Exposé One

An Overview of Thurston's Theorems on Surfaces

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1.1 INTRODUCTION

Thurston's theory ([Thu88], see also [Thu80], [Poé80]) is concerned with the following three problems:

1. Describing all simple closed curves on a surface up to isotopy.
2. Describing all diffeomorphisms of a surface up to isotopy.
3. Giving a boundary for Teichmüller space that is natural with respect to the action of diffeomorphisms.

Every closed surface admits a Riemannian metric of constant curvature [Gau65]. Table 1.1 summarizes the possibilities and at the same time establishes a parallel between geometric and the topological properties.

Table 1.1. The three possible geometries on surfaces

Surface	K **(curvature)**	χ **(Euler characteristic)**	**Remarks**
S^2, $\mathbb{R}P^2$	$K = 1$ (Elliptic geometry)	$\chi > 0$	π_1 is finite, $\pi_2 \neq 0$
T^2, K^2 (Torus, Klein bottle)	$K = 0$ (Euclidean geometry)	$\chi = 0$	These are $K(\pi, 1)$'s and their universal covering space is $\mathbb{R}^2$
Genus > 1	$K = -1$ (Hyperbolic geometry)	$\chi < 0$	

Most of Thurston's theorems hold for any compact surface, but in what follows, we restrict ourselves to compact orientable surfaces, possibly with boundary.

1.2 THE SPACE OF SIMPLE CLOSED CURVES

Let M be a compact, connected, orientable surface. We write $\mathcal{S}(M)$ (or, briefly, $\mathcal{S}$) for the set of isotopy classes of simple, closed, connected curves in M that are not homotopic to a point or homotopic to a boundary component of M.

(1) The elements of $\mathcal{S}$ are *not* oriented.

(2) Since two simple closed curves that are homotopic are also isotopic (see [Eps66]), we may replace "isotopy classes" in the above definition with "homotopy classes."

Consider the symmetric map

$$i\colon \mathcal{S}\times\mathcal{S}\to\mathbb{Z}^+=\{0,1,2,\dots\}$$

defined in the following fashion: $i(\alpha,\beta)$ is the minimum number of intersections of a representative for α with a representative for β. This is the *geometric intersection number* (as opposed to the algebraic intersection number).

Example. On the torus T^2, we choose two oriented generators x and y for $\pi_1(T^2)$. Then all elements of $\mathcal{S}$ may be represented by $\gamma(a,b)=ax+by$, where $a,b\in\mathbb{Z}$ and $\gcd(a,b)=1$; in $\mathcal{S}$, we have $\gamma(a,b)=\gamma(-a,-b)$. The following formula is easy to verify:

$$i\left(\gamma(a,b),\gamma(c,d)\right)=\left|\det\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right|.$$

LEMMA 1.1. *Let M and $\mathcal{S}=\mathcal{S}(M))$ be as above.*
(1) If $\alpha\in\mathcal{S}$, there is a $\beta\in\mathcal{S}$ such that $i(\alpha,\beta)\neq 0$.
(2) If $\alpha_1\neq\alpha_2$ in $\mathcal{S}$, there is a $\beta\in\mathcal{S}$ such that $i(\alpha_1,\beta)=0$ and $i(\alpha_2,\beta)\neq 0$.

The proof is given in Exposé 3.

The space of functionals. We consider the set $\mathbb{R}_+^{\mathcal{S}}$ of functions from $\mathcal{S}$ to the nonnegative reals, with the weak topology. The usual multiplication by the positive reals defines rays in $\mathbb{R}_+^{\mathcal{S}}$. The set of these rays is the projective space $P(\mathbb{R}_+^{\mathcal{S}})$; it is given the quotient topology. We have the natural maps

$$\mathcal{S}\xrightarrow{i_*}\mathbb{R}_+^{\mathcal{S}}\setminus 0\xrightarrow{\pi}P(\mathbb{R}_+^{\mathcal{S}})$$

where the map i_* is defined by $i_*(\alpha)(\beta)=i(\alpha,\beta)$. By statement *(1)* of Lemma 1.1, $i_*(\mathcal{S})$ does not contain 0, and by statement *(2)*, the map $\pi\circ i_*$ is injective.

Consider the completion of $\mathcal{S}$, denoted $\overline{\mathcal{S}}$, which is the closure of $\pi\circ i_*(\mathcal{S})$ in $P(\mathbb{R}_+^{\mathcal{S}})$. The elements of $\overline{\mathcal{S}}$ are represented by sequences $\{(t_n,\alpha_n)\}$, $t_n>0$, $\alpha_n\in\mathcal{S}$, such that for all β in $\mathcal{S}$, the sequence of real numbers $t_n i(\alpha_n,\beta)$ converges.

Thus, within $P(\mathbb{R}_+^{\mathcal{S}})$, the set $\mathcal{S}$ has a nontrivial topology. Intuitively, we may give a meaning to the notion that "two curves γ, γ' are close to each other." This "proximity" has nothing to do with the respective homotopy classes of the curves, but with the fact that, up to a multiple, in every region of the surface, γ and γ' are more or less made up of the same number of strands, going in more or less the same direction. All of this will be discussed in greater detail in Exposé 4.

We also need to introduce the space $\mathcal{S}'$ of isotopy classes of simple, closed, but not necessarily connected curves in M, whose every component represents an element of $\mathcal{S}$. Two distinct components of the same curve are allowed to be isotopic to each other, so that we may consider scalar multiplication: for an integer $n > 0$ and $\gamma \in \mathcal{S}$, $n\gamma$ is represented by n parallel curves.

As before, we define $i\colon \mathcal{S}' \times \mathcal{S} \to \mathbb{Z}_+$, and obtain the diagram

$$\mathcal{S}' \xrightarrow{i_*} \mathbb{R}_+^{\mathcal{S}} \setminus 0 \xrightarrow{\pi} P(\mathbb{R}_+^{\mathcal{S}}).$$

Clearly, i_* respects multiplication by scalars, hence $\pi \circ i_*$ is not injective on $\mathcal{S}'$. But one may easily show that $\pi \circ i_*(\mathcal{S}')$ admits $\overline{\mathcal{S}}$ as its closure (see Exposé 4). In the following, we denote by $\mathbb{R}_+ \times \mathcal{S}$ the cone on $i_*(\mathcal{S})$ in $\mathbb{R}_+^{\mathcal{S}}$. Also, we denote by $M_{g,b}^2$ the surface $\underset{g}{\#}(T^2) - \bigcup\limits_{b} D^2$.

THEOREM 1.2. *If M is a closed surface of genus $g \geq 2$, then $\overline{\mathcal{S}}$ is homeomorphic to S^{6g-7} (this is proven in Exposé 4). If $M = M_{g,b}^2$ and $\chi(M) < 0$, then $\overline{\mathcal{S}(M)}$ is homeomorphic to $S^{6g+2b-7}$ (see Exposé 11). Last, $\overline{\mathcal{S}(T^2)} \simeq S^1$ and $\overline{\mathcal{S}(D^2)} = \overline{\mathcal{S}(S^2)} = \overline{\mathcal{S}(S^1 \times [0,1])} = \emptyset$.*

1.3 MEASURED FOLIATIONS

For simplicity, M will be closed. A *measured foliation* on M is a foliation $\mathcal{F}$ with singularities (of the type of a holomorphic quadratic differential $z^{p-2}\,dz^2$, where $p = 3, 4, \dots$) together with a transverse measure that is invariant under holonomy. In the neighborhood of a nonsingular point, there exists a chart $\varphi\colon U \to \mathbb{R}_{x,y}^2$ such that $\varphi^{-1}(y = \text{constant})$ consists of the leaves of $\mathcal{F}|_U$. If $U_i \cap U_j$ is nonempty, there exist transition functions φ_{ij} of the form

$$\varphi_{ij}(x,y) = (h_{ij}(x,y), c_{ij} \pm y)$$

where c_{ij} is a constant. In these charts, the transverse measure is given by $|dy|$.

Remark. The foliations that admit transition functions of the form $(f(x,y), c + y)$ are those that are defined by a closed 1-form ω; away from singularities, y is a local primitive for ω. The singularities of $\mathcal{F}$ are p-saddles ($p \geq 3$) as in Figure 1.1.

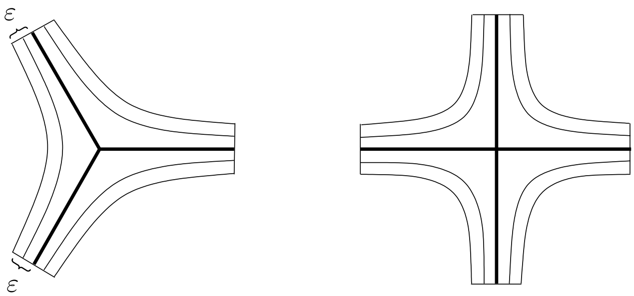

Figure 1.1. p-saddles, for $p = 3$, $p = 4$

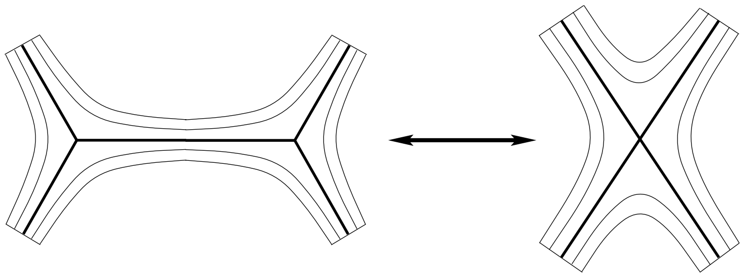

Figure 1.2. Whitehead equivalence

If γ is a simple closed curve in M, we call $\int_\gamma \mathcal{F}$ the *total variation* of the y-coordinate of $p \in \gamma$ as p traverses γ. For $\alpha \in \mathcal{S}$, define

$$I(\mathcal{F}, \alpha) = \inf_{\gamma \in \alpha} \int_\gamma \mathcal{F}.$$

$\mathcal{F}_1$ and $\mathcal{F}_2$ are *Whitehead equivalent* if one may be transformed to the other by isotopies and elementary deformations of the type suggested by Figure 1.2. (Observe that these deformations allow the transverse measure to be carried over.)

Denote by $\mathcal{MF}$ the set of Whitehead equivalence classes. Define

$$I_* \colon \mathcal{MF} \to \mathbb{R}_+^{\mathcal{S}}$$

by

$$I_*(\mathcal{F})(\alpha) = I(\mathcal{F}, \alpha).$$

$\mathcal{F}_1$ and $\mathcal{F}_2$ are *m-equivalent* (or *Schwartz equivalent*) if $I_*(\mathcal{F}_1) = I_*(\mathcal{F}_2)$. Note that Schwartz equivalence is an immediate consequence of Whitehead equivalence.

THEOREM 1.3. *The map I_* injects $\mathcal{MF}$ into $\mathbb{R}_+^{\mathcal{S}}$. What is more, we have $I_*(\mathcal{MF}) \cup 0 = \overline{\mathbb{R}_+ \times \mathcal{S}}$, and if $g > 1$, this set is homeomorphic with $\mathbb{R}^{6g-6}$. In particular, Schwartz equivalence is the same thing as Whitehead equivalence.*

The proof of this theorem is dealt with in Exposés 5 and 6. What is more, since $I_*(\mathcal{MF})$ misses 0, the theorem says that in $P(\mathbb{R}_+^{\mathcal{S}})$ we have $\overline{\mathcal{S}} = \pi \circ I_*(\mathcal{MF})$. This gives a nice geometric representation of the functionals in $\overline{\mathbb{R}_+ \times \mathcal{S}}$.

1.4 TEICHMÜLLER SPACE

We will consider a surface M with $\chi(M) < 0$. Consider the space $\mathcal{H}$ of all metrics on M with constant curvature $K = -1$ such that every component of the boundary of M is a geodesic. Let $\mathrm{Diff}_0(M)$ be the group of diffeomorphisms isotopic to the identity, with the C^∞ topology. As we shall see later, this group acts freely and continuously on $\mathcal{H}$. The orbit space under this action, equipped with the quotient topology, is called the *Teichmüller space* $\mathcal{T}(M)$ (briefly, $\mathcal{T}$). If M is orientable, there is another definition in terms of complex structures on M. The equivalence of the two definitions is a consequence of the uniformization theorem [Wey97].

Remarks. Consider a fixed M, together with another surface $X_\rho = X$ with a hyperbolic metric ρ. If $\varphi\colon M \to X$ is a diffeomorphism, the pair (X, φ) is called a *Teichmüller surface*. Two Teichmüller surfaces (X, φ) and (X', φ') are said to be *equivalent* if there is an isometry $f\colon X \to X'$ such that φ' and $f \circ \varphi$ are isotopic. It is convenient to identify the points of $\mathcal{T}$ with equivalence classes of Teichmüller surfaces.

We also remark here that two diffeomorphisms of M are homotopic if and only if they are isotopic (see [Eps66]).

If M is closed, of genus $g > 1$, a classical theorem of Teichmüller theory asserts that

$$\mathcal{T}(M) \simeq \mathbb{R}^{6g-6}.$$

This result, due to Fricke and Klein, will be proven in Exposé 7. Further, we have

$$\mathcal{T}(M^2_{g,b}) \simeq \mathbb{R}^{6g-6+2b}.$$

For all $\theta \in \mathcal{T}$ and $\alpha \in \mathcal{S}$, we define

$$\ell(\theta, \alpha) = \inf_{\gamma \in \alpha} (\theta(\gamma))$$

where $\theta(\gamma)$ denotes the length of γ computed in the metric θ, which is prescribed up to isotopy on M. The metric being fixed, the infimum is attained for a unique geodesic. From the above formula, we obtain the map

$$\ell_*\colon \mathcal{T} \to \mathbb{R}_+^{\mathcal{S}};$$

it can be easily seen that the image of the map misses $I_*(\mathcal{MF}) \cup 0$. The *mapping class group* $\pi_0(\mathrm{Diff}(M))$ acts on Teichmüller space as well as on $\mathcal{S}$, and thus on $\mathbb{R}_+^{\mathcal{S}}$; the map ℓ_* is clearly equivariant.

In Exposé 7, we prove the following theorem.

THEOREM 1.4. *The map ℓ_* is a homeomorphism onto its image.*

It is thus possible to put a natural topology on $\mathcal{T} \cup \overline{\mathcal{S}}$; we consider the topological space $\ell_*(\mathcal{T}) \cup I_*(\mathcal{MF})$, in which the rays in $I_*(\mathcal{MF})$ are identified to points, and we take the quotient topology.

In Exposé 8, we prove the following, in the case where M has no boundary.

THEOREM 1.5. *Let $M = M^2_{g,b}$.*

1. The topological space $\mathcal{T} \cup \overline{\mathcal{S}}$ is homeomorphic to D^{6g-6} if M is closed and $g > 1$; it is homeomorphic to $D^{6g-6+2b}$ if $\chi(M) < 0$.

2. The canonical map $\mathcal{T} \cup \overline{\mathcal{S}} \to P(\mathbb{R}_+^{\mathcal{S}})$ is an embedding.

The space $\mathcal{T} \cup \overline{\mathcal{S}}$, denoted $\overline{\mathcal{T}}$, is the *Thurston compactification* of Teichmüller space. It follows immediately from the definitions that for any diffeomorphism φ of M, the natural action of φ on $\overline{\mathcal{T}}$ is continuous.

If φ is a diffeomorphism of M, and $[\varphi]$ denotes the homeomorphism induced by φ on $\overline{\mathcal{T}}$, then $[\varphi]$ has a fixed point, by the Brouwer fixed point theorem. There are two possibilities.

(i) If $[\varphi]$ has a fixed point in $\mathcal{T}$, then φ is isotopic to an isometry φ' in a hyperbolic metric; in particular, φ' is periodic.

(ii) If $[\varphi]$ fixes a point in $\overline{\mathcal{S}}$, there is a foliation $\mathcal{F}$ such that $\varphi(\mathcal{F})$ is Whitehead equivalent to $\lambda\mathcal{F}$, $\lambda \in \mathbb{R}_+$, where $\lambda\mathcal{F}$ has the same underlying foliation as $\mathcal{F}$, with a transverse measure λ times that for $\mathcal{F}$.

This cursory analysis will be made more precise in what follows.

1.5 PSEUDO-ANOSOV DIFFEOMORPHISMS

We begin with an elementary example. Let $\varphi \in \mathrm{Diff}^+(T^2)$. Up to isotopy, φ is in $\mathrm{SL}(2,\mathbb{Z})$. There are three distinct possibilities for the eigenvalues λ_1 and λ_2 of φ, as follows:

(a) λ_1 and λ_2 are complex ($\lambda_1 = \overline{\lambda_2}$, $\lambda_1 \neq \lambda_2$, $|\lambda_1| = |\lambda_2| = 1$). In this case, φ is of finite order.

(b) $\lambda_1 = \lambda_2 = 1$ (respectively, $\lambda_1 = \lambda_2 = -1$). Up to a change of coordinates,

$$\varphi = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \left[\text{respectively,} \quad \varphi = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix} \right],$$

which is a *Dehn twist* (respectively, the product of a Dehn twist with the "hyperelliptic involution"). In either case, φ leaves invariant a simple curve.

(c) λ_1 and λ_2 are distinct irrationals. Then φ is an *Anosov* diffeomorphism [Ano69, Sma67].

This analysis is generalized by Thurston to any compact surface:

THEOREM 1.6. *Any diffeomorphism φ on M is isotopic to a map φ' satisfying one of the following three conditions:*

(i) φ' fixes an element of $\mathcal{T}$ and is of finite order.

(ii) φ' is "reducible," in the sense that it preserves a simple curve (representing an element of $\mathcal{S}'$); in this case, one pursues the analysis of φ' by cutting M open along this curve.

(iii) There exists $\lambda > 1$ and two transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^s$ such that

$$\varphi'(\mathcal{F}^s) = \frac{1}{\lambda}\mathcal{F}^s$$

and

$$\varphi'(\mathcal{F}^u) = \lambda\mathcal{F}^u.$$

The equalities in *(iii)* mean that the underlying foliations are equal, and the measures are scaled.

Aside from the obvious, saying that $\mathcal{F}^s$ and $\mathcal{F}^u$ are transverse means that their singularities are the same, and that in a neighborhood of the singularities the configuration is similar to that in Figure 1.3. A diffeomorphism that satisfies condition *(iii)* is called *pseudo-Anosov*.

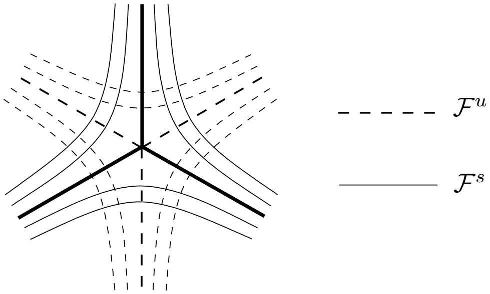

Figure 1.3. Pseudo-Anosov singularities

Theorem 1.6 is proved in Exposé 9. To apply this theorem inductively, we need to extend the theory to the case of surfaces with boundary. This is done in Exposé 11.

In Exposé 12, we show that, for a pseudo-Anosov diffeomorphism φ, the measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ represent the only fixed points of $[\varphi]$ in $\overline{\mathcal{T}}$, and two homotopic pseudo-Anosov diffeomorphisms are conjugate by a diffeomorphism isotopic to the identity. The key to these theorems is the following "mixing" property that the pseudo-Anosov diffeomorphism φ possesses: for all $\alpha, \beta \in \mathcal{S}$, we have

$$\lim_{n\to\infty} \frac{i(\varphi^n(\alpha), \beta)}{\lambda^n} = I(\mathcal{F}^s, \alpha) I(\mathcal{F}^u, \beta).$$

Spectral properties of pseudo-Anosov diffeomorphisms. For $\theta \in \mathcal{T}$ and $\alpha \in \mathcal{S}$, we defined in Section 1.4 the positive number $\ell(\theta, \alpha)$. Diffeomorphisms have eigenvalues in the following sense.

THEOREM 1.7. *Let $\varphi \in \mathrm{Diff}(M^2)$. There exists a finite set of algebraic integers $\lambda_1, \ldots, \lambda_k \geq 1$ such that, for each $\alpha \in \mathcal{S}$, there exists λ_j with*

$$\lim_{n\to\infty} \ell(\theta, \varphi^n(\alpha))^{1/n} = \lambda_j$$

for all $\theta \in \mathcal{T}$. Furthermore, φ is pseudo-Anosov if and only if $k = 1$ and $\lambda_1 > 1$; in this case $\lambda_1 = \lambda$ (see Exposés 9 and 11).

Entropy. On a compact metric space X with a continuous map $f \colon X \to X$, we may define the *topological entropy* $h(f)$ (see Exposé 10). If φ is a pseudo-Anosov diffeomorphism, one proves that $h(\varphi) = \log(\lambda)$. Moreover, φ possesses an obvious invariant measure and $h(\varphi)$ is its metric entropy [Sin76].

THEOREM 1.8. *A pseudo-Anosov diffeomorphism minimizes the topological entropy in its isotopy class.*

The list of Thurston's results is much longer, but we end this overview here to get to the heart of the subject.

1.6 THE CASE OF THE TORUS

This case is particularly simple and is treated separately. On the torus T^2, consider the three elements e_1, e_2, e_3 of $\mathcal{S}(T^2)$ shown in Figure 1.4. Let these be oriented for the time being.

Let x_1 and x_2 be the canonical generators e_1 and e_2 with the orientations shown in Figure 1.4.

If γ is a simple oriented curve representing $mx_1 + nx_2$, we find

$$i(e_1, \gamma) = |n|, \quad i(e_2, \gamma) = |m|, \quad i(e_3, \gamma) = |n - m|.$$

These three numbers determine γ in $\mathcal{S}$, but the first two are not sufficient. The three numbers form a degenerate triangle, in the sense that one of them is equal to the sum of the other two.

We now consider the standard simplex with barycentric coordinates X_1, X_2, X_3 (where $X_i \geq 0$, $\sum X_i = 1$). The simplex decomposes into the four regions indicated in Figure 1.5.

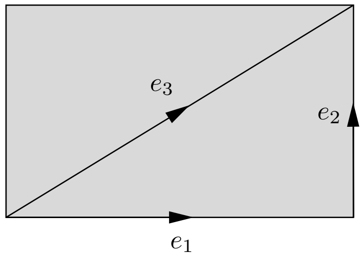

Figure 1.4. The torus T^2

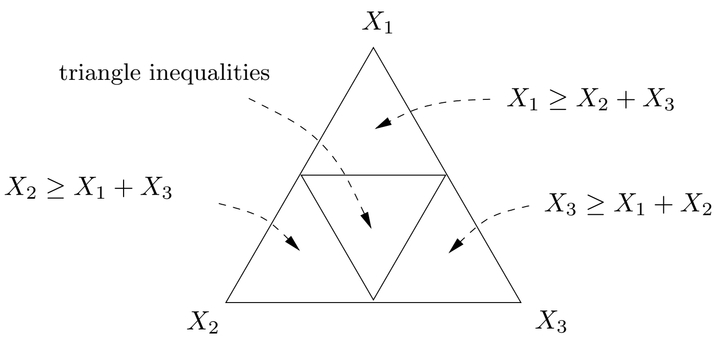

Figure 1.5.

Let $(\nabla \leq)$ be the domain where the triangle inequality holds; the boundary $\partial(\nabla \leq)$ corresponds to degenerate triangles. We think of the standard simplex as being in $\mathbb{R}^3_+$, and we denote by $\text{cone}(\partial(\nabla \leq))$ the cone of half-lines that start at 0 and pass through $\partial(\nabla \leq)$.

To each $\gamma \in \mathcal{S}$, we associate the numbers

$$X_j = \frac{i(e_j, \gamma)}{\sum_{i=1}^{3} i(e_i, \gamma)}, \quad j = 1, 2, 3;$$

a simple calculation shows that we can thus identify $\mathcal{S}$ with the set of rational points of $\partial(\nabla \leq)$.

LEMMA 1.9. *Let $\beta \in \mathcal{S}$. There exists a continuous function*

$$\Phi_\beta \colon \text{cone}(\partial(\nabla \leq)) \to \mathbb{R}_+$$

that is homogeneous of degree 1 (that is, $\Phi_\beta(kv) = k\Phi_\beta(v)$), and that satisfies

$$i(\alpha, \beta) = \Phi_\beta\left(i(\alpha, e_1), i(\alpha, e_2), i(\alpha, e_3)\right)$$

for all $\alpha \in \mathcal{S}$.

Proof. We can give an explicit construction as follows. Suppose that β corresponds to $mx_1 + nx_2$, $n, m \in \mathbb{Z}$, $\gcd(m, n) = 1$. (The only ambiguity is that $-mx_1 - nx_2$ also corresponds to β.) On the surface of the subcone $X_3 = X_1 + X_2$, we set

$$\Phi_\beta(X_1, X_2, X_3) = \left|\det \begin{pmatrix} X_2 & -X_1 \\ m & n \end{pmatrix}\right|$$

On the other two faces, we set

$$\Phi_\beta(X_1, X_2, X_3) = \left|\det \begin{pmatrix} X_2 & X_1 \\ m & n \end{pmatrix}\right|$$

At the intersection of these faces, these formulas agree and Φ_β has the stated property. □

Remark. Φ_β is piecewise linear, a property that we will recover from the other "explicit formulas" of the theory.

Consider now a sequence (λ_n, α_n) with $\lambda_n \in \mathbb{R}_+$ and a sequence $\alpha_n \in \mathcal{S}$ such that, for all $\beta \in \mathcal{S}$, the sequence $\lambda_n i(\alpha_n, \beta)$ converges. Denote by $\lim(\lambda_n, \alpha_n)$ the functional

$$\lim(\lambda_n, \alpha_n)(\beta) = \lim \lambda_n i(\alpha_n, \beta).$$

Since Φ_β is homogeneous, we have

$$\lim(\lambda_n, \alpha_n)(\beta) = \Phi_\beta(\lim(\lambda_n, \alpha_n)(e_1), \lim(\lambda_n, \alpha_n)(e_2), \lim(\lambda_n, \alpha_n)(e_3)).$$

This implies that the bijection of $\mathbb{R}_+ \times \mathcal{S}$, regarded as part of $\mathbb{R}_+^{\mathcal{S}}$, onto the rational rays of $\operatorname{cone}(\partial(\nabla \leq))$ extends to a homogeneous homeomorphism:

$$\overline{\mathbb{R}_+ \times \mathcal{S}} \simeq \operatorname{cone}(\partial(\nabla \leq)) \simeq \mathbb{R}^2.$$

Thus, in $P(\mathbb{R}_+^{\mathcal{S}})$, we have $\overline{\mathcal{S}} \simeq S^1$.

Consider a measured foliation $\mathcal{F}$ of T^2. One can show that $\mathcal{F}$ has no singularities and that it is transversely orientable (this is a consequence of a simple Euler–Poincaré type formula). We can identify $\mathcal{F}$ with a closed nonsingular 1-form. This form is then isotopic to a unique *linear form* (a 1-form with constant coefficients in the canonical coordinates on T^2) [Ste69].

If ω is linear, every curve $\gamma = mx_1 + nx_2$ is transverse to ω, or else is contained in a leaf; thus

$$\left|\int_\gamma \omega\right|^2 = I(\omega, \gamma).$$

The form ω is determined up to sign by $I(\omega, e_1)$, $I(\omega, e_2)$, and $I(\omega, e_3)$. The following lemma is now clear.

LEMMA 1.10. *Let $\mathcal{F}$ be a measured foliation on T^2.*

1. *The numbers $I(\mathcal{F}, e_1)$, $I(\mathcal{F}, e_2)$, and $I(\mathcal{F}, e_3)$ form a degenerate triangle.*
2. *If $\beta \in \mathcal{S}$, we have*

$$I(\mathcal{F}, \beta) = \Phi_\beta(I(\mathcal{F}, e_1), I(\mathcal{F}, e_2), I(\mathcal{F}, e_3))$$

where Φ_β is the function from Lemma 1.9.

The first point is clear from Figure 1.6.

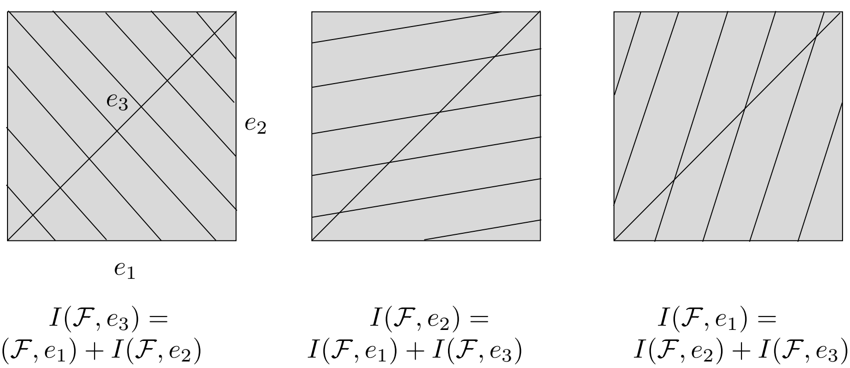

Figure 1.6. Proof of Lemma 1.10 part (1)

As a consequence, in $P(\mathbb{R}_+^{\mathcal{S}})$, we have $\pi \circ I_*(\mathcal{MF}) = \overline{\mathcal{S}}$.

In Section 1.4, we defined Teichmüller space for surfaces of negative Euler characteristic. For T^2, one may give an analogous definition by considering flat metrics ($K = 0$) such that $\text{Area}(T^2) = 1$. (This normalization condition is not needed in the hyperbolic case, since, by the Gauss–Bonnet theorem, the area of a surface is determined by its topology.)

Remark 1. Instead of this normalization, one may work with flat metrics up to scaling. What is more, if T^2 is given a complex structure, its universal cover $\widetilde{T^2}$ is isomorphic to $\mathbb{C}$ and the group of linear automorphisms of $\mathbb{C}$, namely $\{z \mapsto \alpha z + \beta : \alpha, \beta \in \mathbb{C}\}$, coincides with the group of orientation-preserving maps of $\mathbb{R}^2$ preserving the Euclidean metric up to a scalar. From this, one easily deduces the equivalence of our definition of $\mathcal{T}$ with the classical definition as the set of complex marked structures on T^2, up to isotopy.

Remark 2. A flat structure on T^2 has an underlying affine structure. If we fix two generators e_1 and e_2 for $\pi_1(T^2)$, the affine structure underlying the metric

ρ is determined by the data of all the geodesics in the class e_i, which are parallel closed curves, as well as all of the numbers

$$\operatorname{dist}\left(\frac{\Delta}{\Delta'}\right) \Big/ \operatorname{dist}\left(\frac{\Delta'}{\Delta''}\right) \in \mathbb{R}_+$$

where $\{\Delta, \Delta', \Delta''\}$ is an arbitrary triple of geodesics all parallel to e_1 or all parallel to e_2. It is easy to see that any affine structures on T^2 with the same data are isotopic to each other. Thus we may always represent an element of $\mathcal{T}$ by a flat metric ρ whose underlying affine structure is the canonical structure (this choice will always be made in what follows). In other words, the usual straight lines are the geodesics for ρ.

To an element $\rho \in \mathcal{T}$, we may associate the triple (X_1, X_2, X_3), where $X_j = \rho(e_j)/\sum_k \rho(e_k)$, and where $\rho(e_j)$ is the length of the geodesic e_j in the metric ρ.

LEMMA 1.11. *The above map is a homeomorphism* $\mathcal{T} \to \operatorname{int}(\nabla \leq)$.

Proof. It is clear that (X_1, X_2, X_3) satisfies the triangle inequality. Let Δ be a triangle in $\mathbb{R}^2$; every assignment of lengths to the sides satisfying the triangle inequality determines a flat metric on $\mathbb{R}^2$ compatible with the affine structure; this is invariant under the group of translations, hence induces a metric on T^2. This shows surjectivity. For injectivity, we note that two flat metrics with standard affine structures giving the same lengths to the sides of Δ are identical. The topology is left for the reader.

In other words, the composition

$$\mathcal{T} \xrightarrow{\ell_*} \mathbb{R}_+^{\mathcal{S}} \xrightarrow{\text{proj}} \mathbb{R}_+^{(e_1,e_2,e_3)}$$

is a homeomorphism of $\mathcal{T}$ onto its image. To see that ℓ_* is a homeomorphism onto its image, note that the length of a given line segment depends continuously on the lengths assigned to e_1, e_2, e_3 (classical trigonometry!).

We have

$$\ell_*(\mathcal{T}) \bigcap I_*(\mathcal{MF}) = \emptyset.$$

Indeed, if ω is a differential form, there exists a sequence γ_n of simple closed curves such that $\int_{\gamma_n} \omega \to 0$; if α_n denotes the class of γ_n in $\mathcal{S}$, we have $I_*(\omega)(\alpha_n) \to 0$, while for a given metric the lengths of the closed geodesics do not approach zero. □

To prove the analog of Theorem 1.5 for the torus T^2, it remains to prove the following lemma.

LEMMA 1.12. *Let ρ_n be a sequence of flat metrics (normalized to the canonical affine structure), λ_n a sequence of positive reals, and ω a linear form. Suppose that, for $j = 1, 2, 3$, we have*

$$\lambda_n \rho_n(e_j) \to \left| \int_{e_j} \omega \right|.$$

Then, for all closed geodesics α, we have

$$\lambda_n \rho_n(\alpha) \to \left| \int_\alpha \omega \right|.$$

Proof. Let ρ'_n denote the metric $\lambda_n \rho_n$. We treat the case where ω is on the face $X_3 = X_1 + X_2$ of cone($\partial(\nabla \leq)$) (Figure 1.7) and $\int_{e_i} \omega \neq 0$ for $i = 1, 2$.

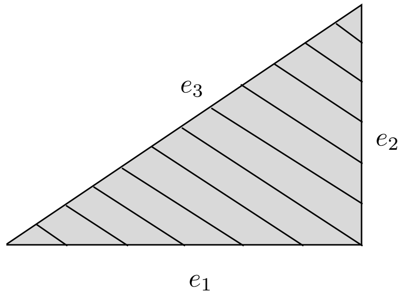

Figure 1.7.

For $j = 1, 2, 3$, we orient e_j so that $\int_{e_j} \omega \geq 0$. Now, let θ_n be the measure of the angle between e_1 and e_2 in the metric ρ'_n.

By the law of cosines, we have

$$[\rho'_n(e_3)]^2 = [\rho'_n(e_1)]^2 + [\rho'_n(e_2)]^2 - 2\rho'_n(e_1)\rho'_n(e_2)\cos\theta_n.$$

The hypothesis then implies that $\cos\theta_n$ tends to -1. If α is a linear segment, say $\alpha = a_1 e_1 + a_2 e_2$, where $a_1, a_2 \in \mathbb{R}$, we have

$$[\rho'_n(\alpha)]^2 = a_1^2[\rho'_n(e_1)]^2 + a_2^2[\rho'_n(e_2)]^2 - 2a_1 a_2 \rho'_n(e_1)\rho'_n(e_2)\cos\theta_n.$$

Thus,

$$[\rho'_n(\alpha)]^2 \to \left[a_1 \int_{e_1} \omega + a_2 \int_{e_2} \omega\right]^2 = \left[\int_\alpha \omega\right]^2.$$

For T^2 the analysis of Theorem 1.6 is trivial. As for Theorem 1.7, it reduces in the case of the torus to a spectral property well-known in linear algebra. □