# CHAPTER ONE

# Introduction

## 1.1 What Is Dynamics?

Dynamics is the science that describes the motion of bodies. Also called *mechanics* (we use the terms interchangeably throughout the book), its development was the first great success of modern physics. Much notation has changed, and physics has grown more sophisticated, but we still use the same fundamental ideas that Isaac Newton developed more than 300 years ago (using the formulation provided by Leonhard Euler and Joseph Louis Lagrange). The basic mathematical formulation and physical principles have stood the test of time and are indispensable tools of the practicing engineer.

Let's be more precise in our definition. Dynamics is the discipline that determines the position and velocity of an object under the action of forces. Specifically, it is about finding a set of differential equations that can be solved (either exactly or numerically on a computer) to determine the trajectory of a body.

In only the second paragraph of the book we have already introduced a great number of terms that require careful, mathematical definitions to proceed with the physics and eventually solve problems (and, perhaps, understand our admittedly very qualitative definition): *position*, *velocity*, *orientation*, *force*, *object*, *body*, *differential equation*, and *trajectory*. Although you may have an intuitive idea of what some of these terms represent, all have rigorous meanings in the context of dynamics. This rigor—and careful notation—is an essential part of the way we approach the subject of dynamics in this book. If you find some of the notation to be rather burdensome and superfluous early on, trust us! By the time you reach Part Two, you will find it indispensable.

We begin in this chapter and the next by providing qualitative definitions of the important concepts that introduce you to our notation, using only relatively simple ideas from geometry and calculus. In Chapter 3, we are much more careful and present the precise mathematical definitions as well as the full vector formulation of dynamics.

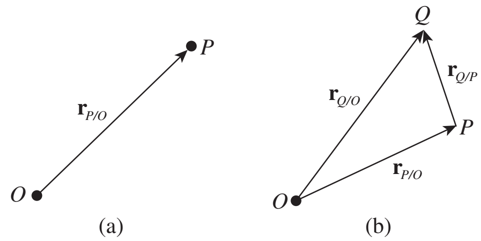


**Figure 1.1** (a) Vector $\mathbf{r}_{P/O}$ from reference point $O$ to point $P$ represents the position of the point $P$ relative to $O$. (b) The addition of two vectors, $\mathbf{r}_{P/O}$ and $\mathbf{r}_{Q/P}$, to get the resultant vector $\mathbf{r}_{Q/O}$.

### 1.1.1 Vectors

We live in a three-dimensional Euclidean[1] universe; we can completely locate the *position* of a point $P$ relative to a reference point $O$ in space by its relative distance in three perpendicular directions. (In Part One we talk about points rather than extended bodies and, consequently, don't have to keep track of the orientation of a body, as is necessary when discussing rigid bodies in Parts Three and Four.) We often call the reference point $O$ the *origin*. An abstract quantity, the *vector*, is defined to represent the position of $P$ relative to $O$, both in distance and direction.

**Qualitative Definition 1.1** A **vector** is a geometric entity that has both magnitude and direction in space.[2]

A position vector is denoted by a boldface, roman-type letter with subscripts that indicate its head and tail. For example, the position $\mathbf{r}_{P/O}$ of point $P$ relative to the origin $O$ is a vector (Figure 1.1). An important geometric property of vectors is that they can be added to get a new vector, called the *resultant* vector. Figure 1.1b illustrates how two vectors are added to obtain a new vector of different magnitude and direction by placing the summed vectors "head to tail."

When the position of point $P$ changes with time, the position at time $t$ is denoted by $\mathbf{r}_{P/O}(t)$. In this case, the *velocity* of point $P$ with respect to $O$ is also a vector. However, to define the velocity correctly, we need to introduce the concept of a *reference frame*.

### 1.1.2 Reference Frames, Coordinates, and Velocity

We have all heard about reference frames since high school, and you may already have an idea of what one is. For example, on a moving train, objects that are stationary on the train—and thus with respect to a reference frame fixed to the train—move with respect to a reference frame fixed to the ground (as in Figure 1.2). To successfully use dynamics, such an intuitive understanding is essential. Later chapters discuss how reference frames fit into the physics and how to use them mathematically; for that

[1] Euclid of Alexandria (ca. 325–265 BCE) was a Greek mathematician considered to be the father of geometry. In his book *The Elements*, he laid out the basic foundations of geometry and the axiomatic method.
[2] In this book, a *qualitative* definition is typically followed by an *operational* or *mathematical* definition of the same term, although the latter definition may come in a later chapter.

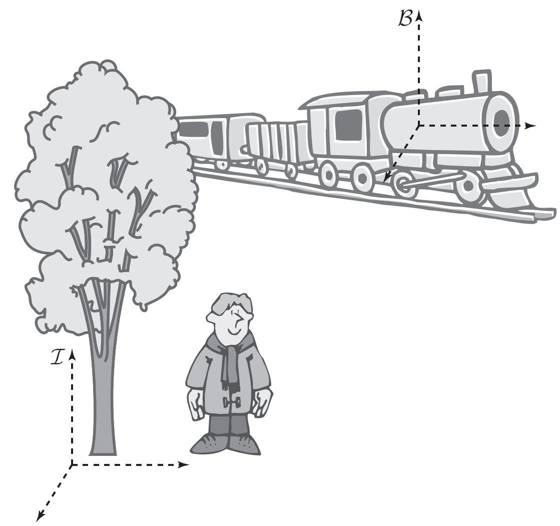


**Figure 1.2** Qualitative definition of a reference frame.

reason, we revisit the topic again in Chapter 3. For now, we summarize our intuition in the following qualitative definition of a reference frame.

**Qualitative Definition 1.2** A **reference frame** is a point of view from which observations and measurements are made regarding the motion of a system.

It is impossible to overemphasize the importance of this concept. Solving a problem in dynamics always starts with defining the necessary reference frames.

From basic geometry, you may be used to seeing a reference frame written as three perpendicular axes meeting at an origin $O$, as illustrated in Figure 1.3. This representation is standard, as it highlights the three orthogonal Euclidean directions. However, this recollection should not be confused with a *coordinate system*. The reference frame and the coordinate system are not the same concept, but rather complement one another. It is necessary to introduce the reference frame to define a coordinate system, which we do next.

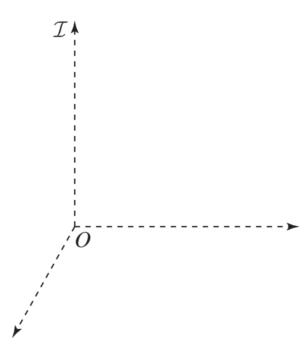


**Figure 1.3** Reference frame $\mathcal{I}$ is represented by three mutually perpendicular axes meeting at origin $O$.

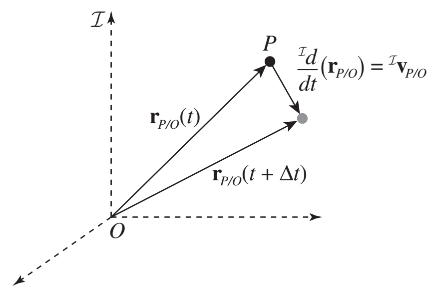


**Figure 1.4** Velocity ${}^{\mathcal{I}}\mathbf{v}_{P/O}$ is the instantaneous rate of change of position $\mathbf{r}_{P/O}$ with respect to frame $\mathcal{I}$. That is, ${}^{\mathcal{I}}\mathbf{v}_{P/O} = (\mathbf{r}_{P/O}(t + \Delta t) - \mathbf{r}_{P/O}(t))/\Delta t$, in the limit $\Delta t \to 0$.

**Definition 1.3** A **coordinate system** is the set of scalars that locate the position of a point relative to another point in a reference frame.

In our three-dimensional Euclidean universe, it takes three scalars to specify the position of a point $P$ in a reference frame. The most natural set of scalars (the three numbers usually labeled $x$, $y$, and $z$) are *Cartesian coordinates*.[3] These coordinates represent the location of $P$ in each of the three orthogonal directions of the reference frame. (Recall the discussion of vectors in the previous section stating that the position of $P$ relative to $O$ is specified in three perpendicular directions.) Cartesian coordinates, however, are only one possible set of the many different scalar coordinates, a number of which are discussed later in the book. Nevertheless, we begin the study of dynamics with Cartesian coordinates because they have a one-to-one correspondence with the directions of a reference frame. It is for this reason that the Cartesian-coordinate directions are often thought to define the reference frame (but don't let this lure you into forgetting the distinction between a coordinate system and a reference frame). We return to the concepts of reference frames and coordinate systems and discuss the relationship between a coordinate system and a vector in Chapter 3.

Throughout the book, reference frames are always labeled. Later we will be solving problems that employ many different frames, and these labels will become very important. Thus we often write the three Cartesian coordinates as $(x, y, z)_{\mathcal{I}}$, explicitly noting the reference frame—here labeled $\mathcal{I}$—in which the coordinates are specified. (The reason for the letter $\mathcal{I}$ will become apparent later.)

Likewise, the change in time of a point's position (i.e., the velocity) only has meaning when referred to some reference frame (recall the train example). For that reason, we always explicitly point out the appropriate reference frame when writing the velocity. A superscript calligraphic letter is used to indicate the frame. Figure 1.4 shows a schematic picture of the velocity ${}^{\mathcal{I}}\mathbf{v}_{P/O} \triangleq \frac{^{\mathcal{I}}d}{dt}(\mathbf{r}_{P/O})$ as the instantaneous rate of change in time of the position $\mathbf{r}_{P/O}$ with respect to the frame $\mathcal{I}$.[4]

We can also express the velocity of point $P$ with respect to $O$ as the rate of change $(\dot{x}, \dot{y}, \dot{z})_{\mathcal{I}} \triangleq \frac{d}{dt}(x, y, z)_{\mathcal{I}}$, where $\dot{x} \triangleq \frac{dx}{dt}$, $\dot{y} \triangleq \frac{dy}{dt}$, and $\dot{z} \triangleq \frac{dz}{dt}$. (Appendix A reviews

[3] Named after René Descartes (1596–1650), the celebrated French philosopher, who founded analytic geometry and invented the notation.
[4] In this book, the symbol $\triangleq$ denotes a definition as opposed to an equality.

some basic rules of calculus if you are rusty.) Because the variables $x$, $y$, and $z$ are scalars, their time derivatives do not need a frame identification. We maintain the notation $(\dot{x}, \dot{y}, \dot{z})_{\mathcal{I}}$, however, to remind you that these three scalars are the rates of change of the three position coordinates in frame $\mathcal{I}$. The rate of change of a scalar Cartesian coordinate is called the *speed* to distinguish it from the velocity. We return to this topic and discuss it in depth and more formally in Chapter 3.

### 1.1.3 Equations of Motion

We now return to the definition of dynamics. *Trajectory* signifies the complete specification of the three positions and three speeds of a point in a reference frame as a function of time. It takes six quantities in our three-dimensional universe to completely specify the motion of a point. This is not necessarily obvious. Why six quantities and not three? Isn't the position enough (since we can always find the velocity by differentiating)? The answer is no, because dynamics is about more than just specifying the position and velocity. It is about finding equations, based on Newton's laws, that allow us to predict the complete trajectory of an object given only its *state* at a single moment in time. By state we mean the three positions and three speeds of the point. These six quantities, defined at a single moment in time, are called the *initial conditions*. The tools of dynamics allow us to find a set of differential equations that can be solved—using these initial conditions—for the position and velocity at any later time. These differential equations are called *equations of motion*.[5]

**Definition 1.4** The **equations of motion of a point** are three second-order differential equations[6] whose solution is the position and velocity of the point as a function of time.

To see this a bit more clearly, imagine that we know the three position variables $x(t)$, $y(t)$, and $z(t)$ of a point in frame $\mathcal{I}$ at some time $t$ and wish to know the position some short time later, $t + \Delta t$. Without the velocity at $t$ we are lost; the point could move anywhere. However, with the three speeds $\dot{x}(t)$, $\dot{y}(t)$, and $\dot{z}(t)$, we know everything; the new position of the point in $\mathcal{I}$ is $(x(t) + \dot{x}(t)\Delta t,\ y(t) + \dot{y}(t)\Delta t,\ z(t) + \dot{z}(t)\Delta t)_{\mathcal{I}}$. The equations of motion allow us to find the speeds at time $t + \Delta t$. The six positions and speeds are sufficient to find the complete trajectory.

As an example, one of the simplest equations of motion is that for a mass on a spring. The position of the mass is given by the Cartesian coordinate $x$, and the force due to the spring is given by $-kx$ (see Figure 2.7c). The position thus satisfies the following second-order differential equation, obtained by equating the force with the mass times acceleration and solving for the acceleration:

$$\ddot{x} = -\frac{k}{m}x.$$

This differential equation is an equation of motion. Its solution gives $x(t)$ and $\dot{x}(t)$, the trajectory of the mass point. Don't worry if you didn't follow how the equation was obtained; that is covered in Chapter 2.

[5] Appendix C supplies a brief review of differential equations.
[6] Or, equivalently, six first-order differential equations.

Many equations of motion cannot be solved exactly; a computer is required to find numerical trajectories. You will have an opportunity to do this many times in this book. However, often we skip solving for the trajectory and find special solutions or conditions on the states by setting the time equal to a specific value, finding certain conditions on the forces, or setting the acceleration to a constant or zero (sometimes called a *steady state*). One particularly useful such solution is known as an *equilibrium point.* The mathematical details of equilibrium solutions are presented in Chapter 12, but it is useful to have a qualitative understanding now, as we will be finding equilibrium solutions of many systems here and there throughout the book.

**Qualitative Definition 1.5** An **equilibrium point** of a dynamic system is a specific solution of the equations of motion in which the rates of change of the states are all zero.

In other words, an equilibrium point is a configuration in which the system is at rest. For the mass-spring system, for example, there is one equilibrium point, which corresponds to the mass situated at precisely the rest length of the spring. Mathematically, if $x(t)$ is an equilibrium point, then $\dot{x}(t) = 0$ and $\ddot{x}(t) = 0$. Thus $x(t) = x(0)$, where $x(0)$ is the initial condition at time $t = 0$. So an equilibrium point is a solution whose value over time remains equal to its initial value.

In summary, dynamics is about finding three second-order differential equations that can be solved for the complete trajectory of an object. The equations can be solved—using the six initial conditions—either analytically (by hand) or numerically (by a computer). It is true that other scalar quantities can be used to specify the position rather than Cartesian coordinates; we will begin to study alternate coordinate systems in detail in Chapter 3. However, we will always need six independent scalars. The remainder of this book describes methods for finding equations of motion—first for a point (particle) and later for extended (rigid) bodies—and presents various techniques for completely or partially solving them.

## 1.2 Organization of the Book

The next chapter reviews the physics of mechanics, covering Newton's laws in depth. We also start to solve simple problems. All the essential physical concepts that form the foundation for the rest of the book are presented in that chapter. Our approach is slightly unconventional in that we begin solving dynamics problems at the outset—in Chapter 2—to highlight the meaning of Newton's laws and how we incorporate the underlying postulates[7] into our methodology.

The remainder of the book is divided into five parts plus a set of four appendices. We divide the book into parts to highlight the logical separation of main topics and show how rigid-body motion builds on the key concepts of particle motion. The material could be covered in one semester by leaving out certain topics or stretched over multiple semesters or quarters. In Part One we restrict ourselves to studying only the planar motion of single particles. Thus motion in only two dimensions is studied; we thus need only four scalars to specify a particle's state

[7] A postulate is a basic assumption that is accepted without proof.

rather than six. We do this to simplify the mathematics and focus on the key physical concepts, allowing you to develop an understanding of the procedures used to solve dynamics problems. You will solve an amazing array of real and important problems in Part One.

Chapter 3 returns to first principles and lays out the mathematical framework for a full vector treatment of kinematics and dynamics in the plane. Our focus is on the use of various coordinate systems and approaches to treating velocity and acceleration. Throughout the chapter we return to the same example: the simple pendulum. While this example may seem a bit academic, our approach is to focus repeatedly on this relatively simple system to emphasize the various new techniques presented and explain how they interrelate and add value. At the end of the chapter these new concepts are used to solve a selection of more difficult problems.

Chapters 4 and 5 present the concepts of momentum and energy, respectively, for a particle. It is here that we begin to solve equations of motion for the *characteristics* of trajectories (also called *integrals of the motion*). These ideas will be useful throughout the remainder of your study of dynamics and form the foundation of modern physics.

Part Two presents an introduction to multiparticle systems (Chapters 6 and 7). The previous concepts are generalized to simultaneously study many, possibly interacting, particles. In Chapter 6 we introduce two important examples of multiparticle systems—collisions and variable-mass systems. Chapter 7 sets the stage for the rigid-body discussions in Parts Three and Four by analyzing angular momentum and energy for many particles.

Part Three introduces rigid-body dynamics in the plane. We show (Chapters 8 and 9) how to specialize our tools to study a rigid collection of particles (i.e., particles whose relative positions are fixed). In particular, the definition of equations of motion is expanded to include the differential equations that describe the orientation of a rigid body. We use these ideas to study a variety of important engineering systems. We still confine our study to motion in the plane, however, to focus on the physical concepts without being burdened by the complexity of three-dimensional kinematics. It is here that we introduce the moment of inertia and, most importantly, the separation of angular momentum.

Part Four develops the full three-dimensional equations that describe the motion of multiparticle systems and rigid bodies. Part Four (Chapter 10) begins with the study of the general orientation of reference frames, three-dimensional angular velocity, and the full vector kinematics of particles and rigid bodies. Chapter 11 completes the discussion by developing the equations of motion for three-dimensional rigid-body motion. It is here that we find the amazing and beautiful motion associated with rotation and spin, such as the gyroscope and the bicycle wheel.

Part Five—Advanced Topics—allows for greater exploration of important ideas and serves to whet the appetite for later courses in dynamics. Chapter 12 treats three important problems in dynamics more deeply, exploring how the concepts in the book are used to understand and synthesize engineered systems. This introduction is useful for future coursework in dynamics and dynamical systems. Chapter 13 includes a brief introduction to Lagrange's method and Kane's method. It serves as a bridge to your later, more advanced classes in dynamics and provides a first look at alternative techniques for finding equations of motion.

We have organized the book in a way that maximizes the use of problems and examples to enhance learning. Throughout the text we solve specific *examples*—sometimes repeated using different methods—to illustrate key concepts. Toward the end of each chapter we include a *tutorials* section. Tutorials are slightly longer than examples; they synthesize the material of the chapter and illustrate the important ideas on real systems. The tutorials are an essential learning tool to introduce useful techniques that may reappear later in the book. The tutorials vary widely in length, depth, and difficulty. You may want to skim the longer or more difficult tutorials on the first read and return later for reinforcement of key concepts or for practice on difficult problems. We have intentionally incorporated this range of tutorials to maximize the utility of the text for the widest possible audience and to make it a practical and helpful reference throughout your career.[8]

We also include computation in many of our examples, tutorials, and problems. Computation is central to modern engineering and an important skill to be learned. It is integral to the learning and practice of dynamics. To simplify our presentation and make it consistent throughout the book, we have exclusively used MATLAB for all numerical work. There are many excellent numerical packages available (and some students may want to code their own). We chose MATLAB because of its ubiquity, its ease of use, and the transparent nature of its programming language. Our goal, however, is not to teach the use of a particular programming tool but for you to become comfortable with the full problem-solving process, from model building through solution.

We end each chapter with a summary of *key ideas*, which contains a short list of the main topics of the chapter. We intentionally minimize the prose in these sections to make it as easy as possible to use for reference and review. Reading these sections does not replace reading the chapters; they are meant only to serve as helpful references.

We used many sources in preparing this book and are indebted to a large number of authors that preceded us. Our primary references are listed in the Bibliography. In some cases, however, we highlight a particularly important result and direct you to other references with more in-depth discussions or additional insights. Thus each chapter has a Notes and Further Reading section, where we point out these sources.

Finally, we end each chapter in Parts One to Four with a problems section that includes problems that address each of the topics of the chapter. We have tried to provide problems of varying levels of difficulty and those that require computation. We have not included problems sections in Part Five, as Chapters 12 and 13 are intended as only an introduction to more advanced topics.

## 1.3 Key Ideas

- A **vector** is a quantity with both magnitude and direction in space. The position of point $P$ relative to point $O$ is the vector $\mathbf{r}_{P/O}$.
- A **reference frame** provides the perspective for observations regarding the motion of a system. A reference frame contains three orthogonal directions.

[8] Because Chapters 12 and 13 are similar to extended tutorials and are meant as only an introduction to more advanced material, we do not include tutorials or problems in them.

- The **velocity** is the change in time of a position with respect to a particular reference frame. The velocity of point $P$ relative to frame $\mathcal{I}$ is ${}^{\mathcal{I}}\mathbf{v}_{P/O} \triangleq \frac{{}^{\mathcal{I}}d}{dt}(\mathbf{r}_{P/O})$.
- A **coordinate system** is the set of scalars used to locate a point relative to another point in a reference frame. **Cartesian coordinates** $x$, $y$, and $z$ constitute the most common coordinate system. We usually use $(x, y, z)_{\mathcal{I}}$ to represent the Cartesian coordinates with respect to frame $\mathcal{I}$. The rates of change $(\dot{x}, \dot{y}, \dot{z})_{\mathcal{I}}$ of the Cartesian coordinates are called *speeds*.
- The **state** of a particle consists of its position and velocity in a reference frame at a given time.
- The **equations of motion** are the three second-order differential equations for the particle state whose solution provides the trajectory of a point.
- An **equilibrium point** is a special solution of the equations of motion for which the rates of change of all states are zero.

## 1.4 Notes and Further Reading

The modern formulation of dynamics is the culmination of more than two centuries of development. For instance, while Newton presented the fundamental physics, the concept of equations of motion and the formulation of the second law we know today were given by Euler.[9] The modern concept of a vector was introduced by Hamilton in the mid-nineteenth century.[10] A good, concise discussion of the early history of dynamics can be found in Tenenbaum (2004). A more thorough treatment of the history of mechanics is in Dugas (1988). We also recommend the book of essays by Truesdell (1968) for insightful discussions of important historical developments.

Careful notation is essential for both learning dynamics and solving problems in your professional career. Unfortunately, no universally accepted notation is in use. In fact, there is much discussion among educators and practitioners over how to balance simplicity and clarity. Our notation—particularly the use of reference frames in derivatives—is closest to that of Kane (1978) and Kane and Levinson (1985). A similar notational approach is used by Tenenbaum (2004) and Rao (2006). Our notation for position is also used in Tongue and Sheppard (2005) with a variation in Beer et al. (2007). Our qualitative definition of reference frames is similar to that in Rao (2006). Other good discussions of the importance of reference frames in dynamics can be found in Greenwood (1988), Kane and Levinson (1985), and Tenenbaum (2004). Tenenbaum also has a similar and insightful discussion regarding the distinction between coordinate systems and reference frames.

[9] Leonhard Euler (1707–1783) was a Swiss mathematician and physicist. He is known for his seminal contributions in mathematics, dynamics, optics, and astronomy. Much of our current notation is attributable to Euler. He is probably best known for the identity $e^{i\pi} + 1 = 0$, often called the most beautiful equation in mathematics.

[10] Sir William Rowan Hamilton (1805–1865) was an Irish mathematician and physicist. He made fundamental contributions to dynamics and other related fields. His energy-based formulation is the foundation of modern quantum mechanics.

## 1.5 Problems

**1.1** What are the Cartesian coordinates of point $P$ in frame $\mathcal{I}$, as shown in Figure 1.5?

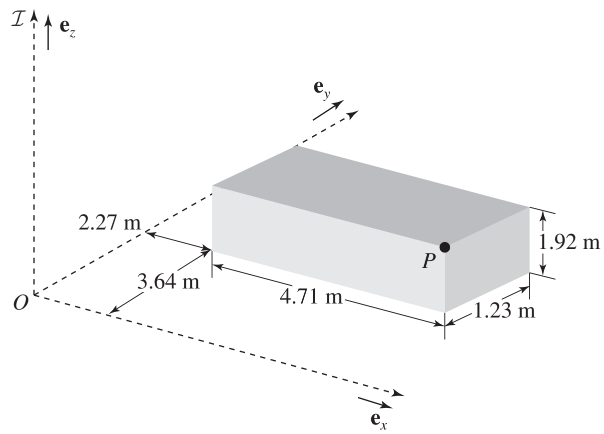


**Figure 1.5** Problem 1.1.

**1.2** Sketch and label the vectors $\mathbf{r}_{P/O}$, $\mathbf{r}_{P/Q}$, $\mathbf{r}_{Q/P}$ in Figure 1.6.

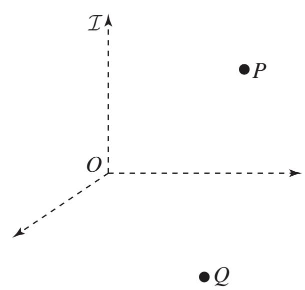


**Figure 1.6** Problem 1.2.

**1.3** Match each of the following definitions to the appropriate term below:

a. A perspective for observations regarding the motion of a system
b. A mathematical quantity with both magnitude and direction
c. Second-order differential equations whose solution is the trajectory of a point
d. A set of scalars used to locate a point relative to another point

- Vector
- Reference frame
- Coordinate system
- Equations of motion