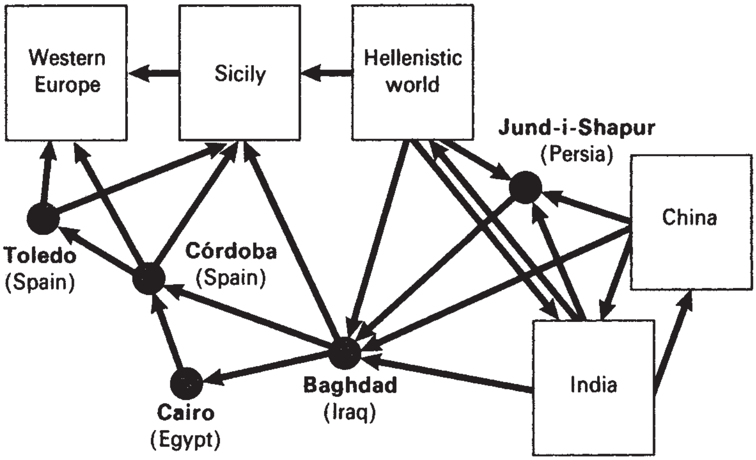

FIGURE 1.3: An alternative trajectory for the "Dark Ages"

activity and exchange between a number of cultural areas that went on while Europe was in deep slumber. A trajectory for the fifteenth century onward would show that mathematical cross-fertilization and creativity were more or less confined to countries within Europe until the emergence of the truly international character of modern mathematics during the twentieth century.

The role of the Islamic civilization is brought out in figure 1.3. Scientific knowledge that originated in India, China, and the Hellenistic world was sought out by Islamic scholars and then translated, refined, synthesized, and augmented at different centers of learning, starting at Jund-i-Shapur[9] in Persia around the sixth century (even before the coming of Islam) and then moving to Baghdad, Cairo, and finally to Toledo and Córdoba in Spain, from where this knowledge spread into western Europe. Considerable resources were made available to the scholars through the benevolent patronage of the caliphs, the Abbasids (the rulers of the eastern Arab empire, with its capital at Baghdad) and the Umayyads (the rulers of the western Arab empire, with its capital first at Damascus and later at Córdoba).

The role of the Abbasid caliphate was particularly important for the future development of mathematics. The caliphs, notably al-Mansur (754–775), Harun al-Rashid (786–809), and al-Mamun (809–833), were in the forefront of promoting the study of astronomy and mathematics in

Baghdad. Indian scientists were invited to Baghdad. When Plato's Academy was closed in 529, some of its scholars found refuge at Jund-i-Shapur, which a century later became part of the Islamic world. Greek manuscripts from the Byzantine empire, the translations of the Syriac schools of Antioch and Damascus, and the remains of the Alexandrian library in the hands of the Nestorian Christians at Edessa were all eagerly sought by Islamic scholars, aided by the rulers who had control over or access to men and materials from the Byzantine empire, Persia, Egypt, Mesopotamia, and places as far east as India and China.

Caliph al-Mansur built at Baghdad a Bait al-Hikma (House of Wisdom), which contained a large library for the manuscripts that had been collected from various sources; an observatory that became a meeting place of Indian, Babylonian, Hellenistic, and probably Chinese astronomical traditions; and a university where scientific research continued apace.[10] A notable member of the institution, Muhammad ibn Musa al-Khwarizmi (fl. AD 825), wrote two books that were of crucial importance to the future development of mathematics. One of them, the Arabic text of which is extant, is titled *Hisab al-jabr w'al-muqabala* (which may be loosely translated as Calculation by Reunion and Reduction). The title refers to the two main operations in solving equations: "reunion," the transfer of negative terms from one side of the equation to the other, and "reduction," the merging of like terms on the same side into a single term.[11] In the twelfth century the book was translated into Latin under the title *Liber algebrae et almucabola*, thus giving a name to a central area of mathematics. A traditional meaning of the Arabic word *jabr* is "the setting of a broken bone" (and hence "reunion" in the title of al-Khwarizmi's book). Some decades ago it was not an uncommon sight on Spanish streets to come across a sign advertising "Algebrista y Sangrador" (i.e., someone dedicated to setting dislocated bones) at the entrance of barbers' shops.[12]

Al-Khwarizmi wrote a second book, of which only a Latin translation is extant: *Algorithmi de numero indorum*, which explained the Indian number system. While al-Khwarizmi was at pains to point out the Indian origin of this number system, subsequent translations of the book attributed not only the book but the numerals to the author. Hence, in Europe any scheme using these numerals came to be known as an "algorism" or, later, "algorithm" (a corruption of the name al-Khwarizmi) and the numerals themselves as Arabic numerals.

Figure 1.3 shows the importance of two areas of southern Europe in the transmission of mathematical knowledge to western Europe. Spain and

Sicily were the nearest points of contact with Islamic science and had been under Arab hegemony, Córdoba succeeding Cairo as the center of learning during the ninth and tenth centuries. Scholars from different parts of western Europe congregated in Córdoba and Toledo in search of ancient and contemporary knowledge. It is reported that Gherardo of Cremona (c. 1114–1187) went to Toledo, after its recapture by the Christians, in search of Ptolemy's *Almagest*, an astronomical work of great importance produced in Alexandria during the second century AD. He was so taken by the intellectual activity there that he stayed for twenty years, during which time he was reported to have copied or translated eighty-seven manuscripts of Islamic science or Greek classics, which were later disseminated across western Europe. Gherardo was just one of a number of European scholars, including Plato of Tivoli, Adelard of Bath, and Robert of Chester, who flocked to Spain in search of knowledge.[13]

The main message of figure 1.3 is that it is dangerous to characterize the history of mathematics solely in terms of European developments. The darkness that was supposed to have descended over Europe for a thousand years before the illumination that came with the Renaissance did not interrupt mathematical activity elsewhere. Indeed, as we shall see in later chapters, the period saw not only a mathematical renaissance in the Islamic world but also high points of Indian and Chinese mathematics.

Mathematical Signposts and Transmissions across the Ages

Alternative trajectories to the ones shown in figures 1.1 and 1.2 should highlight the following three features of the plurality of mathematical development:

1. The global nature of mathematical pursuits of one kind or another
2. The possibility of independent mathematical development within each cultural tradition followed or not followed by cross-fertilization
3. The crucial importance of diverse transmissions of mathematics across cultures, culminating in the creation of the unified discipline of modern mathematics

However, to construct a feasible diagram we must limit the number of geographical areas of mathematical activity we wish to include. Selection inevitably introduces an element of arbitrariness, for some areas that may merit inclusion are excluded, while certain inclusions may be controversial. Two considerations have influenced the choice of the cultural areas represented in figure 1.4. First, a judgment was made, on the basis of existing evidence, as to which places saw significant developments in mathematics. Second, an assessment of the nature and direction of the transmission of mathematical knowledge also helped to identify the areas of interest.

On the basis of these two criteria, ancient Egypt and Mesopotamia, Greece (and the Hellenistic world), India, China, the Islamic (or "Arab") world, and Europe were selected as being important in the historical development of mathematics. For one cultural area, the application of the two selection criteria produced conflicting results: from existing evidence, the Maya of Central America were isolated from other centers of mathematical activity, yet their achievements in numeration and calendar construction were quite remarkable by any standards. I therefore decided to include the Maya in figure 1.4, and to examine their contributions briefly in chapter 2.

The limited scope of this book and the application of the above criteria make it impossible to examine the mathematical experience of Africa, Korea, and Japan in greater detail. However, chapter 2 contains a discussion of the Ishango bone and the Yoruba numerals, and chapter 3 a detailed examination of Egyptian mathematics, all of which were products of Africa. Further information on the mathematical traditions of Korea and Japan is available in the second of the two chapters on Chinese mathematics (chapter 7), since these traditions were both heavily influenced by China.

Figure 1.4, together with its detailed legend, emphasizes the following features of mathematical activity through the ages:

1. The continuity of mathematical traditions until the last few centuries in most of the selected cultural areas
2. The extent of transmissions between different cultural areas that were geographically or otherwise separated from one another
3. The relative ineffectiveness of cultural barriers (or "filters") in inhibiting the transmission of mathematical knowledge (In a number of other areas of human knowledge, notably in philosophy and the arts, the barriers are often insurmountable unless filters can be devised to make foreign "products" more palatable.)

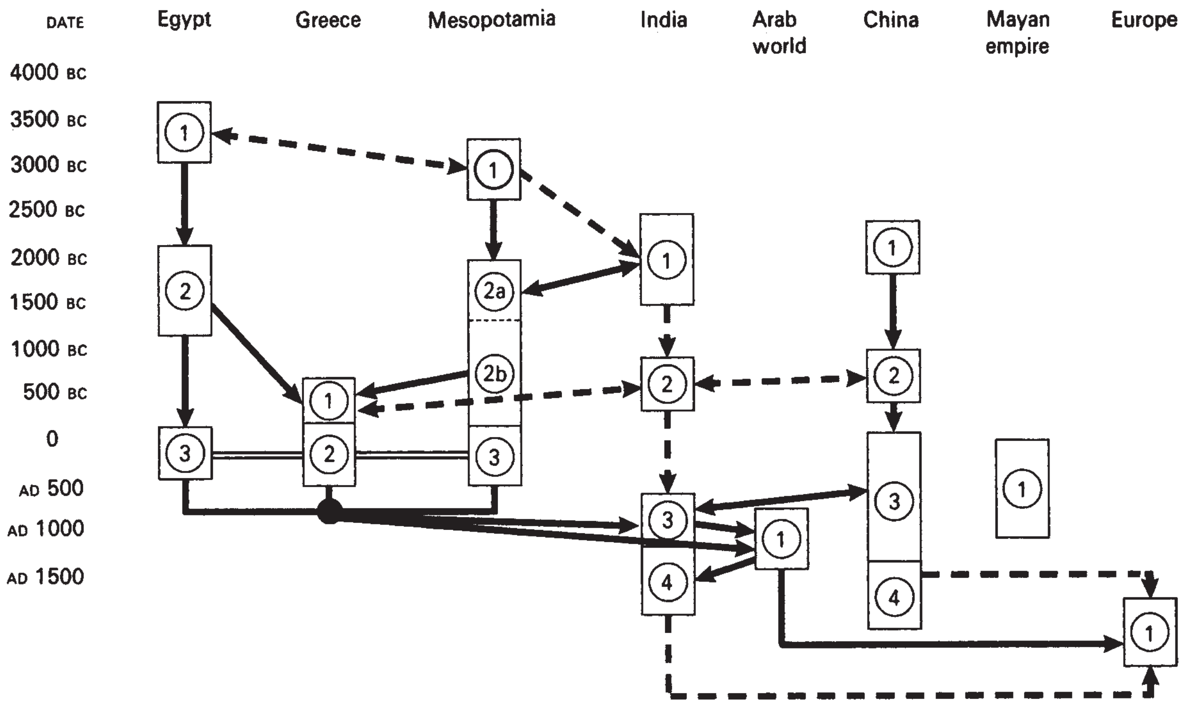

FIGURE 1.4: The spread of mathematical ideas down the ages

EGYPT 1 *Predynastic period*: Appearance of the earliest forms of writing and hieroglyphic numerals

2 *Middle Kingdom to New Kingdom*: Egyptian mathematics mainly contained in the Moscow and Ahmes papyri

3 *Greek and Roman period*: Flowering of mathematics at Alexandria

GREECE 1 *Classical period*: Beginnings of deductive geometry and number theory

2 *Hellenistic period*: Growing synthesis of classical, Egyptian, and Babylonian mathematics

MESOPOTAMIA 1 *Sumerian period*: Beginnings of cuneiform numerals

2a *First Babylonian dynasty*: Early algebra, commerical arithmetic, and geometry from clay tablets

2b *New Babylonian and Persian periods*: Mathematics and astronomy

3 *Seleucid dynasty*: Hellenistic mathematics

INDIA 1 *Harappan period*: Protomathematics from bricks, baths, etc.

2 *Vedic period*: Ritual geometry

3 *Classical period*: Indian numerals, computing algorithms, algebra, and trigonometry

4 *Medieval period*: Kerala mathematics

ARAB WORLD 1 Preservation and synthesis of mathematical traditions from different areas, laying the foundations of modern mathematics

CHINA 1 *River valley civilization*: Beginnings of practical mathematics

2 *Shang and Zhou dynasties*: Rod numerals, *Zhou Bi* (the earliest extant mathematics textbook)

3 *Han to Tang dynasties*: "Arithmetic in Nine Sections" (the most important text in Chinese mathematics)

4 *Song to Ming dynasties*: Golden age of Chinese mathematics

MYAN EMPIRE 1 Construction of a highly accurate calendar and the development of a place-value number system (base 20) with zero

EUROPE 1 Development of modern mathematics, building on mathematics from other sources

═══ Hellenistic cultural areas (Egypt, Greece, Mesopotamia)

←——→ *Confirmed lines of transmission (two-way)*

(I) Harappan culture *and* first Babylonian dynasty
(II) Han to Tang dynasties *and* classical period (India)

——→ *Confirmed lines of transmission (one-way)*

(I) Middle Kingdom to New Kingdom (Egypt) *to* classical period (Greece)
(II) New Babylonian and Persian periods *to* classical period (Greece)
(III) Hellenistic cultural areas *to* classical period (India)
(IV) Hellenistic cultural areas *to* Arab world
(V) Classical period (India) *to* Arab world
(VI) Arab world *to* Europe and India

←- - -→ *Unconfirmed or tentative lines of transmission (two-way)*

(I) Predynastic period (Egypt) *and* Sumerian period (Mesopotamia)
(II) Classical period (Greece) *and* Vedic period (India)
(III) Vedic period (India) *and* Shang and Zhou dynasties (China)

- - - → *Unconfirmed or tentative lines of transmission (one-way)*

(I) Sumerian culture *to* Harappan culture
(II) Song to Ming dynasties *to* Europe
(III) Kerala mathematics *to* Europe

In both Egypt and Mesopotamia there existed well-developed written number systems as early as the third millennium BC. The peculiar character of the Egyptian hieroglyphic numerals led to the creation of special types of algorithms for basic arithmetic operations. Both these developments and subsequent work in the area of algebra and geometry, especially during the period between 1800 and 1600 BC, will form the subject matter of chapter 3. Figure 1.4 brings out another impressive aspect of Egyptian mathematics—the continuity of a tradition for over three thousand years, culminating in the great period of Alexandrian mathematics around the beginning of the Christian era. We shall not examine the content and personalities of this mature phase of Egyptian mathematics in any detail, since its coverage in standard histories of mathematics is more than adequate. There is, however, a widespread tendency in many of these texts to view Alexandrian mathematics as a mere extension of Greek mathematics, in spite of the distinctive character of the mathematics of Archimedes, Heron, Diophantus, and Pappus, to mention a few notable names of the Alexandrian period.

The other early contributor to mathematics was the civilization that grew around the twin rivers, the Tigris and the Euphrates, in Mesopotamia. There mathematical activity flourished, given impetus by the establishment of a place-value sexagesimal (i.e., base 60) system of numerals, which must surely rank as one of the most significant developments in the history of mathematics. However, the golden period of mathematics in this area (or at least the period for which considerable written evidence exists) came during the First Babylonian period (c. 1800–1600 BC), which saw not only the introduction of further refinements to the existing numeral system but also the development of an algebra more advanced than that in use in Egypt. The period is so important that the mathematics that developed in Mesopotamia is often simply referred to as Babylonian mathematics. As with Egypt, the next period of significant advance followed Alexander's conquest and the establishment of the Seleucid dynasty. Babylonian mathematics (a term that will be used interchangeably with Mesopotamian hereafter to describe the mathematics of this cultural area) is discussed in chapter 4.

There is growing evidence of mathematical links between Egypt and Mesopotamia before the Hellenistic period, which we would expect, given their proximity and the records we have of their economic and political contacts. Earlier, Parker (1972) had examined the evidence for a spread of

Mesopotamian algebra and geometry to Egypt. He pointed out that certain parallel developments in both geometry and algebra provided at least some support for links between the two cultural areas. This has now been reinforced by Friberg (2005), who examined "Egyptian mathematics against an up-to-date background in the history of Mesopotamian mathematics." We will discuss Friberg's work in greater detail in chapter 5. However, given that there is more evidence than hitherto believed, we represent the contacts between Egypt and Mesopotamia by a two-headed arrow in figure 1.4.

There is also evidence of the great debt that Greece owed to Egypt and Mesopotamia for its earlier mathematics and astronomy. We have mentioned the acknowledgment of this debt by the Greeks themselves, who believed that mathematics originated in Egypt. The travels of the early Greek mathematicians such as Thales, Pythagoras, and Eudoxus to Egypt and Mesopotamia in search of knowledge have been attested to both by their contemporaries and by later historians writing on the period. The period of greatest Egyptian influence on the Greeks may have been the first half of the first millennium BC. The Greek colonies scattered across the Mediterranean provided a wide channel of interchange. It is at the time of their heyday that we hear of Anaximander of Miletus (610–546 BC) introducing the gnomon (a geometric shape of both mathematical and astronomical significance)[14] from Babylon. During the same period, contacts with the Greeks were maintained through the campaigns of the Assyrian king Sargon II (722–705 BC), and later through Ashurbanipal's occupation of Egypt and his meeting with Gyges of Lydia toward the middle of the seventh century BC. Even when Assyria ceased to exist, the Jewish captivity played a significant part in disseminating Babylonian learning. This was followed by the Persian invasion of Greece at the beginning of the fifth century and the final defeat of the Persians at the end of the fifth century. Thus continuous contacts were maintained throughout a period in which Greek mathematics was still in its infancy, as the foundations were being laid for the flowering of Greek creativity in a couple of centuries. In the next five hundred years, the pupil would learn and develop sufficiently to teach the teachers.

Adding to these historical conjectures, there is now stronger evidence of links between the mathematical traditions of Egypt, Mesopotamia, and Greece. In a recent book Friberg (2007b) has argued as a sequel to his earlier thesis (Friberg 2005) of "unexpected links between Egyptian and Babylonian mathematics" that there are "amazing traces of a Babylonian origin in Greek mathematics."[15] These "traces" (discussed in chapter 5) are found in the fact that several of the famous Greek mathematicians showed an easy familiarity with what Friberg describes as Babylonian "metric algebra," that is, a characteristic approach that combines geometry, metrology, and the solution of quadratic equations.

The transmissions to Greece from the two areas are shown in figure 1.4 by the arrows from 2 in Egypt and 2b in Mesopotamia to 1 in Greece. All three areas then became part of the Hellenistic world, and during the period between the third century BC and the third century AD, and partly due to the interaction between the three mathematical traditions, there emerged one of the most creative periods in mathematics. We usually associate this period with names such as Euclid, Archimedes, Apollonius and Diophantus. But if Friberg's thesis is sustained, there was a 'non-Euclidean lower level' of mathematics present in these traditions. These links are represented by the double lines between 3 in Egypt, 2 in Greece and 3 in Mesopotamia.

The geographical location of India made it throughout history an important meeting place of nations and cultures. This enabled India from the very beginning to play an important role in the transmission and diffusion of ideas. The traffic was often two-way, with Indian ideas and achievements traveling abroad as easily as those from outside entered. Archaeological evidence shows both cultural and commercial contacts between Mesopotamia and the Indus Valley. While there is no direct evidence of mathematical exchange between the two cultural areas, certain astronomical calculations of the longest and shortest day included in the *Vedanga Jyotisa*, the oldest extant Indian astronomical/astrological text, as well as the list of twenty-eight *nakshatras* found in the early Vedic texts, have close parallels with those used in Mesopotamia. And hence the tentative link, shown by broken lines in figure 1.4, between 1 in Mesopotamia and 1 in India.[16]

The relative seclusion that India had enjoyed for centuries was broken by the invasion of the Persians under Darius around 513 BC. In the ensuing six centuries, except for a century and a half of security under the Mauryan dynasty, India was subjected to incursions by the Greeks, the Sakas, the Pahlavas, and the Kusanas. Despite the turbulence, the period offered an opportunity for a close and productive contact between India and the West. Beginning with the appearance of the vast Persian empire, which touched Greece at one extremity and India at the other, tributes from Greece and from the frontier hills of India found their way to the same imperial treasure houses at Ecbatana or Susa. Soldiers from Mesopotamia, the Greek cities of Asia Minor, and India served in the same armies. The word *indoi* for Indians began to appear in Greek literature. Certain interesting parallels between Indian and Pythagorean philosophy have already been pointed out. Indeed, according to some Greek sources, Pythagoras had ventured as far afield as India in his search for knowledge.

By the time Ptolemaic Egypt and Rome's Eastern empire had established themselves just before the beginning of the Christian era, Indian civilization was already well developed, having founded three great religions—Hinduism, Buddhism, and Jainism—and expressed in writing some subtle currents of religious thought and speculation as well as fundamental theories in science and medicine. There are scattered references to Indian science in literary sources from countries to the west of India after the time of Alexander. The Greeks had a high regard for Indian "gymnosophists" (i.e., philosophers) and Indian medicine. Indeed, there are various expressions of nervousness about the Indian use of poison in warfare. In a letter to his pupil Alexander in India, Aristotle warns of the danger posed by intimacy with a "poison-maiden," who had been fed on poison from her infancy so that she could kill merely by her embrace!

There is little doubt that the Mesopotamian influence on Indian astronomy continued into the Hellenistic period, when the astronomy and mathematics of the Ptolemaic and Seleucid dynasties became important forces in Indian science, readily detectable in the corpus of astronomical works known as *Siddhantas*, written around the beginning of the Christian era. Evidence of such contacts (especially in the field of medicine) has been found in places such as Jund-i-Shapur in Persia dating from between AD 300 and 600. As mentioned earlier, Jund-i-Shapur was an important meeting place of scholars from a number of different areas, including Indians and, later, Greeks who sought refuge there with the demise of Alexandria as a center of learning and the closure of Plato's Academy. All such contacts are shown in figure 1.4 by lines linking 2 in India to 1 in Greece and 3 in India to the Hellenistic cultural areas.

By the second half of the first millennium AD, the most important contacts for the future development of mathematics were those between India and the Islamic world. This is shown by the arrow from 3 in India to 1 in the "Arab" world. As we saw in figure 1.3, the other major influence on the Islamic world was from the Greek cultural areas, and the nature of these influences has been discussed in some detail. As far as Indian influence via the Islamic world on the future development of mathematics is concerned, it is possible to identify three main areas:

1. The spread of Indian numerals and their associated algorithms, first to the Islamic world and later to Europe
2. The spread of Indian trigonometry,[17] especially the use of the sine function
3. The solutions of equations in general, and of indeterminate equations in particular[18]

These contributions will be discussed in chapters 8–10, which deal with Indian mathematics.

We have already looked briefly at the contributions of Islamic scholars as producers, transmitters, and custodians of mathematical learning. Their role as teachers of mathematics to Europe is not sufficiently acknowledged. The arrow from 1 in the "Arab" world to 1 in Europe represents the crucial role of the Islamic world in the creation and spread of mathematics, which culminated in the birth of modern mathematics. These contributions will be discussed in the final chapter of this book.

Figure 1.4 shows another important cross-cultural contact, between India and China. There is very fragmentary evidence (as shown by the broken line between 2 in India and 2 in China) of contacts between the two countries before the spread of Buddhism into China. After this, from around the first century AD, India became the center for pilgrimage of Chinese Buddhists, opening the way for a scientific and cultural exchange that lasted for several centuries. In a catalogue of publications during the Sui dynasty (c. 600), there appear Chinese translations of Indian works on astronomy, mathematics, and medicine. Records of the Tang dynasty indicate that from 600 onward Indian astronomers were hired by the Astronomical Board of Changan to teach the principles of Indian astronomy. The solution of indeterminate equations, using the method of *kuttaka* in India and of *qiuyishu* in China, was an abiding passion in both countries. The nature and direction of transmission of mathematical ideas between the two areas is a complex but interesting problem, one to which we shall return in later chapters. The two-headed arrow linking 3 in India with 3 in China is a recognition of the existence of such transmission. Also, there is some evidence of a direct transmission of mathematical (and astronomical) ideas between China and the Islamic world, around the beginning of the second millennium AD.[19] Numerical methods of solving equations of higher order such as quadratics and cubics, which attracted the interest of later Islamic mathematicians, notably al-Kashi (c. 1400), may have been influenced by Chinese work in this area. There is every likelihood that some of the important trigonometric concepts introduced into Chinese mathematics around this period may have an Islamic origin.

There are broken lines of transmission in figure 1.4 that need some explanation. One of the conjectures posed and elaborated in chapter 10 is the possibility that mathematics from medieval India, particularly from the southern state of Kerala, may have had an impact on European mathematics of the sixteenth and seventeenth centuries. While this cannot be substantiated at present by existing direct evidence, the circumstantial evidence has become much stronger as a result of some recent archival research. The fact remains that around the beginning of the fifteenth century Madhava of Kerala derived infinite series for p and for certain trigonometric functions, thereby contributing to the beginnings of mathematical analysis about 250 years before European mathematicians such as Leibniz, Newton, and Gregory were to arrive at the same results from their work on infinitesimal calculus. The possibility of medieval Indian mathematics influencing Europe is indicated by the arrow linking 4 in India with 1 in Europe.

During the medieval period in India, especially after the establishment of Mughal rule in North India, the Arab and Persian mathematical sources became better known there. From about the fifteenth century onward there were two independent mathematical developments taking place, one Sanskrit-derived and constituting the mainstream tradition of Indian mathematics, then best exemplified in the work of Kerala mathematicians in the South, and the other based in a number of Muslim schools (or *madrassahs*) located mainly in the North. We recognize this transmission by constructing an arrow linking 1 in the Arab world to 4 in India. A discussion of the flourishing mathematical tradition introduced into India during the medieval times, where the sources were Persian and Arabic texts, will be found in chapter 9.

The medieval period also saw a considerable transfer of technology and products from China to Europe, which has been thoroughly investigated by Lach (1965) and Needham (1954). The fifteenth and sixteenth centuries witnessed the culmination of a westward flow of technology from China that had started as early as the first century AD. It included, from the list given by Needham (1954, pp. 240–41), the square-pallet chain pump, metallurgical blowing engines operated by water power, the wheelbarrow, the sailing carriage, the wagon mill, the crossbow, the technique of deep drilling, the so-called Cardan suspension for a compass, the segmental arch-bridge, canal lock-gates, numerous inventions in ship construction (including watertight compartments and aerodynamically efficient sails), gunpowder, the magnetic compass for navigation, paper and printing, and porcelain. The conjecture here is that with the transfer of technology went certain mathematical ideas, including different algorithms for extracting square and cube roots, the "Chinese remainder theorem," solutions of cubic and higher-order equations by what is known as Horner's method, and indeterminate analysis. Such a transmission from China need not have been a direct one but may have taken place through India and the Islamic world. We shall return to the question of influences and transmission from China to the rest of the world in chapter 7.

During the first half of the first millennium of the Christian era, the Central American Mayan civilization attained great heights in a number of different fields including art, sculpture, architecture, mathematics, and astronomy. In the field of numeration, the Maya shared in two fundamental discoveries: the principle of place value and the use of zero. Present evidence indicates that the principle of place value was discovered independently four times in the history of mathematics. At the beginning of the second millennium BC, the Mesopotamians were working with a place-value notational system to base 60. Around the beginning of the Christian era, the Chinese were using positional principles in their rod numeral computations. Between the third and fifth centuries AD, Indian mathematicians and astronomers were using a place-value decimal system of numeration that would eventually be adopted by the whole world. And finally, the Maya—apparently cut off from the rest of the world—had developed a positional number system to base 20. As regards zero, there are only two original instances of its modern use in a number system: by the Maya and by the Indians around the beginning of the Christian era.

But mathematics is not the only area in which the Maya surprise us. With the most rudimentary instruments at their disposal they undertook astronomical observations and calendar construction with a precision that went beyond anything available in Europe at that time. They had accurate estimates of the duration of solar, lunar, and planetary movements. They estimated the synodic period of Venus (i.e., the time between one appearance at a given point in the sky and its next appearance at that point) to be 584 days, which is an underestimate of 0.08 days. They achieved these discoveries with no knowledge of glass or, consequently, of any sort of optical device. Neither did they apparently have any device for measuring the passage of time, such as clocks or sandglasses, without which it would now seem impossible to produce astronomical data.

Figure 1.5 shows the geographical areas whose mathematics form the subject matter of this book. I am conscious of not having examined in sufficient detail the mathematical pursuits of other groups, notably the Africans south of the Sahara, the Amerindians of North America, and the indigenous Australasians, although the topics treated in chapter 2 should go some way in making up for this neglect.[20] Much research still needs to be done on mathematical activities in these areas, despite some promising work on ethnomathematics in recent years, notably by Gerdes (1995, 1999, 2002) and Zaslavsky (1973a) on African mathematics.[21]

Since the publication of the first edition of this book in 1991, there has been an increase in interest in ethnomathematics, or the study of mathematical concepts in their cultural context, often within socially cohesive and small-scale indigenous groups. Within the definition of mathematics given earlier, the emphasis is on how structures and systems of ideas involving number, pattern, logic, and spatial configuration arose in different cultures. This view has had to contend with the strongly entrenched notion that mathematics, having originated in some primitive unformed state, advanced in a linear direction to the current state of modern mathematics and will continue to grow in that direction. A mathematical system that emerges in a culture removed from this "mainstream" would then be perceived as a mere distraction of little relevance to the ideas and activities supported by modern mathematics.

A telling criticism of the first edition of *The Crest of the Peacock* is that it implicitly subscribed to this "linear" view, being "epistemologically based on the idea of direct literal translations of non-western mathematics to the western tradition" (Eglash 1997, p. 79). In response to this criticism and in subsequent editions, the coverage has been extended to include areas in the Pacific and elaborate further on the mathematical activities in the African and American continents.

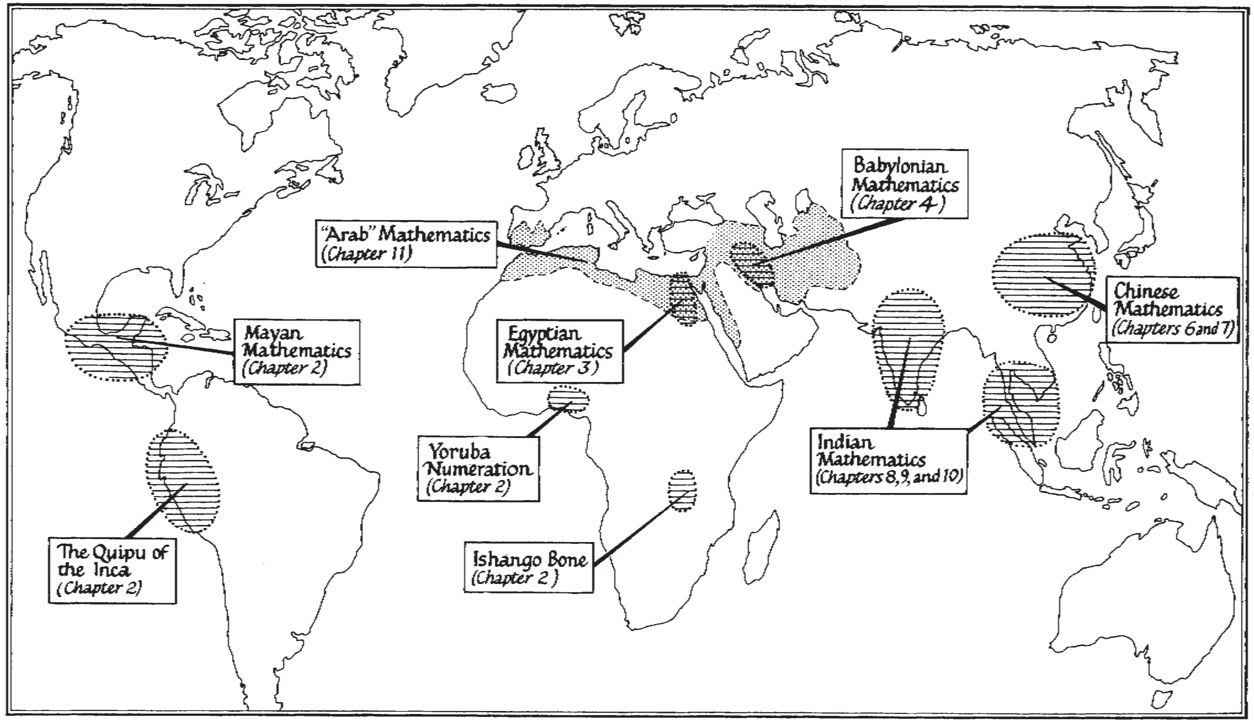

FIGURE 1.5: Cultures whose mathematics form the subject of this book

Notes

1. The term “cultural dependencies” is used here to describe those countries—notably the United States, Canada, Australia, and New Zealand—which are inhabited mainly by populations of European origin or which have historical and cultural roots similar to those of European peoples. For the sake of brevity, the term “Europe” is used hereafter to include these cultural dependencies as well.

2. This is a variation on the famous Needham question (named after Joseph Needham, the well-known twentieth-century British scientist and sinologist): Why did modern science develop in Europe when China with its momentous inventions like printing and gunpowder seemed so much better placed to achieve it? A similar question may be asked substituting instead of China the names of India or the Islamic world. For further discussion, see Bala (2006) and Bala and Joseph (2007).

3. See Brohman (1995a, 1995b) for further details.

4. The shift has occurred not only in the history of mathematics. The traditional Eurocentric world history presupposed the existence of an imaginary line of “civilizational apartheid” between the European and the non-European world whereby the former had single-handedly propelled the whole world from tradition into modernity while the latter remained stagnant. In recent years, spurred by a non-Eurocentric global history focusing on the historical resource portfolios (i.e., ideas, institutions, and technologies) diffused from the East across to the West, one discerns the emergence of what may be described as a neo-Eurocentric approach: one that acknowledges the borrowing of non-Western resources in the rise of the West but recasts Europe as “cosmopolitan, tolerant, open to others ideas, and highly adaptive insofar as it put all these non-Western sources together in a unique way to produce modernity.” I am grateful to John Hobson for making this point in a private communication. It follows logically from his book *The Eastern Origins of Western Civilisation* (2004).

5. These parallels include (a) a belief in the transmigration of souls; (b) the theory of four elements constituting matter; (c) the reasons for not eating beans; (d) the structure of the religio-philosophical character of the Pythagorean fraternity, which resembled Buddhist monastic orders; and (e) the contents of the mystical speculations of the Pythagorean schools, which bear a striking resemblance to the Hindu *Upanishad*s. According to Greek tradition, Pythagoras, Thales, Empedocles, Anaxagoras, Democritus, and others undertook journeys to the East to study philosophy and science. While it is far-fetched to assume that all these individuals reached India, there is a strong possibility that some of them became aware of Indian thought and science through Persia.

6. It is interesting to note that the terminology used in modern mathematics has a mixed origin consisting mainly of Greek, Latin, and modern European languages. The terms used in both Egyptian and Mesopotamian texts date back to the period before

the Greeks. Given the nature and scope of this book, we will continue to use modern terminology and avoid literal translations of the technical terms given in the ancient texts. Thus, for example, we use the modern term "triangle" (three angles) rather than the Babylonian term translated as "wedge" (three sides). The concept of an "angle" came only with the Greeks. A right-angled triangle in Old Babylonian mathematics had no angle connotation and was literally transliterated as one of two triangles into which a rectangle was divided by the longer diagonal. Similarly, although we use a modern term such as a "square" in the presentation of the ancient mathematics of Egypt and Mesopotamia, it should be noted that the corresponding term in these texts is "equal side" (or "same side").

7. In terms of historical, geographical, as well as intellectual proximity, Islamic science could be regarded as the most immediate predecessor of modern Western science. Some of the more recent studies (Bala 2006; Saliba 2007) show the existence of epistemological links between the two sciences. The "mathematicization" of nature, the centrality of the empirical method in scientific methodology, and the rationality of scientific discourses are features of Islamic science inherited by founders of modern Western science.

8. They include (a) an early description of pulmonary circulation of the blood, by ibn al-Nafis, usually attributed to Harvey, though there are records of an even earlier explanation in China; (b) the first known statement about the refraction of light, by ibn al-Hayatham, usually attributed to Newton; (c) the first known scientific discussion of gravity, by al-Khazin, again attributed to Newton; (d) the first clear statement of the idea of evolution, by ibn Miskawayh, usually attributed to Darwin; and (e) the first exposition of the rationale underlying the "scientific method," found in the works of ibn Sina, ibn al-Hayatham, and al-Biruni but usually credited to Roger Bacon. A general discussion of the Western debt to the Middle East is given by Savory (1976), while detailed references to specific contributions of Islamic science are given by Gillespie (1969–).

9. Jund-i-Shapur (or Guneshahpuhr) was founded around AD 260 by Shahpuhr I (241–272) to settle Roman prisoners captured in the war against Valerian and was located in Khuzistan in southwestern Iran. Early settlers included Roman engineers and physicians, and doubtless others who may have been acquainted with Greek, Egyptian, and Mesopotamian mathematics. The Christian bishop Demetrianus from Antioch founded a bishopric there, and during the fifth and sixth centuries Nestorianism was the only form of Christianity permitted in Iran. This intolerance contrasted with the openness and tolerance exhibited toward other religious immigrants, for when Zeno closed the School of the Persians in Edessa (AD 489), its intellectual and spiritual center moved to Persian Nisibis, where the exiles re-created their famous seat of learning. The Medical School of Jund-i-Shapur was founded on Greek medical knowledge (itself from Egyptian and Babylonian) by these Nestorian physicians. In the realms of philosophy, it is often forgotten that the Sasanian king Khusro I welcomed the major seven Neoplatonist Greek philosophers who fled Athens in 529 when the Academy there was

closed on the orders of the Byzantine Justinian. Some of these scholars worked for some time at Jund-i-Shapur but became homesick; Khusro negotiated their safe conduct and pardon for their return to Athens. Indeed, it was said of the enlightened Khusro that he was "a disciple of Plato seated on the Persian throne." The Jund-i-Shapur Medical School remained a center of excellence right through to Islamic times and indeed well past the mid-ninth century. While there are no extant records relating to mathematical activities in Jund-i-Shapur, we have evidence to indicate that during the reign of Shahpuhr I and later Khusro I, translations into Middle Persian (Pahlavi) were made in Iran from Greek and Sanskrit texts. It is more than likely that these included texts in astronomy, mathematics, and other sciences. After the downfall of the Sasanians, the Islamic regimes of the caliphs were by turns favorable or otherwise to the ancient learning enshrined at Jund-i-Shapur. Either way, Islamic knowledge was vastly increased through such deep and enduring exchanges.

10. This familiar story (or even some believe a caricature) about the role played by the House of Wisdom is now being reassessed. For further details see Gutas (1998) and Saliba (2007). See also endnote 2 of chapter 11.

11. But see the comment and reference given in endnote 24 of chapter 11 for further clarification.

12. A Spanish dictionary gives the following meanings: *álgebra*. 1. f. Parte de las matemáticas en la cual las operaciones aritméticas son generalizadas empleando números, letras y signos. 2. f. desus. Arte de restituir a su lugar los huesos dislocados (translation: the art of restoring broken bones to their correct positions).

13. For further details of these transmissions, see Zaimeche (2003, p. 10).

14. *Gnomon* is an ancient Greek word meaning "indicator" or "that which reveals." There are references to the gnomon in other traditions, for example, the seminal Chinese text *Nine Chapters on the Mathematical Art,* and it was referred to earlier by the Duke of Zhou (eleventh century BC). "Gnomon" also refers to the triangular part of a sundial that casts the shadow.

15. In the concluding paragraph Friberg (2005, p. 270) writes: "The observation that Greek *ostraca* [i.e., limestone chippings and pottery used as writing material] and papyri with Euclidean mathematics existed side by side with demotic and Greek papyri with Babylonian style mathematics is important for the reason that this surprising circumstance is an indication that when the Greeks themselves claimed that they got their mathematics from Egypt, they can really have meant that they got their mathematical inspiration from Egyptian texts with the mathematics of the Babylonian type. To make this thought more explicit would be a natural continuation of the present investigation." Friberg (2007) is the continuation of the investigation alluded to and provides the material for the Greek links with the two earlier civilizations.

16. In the case of Indian astronomy and the mathematics associated with it, the early influences from Mesopotamia came through the mediation of the Greeks. Probably in the fifth century BC, India acquired Babylonian astronomical period relations and arithmetic (e.g., representing continuously changing quantities with "zigzag" functions). Around the early centuries AD, the Babylonian arithmetical procedures were combined with Greek geometrical methods to determine solar and lunar positions, as reported in the Indian astronomical treatises *Romaka-siddhanta* and *Paulisa-siddhanta*. For further details, see Pingree (1981).

17. Since this is the first time we use the term "trigonometry," a word of caution is necessary. Trigonometry (meaning "triangle measurement") is a relatively modern term dating back to the sixteenth century. While today we have difficulty disentangling the concept of trigonometry from the ratio of sides in a right-angled triangle, for a long period of history the concept related only to circles and their arcs. And this was particularly so for the Greeks and the Indians. It was a search for a measure of the angle (or the inclination) of one line to another, an interest (and ability) to estimate the lengths of line segments, and a "systematic ability to convert back and forth between measures of angles and of lengths" that gave rise to modern trigonometry. I am grateful to Van Brummelen (2008) for this insight.

18. An example of an indeterminate equation in two unknowns (x and y) is $3x + 4y = 50$, which has a number of positive whole-number (or integer) solutions for (x, y). For example, $x = 14$, $y = 2$ satisfies the equation, as do the solution sets (10, 5), (6, 8) and (2, 11).

19. An exchange of astronomical knowledge took place between the Islamic world and the Yuan dynasty in China in the latter part of the thirteenth century, when both territories were part of the Mongol empire. A few Chinese astronomers were employed at the observatory in Maragha (set up by Hulegu Khan in 1258) and probably helped in the construction of the Chinese-Uighur calendar (a type of a lunisolar calendar or a calendar whose date indicates both the phase of the moon and the time of the solar year). This calendar was widely used in Iran from the late thirteenth century onward. There were at least ten Islamic astronomers working in the Islamic Astronomical Bureau in Beijing founded by the first Mongol emperor of China, Kublai Khan, in 1271. At this bureau, continuous observations were made and a *zij* (or astronomical handbook with tables) was compiled in Persian. This work was then translated into Chinese during the early Ming dynasty (1383) and, together with Kushayar's influential Islamic text, *Introduction to Astrology*, served for a number of years as important sources for further research and study by Chinese scholars. For further details, see van Dalen (2002).

20. It could be argued that in the examples discussed in chapter 2 there is undue emphasis on the role of number systems and insufficient attention paid to what Gerdes (1995) describes as "frozen geometry." These would include geometric or logical relationships embedded in diverse activities such as basket weaving, knitting, and sand

drawings highlighted by scholars such as Gerdes (1999) and Harris (1997). The problem in including such ethnomathematical activities is partly one of determining their historical origins and partly one of deciding what are to be included/excluded given the scope of this book.

21. The burgeoning study of African mathematics in recent years has highlighted a variety of mathematics that goes under the blanket term "ethnomathematics." Rambane and Mashige (2007, 184–85) have constructed the following list, with references to those who have worked in these areas.

1. *Oral mathematics*. The mathematical knowledge that is transmitted orally from one generation to another.
2. *Oppressed mathematics*. The mathematical elements in daily life that remain unrecognized by the dominant (colonial and neocolonial) ideologies (Gerdes 1985b).
3. *Indigenous mathematics.* A mathematical curriculum that uses everyday indigenous mathematics as the starting point. The origin of this concept is found in Gay and Cole (1967), who criticized the teaching of Kpelle children in Liberia in Western-oriented schools "things that have no point or meaning within their culture."
4. *Sociomathematics of Africa*. "The applications of mathematics in the lives of African people, and, conversely, the influence that African institutions had upon the evolution of their mathematics" (Zaslavsky 1973b, 1991).
5. *Informal mathematics*. Mathematics that is transmitted and learned outside the formal system of education, sometimes referred to as "street mathematics" (Posner 1982; Nunes et al. 1993).
6. *Nonstandard mathematics.* A distinctive mathematics beyond the standard form, found outside the school and university (Gerdes 1985b).
7. *Hidden* or *frozen mathematics*. Mathematics that has to be unfrozen from "hidden" or frozen objects or techniques, such as basket making, weaving, or traditional architecture (Gerdes 1985b).