Chapter One

Type A Weyl Group Multiple Dirichlet Series

We begin by defining the basic shape of the class of Weyl group multiple Dirichlet series. To do so, we choose the following parameters.

- Φ, a reduced root system. Let r denote the rank of Φ.
- n, a positive integer,
- F, an algebraic number field containing the group μ_{2n} of $2n$-th roots of unity,
- S, a finite set of places of F containing all the archimedean places, all places ramified over $\mathbb{Q}$, and large enough so that the ring
$$\mathfrak{o}_S = \{x \in F \mid |x|_v \leqslant 1 \text{ for } v \notin S\}$$
of S-integers is a principal ideal domain,
- $\boldsymbol{m} = (m_1, \cdots, m_r)$, an r-tuple of nonzero S-integers.

We may embed F and $\mathfrak{o}_S$ into $F_S = \prod_{v \in S} F_v$ along the diagonal. Let $(d, c)_{n,S}$ denote the S-Hilbert symbol, the product of local Hilbert symbols $(d, c)_{n,v} \in \mu_n$ at each place $v \in S$, defined for $c, d \in F_S^\times$. Let $\Psi : (F_S^\times)^r \longrightarrow \mathbb{C}$ be any function satisfying

$$\Psi(\varepsilon_1 c_1, \cdots, \varepsilon_r c_r) = \prod_{i=1}^{r} (\varepsilon_i, c_i)_{n,S} \prod_{1 \leq j < k \leq r} (\varepsilon_j, c_k)_{n,S}^{-1} \Psi(c_1, \cdots, c_r) \tag{1.1}$$

for any $\varepsilon_1, \cdots, \varepsilon_r \in \mathfrak{o}_S^\times (F_S^{\times, n})$ and $c_1, \cdots, c_r \in F_S^\times$. Here $(F_S^{\times, n})$ denotes the set of n-th powers in $F_S^\times$. It is proved in [12] that the set $\mathcal{M}$ of such functions is a finite-dimensional (nonzero) vector space.

To any such function Ψ and data chosen as above, Weyl group multiple Dirichlet series are functions of r complex variables $\boldsymbol{s} = (s_1, \cdots, s_r) \in \mathbb{C}^r$ of the form

$$Z_\Psi^{(n)}(\boldsymbol{s}; \boldsymbol{m}; \Phi) = Z_\Psi(\boldsymbol{s}; \boldsymbol{m}) = \sum_{\substack{\boldsymbol{c} = (c_1, \cdots, c_r) \in (\mathfrak{o}_S / \mathfrak{o}_S^\times)^r \\ c_i \neq 0}} \frac{H(\boldsymbol{c}; \boldsymbol{m}) \Psi(\boldsymbol{c})}{\mathbb{N}c_1^{2s_1} \cdots \mathbb{N}c_r^{2s_r}}, \tag{1.2}$$

where $\mathbb{N}c$ is the cardinality of $\mathfrak{o}_S / c\mathfrak{o}_S$, and it remains to define the coefficients $H(\boldsymbol{c}; \boldsymbol{m})$ in the Dirichlet series. In particular, the function Ψ is not independent of the choice of representatives in $\mathfrak{o}_S / \mathfrak{o}_S^\times$, so the function H must possess complementary transformation properties for the sum to be well-defined.

Indeed, the function H satisfies a "twisted multiplicativity" in $\boldsymbol{c}$, expressed in terms of n-th power residue symbols and depending on the root system Φ, which specializes to the usual multiplicativity when $n = 1$. Recall that the n-th power residue symbol $\left(\frac{c}{d}\right)_n$ is defined when c and d are coprime elements of $\mathfrak{o}_S$ and $\gcd(n, d) = 1$. It depends only on c modulo d, and satisfies the reciprocity law

$$\left(\frac{c}{d}\right)_n = (d, c)_{n,S} \left(\frac{d}{c}\right)_n .$$

(The properties of the power residue symbol and associated S-Hilbert symbols in our notation are set out in [12].) Then given $\boldsymbol{c} = (c_1, \cdots, c_r)$ and $\boldsymbol{c}' = (c'_1, \cdots, c'_r)$ in $\mathfrak{o}_S^r$ with $\gcd(c_1 \cdots c_r, c'_1 \cdots c'_r) = 1$, the function H satisfies

$$\frac{H(c_1c'_1, \cdots, c_rc'_r; \boldsymbol{m})}{H(\boldsymbol{c}; \boldsymbol{m})\, H(\boldsymbol{c}'; \boldsymbol{m})} = \prod_{i=1}^{r} \left(\frac{c_i}{c'_i}\right)_n^{||\alpha_i||^2} \left(\frac{c'_i}{c_i}\right)_n^{||\alpha_i||^2} \prod_{i<j} \left(\frac{c_i}{c'_j}\right)_n^{2\langle\alpha_i,\alpha_j\rangle} \left(\frac{c'_i}{c_j}\right)_n^{2\langle\alpha_i,\alpha_j\rangle}, \tag{1.3}$$

where α_i, $i = 1, \cdots, r$ denote the simple roots of Φ and we have chosen a Weyl group invariant inner product $\langle \cdot, \cdot \rangle$ for our root system embedded into a real vector space of dimension r. The inner product should be normalized so that for any $\alpha, \beta \in \Phi$, both $||\alpha||^2 = \langle \alpha, \alpha \rangle$ and $2\langle \alpha, \beta \rangle$ are integers. We will devote the majority of our attention to Φ of Type A, in which case we will assume the inner product is chosen so that all roots have length 1.

The function H possesses a further twisted multiplicativity with respect to the parameter $\boldsymbol{m}$. Given any

$$\boldsymbol{c} = (c_1, \cdots, c_r), \qquad \boldsymbol{m} = (m_1, \cdots, m_r), \qquad \boldsymbol{m}' = (m'_1, \cdots, m'_r)$$

with $\gcd(m'_1 \cdots m'_r, c_1 \cdots c_r) = 1$, H satisfies the twisted multiplicativity relation

$$H(\boldsymbol{c}; m_1m'_1, \cdots, m_rm'_r) = \left(\frac{m'_1}{c_1}\right)_n^{-||\alpha_1||^2} \cdots \left(\frac{m'_r}{c_r}\right)_n^{-||\alpha_r||^2} H(\boldsymbol{c}; \boldsymbol{m}). \tag{1.4}$$

As a consequence of properties (1.3) and (1.4) the specification of H reduces to the case where the components of $\boldsymbol{c}$ and $\boldsymbol{m}$ are all powers of the same prime. Given a fixed prime p of $\mathfrak{o}_S$ and any $\boldsymbol{m} = (m_1, \cdots, m_r)$, let $l_i = \mathrm{ord}_p(m_i)$. Then we must specify $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ for any r-tuple of nonnegative integers $\boldsymbol{k} = (k_1, \cdots, k_r)$. For brevity, we will refer to these coefficients as the "p-part" of H. To summarize, specifying a multiple Dirichlet series $Z_\Psi^{(n)}(\boldsymbol{s}; \boldsymbol{m}; \Phi)$ with chosen data is equivalent to specifying the p-parts of H.

Remark. Both the transformation property of Ψ in (1.1) and the definition of twisted multiplicativity in (1.3) depend on an enumeration of the simple roots of Φ. However, the product $H \cdot \Psi$ is independent of this enumeration of roots and furthermore well-defined modulo units, according to the reciprocity law. The p-parts of H are also independent of this enumeration of roots.

The definitions given above apply to any root system Φ. In most of this text, we will take Φ to be of Type A. In this case we will give two combinatorial definitions of the p-part of H. These two definitions of H will be referred to as H_Γ and H_Δ, and eventually shown to be equal. Thus either may be used to define the multiple Dirichlet series $Z(\boldsymbol{s}; \boldsymbol{m}; A_r)$. Both definitions will be given in terms of Gelfand-Tsetlin patterns.

By a *Gelfand-Tsetlin pattern of rank* r we mean an array of integers

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} a_{00} & & a_{01} & & a_{02} & & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & & a_{1r} & \\ & & \ddots & & & & \iddots & & \\ & & & & a_{rr} & & & & \end{array} \right\} \tag{1.5}$$

where the rows interleave; that is, $a_{i-1,j-1} \geqslant a_{i,j} \geqslant a_{i-1,j}$. Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a dominant integral element for SL_{r+1}, so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$. In the next chapter, we will explain why Gelfand-Tsetlin patterns with top row $(\lambda_1, \cdots, \lambda_r, 0)$ are in bijection with basis vectors for the highest weight module for $SL_{r+1}(\mathbb{C})$ with highest weight λ.

The coefficients $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ in both definitions H_Γ and H_Δ will be described in terms of Gelfand-Tsetlin patterns with top row (or equivalently, highest weight vector)

$$\lambda + \rho = (l_1 + l_2 + \cdots + l_r + r, \cdots, l_{r-1} + l_r + 2, l_r + 1, 0). \tag{1.6}$$

We denote by $\mathrm{GT}(\lambda+\rho)$ the set of all Gelfand-Tsetlin patterns having this top row. Here

$$\rho = (r, r-1, \cdots, 0) \quad \text{and} \quad \lambda = (\lambda_1, \cdots, \lambda_{r+1}) \quad \text{where} \quad \lambda_i = \sum_{j \geqslant i} l_j. \tag{1.7}$$

To any Gelfand-Tsetlin pattern $\mathfrak{T}$, we associate the following pair of functions with image in $\mathbb{Z}^r_{\geqslant 0}$:

$$k_\Gamma(\mathfrak{T}) = (k_{\Gamma,1}(\mathfrak{T}), \cdots, k_{\Gamma,r}(\mathfrak{T})), \qquad k_\Delta(\mathfrak{T}) = (k_{\Delta,1}(\mathfrak{T}), \cdots, k_{\Delta,r}(\mathfrak{T})),$$

where

$$k_{\Gamma,i}(\mathfrak{T}) = \sum_{j=i}^{r} (a_{i,j} - a_{0,j}) \quad \text{and} \quad k_{\Delta,i}(\mathfrak{T}) = \sum_{j=r+1-i}^{r} (a_{0,j-r-1+i} - a_{r+1-i,j}). \tag{1.8}$$

In the language of representation theory, the weight of the basis vector corresponding to the Gelfand-Tsetlin pattern $\mathfrak{T}$ can be read from differences of consecutive row sums in the pattern, so both k_Γ and k_Δ are expressions of the weight of the pattern up to an affine linear transformation.

Then given a fixed r-tuple of nonnegative integers $(l_1, \cdots, l_r)$, we make the following two definitions for p-parts of the multiple Dirichlet series:

$$H_\Gamma(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{\substack{\mathfrak{T} \in \mathrm{GT}(\lambda+\rho) \\ k_\Gamma(\mathfrak{T}) = (k_1, \cdots, k_r)}} G_\Gamma(\mathfrak{T}) \tag{1.9}$$

and

$$H_\Delta(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{\substack{\mathfrak{T} \in \mathrm{GT}(\lambda+\rho) \\ k_\Delta(\mathfrak{T}) = (k_1, \cdots, k_r)}} G_\Delta(\mathfrak{T}), \tag{1.10}$$

where the functions G_Γ and G_Δ on Gelfand-Tsetlin patterns will now be defined.

We will associate with $\mathfrak{T}$ two arrays $\Gamma(\mathfrak{T})$ and $\Delta(\mathfrak{T})$. The entries in these arrays are

$$\Gamma_{i,j} = \Gamma_{i,j}(\mathfrak{T}) = \sum_{k=j}^{r} (a_{i,k} - a_{i-1,k}), \qquad \Delta_{i,j} = \Delta_{i,j}(\mathfrak{T}) = \sum_{k=i}^{j} (a_{i-1,k-1} - a_{i,k}), \tag{1.11}$$

with $1 \leqslant i \leqslant j \leqslant r$, and we often think of attaching each entry of the array $\Gamma(\mathfrak{T})$ (resp. $\Delta(\mathfrak{T})$) with an entry of the pattern $a_{i,j}$ lying below the fixed top row. Thus we think of $\Gamma(\mathfrak{T})$ as applying a kind of *right-hand rule* to $\mathfrak{T}$, since $\Gamma_{i,j}$ involves entries above and to the right of $a_{i,j}$ as in (1.11); in Δ we use a *left-hand rule* where $\Delta_{i,j}$ involves entries above and to the left of $a_{i,j}$ as in (1.11). When we represent these arrays graphically, we will right-justify the Γ array and left-justify the Δ array. For example, if

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} 12 & & 9 & & 4 & & 0 \\ & 10 & & 5 & & 3 & \\ & & 7 & & 4 & & \\ & & & 6 & & & \end{array} \right\}$$

then

$$\Gamma(\mathfrak{T}) = \begin{bmatrix} 5 & 4 & 3 \\ & 3 & 1 \\ & & 2 \end{bmatrix} \quad \text{and} \quad \Delta(\mathfrak{T}) = \begin{bmatrix} 2 & 6 & 7 \\ 3 & 4 & \\ 1 & & \end{bmatrix}.$$

To provide the definitions of G_Γ and G_Δ corresponding to each array, it is convenient to *decorate* the entries of the Γ and Δ arrays by boxing or circling certain of them. Using the *right-hand rule* with the Γ array, if $a_{i,j} = a_{i-1,j-1}$ then we say $\Gamma_{i,j}$ is *boxed*, and indicate this when we write the array by putting a box around it, while if $a_{i,j} = a_{i-1,j}$ we say it is *circled* (and we circle it). Using the *left-hand rule* to obtain the Δ array, we box $\Delta_{i,j}$ if $a_{i,j} = a_{i-1,j}$ and we circle it if $a_{i,j} = a_{i-1,j-1}$. For example, if

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} 12 & & 10 & & 4 & & 0 \\ & 10 & & 5 & & 3 & \\ & & 7 & & 5 & & \\ & & & 6 & & & \end{array} \right\} \tag{1.12}$$

then the decorated arrays are

$$\Gamma(\mathfrak{T}) = \begin{bmatrix} \textcircled{4} & 4 & 3 \\ & 4 & \boxed{2} \\ & & 1 \end{bmatrix}, \qquad \Delta(\mathfrak{T}) = \begin{bmatrix} \boxed{2} & 7 & 8 \\ 3 & \textcircled{3} & \\ 1 & & \end{bmatrix}. \tag{1.13}$$

We sometimes use the terms *right-hand rule* and *left-hand rule* to refer to both the direction of accumulation of the row differences, and to the convention for decorating these accumulated differences.

If $m, c \in \mathfrak{o}_S$ with $c \neq 0$ define the Gauss sum

$$g(m,c) = \sum_{a \bmod c} \left(\frac{a}{c}\right)_n \psi\left(\frac{am}{c}\right), \tag{1.14}$$

where ψ is a character of F_S that is trivial on $\mathfrak{o}_S$ and no larger fractional ideal. With p now fixed, for brevity let

$$g(a) = g(p^{a-1}, p^a) \quad \text{and} \quad h(a) = g(p^a, p^a). \tag{1.15}$$

These functions will only occur with $a > 0$. The reader may check that $g(a)$ is nonzero for any value of a, while $h(a)$ is nonzero only if $n|a$, in which case $h(a) = (q-1)q^{a-1}$, where q is the cardinality of $\mathfrak{o}_S/p\mathfrak{o}_S$. Thus if $n|a$ then $h(a) = \phi(p^a)$, the Euler phi function for $p^a\mathfrak{o}_S$.

Let

$$G_\Gamma(\mathfrak{T}) = \prod_{1\leqslant i\leqslant j\leqslant r} \begin{cases} g(\Gamma_{ij}) & \text{if } \Gamma_{ij} \text{ is boxed but not circled in } \Gamma(\mathfrak{T}); \\ q^{\Gamma_{ij}} & \text{if } \Gamma_{ij} \text{ is circled but not boxed;} \\ h(\Gamma_{ij}) & \text{if } \Gamma_{ij} \text{ is neither circled nor boxed;} \\ 0 & \text{if } \Gamma_{ij} \text{ is both circled and boxed.} \end{cases}$$

We say that the pattern $\mathfrak{T}$ is *strict* if $a_{i,j} > a_{i,j+1}$ for every $0 \leqslant i \leqslant j < r$. Thus the rows of the pattern are a strictly decreasing sequence. Otherwise we say that the pattern is *non-strict*. It is clear from the definitions that $\mathfrak{T}$ is non-strict if and only if $\Gamma(\mathfrak{T})$ has an entry that is both boxed and circled, so $G_\Gamma(\mathfrak{T}) = 0$ for non-strict patterns. Similarly let

$$G_\Delta(\mathfrak{T}) = \prod_{1\leqslant i\leqslant j\leqslant r} \begin{cases} g(\Delta_{ij}) & \text{if } \Delta_{ij} \text{ is boxed but not circled in } \Delta(\mathfrak{T}); \\ q^{\Delta_{ij}} & \text{if } \Delta_{ij} \text{ is circled but not boxed;} \\ h(\Delta_{ij}) & \text{if } \Delta_{ij} \text{ is neither circled nor boxed;} \\ 0 & \text{if } \Delta_{ij} \text{ is both circled and boxed.} \end{cases}$$

For example, we may use the decorated arrays as in (1.13) to write down $G_\Gamma(\mathfrak{T})$ and $G_\Delta(\mathfrak{T})$ for the pattern $\mathfrak{T}$ appearing in (1.12) as follows:

$$G_\Gamma(\mathfrak{T}) = q^4 h(4)h(3)h(4)g(2)h(1), \quad \text{and} \quad G_\Delta(\mathfrak{T}) = g(2)h(7)h(8)h(3)q^3 h(1).$$

Inserting the respective definitions for G_Γ and G_Δ into the formulas (1.9) and (1.10) completes the two definitions of the p-parts of H_Γ and H_Δ, and with it two definitions for a multiple Dirichlet series $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$. For example, the pattern $\mathfrak{T}$ in (1.12) would appear in the p-part of $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ if

$$(\mathrm{ord}_p(m_1), \cdots, \mathrm{ord}_p(m_r)) = (1, 5, 3)$$

according to the top row of $\mathfrak{T}$. Moreover, this pattern would contribute $G_\Gamma(\mathfrak{T})$ to the term $H_\Gamma(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) = H_\Gamma(p^4, p^8, p^6; p^1, p^5, p^3)$ according to the definitions in (1.8) and (1.9).

In [17], the definition H_Γ was used to define the series, and so we will state our theorem on functional equations and analytic continuation of $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ using this choice. Before stating the result precisely, we need to define certain normalizing factors for the multiple Dirichlet series. These have a uniform description for all root systems (see Section 3.3 of [12]), but for simplicity we state them only for Type A here.

Let

$$\boldsymbol{G}_n(s) = (2\pi)^{-2(n-1)s} n^{2ns} \prod_{j=1}^{n-1} \Gamma\left(2s - 1 + \frac{j}{n}\right). \tag{1.16}$$

We will identify the weight space for $\mathrm{GL}(r+1,\mathbb{C})$ with $\mathbb{Z}^{r+1}$ in the usual way. For any $\alpha \in \Phi^+$, the set of positive roots, there exist $1 \leq i < j \leq r+1$ such that $\alpha = \alpha_{i,j}$ is the root $(0,\cdots,0,1,0,\cdots,-1,0,\cdots)$ with the 1 in the i-th place and the -1 in the j-th place. If $\alpha = \alpha_{i,j}$ is a positive root, then define

$$\boldsymbol{G}_\alpha(\boldsymbol{s}) = \boldsymbol{G}_n\left(\frac{1}{2} + \left(s_i + s_{i+1} + \cdots + s_{j-1} - \frac{j-i}{2}\right)\right). \tag{1.17}$$

Further let

$$\zeta_\alpha(\boldsymbol{s}) = \zeta\left(1 + 2n\left(s_i + s_{i+1} + \cdots + s_{j-1} - \frac{j-i}{2}\right)\right)$$

where ζ is the Dedekind zeta function attached to the number field F. Then the normalized multiple Dirichlet series is given by

$$Z^*_\Psi(\boldsymbol{s};\boldsymbol{m}) = \left[\prod_{\alpha\in\Phi^+} \boldsymbol{G}_\alpha(\boldsymbol{s})\zeta_\alpha(\boldsymbol{s})\right] Z_\Psi(\boldsymbol{s},\boldsymbol{m}). \tag{1.18}$$

THEOREM 1.1 *The Weyl group multiple Dirichlet series $Z^*_\Psi(\boldsymbol{s};\boldsymbol{m})$ with coefficients H_Γ as in (1.9) has meromorphic continuation to $\mathbb{C}^r$ and satisfies functional equations*

$$Z^*_\Psi(\boldsymbol{s};\boldsymbol{m}) = |m_i|^{1-2s_i} Z^*_{\sigma_i\Psi}(\sigma_i\boldsymbol{s};\boldsymbol{m}) \tag{1.19}$$

for all simple reflections $\sigma_i \in W$, where

$$\sigma_i(s_i) = 1 - s_i, \qquad \sigma_i(s_j) = \begin{cases} s_i + s_j - 1/2 & \text{if } i,j \text{ adjacent,} \\ s_j & \text{otherwise.} \end{cases}$$

Here $\sigma_i : \mathcal{M} \to \mathcal{M}$ is a linear map defined in [14].

The endomorphisms σ_i of the space $\mathcal{M}$ of functions satisfying (1.1) are the simple reflections in an action of the Weyl group W on $\mathcal{M}$. See [12] and [14] for more information.

This proves Conjecture 2 of [17]. An explicit description of the polar hyperplanes of Z^*_Ψ can be found in Section 7 of [14]. As we demonstrate in Chapter 6, this theorem ultimately follows from proving the equivalence of the two definitions of the p-part H_Γ and H_Δ offered in (1.9) and (1.10). Because of this implication, and because it is of interest to construct such functions attached to a representation but independent of choices of coordinates (a notion we make precise in subsequent chapters using the crystal description), we consider the equivalence of these two descriptions to be our main theorem.

THEOREM 1.2 *We have $H_\Gamma = H_\Delta$.*

The proof is outlined in Chapter 6 and completed in the subsequent chapters.

To give a flavor for this result, we give one example. Suppose that $r = 2$, $(k_1,k_2) = (6,6)$, and $(l_1,l_2) = (2,5)$. Then the following strict Gelfand-Tsetlin patterns contribute to $H_\Gamma(p^6,p^6;p^2,p^5)$:

$$\left\{\begin{matrix} 9 & & 6 & & 0 \\ & 9 & & 3 & \\ & & 6 & & \end{matrix}\right\}, \quad \left\{\begin{matrix} 9 & & 6 & & 0 \\ & 8 & & 4 & \\ & & 6 & & \end{matrix}\right\}, \quad \left\{\begin{matrix} 9 & & 6 & & 0 \\ & 7 & & 5 & \\ & & 6 & & \end{matrix}\right\}.$$

These give

$$H_\Gamma(p^6, p^6; p^2, p^5) = g(6)h(3)^2 + h(2)h(4)h(6) + h(1)h(5)h(6). \tag{1.20}$$

Similarly the following Gelfand-Tsetlin patterns contribute to H_Δ:

$$\left\{\begin{array}{ccccc} 9 & & 6 & & 0 \\ & 9 & & 0 & \\ & & 3 & & \end{array}\right\}, \left\{\begin{array}{ccccc} 9 & & 6 & & 0 \\ & 8 & & 1 & \\ & & 3 & & \end{array}\right\},$$

$$\left\{\begin{array}{ccccc} 9 & & 6 & & 0 \\ & 7 & & 2 & \\ & & 3 & & \end{array}\right\}, \left\{\begin{array}{ccccc} 9 & & 6 & & 0 \\ & 6 & & 3 & \\ & & 3 & & \end{array}\right\}.$$

These give

$$H_\Delta(p^6, p^6; p^2, p^5) = g(6)h(6) + h(1)h(5)h(6) + h(2)h(4)h(6) + g(3)^2h(6). \tag{1.21}$$

In this special case, two terms in the expressions for H_Γ and H_Δ match exactly, a point that we will return to in (1.22). Theorem 1.2 is true for all n, since

$$g(6)h(3)^2 = g(6)h(6) + g(3)^2h(6).$$

Although this identity is true for all n, its meaning is different for different values of n. Indeed, both sides are zero unless n divides 6. When $n = 2$ or 6, the left-hand side vanishes since $h(3) = 0$, while the right-hand side vanishes since, by elementary properties of Gauss sums (see Proposition 8.1), $g(6) = -q^5$ and $g(3)^2 = q^5$. When $n = 1$ or $n = 3$, both sides are nonzero, and the equality is the identity

$$-q^5(q^3 - q^2)^2 = -q^5(q^6 - q^5) + (-q^2)^2(q^6 - q^5).$$

The identity of Theorem 1.2 equates two sums. The example that we have just given shows that it is not possible to give a term-by-term comparison of the two sums (1.9) and (1.10). That is, there is no bijection $\mathfrak{T} \mapsto \mathfrak{T}'$ between the set of Gelfand-Tsetlin patterns with top row $\lambda + \rho$ and itself such that $k_\Gamma(\mathfrak{T}) = k_\Delta(\mathfrak{T}')$ and $G_\Gamma(\mathfrak{T}) = G_\Delta(\mathfrak{T}')$. Still there is a bijection such that $k_\Gamma(\mathfrak{T}) = k_\Delta(\mathfrak{T}')$ and such that the identity $G_\Gamma(\mathfrak{T}) = G_\Delta(\mathfrak{T}')$ is *often* true. In other words, many terms in (1.9) are equal to corresponding terms in (1.10).

The involution on the set of Gelfand-Tsetlin patterns is called the *Schützenberger involution*. It was originally introduced by Schützenberger [66] in the context of tableaux. The involution was transported to the setting of Gelfand-Tsetlin patterns by Kirillov and Berenstein [51], and defined for general crystals (to be discussed in Chapter 2) by Lusztig [58]. We give its definition now.

Given a Gelfand-Tsetlin pattern (1.5), the condition that the rows interleave means that each $a_{i,j}$ is constrained by the inequalities

$$\min(a_{i-1,j-1}, a_{i+1,j}) \quad a_{i,j} \quad \max(a_{i-1,j}, a_{i+1,j+1}).$$

This means that we can reflect the entry $a_{i,j}$ across the midpoint of this interval and obtain another Gelfand-Tsetlin pattern. Thus we replace every entry $a_{i,j}$ in the i-th row by

$$a'_{i,j} = \min(a_{i-1,j-1}, a_{i+1,j}) + \max(a_{i-1,j}, a_{i+1,j+1}) - a_{i,j}.$$

This requires interpretation if $j = i$ or $j = r$. For these cases, we will set

$$a'_{i,i} = a_{i-1,i-1} + \max(a_{i-1,i}, a_{i+1,i+1}) - a_{i,i}$$

and

$$a'_{i,r} = \min(a_{i-1,r-1}, a_{i+1,r}) + a_{i-1,r} - a_{i,r},$$

unless $i = j = r$, and then we set $a'_{r,r} = a_{r-1,r-1} + a_{r-1,r} - a_{r,r}$. This operation on the entire row will be denoted by t_{r+1-i}. Note that it only affects this lone row in the pattern. Further involutions on patterns may be built out of the t_i, and will be called q_i following Berenstein and Kirillov. Let q_0 be the identity map, and define recursively $q_i = t_1 t_2 \cdots t_i q_{i-1}$. The t_i have order two. They do not satisfy the braid relation, so $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$. However $t_i t_j = t_j t_i$ if $|i - j| > 1$ and this implies that the q_i also have order two. The operation q_r is the *Schützenberger involution* that we mentioned earlier.

For example, let $r = 2$, and let us compute q_2 of a typical Gelfand-Tsetlin pattern. Following the algorithm outlined above, if

$$\mathfrak{T} = \left\{ \begin{array}{ccccc} 9 & & 6 & & 0 \\ & 7 & & 5 & \\ & & 6 & & \end{array} \right\},$$

then

$$q_2(\mathfrak{T}) = \left\{ \begin{array}{ccccc} 9 & & 6 & & 0 \\ & 8 & & 1 & \\ & & 3 & & \end{array} \right\}.$$

Indeed, $q_2 = t_1 t_2 t_1$ and we compute:

$$\left\{ \begin{array}{ccccc} 9 & & 6 & & 0 \\ & 7 & & 5 & \\ & & 6 & & \end{array} \right\} \xrightarrow{t_1} \left\{ \begin{array}{ccccc} 9 & & 6 & & 0 \\ & 7 & & 5 & \\ & & 6 & & \end{array} \right\} \xrightarrow{t_2}$$

$$\left\{ \begin{array}{ccccc} 9 & & 6 & & 0 \\ & 8 & & 1 & \\ & & 6 & & \end{array} \right\} \xrightarrow{t_1} \left\{ \begin{array}{ccccc} 9 & & 6 & & 0 \\ & 8 & & 1 & \\ & & 3 & & \end{array} \right\}.$$

Now observe that

$$G_\Gamma(\mathfrak{T}) = h(6)\, h(5)\, h(1) = G_\Delta(q_2 \mathfrak{T}). \tag{1.22}$$

This accounts for the equality of one of the terms in the sum (1.20) and one of the terms in the sum (1.21), and illustrates the point that *often* $G_\Gamma(\mathfrak{T}) = G_\Delta(q_r \mathfrak{T})$. The difficulty of Theorem 1.2 is the problem of accounting for the exceptions to this rule in a systematic way.

It is sometimes useful to modify the coefficients in the Dirichlet series as follows. If a is a positive integer let

$$g^\flat(a) = q^{-a} g(a), \qquad h^\flat(a) = q^{-a} h(a). \tag{1.23}$$

These "reduced" coefficients have the property that they depend only on a modulo n. Let

$$G^\flat_\Gamma(\mathfrak{T}) = \;_{1\leqslant i\leqslant j\leqslant r} \left\{ \begin{array}{ll} g^\flat(\Gamma_{ij}) & \text{if } \Gamma_{ij} \text{ is boxed but not circled in } \Gamma(\mathfrak{T}); \\ 1 & \text{if } \Gamma_{ij} \text{ is circled but not boxed;} \\ h^\flat(\Gamma_{ij}) & \text{if } \Gamma_{ij} \text{ is neither circled nor boxed;} \\ 0 & \text{if } \Gamma_{ij} \text{ is both circled and boxed,} \end{array} \right.$$

and similarly define

$$G^{\flat}_{\Delta}(\mathfrak{T}) = \underset{1\leqslant i\leqslant j\leqslant r}{} \begin{cases} g^{\flat}(\Delta_{ij}) & \text{if } \Delta_{ij} \text{ is boxed but not circled in } \Delta(\mathfrak{T}); \\ 1 & \text{if } \Delta_{ij} \text{ is circled but not boxed;} \\ h^{\flat}(\Delta_{ij}) & \text{if } \Delta_{ij} \text{ is neither circled nor boxed;} \\ 0 & \text{if } \Delta_{ij} \text{ is both circled and boxed.} \end{cases}$$

It is not hard to check that

$$G_{\Gamma}(\mathfrak{T}) = q^{\sum_{i=1}^{r} k_{\Gamma,i}(\mathfrak{T})} G^{\flat}_{\Gamma}(\mathfrak{T}), \qquad G_{\Delta}(\mathfrak{T}) = q^{\sum_{i=1}^{r} k_{\Delta,i}(\mathfrak{T})} G^{\flat}_{\Delta}(\mathfrak{T}).$$

It follows that if we define

$$H^{\flat}_{\Gamma}(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \underset{k_{\Gamma}(\mathfrak{T})=(k_1,\cdots,k_r)}{} G^{\flat}_{\Gamma}(\mathfrak{T}) \tag{1.24}$$

and similarly let

$$H^{\flat}_{\Delta}(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \underset{k_{\Delta}(\mathfrak{T})=(k_1,\cdots,k_r)}{} G^{\flat}_{\Delta}(\mathfrak{T}), \tag{1.25}$$

then extend these reduced coefficients $H^{\flat}$ by the same multiplicativities (1.3) and (1.4) as H, we have

$$Z_{\Psi}(\boldsymbol{s}; \boldsymbol{m}) = \underset{\substack{\boldsymbol{c}=(c_1,\cdots,c_r)\in(\mathfrak{o}_S/\mathfrak{o}_S^{\times})^r \\ c_i=0}}{} \frac{H^{\flat}(\boldsymbol{c}; \boldsymbol{m})}{\mathbb{N}c_1^{2s_1-1} \cdots \mathbb{N}c_r^{2s_r-1}}.$$