Chapter 1

MATHEMATICS IN THE AIR

Most mathematical tricks make for poor magic and in fact have very little mathematics in them. The phrase "mathematical card trick" conjures up visions of endless dealing into piles and audience members wondering how long they will have to sit politely. Our charge is to present entertaining tricks that are easy to perform and yet have interesting mathematics inside them. We cannot do this without your help. To get started, please go find four playing cards. They can be any four cards, all different or the four aces. It doesn't matter. Let us begin by performing the trick for you. Since we can do it without being present, you'll be able to do it for a friend on the phone. After practicing, try calling your kid brother or your mom and perform the following.

Figure 1. Four cards

Figure 2. Four cards in a packet

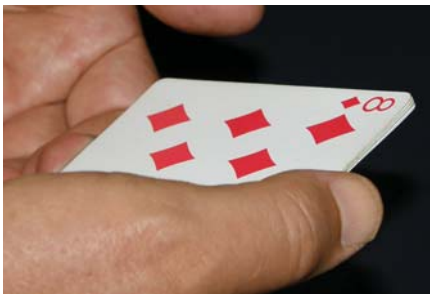

Figure 3. Looking at bottom card

Have a look at the bottom card of the packet. That's your card and you have to remember it.

Next, the cards are going to be mixed by some simple instructions. Put the top card on the bottom of the packet. Turn the current top card face-up and place it back on top.

Figure 4. Top card placed on bottom

Figure 5. Current top card turned face-up

Now, give the packet a cut. It doesn't matter how many cards you cut from top to bottom: one, two, three, or four (which is the same as none). Next, spread off the top two cards, keeping them together, and turn them over, placing them back on top.

Figure 6. Cutting the deck

Figure 7. Spreading off and turning over the top two

Cut the cards at random again and then turn the top two over. Give them another cut and turn two over.

Give them a final cut. This cutting and turning has mixed the cards in a random fashion. There is no way anyone can know the order. Remember the name of your card! We're going to find it together.

Figure 8. Cutting again

Figure 9. Turning over two again

Figure 10. Another cut

Figure 11. Turning over two

Figure 12. Turning over the top card

Figure 13. Putting it on the bottom

Turn the top card over (if it's face-down, turn it face-up; if it's face-up, turn it face-down). Put this card on the bottom of the packet.

Figure 14. Putting the top card on the bottom

Figure 15. Turning over the top card

Put the current top card on the bottom of the packet without turning it over. Finally, turn the top card over and place it back on top.

Figure 16. The "oddball" card

Figure 17. The chosen card

Now, we're done. Name your card. Spread out the packet of four. You'll find three cards facing one way and your card facing the opposite way!

When we perform this trick with a live audience in the same room, we try to work it on a man with a tie or a woman with a scarf. We give him or her the four cards with instructions to shuffle, peek at the bottom card, and follow the instructions above until he or she has cut and turned over two a few times. We then ask our subject to put the four cards behind his or her back. The rest of the instructions are carried out with the cards concealed this way. When the cutting and turning

phase is finished, we stare intently at the person's midsection in giving the final two steps of instructions as if we were looking through our subject. Before the final line of instruction we reach over and move the tie or scarf as if it were blocking our view. We have him or her name the card before bringing out the packet.

We have used this trick for an audience of a hundred high school students—each student received a packet of four cards, and the trick was worked simultaneously for all of them. It's a charming trick and really seems to surprise people.

Okay. How does it work? Let's start by making that your problem: How does it work? You'll find it curiously difficult to give a clear explanation. In twenty years of teaching, asking students to try to explain this trick, we have yet to have anyone give a truly clear story. The plan is to lead you through this in stages (it has some math in it). The solution comes later in this chapter. Before proceeding, let's generalize.

The trick is known as Baby Hummer in magic circles. It was invented by magician Charles Hudson as a variation on an original trick by a truly eccentric genius named Bob Hummer. We'll learn a lot more about Hummer as we go along. Here is his original use of the principle we're trying to explain.

Take any ten cards. Have them all face-down and hold them as if you were about to deal in a card game. Go through the following procedure, which mixes the cards face-up and face-down: Spread the top two cards off and turn them over, placing them back on top. Give the cards a straight cut (see figure 6). Repeat this "turn two and cut at random" procedure as often as you like. The cards will be in an unpredictable mess. To find the order in the mess, proceed as follows: Go through the packet, reversing every second card (the cards in positions 2, 4, 6, 8, and 10). You will find exactly five cards face-up, no matter how many times the "turn two and cut at random" procedure was repeated.

Hummer marketed this trick in a privately printed manuscript called "Face-up/Face-down Mysteries" (1942).[1] This ten-card trick does not play as well for audiences as the Baby Hummer we started with. Hummer introduced a kind of swindle as a second phase. After showing that five cards are face-up and five cards are face-down, casually rearrange the cards so that the face-up and face-down cards alternate up, down, up, down, and so on. Hand the ten cards to a spectator who is instructed to put the cards under the table (or behind his or

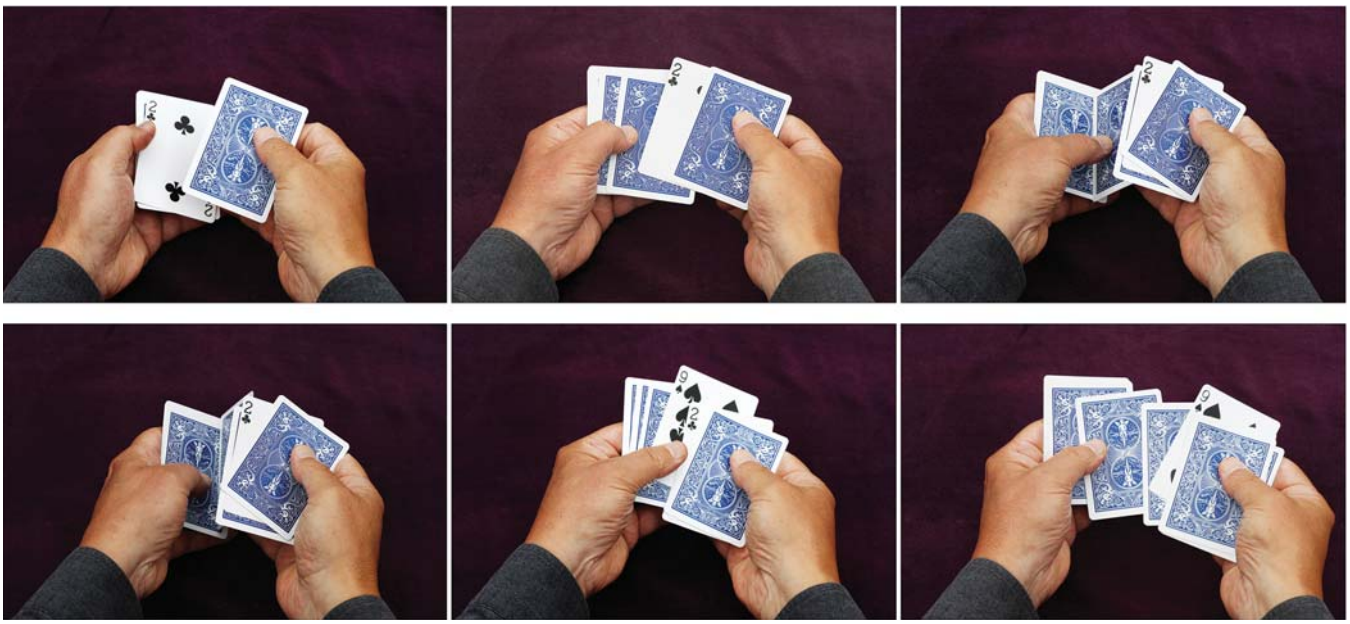

Figure 18. Reversing every second card

her back). Have the spectator repeat the "turn two and cut at random" procedure a few times. Take the cards back without looking at them. Now, with the cards under the table (or behind your back), remove every second card as before and turn them over. You will find that the cards all face the same way.

Again, one may ask, why does this work? *Just what properties of the arrangement are preserved by Hummer's "turn two and cut at random" procedure?* To think about Hummer's "turn two and cut at random" mixing scheme, we find it helpful to have a way of writing down all the possible arrangements that can occur. Instead of working with a deck of four or ten cards, one can just as easily work with a general deck of even size. We work with $2n$ cards (so, if $n = 2$ then $2n = 4$, or if $n = 5$ then $2n = 10$). As will be seen in a while, decks of odd size are a different kettle of fish. We can indicate the exact arrangement of $2n$ cards, some face-up and some face-down, by writing the numbers on the cards in order and identifying face-up with a bar on top of a number. Thus, a four-card deck with a face-up 3 on top, a face-down 1 next, a face-down 4 next, and a face-up 2 at the bottom is denoted $\overline{3}, 1, 4, \overline{2}$. For a deck of ten cards, a possible arrangement is $2, \overline{1}, \overline{4}, 8, 6, \overline{5}, \overline{3}, \overline{10}, \overline{7}, 9$.

The symbols 1, 2, 3, . . . , $2n$ can be arranged in $1\times2\times3\times4\times\cdots\times2n$ ways. This number is often denoted as $(2n)!$ (read "$2n$ factorial"). Each such arrangement can be decorated with bars in $2\times2\times2\times\cdots\times2=2^{2n}$ ways (each of the $2n$ symbols can be barred or not). In all, this makes for $2^{2n}\times(2n)!$ distinct arrangements. This is a huge number even for a moderate n. For $2n = 4$, it is $2^4 \times 4! = 16 \times 24 = 384$. For $2n = 10$, it is 3,715,391,200 (close to four billion). This is the maximum possible number of arrangements. As we will see, not all of these are achievable if we start with a face-down deck using Hummer's "turn two and cut at random" process.

Before we give the general answer, here is a starter result that shows that many of the $2^{2n}\times(2n)!$ arrangements are not achievable. This result also clearly explains why Hummer's ten-card trick works. We present it as a simple theorem to show that theorems can grow anywhere.

THEOREM. Let a deck of $2n$ cards start all face-down. After any number of "turn two and cut at random" operations, the following regularity is forced:

The number of face-up cards at *even* positions
equals
the number of face-up cards at *odd* positions.

Normally, we will put our proofs at the end of each chapter. However, we give the proof for this here. What we want to prove is certainly true when we start—there are no face-up cards in either even or odd positions at the start. Suppose the statement of the theorem holds after some fixed number of shuffles. Observe that it still holds after a single card is cut from top to bottom. Therefore, it holds if any number of cards is cut from top to bottom. So the result to be proved holds for any number of cuts. Finally, suppose that the result to be proved holds for the current deck. Note that the current deck may well have cards face-up and face-down. Let us argue that it continues to hold after the top two cards are turned over and put back on top. We see this by considering all possible arrangements of the top two cards. They may be:

down, down down, up up, down up, up.

After turning two, these four possibilities become:

up, up down, up up, down down, down.

In the middle two cases, the up-down pattern hasn't changed, so the statement holds after turning two if it held at the start. In the first case, the odd positions and the even positions each have one more up card. Since the numbers of face-ups in even and odd positions were equal before we turned two, they are equal after. The same argument works in the last case. This covers all cases and proves the theorem.

From the theorem, it is a short step to see why Hummer's trick works. Start with $2n$ cards face-down ($2n = 10$ for Hummer). After any number of "turn two and cut at random" shuffles, there will be some number of face-up cards. Let A be the number of face-up cards among the n cards at even positions. There must be $n - A$ face-down cards among the even positions since there are n cards in even positions. By the theorem, the same holds for the n cards at odd positions—A face-up and $n - A$ face-down. If you remove the cards at odd positions and turn them over, this gives $n - A$ face-up cards to add to the A face-up cards at even positions. This makes $(n - A) + A = n$ face-up cards in all. Of course, the other n cards are face-down. The conclusion is forced.

Did the proof we just gave ruin the trick? For us, it is a beam of light illuminating a fuzzy mystery. It makes us just as happy to see clearly as to be fooled.

To check your understanding, we mention that, in magic circles, Hummer's principle is sometimes called CATO for "cut and turn over two." This is in opposite order to the "turn over two and cut." The theorem holds for CATO as well as "cut and turn over four" or "turn over an even number and cut."

Later in this chapter we show that exactly $2 \times (2n)!$ arrangements are achievable and just which ones these are. This more general result implies the theorem we just proved and, indeed, all possible theorems about Hummer's mixing process.

In the meantime we turn to the question: How can a really good trick be twisted out of this math? We give as an answer a closely guarded secret of one of the great card men of the present era. Steve Freeman has given us permission to explain what we think is an amazing

amplification of Hummer's shuffles. We explain it by first describing the effect and then the modus operandi. Those wishing to understand why it works will have to study the math at the end of the chapter.

ROYAL HUMMER

First, the effect as the audience sees it. The performer hands the spectator about one-third of the deck, asking that the cards be thoroughly shuffled. Taking the cards back from the spectator, the performer explains that the cards will be further mixed, face-up and face-down, at the spectator's discretion, to make a real mess. The cards are dealt off in pairs, the spectator deciding each time if they should be left as is or turned over. This is repeated with the cards in groups of four. At this point, there is a pile of face-up/face-down cards on the table. The performer says, "I think you must agree that the cards are truly randomly distributed." The spectator gets one more decision—after the performer deals the cards into two piles (left, right, left, right, and so on) the spectator chooses a pile, turns it over, and puts it on top of the other pile. For the denouement, the performer explains that the highest hand in poker, the perfect poker hand, is a royal flush—ace, king, queen, jack, and ten, all of the same suit. The cards are spread and there are exactly five face-down cards. "Five cards—that just makes a poker hand." The five are turned over one at a time—they form a royal flush.

That's the way the trick looks. Here is how it works. Before you begin, look through the deck of cards, as if checking to see if the deck is complete, and place one of the royal flushes on top (they do not have to be in order). Remove the top twenty or so cards. The exact number doesn't matter as long as it's even and contains the royal flush. Have the spectator shuffle these cards. Take the cards back, turn them all face-up, and start spreading through them as you explain the next phase. Look at the first two cards.

Figure 19. Neither card is in the flush

Figure 20. Turning the second card

1. If neither one is in the royal flush, leave the first card face-up and flip the second card face-down, keeping both in their original position (you may use the first card to flip over the second one).

Figure 21. Second card is turned over

Figure 22. Pair is placed on the table

Figure 23. Only the first card is in the flush

Figure 24. First card is turned over

2. If the first one is in the royal flush and the second one is not, flip the first one face-down and then flip the second one face-down (they stay in their original positions).

Figure 25. Second card is also turned over, with the cards kept in order

Figure 26. Pair is placed on the table

3. If the second one is in the royal flush and the first one is not, leave both face-up.

Figure 27. Only the second card is in the flush

Figure 28. Placing the pair face-up in order on the table

4. If both are in the royal flush, flip the first one face-down and leave the second one face-up.

Figure 29. Both cards are in the flush

Figure 30. Turning the first card over

Figure 31. Placing the pair on the table

Figure 32. The completed arrangement

The pairs may be dropped onto the table in a pile after each is adjusted or passed into the other hand. Work through the packet a pair at a time, using the same procedure for each pair. If, by chance, you wind up with an odd number of cards, add an extra card from the rest of the deck.

Now, take off the cards in pairs, asking the spectator to decide, for each pair, whether to "leave them or turn them," and put them into a

pile on the table as dictated. When done, you can pick up the pile and go through the "leave them or turn them" process for pairs as before (or in sets of four, if desired). To finish, deal the cards into two piles (left, right, left, right, . . .). Have the spectator pick up either pile, turn it over, and place it on the other one. If the royal flush cards are not facing down, turn the whole packet over before spreading.

This is a wonderful trick. It really seems as if the mixing is haphazard. The ending shocks people. It does take some practice but it's worth it—a self-working trick done with a borrowed deck (which doesn't have to be complete).

Perhaps the most important lesson to be learned is how a simple mathematical principle, introduced via a fairly weak trick, can be built into something special. This is the result of fifty years of sustained development by the magic community. People from all walks of life spent time turning the trick over, suggesting variations, and being honest about their success or failure. At the beginning and end were two brilliant contributors—Bob Hummer and Steve Freeman. We are in their debt.

A word about practice. The first times you run through, following the procedures (1)–(4) above, will be awkward and slow. After a hundred or so practice runs, you should be able to do it almost subconsciously, without really looking at the cards. A skillful performer must be able to patter along ("We will be turning cards face-up and face-down as we go. You will decide which is which . . ."). The whole proceeding must have a casual, unstudied feel to it. All of this takes practice.

In the rest of this chapter, we explain some math. As a warmup, let us argue that the Baby Hummer trick that begins this chapter always works. To begin with, in the original setup we have three cards facing one way and one card (which we'll call the "oddball") facing the other way. We'll say that cards in positions one and three (from the top) are "mates," as are cards in positions two and four. The setup instructions then force the chosen card and the oddball to be mates. It is easy to check that any "turn two and cut randomly" shuffle (or Hummer shuffle, for short) will preserve this relationship (there are basically only two cases to check). Finally, the finishing instructions have the effect of turning over exactly one card and its mate. This has the effect of forcing the chosen card to be the oddball. Again, two cases to check. End of story.

With all the variations, it is natural to ask just what can be achieved from a face-down packet, originally arranged in order 1, 2, 3, . . . , $2n$, after an arbitrary number of Hummer shuffles. The following theorem delineates exactly what can happen.

THEOREM. After any number of Hummer shuffles of $2n$ cards, any arrangement of values is possible. However, the face-up/face-down pattern is constrained as follows: Consider the card at position i. Add one to its value if face-up. Add this to i. This sum is simultaneously even (or odd) for all positions i.

EXAMPLE. Consider a four-card deck in the final arrangement: 4, $\overline{2}$, $\overline{1}$, 3. In position 1, the sum "position + value + (1 if face-up, and 0 if face-down)" is $1 + 4 + 0 = 5$, which is odd. The other three positions give

$$2 + 2 + 1 = 5,\ 3 + 1 + 1 = 5,\ 4 + 3 + 0 = 7,$$

all odd values.

REMARKS. The constraint in the theorem is the only constraint. All arrangements arrived at by the Hummer shuffling are bound by it, and any pattern of cards that satisfies the constraint is achievable by Hummer shuffles. An interesting unsolved problem is to figure out the minimum number of Hummer shuffles it takes to achieve any particular pattern.

Any property of Hummer shuffles is derivable from the theorem. We record some of these as corollaries.

COROLLARY 1. The number of achievable arrangements for a deck of $2n$ cards after Hummer shuffling is $2 \times (2n)!$.

REMARK. In mathematical language, the set of all achievable arrangements of $2n$ cards after Hummer shuffling forms a group.

COROLLARY 2. (Explanation of Hummer's original trick.) After any number of Hummer shuffles, the number of face-up cards at even positions equals the number of face-up cards at odd

positions. Thus, if the even cards are removed and turned over, the total number of face-up cards is n.

PROOF: Consider the cards at even positions. If there are j even values, all of these must face the same way. Similarly, the $n - j$ odd values must face the other way. At the odd positions, there will be $n - j$ even values all facing in the opposite way to the even values at even positions. When the cards at even positions are removed and turned over, there are $j + (n - j) = n$ facing the same way, with the remaining n facing the opposite way.

COROLLARY TO COROLLARY 2. The argument underlying corollary 2 shows that in fact, after any number of Hummer shuffles followed with every other card removed and reversed, the cards originally at even positions all face the same way (likewise, the cards originally at odd positions all face the opposite way). Let us make this into a trick: Take five red cards and five black cards and arrange them in alternate colors in a face-down pile. Hummer shuffle any number of times, remove every other card, and reverse these. All the red cards face one way and all the black cards face the opposite way. This makes for quite a surprising trick. It may be endlessly varied. For example, remove four aces and six other cards. Place the aces in every second position (i.e., in positions two, four, six, and eight). Turn the bottom card face-up. The cards may be Hummer shuffled any number of times. Follow this by reversing every other card. The four aces will face opposite the remaining cards. Charles Hudson derived a number of entertaining tricks built on this idea. His Baby Hummer trick is explained above. Steve Freeman's Royal Hummer trick may be the ultimate version.

Final Notes. We are not done understanding Hummer shuffles. The following two notes record a natural question (does it only work with even-sized decks?) and a new trick that comes from the analysis. There is a lot we still don't know. (For example, what about turning up three?)

Note 1. It is natural to wonder if the trick will work with an odd number of cards. It would be nice to ask the spectator to remove a

random poker hand of five cards and begin the trick from here. We assume below that "turn two and cut at random" is used throughout.

There is one regularity: There will always be an even number of cards face-up. Alas, this is the only regularity. All $2^{n-1} \times n!$ signed arrangements of n cards (with n odd) are achievable with a deck of n cards.

Let us record one proof of this. First, any three cards can be manipulated so: $123 \to \overline{21}3 \to \overline{23}1 \to 321$, thus transposing positions 1 and 3. By doing this, any permutation of the even positions and also any permutation of the odd positions is possible. Consider transposing positions 1 and 3, and then 3 and 5, and then 5 and 7, . . . , and then $n - 2$ and n. This results in 3, 2, 5, 4, 7, 6, . . . , $n - 1$, 1. For example, with seven cards we get 3, 2, 5, 4, 7, 6, 1. Now transpose consecutive pairs in the even positions, moving the card labeled 2 to the right. This results in 3, 4, 5, . . . , $n - 1$, 2, 1. Finally, cut the bottom two cards to the top. This all results in a simple transposition. As usual, this allows us to transpose any two consecutive cards and so finally to achieve any permutation of the labels.

Next, we show how to achieve any face-up/face-down pattern with an even number of face-up cards (where we use 0 to denote a face-down card, and 1 to denote a face-up card). This is achieved "two at a time." The following moves show how this can be done: 000 . . . 0 → 110 . . . 0 → 11110 . . . 0 → 10010 . . . 0 → 1001110 . . . 0 → 1000010 . . . 0 . . . . After cutting, this gives any possible separation of the 1's (since n is odd). This shows that any pair can be turned face-up. Working one pair at a time shows that any pattern of an even number of cards can be turned face-up. Finally, combining our ability to create arbitrary arrangements of values with an arbitrary face-up/face-down pattern gives the final result.

From the above we may conclude that there is no real extension of Hummer's trick to an odd-sized packet. Of course, the two types of parity delineated above may form the basis for tricks.

Note 2. One reason for developing all this theory is the hope of inventing a new trick. Following is one that comes from our analysis.

Here is the effect. Ask a spectator to remove the ace through ten of spades and arrange them in order (ace–ten or ten–ace—it doesn't matter which). Then turn your back and have the spectator Hummer

shuffle the ten-card packet any number of times. You can promise that you don't know anything about the order of the cards. Ask a spectator to name the values one at a time (from the top down) and you tell them if the cards are face-up or not.

From what was developed above, the only mystery is knowing the orientation (face-up or face-down) of the top card (all else follows). You simply guess! If correct, keep going. If wrong, rub your eyes and ask the spectator to concentrate. Try again! The trick as described may be done on the phone. Note you only need to know the odd/even values of consecutive cards to know their orientation.

Let us be the first to admit that, as described, this is a pretty poor trick. We hope that someone someplace will turn it over and around and come up with something performable. Please let us know (we'll shout it from the rooftops or, if you like, keep it as secret as secret can be).

BACK TO MAGIC

To conclude on a high note, here is Steve Freeman's favorite method of getting set for his Royal Hummer trick. This is a replacement for procedures (1)–(4) above. To begin, you have a packet of twenty or so cards that contains a royal flush, with all cards facing the same way. The royal flush is scattered throughout the packet. The cards will be split into two, one face-up packet in each hand. The hands alternately deal into one pile on the table, turning some cards over. At the end, the indifferent cards at even positions will be face-up. Indifferent cards at odd positions will be face-down. The royal flush cards are opposite. When the cards are dealt into two piles and one pile is turned over on the others, all of the indifferent cards face the same way and all of the royal flush cards are opposite.

To get comfortable with this, try a simple exercise: Take two packets of face-up cards, hold one in each hand in dealing position, and deal alternately into one pile, face-up on the table, left, right, left, right, etc. Do this until you can do it easily. Now, with the same start, try turning the left hand's cards face-down as they are dealt, so that the cards are placed down, up, down, up, and so forth. If this is awkward, try also turning the right hand's cards down (with the left's face-up) and then both hands' cards face-down. It is useful to keep the left/right alternation standardized throughout.

Now for the real thing. Begin with an even number of cards, less than half the deck, containing a royal flush, with all cards face-up. Split these into two roughly equal packets, held face-up in each hand. Each time, deal first from the left then from the right into the pile on the table. Observe the following rules:

1. If two indifferent cards show, deal the left face-up followed by the right face-down.
2. If two royal flush cards show, deal the left face-down followed by the right face-up.
3. With a flush card left and an indifferent card right, deal the left face-down followed by the right face-down.
4. With an indifferent card left and a flush card right, deal the left face-up and the right face-up.

If one hand runs out of cards, just split the remaining cards into two packets and continue. The trick continues as described above. Again, this takes practice to do naturally, accurately, and casually. Several dozen run-throughs might suffice.