Chapter One

SYMMETRIC MARKOVIAN SEMIGROUPS AND DIRICHLET FORMS

1.1. DIRICHLET FORMS AND EXTENDED DIRICHLET SPACES

The concepts of Dirichlet form and Dirichlet space were introduced in 1959 by A. Beurling and J. Deny [8] and the concept of the extended Dirichlet space was given in 1974 by M. L. Silverstein [138]. They all assumed that the underlying state space E is a locally compact separable metric space. Concrete examples of Dirichlet forms (bilinear form, weak solution formulations) have appeared frequently in the theory of partial differential equations and Riemannian geometry. However, the theory of Dirichlet forms goes far beyond these.

In this section, we work with a σ-finite measure space $(E, \mathcal{B}(E), m)$ without any topological assumption on E and establish the correspondence of the above-mentioned notions to the semigroups of symmetric Markovian linear operators. The present arguments are a little longer than the usual ones under the topological assumption found in [39] and [73, §1.4] but they are quite elementary in nature.

Only at the end of this section, we shall assume that E is a Hausdorff topological space and consider the semigroups and Dirichlet forms generated by symmetric Markovian transition kernels on E.

Let $(E, \mathcal{B}(E))$ be a measurable space and m a σ-finite measure on it. Let $\mathcal{B}^m(E)$ be the completion of $\mathcal{B}(E)$ with respect to m. Numerical functions f, g on E are said to be m-equivalent ($f = g$ $[m]$ in notation) if $m(\{x \in E : f(x) \neq g(x)\}) = 0$. For $p \geq 1$ and a numerical function $f \in \mathcal{B}^m(E)$, we put

$$\|f\|_p = \left(\int_E |f(x)|^p m(dx) \right)^{1/p}.$$

The family of all m-equivalence classes of $f \in \mathcal{B}^m(E)$ with $\|f\|_p < \infty$ is denoted by $L^p(E; m)$, which is a Banach space with norm $\|\cdot\|_p$, namely, a complete normed linear space. We denote by $L^\infty(E; m)$ the family of all m-equivalence classes of $f \in \mathcal{B}^m(E)$ which are bounded m-a.e. on E. $L^\infty(E; m)$ is a Banach space with norm

$$\|f\|_\infty := \inf_{N:\, m(N)=0} \sup_{x\in E\setminus N} |f(x)|.$$

Note that $L^2(E;m)$ is a real Hilbert space with inner product

$$(f,g) = \int_E f(x)g(x)m(dx), \qquad f,g \in L^2(E;m).$$

For a moment, let us consider an abstract real Hilbert space H with inner product $(\cdot,\cdot)$. $\sqrt{(f,f)}$ for $f \in H$ is denoted by $\|f\|_H$. As is summarized in Section A.4, there are mutual one-to-one correspondences among four objects on the Hilbert space H: the family of all closed symmetric forms $(\mathcal{E},\mathcal{D}(\mathcal{E}))$, the family of all strongly continuous contraction semigroups $\{T_t; t \geq 0\}$, the family of all strongly continuous contraction resolvents $\{R_\alpha; \alpha > 0\}$, and the family of all non-positive definite self-adjoint operators A. Here we mention the correspondences among the first three objects only.

$\mathcal{E}$ or $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is said to be a *symmetric form* on H if $\mathcal{D}(\mathcal{E})$ is a dense linear subspace of H and $\mathcal{E}$ is a non-negative definite symmetric bilinear form defined on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ in the sense that for every $f,g,h \in \mathcal{D}(\mathcal{E})$ and $a,b \in \mathbb{R}$

$$\mathcal{E}(f,g) = \mathcal{E}(g,f), \quad \mathcal{E}(f,f) \geq 0, \quad \text{and}$$

$$\mathcal{E}(af+bg,h) = a\mathcal{E}(f,h) + b\mathcal{E}(g,h).$$

For $\alpha > 0$, we define

$$\mathcal{E}_\alpha(f,g) = \mathcal{E}(f,g) + \alpha(f,g), \quad f,g \in \mathcal{D}(\mathcal{E}).$$

We call a symmetric form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on H *closed* if $\mathcal{D}(\mathcal{E})$ is complete with norm $\sqrt{\mathcal{E}_1(f,f)}$. $\mathcal{D}(\mathcal{E})$ is then a real Hilbert space with inner product $\mathcal{E}_\alpha$ for each $\alpha > 0$.

A family of symmetric linear operators $\{T_t; t > 0\}$ on H is called a *strongly continuous contraction semigroup* if, for any $f \in H$,

$$T_sT_tf = T_{s+t}f, \quad \|T_tf\|_H \leq \|f\|_H, \quad \lim_{t\downarrow 0} \|T_tf - f\|_H = 0.$$

We call a family of symmetric linear operators $\{G_\alpha; \alpha > 0\}$ on H a *strongly continuous contraction resolvent* if for every $\alpha, \beta > 0$ and $f \in H$,

$$G_\alpha f - G_\beta f + (\alpha-\beta)G_\alpha G_\beta f = 0, \quad \alpha\|G_\alpha f\|_H \leq \|f\|_H,$$

$$\lim_{\alpha\to\infty} \|\alpha G_\alpha f - f\|_H = 0.$$

The semigroup $\{T_t; t \geq 0\}$ and the resolvent $\{G_\alpha; \alpha > 0\}$ as above correspond to each other by the next two equations:

$$G_\alpha f = \int_0^\infty e^{-\alpha t} T_t f dt, \quad f \in H, \tag{1.1.1}$$

the integral on the right hand side being defined in Bochner's sense, and

$$T_t f = \lim_{\beta\to\infty} e^{-t\beta} \sum_{n=0}^{\infty} \frac{(t\beta)^n}{n!} (\beta G_\beta)^n f, \quad f \in H. \tag{1.1.2}$$

$\{G_\alpha; \alpha > 0\}$ determined by (1.1.1) from $\{T_t; t > 0\}$ is called the *resolvent of* $\{T_t; t \geq 0\}$.

Given a strongly continuous contraction symmetric semigroup $\{T_t; t > 0\}$ on H, for each $t > 0$,

$$\mathcal{E}^{(t)}(f, g) := \frac{1}{t}(f - T_t f, g), \quad f, g \in H \tag{1.1.3}$$

defines a symmetric form $\mathcal{E}^{(t)}$ on H with domain H. For each $f \in H$, $\mathcal{E}^{(t)}(f, f)$ is non-negative and increasing as $t > 0$ decreases (this can be shown, for example, by using spectral representation of $\{T_t; t > 0\}$). We may then set

$$\mathcal{D}(\mathcal{E}) = \{f \in H\colon \lim_{t\downarrow 0} \mathcal{E}^{(t)}(f, f) < \infty\}, \tag{1.1.4}$$

$$\mathcal{E}(f, g) = \lim_{t\downarrow 0} \mathcal{E}^{(t)}(f, g), \qquad f, g \in \mathcal{D}(\mathcal{E}), \tag{1.1.5}$$

which becomes a closed symmetric form on H called the *closed symmetric form of the semigroup* $\{T_t; t > 0\}$. We call $\mathcal{E}^{(t)}$ of (1.1.3) the *approximating form* of $\mathcal{E}$.

Conversely, suppose that we are given a closed symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on H. For each $\alpha > 0, f \in H$ and $v \in \mathcal{D}(\mathcal{E})$, we have

$$|(f, v)| \leq \|f\|_2 \|v\|_2 \leq (1/\alpha)^{1/2} \|f\|_2 \sqrt{\mathcal{E}_\alpha(v, v)},$$

which means that $\Phi(v) = (f, v)$ is a bounded linear functional on the Hilbert space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha)$. By the Riesz representation theorem, there exists a unique element of $\mathcal{D}(\mathcal{E})$ denoted by $G_\alpha f$ such that for every $f \in H$ and $v \in \mathcal{D}(\mathcal{E})$,

$$G_\alpha f \in \mathcal{D}(\mathcal{E}) \quad \text{and} \quad \mathcal{E}_\alpha(G_\alpha f, v) = (f, v). \tag{1.1.6}$$

$\{G_\alpha; \alpha > 0\}$ so defined is a strongly continuous contraction resolvent on H, which in turn determines a strongly continuous contraction semigroup

$\{T_t; t > 0\}$ on H by (1.1.2). They are called the *resolvent* and *semigroup generated by the closed symmetric form* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, respectively.

The above-mentioned correspondences from $\{T_t; t > 0\}$ to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and from $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ to $\{T_t; t > 0\}$ are mutually reciprocal.

From now on, we shall take as H the space $L^2(E; m)$ on a σ-finite measure space $(E, \mathcal{B}(E), m)$. In this book, we need to consider extensions of the domain $\mathcal{D}(\mathcal{E})$ of a closed symmetric form $\mathcal{E}$ on $L^2(E; m)$. For this purpose, we shall designate $\mathcal{D}(\mathcal{E})$ by $\mathcal{F}$ so that a closed symmetric form on $L^2(E; m)$ will be denoted by $(\mathcal{E}, \mathcal{F})$. We now proceed to introduce the notions of Dirichlet form and extended Dirichlet space.

DEFINITION 1.1.1. For $1 \le p \le \infty$, a linear operator L on $L^p(E; m)$ with domain of definition $\mathcal{D}(L)$ is called *Markovian* if

$$f \in \mathcal{D}(L) \textit{ with } 0 \le f \le 1 \ [m] \quad \Longrightarrow \quad 0 \le Lf \le 1 \ [m].$$

A real function φ, namely, a mapping from $\mathbb{R}$ to $\mathbb{R}$, is said to be a *normal contraction* if

$$\varphi(0) = 0 \quad \text{and} \quad |\varphi(s) - \varphi(t)| \le |s - t| \text{ for every } s, t \in \mathbb{R}.$$

A function defined by $\varphi(t) = (0 \vee t) \wedge 1$, $t \in \mathbb{R}$, is a normal contraction which is called the *unit contraction*. For any $\varepsilon > 0$, a real function φ_ε satisfying the next condition is a normal contraction:

$$\begin{aligned} &\varphi_\varepsilon(t) = t \text{ for } t \in [0, 1]; \quad -\varepsilon \le \varphi_\varepsilon(t) \le 1 + \varepsilon \text{ for } t \in \mathbb{R}, \\ &0 \le \varphi_\varepsilon(t) - \varphi_\varepsilon(s) \le t - s \quad \text{for } s < t. \end{aligned} \tag{1.1.7}$$

DEFINITION 1.1.2. A symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E; m)$ is called *Markovian* if, for any $\varepsilon > 0$, there exists a real function φ_ε satisfying (1.1.7) and

$$f \in \mathcal{D}(\mathcal{E}) \quad \Longrightarrow \quad g := \varphi_\varepsilon \circ f \in \mathcal{D}(\mathcal{E}) \text{ with } \mathcal{E}(g, g) \le \mathcal{E}(f, f). \tag{1.1.8}$$

A closed symmetric form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ is called a *Dirichlet form* if it is Markovian. In this case, the domain $\mathcal{F}$ is said to be a *Dirichlet space*.

THEOREM 1.1.3. *Let* $(\mathcal{E}, \mathcal{F})$ *be a closed symmetric form on* $L^2(E; m)$ *and* $\{T_t\}_{t>0}$, $\{G_\alpha\}_{\alpha>0}$ *be the strongly continuous contraction semigroup and resolvent on* $L^2(E; m)$ *generated by* $(\mathcal{E}, \mathcal{F})$*, respectively. Then the following conditions are mutually equivalent:*

(a) T_t *is Markovian for each* $t > 0$.
(b) αG_α *is Markovian for each* $\alpha > 0$.
(c) $(\mathcal{E}, \mathcal{F})$ *is a Dirichlet form on* $L^2(E; m)$.

(d) *The unit contraction operates on* $(\mathcal{E}, \mathcal{F})$*:*

$$f \in \mathcal{F} \implies g := (0 \vee f) \wedge 1 \in \mathcal{F} \text{ and } \mathcal{E}(g, g) \leq \mathcal{E}(f, f).$$

(e) *Every normal contraction operates on* $(\mathcal{E}, \mathcal{F})$*: for any normal contraction* φ

$$f \in \mathcal{F} \implies g = \varphi \circ f \in \mathcal{F} \text{ and } \mathcal{E}(g, g) \leq \mathcal{E}(f, f).$$

Proof. The implications (**a**) ⇒ (**b**) and (**b**) ⇒ (**a**) follow from (1.1.1) and (1.1.2), respectively. The implication (**e**) ⇒ (**d**) ⇒ (**c**) is obvious.

(**c**) ⇒ (**b**): We fix an $\alpha > 0$ and a function $f \in L^2(E; m)$ with $0 \leq f \leq 1$ $[m]$, and introduce a quadratic form on $\mathcal{F}$ by

$$\Phi(v) = \mathcal{E}(v, v) + \alpha \left(v - \frac{f}{\alpha}, v - \frac{f}{\alpha} \right), \quad v \in \mathcal{F}.$$

It follows from (1.1.6) that

$$\Phi(G_\alpha f) + \mathcal{E}_\alpha(G_\alpha f - v, G_\alpha f - v) = \Phi(v), \quad v \in \mathcal{F},$$

namely, $G_\alpha f$ is a unique element of $\mathcal{F}$ minimizing $\Phi(v)$. Suppose $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(E; m)$. There exists then for any $\varepsilon > 0$ a real function φ_ε satisfying (1.1.7) and (1.1.8). We let $\widetilde{\varphi}_\varepsilon(t) = (1/\alpha)\varphi_{\alpha\varepsilon}(\alpha t)$, $u = \widetilde{\varphi}_\varepsilon \circ G_\alpha f$ to obtain

$$u \in \mathcal{F} \text{ and } \mathcal{E}(u, u) \leq \mathcal{E}(G_\alpha f, G_\alpha f).$$

Since $|\widetilde{\varphi}_\varepsilon(t) - s| \leq |t - s|$ for every $s \in [0, 1/\alpha]$ and $t \in \mathbb{R}$, we have $|u(x) - f(x)/\alpha| \leq |G_\alpha f(x) - f(x)/\alpha|$ $[m]$ and $(u - f/\alpha, u - f/\alpha) \leq (G_\alpha f - f/\alpha, G_\alpha f - f/\alpha)$. Therefore, $\Phi(u) \leq \Phi(G_\alpha f)$ and consequently $u = G_\alpha f$ $[m]$, which means that $-\varepsilon \leq G_\alpha f \leq 1/\alpha + \varepsilon$ $[m]$. Letting $\varepsilon \to 0$, we get **(b)**.

It remains to prove the implication (**a**) ⇒ (**e**), which will follow from a more general theorem formulated below. □

In what follows, we occasionally use for a symmetric form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ the notations

$$\|f\|_{\mathcal{E}} := \sqrt{\mathcal{E}(f, f)}, \quad \|f\|_{\mathcal{E}_\alpha} := \sqrt{\mathcal{E}_\alpha(f, f)}, \quad f \in \mathcal{F},\ \alpha > 0$$

Definition 1.1.4. Let $(\mathcal{E}, \mathcal{F})$ be a closed symmetric form on $L^2(E; m)$. Denote by $\mathcal{F}_e$ the totality of m-equivalence classes of all m-measurable functions f on E such that $|f| < \infty$ $[m]$ and there exists an $\mathcal{E}$-Cauchy sequence $\{f_n, n \geq 1\} \subset \mathcal{F}$ such that $\lim_{n\to\infty} f_n = f$ m-a.e on E. $\{f_n\} \subset \mathcal{F}$ in the above is called an *approximating sequence of* $f \in \mathcal{F}_e$. We call the space $\mathcal{F}_e$ the *extended space* attached to $(\mathcal{E}, \mathcal{F})$. When the latter is a Dirichlet form on $L^2(E; m)$, the space $\mathcal{F}_e$ will be called its *extended Dirichlet space*.

THEOREM 1.1.5. *Let $(\mathcal{E}, \mathcal{F})$ be a closed symmetric form on $L^2(E; m)$ and $\mathcal{F}_e$ be the extended space attached to it. If the semigroup $\{T_t; t > 0\}$ generated by $(\mathcal{E}, \mathcal{F})$ is Markovian, then the following are true:*
(i) *For any $f \in \mathcal{F}_e$ and for any approximating sequence $\{f_n\} \subset \mathcal{F}$ of f, the limit $\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}(f_n,f_n)$ exists independently of the choice of an approximating sequence $\{f_n\}$ of f.*
(ii) *Every normal contraction operates on $(\mathcal{F}_e, \mathcal{E})$: for any normal contraction φ*

$$f \in \mathcal{F}_e \implies g := \varphi \circ f \in \mathcal{F}_e, \quad \mathcal{E}(g, g) \le \mathcal{E}(f, f).$$

(iii) *$\mathcal{F} = \mathcal{F}_e \cap L^2(E; m)$. In particular, $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(E; m)$.*

Assertion (ii) of this theorem implies the implication (**a**) ⇒ (**e**) in Theorem 1.1.3, completing the proof of Theorem 1.1.3.

For $f, g \in \mathcal{F}_e$, clearly both $f + g$ and $f - g$ are in $\mathcal{F}_e$. Define $\mathcal{E}(f, g) = \frac{1}{4}(\mathcal{E}(f + g, f + g) - \mathcal{E}(f - g, f - g))$, which is a symmetric bilinear form over $\mathcal{F}_e$. $(\mathcal{E}, \mathcal{F}_e)$ is called the *extended Dirichlet form* of $(\mathcal{E}, \mathcal{F})$.

If a given closed symmetric form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ is a Dirichlet form, then the corresponding semigroup $\{T_t; t > 0\}$ is Markovian by virtue of the already proven implication (**c**) ⇒ (**a**) of Theorem 1.1.3. So the extended Dirichlet space $\mathcal{F}_e$ satisfies all properties mentioned in Theorem 1.1.5.

Before giving the proof of Theorem 1.1.5, we shall fix a Markovian contractive symmetric linear operator T on $L^2(E; m)$ and make some preliminary observations on T.

By the linearity and the Markovian property of T on $L^2(E; m) \cap L^\infty(E; m)$,

$$f_1, f_2 \in L^2 \cap L^\infty, \ 0 \le f_1 \le f_2 \ [m] \Longrightarrow 0 \le Tf_1 \le Tf_2 \le \|f_2\|_\infty \ [m].$$

Due to the σ-finiteness of m, we can construct a Borel function $\eta \in L^1(E; m)$ which is strictly positive on E. If we put $\eta_n(x) = (n\eta(x)) \wedge 1$, then $0 < \eta_n \le 1$, $\eta_n \uparrow 1$, $n \to \infty$. Hence we can define an extension of T from $L^2(E; m) \cap L^\infty(E; m)$ to $L^\infty(E; m)$ as follows:

$$\begin{cases} Tf(x) := \lim_{n\to\infty} T(f \cdot \eta_n)(x) \ \ [m], & f \in L^\infty_+(E; m), \\ Tf := Tf^+ - Tf^-, & f \in L^\infty(E; m). \end{cases} \tag{1.1.9}$$

By the symmetry of T, $(g, T(f \cdot \eta_n)) = (Tg, f \cdot \eta_n)$ for $g \in bL^1(E; m)$. Letting $n \to \infty$, we see that the function $Tf, f \in L^\infty(E; m)$, defined by (1.1.9), satisfies the identity

$$\langle g, Tf \rangle = \langle Tg, f \rangle \quad \text{for every } g \in bL^1(E; m), \tag{1.1.10}$$

where $\langle g, f \rangle$ denotes the integral $\int_E g f dm$ for $g \in L^1(E; m)$, $f \in L^\infty(E; m)$. Consequently, Tf is uniquely determined up to the m-equivalence for

$f \in L^\infty(E;m)$. T becomes a Markovian linear operator on $L^\infty(E;m)$ and satisfies

$$f_n,\, f \in L^\infty_+(E;m),\, f_n \uparrow f\ [m] \Longrightarrow \lim_{n\to\infty} Tf_n = Tf\ [m]. \tag{1.1.11}$$

Further, if a sequence $\{f_n\} \subset L^\infty(E;m)$ is uniformly bounded and converges to f m-a.e. as $n \to \infty$,

$$\lim_{n\to\infty} \langle g, Tf_n\rangle = \langle g, Tf\rangle \quad \text{for every } g \in bL^1(E;m). \tag{1.1.12}$$

Lemma 1.1.6. (i) *For any $g \in L^\infty(E;m)$,*

$$T(g^2) - 2gTg + g^2T1 \geq 0 \quad [m].$$

(ii) *For any $g \in L^\infty(E;m)$, define*

$$\mathcal{A}_T(g) = \frac{1}{2}\int_E \left(T(g^2) - 2gTg + g^2T1\right) dm + \int_E g^2(1 - T1)dm. \tag{1.1.13}$$

It holds for $g \in L^2(E;m) \cap L^\infty(E;m)$ that

$$\mathcal{A}_T(g) = (g - Tg, g). \tag{1.1.14}$$

(iii) *For any $g \in L^\infty(E;m)$ and for any normal contraction φ,*

$$\mathcal{A}_T(\varphi \circ g) \leq \mathcal{A}_T(g). \tag{1.1.15}$$

(iv) *For any $f, g \in L^\infty(E;m)$,*

$$\mathcal{A}_T(f+g)^{1/2} \leq \mathcal{A}_T(f)^{1/2} + \mathcal{A}_T(g)^{1/2}. \tag{1.1.16}$$

Proof. (i) For a simple function on E expressed by

$$s = \sum_{i=1}^n a_i \mathbf{1}_{B_i}, \tag{1.1.17}$$

where $a_i \in \mathbb{R}$, $B_i \in \mathcal{B}(E)$ with $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^n B_i = E$, we have

$$T(g^2) - 2sTg + s^2T1 = \sum_{i=1}^n \mathbf{1}_{B_i} T\left((g - a_i)^2\right) \geq 0 \quad [m]. \tag{1.1.18}$$

Hence it suffices to choose an increasing sequence of simple functions $\{s_\ell,\ \ell \geq 1\}$ of this type such that

$$\lim_{\ell\to\infty} s_\ell = g \quad [m].$$

(ii) Recall that $\{\eta_n,\ n \geq 1\}$ is an increasing sequence of positive functions that is defined preceding (1.1.9). For $g \in L^2 \cap L^\infty$, we have $(Tg^2, \eta_n) = (g^2, T\eta_n)$

by the symmetry of T. By letting $n \to \infty$, we get $\int_E Tg^2 dm = \int_E g^2 T1 dm < \infty$ and $(g - Tg, g) = \frac{1}{2}\int_E \left(2g^2T1 - 2gTg\right) dm + \int_E g^2(1 - T1)dm = \mathcal{A}_T(g)$.
(iii) For $g \in L^\infty(E; m)$ and $k = 1, 2, \ldots$, we put

$$\mathcal{A}_T^k(g) = \frac{1}{2}\langle T(g^2) - 2gTg + g^2T1, \eta_k\rangle + \langle g^2(1 - T1), \eta_k\rangle.$$

When g is a simple function of the type (1.1.17),

$$\mathcal{A}_T^k(s) = \frac{1}{2}\sum_{1\le i,j\le n}(a_i - a_j)^2 J_{ij}^k + \sum_{1\le i\le n} a_i^2 \kappa_i^k.$$

Here $J_{ij}^k = \int_E (T\mathbf{1}_{B_i})\mathbf{1}_{B_j}\eta_k dm$, $\kappa_i^k = \int_E \mathbf{1}_{B_i}(1 - T1)\eta_k dm$.

For any normal contraction φ, it holds that

$$(\varphi(a_i) - \varphi(a_j))^2 \le (a_i - a_j)^2 \quad \text{and} \quad \varphi(a_i)^2 \le a_i^2.$$

Thus for a simple function s, $\mathcal{A}_T^k(\varphi \circ s) \le \mathcal{A}_T^k(s)$. For any $g \in L^\infty(E; m)$, we can take uniformly bounded simple functions s_ℓ with $\lim_{\ell\to\infty} s_\ell = g$ $[m]$ to obtain $\mathcal{A}_T^k(\varphi \circ s_\ell) \le \mathcal{A}_T^k(s_\ell)$. Letting $\ell \to \infty$ and then $k \to \infty$, we have by (1.1.12) that (1.1.15) holds.
(iv) It suffices to show the triangular inequality (1.1.16) for $\mathcal{A}_T^k$ for each fixed k instead of $\mathcal{A}_T$. Since $0 \le \mathcal{A}_T^k(g) < \infty$, $g \in L^\infty(E; m)$, the bilinear form defined by

$$\mathcal{A}_T^k(f, g) = \frac{1}{4}\left(\mathcal{A}_T^k(f + g) - \mathcal{A}_T^k(f - g)\right), \quad f, g \in L^\infty(E; m),$$

satisfies the Schwarz inequality

$$|\mathcal{A}_T^k(f, g)| \le \mathcal{A}_T^k(f)^{1/2} \cdot \mathcal{A}_T^k(g)^{1/2},$$

from which follows the desired triangular inequality. □

Let $\{\varphi^\ell,\ \ell > 0\}$ be a specific family of normal contractions defined by

$$\varphi^\ell(t) = ((-\ell) \vee t) \wedge \ell, \quad t \in \mathbb{R}. \tag{1.1.19}$$

For any m-measurable function g on E with $|g| < \infty$ $[m]$, $\mathcal{A}_T(\varphi^\ell \circ g)$ is increasing as ℓ increases. This is clear from $\varphi^\ell \circ (\varphi^{\ell+1} \circ g) = \varphi^\ell \circ g$ and Lemma 1.1.6(iii). We can then extend $\mathcal{A}_T(g)$ to g by letting

$$\mathcal{A}_T(g) = \lim_{\ell\to\infty} \mathcal{A}_T(\varphi^\ell \circ g)\ (\le \infty). \tag{1.1.20}$$

Lemma 1.1.7. (i) *For $g \in L^2(E; m)$, $\mathcal{A}_T(g) = (g - Tg, g)$.*
(ii) **(Fatou's property)** *For any m-measurable functions g_n, g on E with $|g_n| < \infty$, $|g| < \infty$ $[m]$, $\lim_{n\to\infty} g_n = g$ $[m]$,*

$$\mathcal{A}_T(g) \le \liminf_{n\to\infty} \mathcal{A}_T(g_n). \tag{1.1.21}$$

(iii) *For any m-measurable function g on E with* $|g| < \infty$ $[m]$ *and for any normal contraction* φ, $\mathcal{A}_T(\varphi \circ g) \le \mathcal{A}_T(g)$.
(iv) *The triangular inequality (1.1.16) holds for every m-measurable functions f and g that are finite m-a.e. on E.*

Proof. (i) follows from Lemma 1.1.6(ii) and the contraction property of T on $L^2(E;m)$.
(ii) We first give a proof when $|g_n| \le M$, $|g| \le M$ for some M and $\lim_{n\to\infty} g_n = g$ $[m]$. From the linearity, the Markovian property of T on $L^\infty(E;m)$, and (1.1.11), we have for $b \in \mathbb{R}$

$$T((g-b)^2) = \lim_{j\to\infty} T\left(\inf_{n\ge j}(g_n - b)^2\right) \le \liminf_n T((g_n - b)^2).$$

Since the identity (1.1.18) holds when s is a simple function like (1.1.17), we get from the above inequality

$$0 \le T(g^2) - 2sTg + s^2T1 \le \liminf_n (T(g_n^2) - 2sTg_n + s^2T1).$$

On the other hand,

$$\begin{aligned} &|\left(T(g_n^2) - 2g_nTg_n + g_n^2T1\right) - \left(T(g_n^2) - 2sTg_n + s^2T1\right)| \\ &\quad \le 2|Tg_n|\,|g_n - s| + |g_n^2 - s^2|\,T1 \le 2M|g_n - s| + |g_n^2 - s^2| \end{aligned}$$

hence

$$\begin{aligned} 0 &\le T(g^2) - 2sTg + s^2T1 \\ &\le \liminf_n \left(T(g_n^2) - 2g_nTg_n + g_n^2T1\right) + 2M|g - s| + |g^2 - s^2|. \end{aligned}$$

Taking a sequence of simple functions s such that $s \to g$ $[m]$,

$$0 \le T(g^2) - 2gTg + g^2T1 \le \liminf_n \left(T(g_n^2) - 2g_nTg_n + g_n^2T1\right).$$

Integrating both sides with respect to m and taking the defining formula (1.1.13) into account, we arrive at the desired (1.1.21) using the Fatou's lemma in the Lebesgue integration theory.

When g_n and g are not necessarily uniformly bounded, we can use the results obtained above to get

$$\mathcal{A}_T(\varphi^\ell \circ g) \le \liminf_n \mathcal{A}_T(\varphi^\ell \circ g_n) \le \liminf_n \mathcal{A}_T(g_n).$$

By letting $\ell \to \infty$, we still have the inequality (1.1.21).

(iii) It holds for $f = \varphi \circ g$, $f_\ell = \varphi \circ \varphi^\ell \circ g$ that $\lim_{\ell\to\infty} f_\ell = f$. Equations (1.1.15) and (1.1.21) then lead us to

$$\mathcal{A}_T(f) \le \liminf_{\ell\to\infty} \mathcal{A}_T(f_\ell) \le \liminf_{\ell\to\infty} \mathcal{A}_T(\varphi^\ell \circ g) = \mathcal{A}_T(g).$$

(iv) If we let $f_n := \varphi^n \circ f$ and $g_n := \varphi^n \circ g$, then $\lim_{n\to\infty}(f_n + g_n) = f + g[m]$ so that (1.1.16) and (1.1.21) yield

$$\begin{aligned}\mathcal{A}_T(f+g)^{1/2} &\le \liminf_{n\to\infty} \mathcal{A}_T(f_n+g_n)^{1/2} \\ &\le \lim_{n\to\infty}\left(\mathcal{A}_T(f_n)^{1/2} + \mathcal{A}_T(g_n)^{1/2}\right) = \mathcal{A}_T(f)^{1/2} + \mathcal{A}_T(g)^{1/2}.\end{aligned}$$

□

Proof of Theorem 1.1.5. (i) For any $f \in \mathcal{F}_e$, take its approximating sequence $\{f_n\} \subset \mathcal{F}$. f_n being $\mathcal{E}$-Cauchy, the triangular inequality guarantees the existence of the limit $\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}(f_n,f_n)$. Let us prove that

$$\frac{1}{t}\mathcal{A}_{T_t}(f) \uparrow \mathcal{E}(f,f) \quad \text{as } t \downarrow 0, \tag{1.1.22}$$

which in particular implies that $\mathcal{E}(f,f)$ does not depend the choice of the approximating sequence.

Since $f - f_\ell \in \mathcal{F}_e$ for each ℓ and $\{f_n - f_\ell\} \subset \mathcal{F}$ is its approximating sequence, we have from Lemma 1.1.7 and (1.1.4)

$$\frac{1}{t}\mathcal{A}_{T_t}(f - f_\ell) \le \liminf_{n\to\infty} \frac{1}{t}\mathcal{A}_{T_t}(f_n - f_\ell) \le \lim_{n\to\infty} \|f_n - f_\ell\|_{\mathcal{E}}^2.$$

Therefore, $\lim_{\ell\to\infty} \mathcal{A}_{T_t}(f - f_\ell) = 0$, and by the triangular inequality $\mathcal{A}_{T_t}(f) = \lim_{\ell\to\infty} \mathcal{A}_{T_t}(f_\ell)$, which particularly implies that $\frac{1}{t}\mathcal{A}_{T_t}(f)$ increases as t decreases to 0. Since $\lim_{t\downarrow 0} \frac{1}{t}\mathcal{A}_{T_t}(f_\ell) = \|f_\ell\|_{\mathcal{E}}^2$, we can get from the triangular inequality and the inequality obtained above that

$$\left|\lim_{t\downarrow 0}\sqrt{\frac{1}{t}\mathcal{A}_{T_t}(f)} - \|f_\ell\|_{\mathcal{E}}\right| \le \lim_{t\downarrow 0}\sqrt{\frac{1}{t}\mathcal{A}_{T_t}(f - f_\ell)} \le \lim_{n\to\infty} \|f_n - f_\ell\|_{\mathcal{E}}.$$

The last term tends to 0 as $\ell \to \infty$. The proof of (1.1.22) is complete.

(ii) For any $f \in \mathcal{F}_e$ and any normal contraction φ, we are led from Lemma 1.1.7(iii) and (1.1.22) to

$$\frac{1}{t}\mathcal{A}_{T_t}(\varphi \circ f) \le \frac{1}{t}\mathcal{A}_{T_t}(f) \le \mathcal{E}(f,f) \quad \text{for every } t > 0.$$

Hence it suffices to show $\varphi \circ f \in \mathcal{F}_e$. For an approximating sequence $\{f_n\} \subset \mathcal{F}$ of f, we obtain by Lemma 1.1.7 and (1.1.4)

$$\frac{1}{t}\mathcal{A}_{T_t}(\varphi \circ f_n) \le \frac{1}{t}\mathcal{A}_{T_t}(f_n) \le \mathcal{E}(f_n,f_n).$$

Thus $\varphi \circ f_n \in \mathcal{F}$ with $\mathcal{E}(\varphi \circ f_n, \varphi \circ f_n) \leq \mathcal{E}(f_n, f_n)$. This means that $\varphi \circ f_n$ are elements of $\mathcal{F}$ with uniformly bounded $\mathcal{E}$-norm. Therefore, the Cesàro mean $g_k = (1/k)\sum_{j=1}^{k} \varphi \circ f_{n_j}$ of its suitable subsequence $\{f_{n_j}\}$ is an $\mathcal{E}$-Cauchy sequence by Theorem A.4.1. Since $\lim_{k\to\infty} g_k = \varphi \circ f$ $[m]$, we arrive at $\varphi \circ f \in \mathcal{F}_e$.
(iii) The first identity follows from (1.1.4), Lemma 1.1.7, and (1.1.22). Since every normal contraction operates on $(\mathcal{E}, \mathcal{F})$ by (ii), $(\mathcal{E}, \mathcal{F})$ is Markovian, namely, a Dirichlet form. □

Remark 1.1.8. Property (1.1.22) in particular implies that if $\{f_k, k \geq 1\} \subset \mathcal{F}$ is an $\mathcal{E}$-Cauchy sequence and $f_k \to 0$ $[m]$, then $\mathcal{E}(f_k, f_k) \to 0$. □

COROLLARY 1.1.9. (Fatou's lemma) *Suppose* $\{f_k, k \geq 1\} \subset \mathcal{F}_e$ *and* $f \in \mathcal{F}_e$. *If* $f_k \to f$ $[m]$, *then*

$$\mathcal{E}(f,f) \leq \liminf_{k\to\infty} \mathcal{E}(f_k, f_k).$$

Proof. It follows from (1.1.21) and (1.1.22) that

$$\mathcal{E}(f,f) \leq \liminf_{t\to 0}\ \liminf_{k\to\infty} \frac{1}{t}\mathcal{A}_{T_t}(f_k, f_k) \leq \liminf_{k\to\infty} \mathcal{E}(f_k, f_k).$$

□

In the remainder of this section, $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(E; m)$.

Exercise 1.1.10. Show that for $f, g \in \mathcal{F}_e \cap L^\infty(E; m)$, $f \cdot g \in \mathcal{F}_e$ and $\|f \cdot g\|_{\mathcal{E}} \leq \|g\|_\infty \cdot \|f\|_{\mathcal{E}} + \|f\|_\infty \cdot \|g\|_{\mathcal{E}}$.

We state two lemmas for later use.

LEMMA 1.1.11. (i) *Let* $\{\psi_\ell\}_{\ell\geq 1}$ *be a sequence of normal contractions satisfying* $\lim_{\ell\to\infty} \psi_\ell(t) = t$ *for every* $t \in \mathbb{R}$. *Then* $\lim_{\ell\to\infty} \|\psi_\ell(f) - f\|_{\mathcal{E}_1} = 0$ *for any* $f \in \mathcal{F}$.
(ii) *Suppose* $\{f_n\} \subset \mathcal{F}$ *is* $\mathcal{E}_1$*-convergent to* $f \in \mathcal{F}$. *Then, for any normal contraction* φ, $\{\varphi(f_n)\}$ *is* $\mathcal{E}_1$*-weakly convergent to* $\varphi(f)$. *If further* $\varphi(f) = f$, *then the convergence is* $\mathcal{E}_1$*-strong.*

Proof. (i) If we let $\psi_\ell(f) = f_\ell$, then $f_\ell \in \mathcal{F}$ and $\|f_\ell\|_{\mathcal{E}_1}$ is uniformly dominated by $\|f\|_{\mathcal{E}_1}$. Since $G_1(L^2)$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ by (1.1.6) and $\mathcal{E}_1(f_\ell, G_1 g) = (f_\ell, g) \to (f, g) = \mathcal{E}_1(f, G_1 g)$ for every $g \in L^2$, we can conclude that f_ℓ converges as $\ell \to \infty$ to f weakly in $(\mathcal{F}, \mathcal{E}_1)$. But $\|f_\ell - f\|_{\mathcal{E}_1}^2 \leq 2\|f\|_{\mathcal{E}_1}^2 - 2\mathcal{E}_1(f_\ell, f)$ means that the convergence is strong as well.

(ii) $\mathcal{E}_1$-norm of $\varphi(f_n)$ is uniformly bounded and, for any $g \in L^2(E;m)$, $\mathcal{E}_1(G_1 g, \varphi(f_n) - \varphi(f)) = (g, \varphi(f_n) - \varphi(f)) \to 0$, $n \to \infty$. Hence the first assertion follows. If $\varphi(f) = f$, then as $n \to \infty$,

$$\mathcal{E}_1(\varphi(f_n) - f, \varphi(f_n) - f) \le \mathcal{E}_1(f_n, f_n) + \mathcal{E}_1(f, f) - 2\mathcal{E}_1(f, \varphi(f_n)) \to 0.$$

□

Lemma 1.1.12. *Let f be an m-measurable function on E with $|f| < \infty$ $[m]$. If, for the contractions φ^ℓ of (1.1.19), $f_\ell := \varphi^\ell \circ f \in \mathcal{F}_e$ for every $\ell \ge 1$, and $\sup_\ell \|f_\ell\|_{\mathcal{E}} < \infty$, then $f \in \mathcal{F}_e$.*

Proof. Without loss of generality, we assume that f is non-negative. For each ℓ, choose an approximating sequence $f_{\ell,k} \in \mathcal{F}$ for f_ℓ such that $\sup_k \|f_{\ell,k}\|_{\mathcal{E}}^2 \le \|f_\ell\|_{\mathcal{E}}^2 + 1$. We put $v_{\ell,k} = f_{\ell,k}^+ \wedge f_\ell$. Then $v_{\ell,k} \in \mathcal{F}_e \cap L^2(E;m) = \mathcal{F}$ and it converges to f_ℓ m-a.e. as $k \to \infty$ for each ℓ. Furthermore,

$$\|v_{\ell,k}\|_{\mathcal{E}}^2 \le \|f_{\ell,k}\|_{\mathcal{E}}^2 + \|f_\ell\|_{\mathcal{E}}^2 \le 2 \sup_\ell \|f_\ell\|_{\mathcal{E}}^2 + 1 < \infty.$$

Take a strictly positive m-measurable function g with $\int_E g dm \le 1$ and put $\widetilde{g}(x) = g(x)/(f(x) \vee 1)$. Since $0 \le v_{\ell,k}(x) \le f_\ell(x)$ for $x \in E$, $v_{\ell,k}$ is convergent to f_ℓ in $L^1(E;\widetilde{g}dx)$ as $k \to \infty$ and the latter converges to f in $L^1(E;\widetilde{g}dx)$ as $\ell \to \infty$. Hence $w_\ell = v_{\ell,k_\ell}$ converges to f in $L^1(E;\widetilde{g}dx)$ as well as m-a.e. on E if we choose a suitable subsequence $\{k_\ell\}$ of $\{k\}$. According to the boundedness of $\|v_{\ell,k}\|_{\mathcal{E}}$ obtained above, we can conclude that $f \in \mathcal{F}_e$ admits the Cesàro mean of a subsequence of $w_\ell \in \mathcal{F}$ as its approximating sequence by Theorem A.4.1. □

A numerical function $K(x,B)$ of two variables $x \in E$, $B \in \mathcal{B}(E)$, is said to be a *kernel* on the measurable space $(E, \mathcal{B}(E))$ if, for each fixed $x \in E$, it is a (positive) measure in B and, for each fixed $B \in \mathcal{B}(E)$, it is a $\mathcal{B}(E)$-measurable function in x. We then put

$$Kf(x) := \int_E f(y) K(x, dy), \quad x \in E. \tag{1.1.23}$$

$Kf \in \mathcal{B}_+(E)$ for $f \in \mathcal{B}_+(E)$ because the latter is an increasing limit of simple functions. A kernel K is called *Markovian* if $K(x,E) \le 1$ for every $x \in E$. A Markovian kernel K defines a linear operator on the space of bounded $\mathcal{B}(E)$-measurable functions by (1.1.23). A Markovian kernel K on E is said to be *conservative* or a *probability kernel* if $K(x, \cdot)$ is a probability measure for each $x \in E$.

We call a kernel $K(x,\cdot)$ (or an operator K) on $(E,\mathcal{B}(E))$ *m-symmetric* if

$$\int_E (Kf)(x)g(x)m(dx) = \int_E f(x)(Kg)(x)m(dx) \quad \text{for } f,g \in \mathcal{B}_+(E). \tag{1.1.24}$$

Let K be an m-symmetric Markovian kernel on $(E,\mathcal{B}(E))$ and $f \in b\mathcal{B}(E) \cap L^2(E;m)$. We then have from (1.1.23) $(Kf)^2(x) \le (Kf^2)(x)$, which yields by integrating with respect to m and using (1.1.24) the contraction property

$$\|Kf\|_2^2 \le \int_E K1(x)f(x)^2 m(dx) \le \|f\|_2^2.$$

This means that K can be regarded as a bounded linear operator on the space of m-essentially bounded m-measurable functions in $L^2(E;m)$, which is dense in $L^2(E;m)$. Hence K is uniquely extended to a linear contraction symmetric operator on $L^2(E;m)$.

So far we have assumed that $(E,\mathcal{B}(E))$ is only a measurable space. In the rest of this section, we assume that E is a Hausdorff topological space. In this case, we shall use the notation $\mathcal{B}(E)$ exclusively for the Borel field, namely, the σ-field of subsets of E generated by open sets. The space of $\mathcal{B}(E)$-measurable real-valued functions will be denoted by $\mathcal{B}(E)$. We sometimes need to consider a larger σ-field $\mathcal{B}^*(E)$; the family of universally measurable subsets of E: $\mathcal{B}^*(E) = \bigcap_{\mu\in\mathcal{P}(E)} \mathcal{B}^\mu(E)$, where $\mathcal{P}(E)$ denotes the family of all probability measures on E and $\mathcal{B}^\mu(E)$ is the completion of $\mathcal{B}(E)$ with respect to $\mu \in \mathcal{P}(E)$.

DEFINITION 1.1.13. (i) A family $\{P_t; t \ge 0\}$ is called a *transition function* on $(E,\mathcal{B}(E))$ (resp. $(E,\mathcal{B}^*(E))$) if P_t is a Markovian kernel on $(E,\mathcal{B}(E))$(resp. $(E,\mathcal{B}^*(E))$) for each $t \ge 0$ and the following four conditions are satisfied:

- **(t.1)** $P_sP_tf = P_{s+t}f$ for $s,t \ge 0$ and $f \in b\mathcal{B}(E)$ (resp. $f \in b\mathcal{B}^*(E)$). Here $P_tf(x) := \int_E f(y)P_t(x,dy)$.
- **(t.2)** For each $B \in \mathcal{B}(E)$, $P_t(x,B)$ is $\mathcal{B}([0,\infty)) \times \mathcal{B}(E)$-measurable (resp. $\mathcal{B}([0,\infty)) \times \mathcal{B}^*(E)$-measurable) in two variables $(t,x) \in [0,\infty) \times E$.
- **(t.3)** For each $x \in E$, $P_0(x,\cdot) = \delta_x(\cdot)$, where δ_x denotes the unit mass concentrated at the one-point set $\{x\}$.
- **(t.4)** $\lim_{t\downarrow 0} P_tf(x) = f(x)$ for any $f \in bC(E)$ and $x \in E$.

A transition function $\{P_t; t \ge 0\}$ is called a *transition probability* if P_t is conservative for every $t > 0$.
(ii) A family $\{R_\alpha; \alpha > 0\}$ is called a *resolvent kernel* on $(E,\mathcal{B}(E))$ (resp. $(E,\mathcal{B}^*(E))$) if, for each $\alpha > 0$, αR_α is a Markovian kernel on $(E,\mathcal{B}(E))$

(resp. $(E, \mathcal{B}^*(E))$) and

$$R_\alpha f - R_\beta f + (\alpha - \beta) R_\alpha R_\beta f = 0, \quad \alpha, \beta > 0, \ f \in b\mathcal{B}(E). \tag{1.1.25}$$

$$\lim_{\alpha \to \infty} \alpha R_\alpha f(x) = f(x), \quad x \in E, \quad f \in bC(E). \tag{1.1.26}$$

Property **(t.1)** is called the *semigroup property* or *Chapman-Kolmogorov equation*. Identity (1.1.25) is called the *resolvent equation*. For a transition function $\{P_t; t \geq 0\}$ on $(E, \mathcal{B}(E))$ (resp. $(E, \mathcal{B}^*(E))$), it is easy to verify that

$$R_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt, \quad \alpha > 0, \quad f \in \mathcal{B}(E), \tag{1.1.27}$$

determines uniquely a resolvent on $(E, \mathcal{B}(E))$, (resp. $(E, \mathcal{B}^*(E))$), which is called the *resolvent kernel of the transition function* $\{P_t; t \geq 0\}$.

A topological space E is called a *Lusin space* (resp. *Radon space*) if it is homeomorphic to a Borel (resp. universally measurable) subset of a compact metric space F. For a topological space E, a measure m on $(E, \mathcal{B}(E))$ is said to be *regular* if, for any $B \in \mathcal{B}(E)$, $m(B) = \inf\{m(U) \colon B \subset U, U \text{ open}\} = \sup\{m(K) \colon K \subset B, K \text{ compact}\}$. Any Radon measure on a locally compact separable metric space is regular. Any finite measure on a Lusin space or on a Radon space is regular.

LEMMA 1.1.14. *Let $\{P_t; t \geq 0\}$ be a family of Markovian kernels on a Lusin space E equipped with the Borel field $\mathcal{B}(E)$ or on a Radon space equipped with the σ-field $\mathcal{B}^*(E)$ of its universally measurable subsets.*
(i) *Suppose $\{P_t; t \geq 0\}$ satisfies* **(t.1), (t.3)** *and*

> **(t.4)$'$** *For every $f \in bC(E)$, $P_t f(x)$ is right continuous in $t \in [0, \infty)$ for each $x \in E$.*

Then $\{P_t; t \geq 0\}$ is a transition function.

(ii) *Suppose $\{P_t; t \geq 0\}$ satisfies* **(t.1), (t.4)** *and, for a σ-finite measure m on E, $\{P_t; t \geq 0\}$ is m-symmetric in the sense that P_t is m-symmetric for each $t > 0$. Let T_t be the symmetric linear operator on $L^2(E; m)$ uniquely determined by P_t. Then $\{T_t; t \geq 0\}$ is a strongly continuous contraction semigroup on $L^2(E; m)$.*

Proof. We give a proof for a family $\{P_t; t \geq 0\}$ of Markovian kernels on a Lusin space $(E, \mathcal{B}(E))$. The proof for a Radon space $(E, \mathcal{B}^*(E))$ is the same.
(i) It suffices to establish **(t.2)**. Let H be the collection of functions in $b\mathcal{B}(E)$ such that $P_t f(x)$ is measurable in two variables (t, x). H is then a linear space closed under the operation of taking uniformly bounded increasing limits. By **(t.4)$'$**, it holds that $bC(E) \subset H$. Hence **(t.2)** follows from Proposition A.1.3.

(ii) We may assume that E is a Borel subset of a compact metric space (F, d) and identify $L^2(E;m)$ with $L^2(F;m)$ by setting $m(F \setminus E) = 0$. We first show that $bC(F) \cap L^2(F;m)$ is dense in $L^2(F;m)$. Since m is σ-finite, it suffices to assume that m is a finite measure and that the indicator function of a set $B \in \mathcal{B}(F)$ can be L^2-approximated. For any ε, there exist a compact set K and an open set U such that $K \subset B \subset U$, $m(U \setminus K) < \varepsilon$. If we let $g(x) = d(x, U^c)/(d(x, U^c) + d(x, K))$, $x \in F$, then $g \in bC(F) \cap L^2(F;m)$ and $\|g - \mathbf{1}_B\|_2 < \sqrt{\varepsilon}$.

For any $f \in L^2(E;m)$ and $\varepsilon > 0$, take a function $g \in bC(F) \cap L^2(E;m)$ such that $\|f - g\|_2 < \varepsilon$. Because of the contraction property of $\{T_t; t > 0\}$, we then have $\|T_t f - f\|_2 \le \|P_t g - g\|_2 + 2\varepsilon$. Further,

$$\|P_t g - g\|_2^2 \le 2\|g\|_2^2 - 2(g, P_t g),$$

which tends to 0 as $t \downarrow 0$ by **(t.4)** and the Lebesgue-dominated convergence theorem. □

By virtue of Lemma 1.1.14, any m-symmetric transition function $\{P_t; t \ge 0\}$ on a Lusin space $(E, \mathcal{B}(E))$ or a Radon space $(E, \mathcal{B}^*(E))$ determines a unique strongly continuous contraction semigroup $\{T_t; t \ge 0\}$ on $L^2(E;m)$, which in turn decides a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$ according to Theorem 1.1.3. $(\mathcal{E}, \mathcal{F})$ is called the *Dirichlet form of the transition function* $\{P_t; t \ge 0\}$. In this case, the resolvent $\{G_\alpha; \alpha > 0\}$ of $\{T_t; t \ge 0\}$ is the unique extension of the resolvent kernel $\{R_\alpha; \alpha > 0\}$ of $\{P_t; t \ge 0\}$ from $b\mathcal{B}(E) \cap L^2(E;m)$ to $L^2(E;m)$. Moreover, we have from (1.1.6) that for $f \in b\mathcal{B}(E) \cap L^2(E;m)$,

$$R_\alpha f \in \mathcal{F} \quad \text{with} \quad \mathcal{E}_\alpha(R_\alpha f, v) = (f, v) \ \text{ for every } v \in \mathcal{F}. \tag{1.1.28}$$

Conversely, if the resolvent kernel $\{R_\alpha; \alpha > 0\}$ of a transition function $\{P_t; t \ge 0\}$ satisfies (1.1.28) for a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$, then $\{P_t; t \ge 0\}$ is m-symmetric and its Dirichlet form coincides with $(\mathcal{E}, \mathcal{F})$.

In the rest of this chapter, we give a quick introduction to the basic theory of quasi-regular Dirichlet forms. The importance of a quasi-regular Dirichlet form is that they are in one-to-one correspondence with symmetric Markov processes having some nice properties. We will show that any quasi-regular Dirichlet form is quasi-homeomorphic to a regular Dirichlet form on a locally compact separable metric space. Thus the study of quasi-regular Dirichlet forms can be reduced to that of regular Dirichlet forms.

1.2. EXCESSIVE FUNCTIONS AND CAPACITIES

In this section, let E be a Hausdorff topological space with the Borel σ-field $\mathcal{B}(E)$ being assumed to be generated by the continuous functions on

E and m be a σ-finite measure with $\text{supp}[m] = E$. Here for a measure ν on E, its *support* $\text{supp}[\nu]$ is by definition the smallest closed set outside which ν vanishes. Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(E; m)$, and $\{T_t; t \geq 0\}$ and $\{G_\alpha; \alpha > 0\}$ be its associated semigroup and resolvents on $L^2(E; m)$.

DEFINITION 1.2.1. For $\alpha > 0$, $u \in L^2(E; m)$ is called *α-excessive* if $e^{-\alpha t} T_t u \leq u$ m-a.e. for every $t > 0$.

Remark 1.2.2. (i) If u is α-excessive, then $u \geq 0$. This is because

$$\|e^{-\alpha t} T_t u\|_2 = e^{-\alpha t} \|T_t u\|_2 \leq e^{-\alpha t} \|u\|_2$$

and so $u \geq \lim_{t \to \infty} e^{-\alpha t} T_t u = 0$.
(ii) The constant function 1 is α-excessive if $m(E) < \infty$. For $f \in L^2_+(E; m)$, $G_\alpha f$ is α-excessive.
(iii) If $u_1 \geq 0$, $u_2 \geq 0$ are α-excessive functions, then so are $u_1 \wedge u_2$ and $u_1 \wedge 1$. □

LEMMA 1.2.3. *Let $u \in L^2_+(E; m)$ be α-excessive for $\alpha > 0$. Assume there is $v \in \mathcal{F}$ such that $u \leq v$. Then $u \in \mathcal{F}$ and $\mathcal{E}_\alpha(u, u) \leq \mathcal{E}_\alpha(v, v)$.*

Proof. By the symmetry and contraction property of T_t in $L^2(E; m)$, for each $t > 0$, $(f, g - e^{-\alpha t} T_t g)$ is a non-negative symmetric bilinear form on $L^2(E; m)$. So it satisfies the following Cauchy-Schwarz inequality:

$$|(f, g - e^{-\alpha t} T_t g)| \leq (f, f - e^{-\alpha t} T_t f)^{1/2} \cdot (g, g - e^{-\alpha t} T_t g)^{1/2}.$$

Thus we have by the α-excessiveness of u,

$$(u - e^{-\alpha t} T_t u, u) \leq (u - e^{-\alpha t} T_t u, v) \leq (u, u - e^{-\alpha t} T_t u)^{1/2} \cdot (v, v - e^{-\alpha t} T_t v)^{1/2},$$

and so

$$(u - e^{-\alpha t} T_t u, u) \leq (v, v - e^{-\alpha t} T_t v).$$

It follows then that

$$\begin{aligned} \lim_{t \to 0} \frac{1}{t}(u - T_t u, u) &= \lim_{t \to 0} \frac{1}{t}(u - e^{-\alpha t} T_t u, u) + \lim_{t \to 0} \frac{1}{t}(e^{-\alpha t} - 1)(T_t u, u) \\ &\leq \lim_{t \to 0} \frac{1}{t}(v - e^{-\alpha t} T_t v, v) - \alpha(u, u) \\ &= \mathcal{E}(v, v) + \alpha(v, v) - \alpha(u, u) < \infty. \end{aligned}$$

We conclude from (1.1.4)–(1.1.5) that $u \in \mathcal{F}$ with $\mathcal{E}_\alpha(u, u) \leq \mathcal{E}_\alpha(v, v)$. □

Lemma 1.2.4. *The following statements are equivalent for $u \in \mathcal{F}$ and $\alpha > 0$:*
(i) *u is α-excessive.*
(ii) *$\mathcal{E}_\alpha(u, v) \geq 0$ for every non-negative $v \in \mathcal{F}$.*

Proof. (i) $\Rightarrow$ (ii): It follow from (1.1.5) that

$$0 \leq \frac{1}{t}(u - e^{-\alpha t} T_t u, v) = \frac{1}{t}(u - T_t u, v) + \frac{1 - e^{-\alpha t}}{t}(T_t u, v) \to \mathcal{E}_\alpha(u, v) \quad (1.2.1)$$

as $t \downarrow 0$.
(ii) $\Rightarrow$ (i): For $v \in L^2_+(E; m)$ and $t > 0$, since

$$G_\alpha v - e^{-\alpha t} T_t G_\alpha v = \int_0^t e^{-\alpha s} T_s v ds \geq 0,$$

we have

$$(u - e^{-\alpha t} T_t u,\ v) = (u,\ v - e^{-\alpha t} T_t v) = \mathcal{E}_\alpha(u,\ G_\alpha(v - e^{-\alpha t} T_t v)) \geq 0.$$

This implies that $u - e^{-\alpha t} T_t u \geq 0$ and so (i) holds. □

For a closed subset F of E, define

$$\mathcal{F}_F := \{f \in \mathcal{F} : f = 0\ m\text{–a.e. } on\ E \setminus F\}. \quad (1.2.2)$$

Theorem 1.2.5. *Let $\alpha > 0$ and f be a non-negative function defined on E. For an open set D, denote $\mathcal{L}_{D,f} = \{u \in \mathcal{F} : u \geq f$ m-a.e. on $D\}$. Suppose $\mathcal{L}_{D,f} \neq \emptyset$. Then*
(i) *there is a unique $f_D \in \mathcal{L}_{D,f}$ such that*

$$\mathcal{E}_\alpha(u, u) \geq \mathcal{E}_\alpha(f_D, f_D) \quad \text{for every } u \in \mathcal{L}_{D,f}.$$

(ii) *f_D is the unique function in $\mathcal{L}_{D,f}$ such that*

$$\mathcal{E}_\alpha(u, f_D) \geq \mathcal{E}_\alpha(f_D, f_D) \quad \text{for every } u \in \mathcal{L}_{D,f}.$$

(iii) *$\mathcal{E}_\alpha(f_D, v) \geq 0$ for every $v \in \mathcal{F}$ with $v \geq 0$ m-a.e. on D. In particular, f_D is α-excessive and $\mathcal{E}_\alpha(f_D, v) = 0$ for every $v \in \mathcal{F}_{D^c}$.*
(iv) *$f_D \leq f$ if and only if $f_D \wedge f$ is an α-excessive function. In this case, $f_D = f$ m-a.e. on D. f_D is the minimum element among α-excessive functions in $\mathcal{L}_{D,f}$ in the sense that, if $u \in \mathcal{L}_{D,f}$ is α-excessive, then $f_D \leq u$.*
(v) *If open sets $D_1 \subset D_2$ and $\mathcal{L}_{D_2,f} \neq \emptyset$, then $f_{D_1} \leq f_{D_2}$ and*

$$\mathcal{E}_\alpha(f_{D_1}, f_{D_1}) \leq \mathcal{E}_\alpha(f_{D_2}, f_{D_2}).$$

(vi) *For open sets* $D_1 \subset D_2$, *if* $f \wedge f_{D_2}$ *is an* α*-excessive function, then* $(f_{D_2})_{D_1} = f_{D_1}$. *If further* $f \wedge f_{D_1}$ *is* α*-excessive, then*

$$\mathcal{E}_\alpha(f_{D_1}, f_{D_2}) = \mathcal{E}_\alpha(f_{D_1}, f_{D_1}).$$

(vii) *For open sets* $D_1 \subset D_2$, $(f_{D_1})_{D_2} = f_{D_1}$.

Proof. (i) Because $\mathcal{L}_{D,f}$ is a closed convex set in the Hilbert space $(\mathcal{F}, \mathcal{E}_\alpha)$, it has a unique minimizer f_D.
(ii) For every $u \in \mathcal{L}_{D,f}$ and $0 < \varepsilon < 1$, $f_D + \varepsilon(u - f_D) = (1-\varepsilon)f_D + \varepsilon u \in \mathcal{L}_{D,f}$ and so $\mathcal{E}_\alpha(f_D + \varepsilon(u - f_D), f_D + \varepsilon(u - f_D)) \geq \mathcal{E}_\alpha(f_D, f_D)$. This implies that $\mathcal{E}_\alpha(f_D, u - f_D) \geq 0$. Now suppose $v \in \mathcal{L}_{D,f}$ is another function such that for every $u \in \mathcal{L}_{D,f}$, $\mathcal{E}_\alpha(v, u - v) \geq 0$. As $f_D \in \mathcal{L}_{D,f}$, $\mathcal{E}_\alpha(v, f_D - v) \geq 0$. But with $\mathcal{E}_\alpha(f_D, v - f_D) \geq 0$, we have $\mathcal{E}_\alpha(f_D - v, f_D - v) \leq 0$. Therefore, $v = f_D$.
(iii) For any $v \in \mathcal{F}$ with $v \geq 0$ m-a.e. on D, $f_D + \varepsilon v \in \mathcal{L}_{D,f}$ for every $\varepsilon > 0$. One immediately deduces from $\mathcal{E}_\alpha(f_D + \varepsilon v, f_D + \varepsilon v) \geq \mathcal{E}_\alpha(f_D, f_D)$ that $\mathcal{E}_\alpha(f_D, v) \geq 0$.
(iv) This follows immediately from (iii) and Lemma 1.2.3.
(v) The first part follows from (iv). The second part follows from (i).
(vi) By (iv), $f = f_{D_2}$ on D_2 and hence by definition, $(f_{D_2})_{D_1} = f_{D_1}$. The second assertion follows from (iii) and (iv).
(vii) For every $u \in \mathcal{L}_{D_2, f_{D_1}}$, $\mathcal{E}_\alpha(f_{D_1}, u - f_{D_1}) \geq 0$ by (iii). We therefore have by (ii) that $f_{D_1} = (f_{D_1})_{D_2}$. □

The function f_D is called the α*-reduced function* of f on D.

Remark 1.2.6. (i) If f is α-excessive in $\mathcal{F}$, then f_D is the $\mathcal{E}_\alpha$-orthogonal projection of f into the $\mathcal{E}_\alpha$-orthogonal complement of $\mathcal{F}_{D^c}$. This is because $f = (f - f_D) + f_D$, where $f - f_D \in \mathcal{F}_{D^c}$ by Theorem 1.2.5(iv) and f_D is $\mathcal{E}_\alpha$-orthogonal to $\mathcal{F}_{D^c}$ by Theorem 1.2.5(iii).
(ii) By (iii) and (iv) of Theorem 1.2.5, if $g \in \mathcal{F}$ is α-excessive, then $\mathcal{E}_\alpha(f_D, g) = \mathcal{E}_\alpha(f_D, g_D)$. □

DEFINITION 1.2.7. **((h, α)-capacity)** Fix $\alpha > 0$. Let $h \geq 0$ be a function on E satisfying one of the following two conditions:
(i) $h \in \mathcal{F}$ and h is α-excessive;
(ii) $h \wedge h_D$ is a α-excessive function for every open set $D \subset E$ with $\mathcal{L}_{D,h} \neq \emptyset$. (This is equivalent to, by Theorem 1.2.5(iv), that $h \geq h_D$ for every open set $D \subset E$ with $\mathcal{L}_{D,h} \neq \emptyset$). Define for open subset $D \subset E$,

$$\mathrm{Cap}_{h,\alpha}(D) := \begin{cases} \mathcal{E}_\alpha(h_D, h_D) & \text{if } \mathcal{L}_{D,h} \neq \emptyset, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.2.3}$$

and for an arbitrary subset $A \subset E$,

$$\mathrm{Cap}_{h,\alpha}(A) := \inf\left\{\mathrm{Cap}_{h,\alpha}(D) : \text{ open set } D \supset A\right\}. \tag{1.2.4}$$

Remark 1.2.8. (i) Important cases are $h = 1$ and $h = G_\alpha \varphi$ for some strictly positive $\varphi \in L^2(E; m)$.
(ii) Under either of conditions (i) and (ii), $h = h_D$ $[m]$ on D whenever $\mathcal{L}_{D,h} \neq \emptyset$.
(iii) When $h > 0$ $[m]$ on E, then $\mathrm{Cap}_{h,\alpha}(A) = 0$ implies that $m(A) = 0$.
(iv) If $0 \leq h^{(1)} \leq h^{(2)}$ are two functions satisfying either condition (i) or (ii) in Definition 1.2.7, we have by Theorem 1.2.5(iv) that $h_D^{(1)} \leq h_D^{(2)}$ for any open set D with $\mathcal{L}_{D,h^{(2)}} \neq \emptyset$. Therefore $\mathrm{Cap}_{h^{(1)},\alpha}(D) \leq \mathrm{Cap}_{h^{(2)},\alpha}(D)$ by Lemma 1.2.3.
(v) We shall use the following comparison in $\alpha > 0$ for the capacity: if h_1 is 1-excessive, h_2 is 2-excessive, and $h_2 \leq h_1$, then

$$\mathrm{Cap}_{h_2,2}(A) \leq 2\mathrm{Cap}_{h_1,1}(A), \quad A \subset E.$$

In fact, we have for an open set D,

$$\begin{aligned}\mathrm{Cap}_{h_2,2}(D) &= \inf_{u \in \mathcal{F},\, u \geq h_2 \text{ on } D} \mathcal{E}_2(u,u) \leq \inf_{u \in \mathcal{F},\, u \geq h_1 \text{ on } D} 2\mathcal{E}_1(u,u) \\ &\leq 2\mathrm{Cap}_{h_1,1}(D).\end{aligned}$$

□

In the remainder of this section $h \geq 0$ is a non-trivial function on E satisfying one of the conditions (i) or (ii) in Definition 1.2.7.

THEOREM 1.2.9. (i) *For open sets* $D_1 \subset D_2$, $\mathrm{Cap}_{h,\alpha}(D_1) \leq \mathrm{Cap}_{h,\alpha}(D_2)$.
(ii) *For open sets* D_1 *and* D_2,

$$\mathrm{Cap}_{h,\alpha}(D_1 \cup D_2) + \mathrm{Cap}_{h,\alpha}(D_1 \cap D_2) \leq \mathrm{Cap}_{h,\alpha}(D_1) + \mathrm{Cap}_{h,\alpha}(D_2).$$

(iii) *For any increasing sequence of open sets* $\{D_k, k \geq 1\}$,

$$\mathrm{Cap}_{h,\alpha}(\cup_{k\geq 1} D_k) = \sup_{k\geq 1} \mathrm{Cap}_{h,\alpha}(D_k).$$

(iv) *For any decreasing sequence of open sets* $\{D_k, k \geq 1\}$ *with* $\mathcal{L}_{D_1,h} \neq \emptyset$, $\{h_{D_k}; k \geq 1\}$ *is decreasing to as well as* $\mathcal{E}_\alpha$*-convergent to a function* $h_\infty \in \mathcal{F}$, *and* $\inf_{k\geq 1} \mathrm{Cap}_{h,\alpha}(D_k) = \mathcal{E}_\alpha(h_\infty, h_\infty)$.

Proof. (i) follows from Theorem 1.2.5(v).

(ii) Without loss of generality, we may assume $\mathrm{Cap}_{h,\alpha}(D_i) < \infty$ for $i = 1, 2$. By the property of h_D,

$$\begin{aligned}
&\mathrm{Cap}_{h,\alpha}(D_1 \cup D_2) + \mathrm{Cap}_{h,\alpha}(D_1 \cap D_2) \\
&\quad \le \mathcal{E}_\alpha(h_{D_1} \vee h_{D_2},\, h_{D_1} \vee h_{D_2}) + \mathcal{E}_\alpha(h_{D_1} \wedge h_{D_2},\, h_{D_1} \wedge h_{D_2}) \\
&\quad = \tfrac{1}{2}\mathcal{E}_\alpha(h_{D_1} + h_{D_2},\, h_{D_1} + h_{D_2}) + \tfrac{1}{2}\mathcal{E}_\alpha(|h_{D_1} - h_{D_2}|,\, |h_{D_1} - h_{D_2}|) \\
&\quad \le \tfrac{1}{2}\mathcal{E}_\alpha(h_{D_1} + h_{D_2},\, h_{D_1} + h_{D_2}) + \tfrac{1}{2}\mathcal{E}_\alpha(h_{D_1} - h_{D_2},\, h_{D_1} - h_{D_2}) \\
&\quad = \mathcal{E}_\alpha(h_{D_1},\, h_{D_1}) + \mathcal{E}_\alpha(h_{D_2},\, h_{D_2}) \\
&\quad = \mathrm{Cap}_{h,\alpha}(D_1) + \mathrm{Cap}_{h,\alpha}(D_2).
\end{aligned}$$

(iii) Without loss of generality, assume that $\sup_{k\ge 1} \mathrm{Cap}_{h,\alpha}(D_k) < \infty$. For $j > k$, we have from Theorem 1.2.5(vi)

$$\begin{aligned}
&\mathcal{E}_\alpha(h_{D_j} - h_{D_k},\, h_{D_j} - h_{D_k}) \\
&\quad = \mathcal{E}_\alpha(h_{D_j}, h_{D_j}) - 2\mathcal{E}_\alpha(h_{D_j}, h_{D_k}) + \mathcal{E}_\alpha(h_{D_k}, h_{D_k}) \\
&\quad = \mathcal{E}_\alpha(h_{D_j}, h_{D_j}) - \mathcal{E}_\alpha(h_{D_k}, h_{D_k}) \\
&\quad = \mathrm{Cap}_{h,\alpha}(D_j) - \mathrm{Cap}_{h,\alpha}(D_k) \to 0 \qquad \text{as } j, k \to \infty.
\end{aligned}$$

So h_{D_k} is $\mathcal{E}_\alpha$-convergent to some $h_\infty \in \mathcal{F}$. As $h_\infty = h_k = h$ $[m]$ on D_k, we have $h_\infty = h$ $[m]$ on $\cup_{k\ge 1} D_k$. For $v \in \mathcal{L}_{\cup_{k\ge 1} D_k, h}$, by Theorem 1.2.5(ii),

$$\mathcal{E}_\alpha(h_\infty, v) = \lim_{k\to\infty} \mathcal{E}_\alpha(h_{D_k}, v) \ge \lim_{k\to\infty} \mathcal{E}_\alpha(h_{D_k}, h_{D_k}) = \mathcal{E}_\alpha(h_\infty, h_\infty).$$

By Theorem 1.2.5(ii) again, $h_\infty = h_{\cup_{k\ge 1} D_k}$ and therefore

$$\begin{aligned}
\sup_{k\ge 1} \mathrm{Cap}_{h,\alpha}(D_k) &= \lim_{k\to\infty} \mathrm{Cap}_{h,\alpha}(D_k) = \lim_{k\to\infty} \mathcal{E}_\alpha(h_{D_k}, h_{D_k}) \\
&= \mathcal{E}_\alpha(h_\infty, h_\infty) = \mathrm{Cap}_{h,\alpha}(\cup_{k\ge 1} D_k).
\end{aligned}$$

(iv) $\{h_{D_k}\}$ is decreasing by Theorem 1.2.5(v). For $j > k$, we have from Theorem 1.2.5(vi)

$$\begin{aligned}
&\mathcal{E}_\alpha(h_{D_j} - h_{D_k},\, h_{D_j} - h_{D_k}) \\
&\quad = \mathcal{E}_\alpha(h_{D_j}, h_{D_j}) - 2\mathcal{E}_\alpha(h_{D_j}, h_{D_k}) + \mathcal{E}_\alpha(h_{D_k}, h_{D_k}) \\
&\quad = \mathcal{E}_\alpha(h_{D_k}, h_{D_k}) - \mathcal{E}_\alpha(h_{D_j}, h_{D_j}),
\end{aligned}$$

which leads us to (iv). □

Observe that the proof of Theorem 1.2.9(iii) shows that h_{D_k} converges to $h_{\cup_{k\ge 1} D_k}$ both monotonously and in $(\mathcal{F}, \mathcal{E}_\alpha)$.

THEOREM 1.2.10. $\mathrm{Cap}_{h,\alpha}$ *is a Choquet $\mathcal{K}$-capacity, where $\mathcal{K}$ denotes all the compact subsets of E; that is,*
(i) *For any subsets $A \subset B$, $\mathrm{Cap}_{h,\alpha}(A) \le \mathrm{Cap}_{h,\alpha}(B)$;*
(ii) *For any increasing sequence of subsets $\{A_j, j \ge 1\}$,*

$$\mathrm{Cap}_{h,\alpha}(\cup_{j\ge1}A_j) = \sup_{j\ge1} \mathrm{Cap}_{h,\alpha}(A_j);$$

(iii) *For any decreasing sequence of compact subsets $\{K_j, j \ge 1\}$,*

$$\mathrm{Cap}_{h,\alpha}(\cap_{j\ge1}K_j) = \inf_{j\ge1} \mathrm{Cap}_{h,\alpha}(K_j).$$

Proof. (i) follows immediately from Theorem 1.2.9(i) and the definition of $\mathrm{Cap}_{h,\alpha}$.
(ii) Without loss of generality, we may assume that $\mathrm{Cap}_{h,\alpha}(A_j) < \infty$ for every $j \ge 1$. In view of (i), it suffices to show

$$\mathrm{Cap}_{h,\alpha}(\cup_{j\ge1}A_j) \le \sup_{j\ge1} \mathrm{Cap}_{h,\alpha}(A_j).$$

For any $\varepsilon > 0$, let an open set $O_j \supset A_j$ be such that $\mathrm{Cap}_{h,\alpha}(O_j) < \mathrm{Cap}_{h,\alpha}(A_j) + 2^{-j}\varepsilon$. Define $D_j := \cup_{k=1}^{j} O_k$. Then $\{D_j, j \ge 1\}$ is an increasing sequence of open sets. We claim that

$$\mathrm{Cap}_{h,\alpha}(D_j) \le \mathrm{Cap}_{h,\alpha}(A_j) + (1 - 2^{-j})\varepsilon \qquad \text{for every } j \ge 1. \tag{1.2.5}$$

We prove this by induction. Clearly this is true for $j = 1$. Suppose it is true for $j \ge 1$. Since $D_{j+1} = D_j \cup O_{j+1}$, we have by Theorem 1.2.9(ii),

$$\mathrm{Cap}_{h,\alpha}(D_{j+1}) + \mathrm{Cap}_{h,\alpha}(D_j \cap O_{j+1}) \le \mathrm{Cap}_{h,\alpha}(D_j) + \mathrm{Cap}_{h,\alpha}(O_{j+1}).$$

But as $A_j \subset D_j \cap O_{j+1}$, we have

$$\begin{aligned}
\mathrm{Cap}_{h,\alpha}(D_{j+1}) &\le \mathrm{Cap}_{h,\alpha}(D_j) + \mathrm{Cap}_{h,\alpha}(O_{j+1}) - \mathrm{Cap}_{h,\alpha}(A_j) \\
&\le \mathrm{Cap}_{h,\alpha}(O_{j+1}) + (1 - 2^{-j})\varepsilon \\
&\le \mathrm{Cap}_{h,\alpha}(A_{j+1}) + 2^{-j-1}\varepsilon + (1 - 2^{-j})\varepsilon \\
&= \mathrm{Cap}_{h,\alpha}(A_{j+1}) + (1 - 2^{-j-1})\varepsilon.
\end{aligned}$$

This proves the claim (1.2.5). Therefore, we have

$$\begin{aligned}
\mathrm{Cap}_{h,\alpha}(\cup_{j\ge1}A_j) &\le \mathrm{Cap}_{h,\alpha}(\cup_{j\ge1}D_j) = \sup_{j\ge1} \mathrm{Cap}_{h,\alpha}(D_j) \\
&\le \sup_{j\ge1} \mathrm{Cap}_{h,\alpha}(A_j) + \varepsilon.
\end{aligned}$$

Passing $\varepsilon \downarrow 0$, we get $\mathrm{Cap}_{h,\alpha}(\cup_{j\ge1}A_j) \le \sup_{j\ge1} \mathrm{Cap}_{h,\alpha}(A_j)$, and so (ii) is established.

(iii) It suffices to show that $\mathrm{Cap}_{h,\alpha}(\cap_{j\geq 1}K_j) \geq \inf_{j\geq 1}\mathrm{Cap}_{h,\alpha}(K_j)$. We may assume that $\mathrm{Cap}_{h,\alpha}(\cap_{j\geq 1}K_j) < \infty$. For any $\varepsilon > 0$, let D be an open set such that $D \supset \cap_{j\geq 1}K_j$ and $\mathrm{Cap}_{h,\alpha}(D) < \mathrm{Cap}_{h,\alpha}(\cap_{j\geq 1}K_j) + \varepsilon$. Since K_j is compact for every $j \geq 1$, there is $n \geq 1$ such that $D \supset K_n$. Therefore, $\mathrm{Cap}_{h,\alpha}(D) \geq \mathrm{Cap}_{h,\alpha}(K_n) \geq \inf_{j\geq 1}\mathrm{Cap}_{h,\alpha}(K_j)$. This yields that, after letting $\varepsilon \downarrow 0$, $\mathrm{Cap}_{h,\alpha}(\cap_{j\geq 1}K_j) \geq \inf_{j\geq 1}\mathrm{Cap}_{h,\alpha}(K_j)$. □

DEFINITION 1.2.11. A subset $A \subset E$ is said to be *C-capacitable* for a set function C on E if

$$C(A) = \sup_{\substack{K\subset A\\ K:\ \text{compact}}} C(K).$$

The celebrated Choquet's theorem says that every $\mathcal{K}$-analytic set is C-capacitable for any Choquet $\mathcal{K}$-capacity C (cf. [37, III: 28]). It is known that any Borel subset of a compact metric space is $\mathcal{K}$-analytic (cf. [37, III: 7,13]). In particular, for a Lusin space E, it holds from Theorem 1.2.10 that

$$\mathrm{Cap}_{h,\alpha}(B) = \sup_{K\subset B,\ K\ \text{compact}} \mathrm{Cap}_{h,\alpha}(K) \quad \text{for } B \in \mathcal{B}(E). \tag{1.2.6}$$

DEFINITION 1.2.12. (i) An increasing sequence $\{F_k, k \geq 1\}$ of closed sets of E is an *$\mathcal{E}$-nest* if $\cup_{k\geq 1}\mathcal{F}_{F_k}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$, where $\mathcal{E}_1 = \mathcal{E} + (\ ,\ )_{L^2(E;m)}$.
(ii) A subset N of E is *$\mathcal{E}$-polar* if there is an $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ such that $N \subset \cap_{k\geq 1}(E \setminus F_k)$.
(iii) A statement depending on $x \in A$ is said to hold *$\mathcal{E}$-quasi-everywhere* ($\mathcal{E}$-q.e. in abbreviation) on A if there is an $\mathcal{E}$-polar set $N \subset A$ such that the statement is true for every $x \in A \setminus N$.
(iv) A function f on E is said to be *$\mathcal{E}$-quasi-continuous* if there is an $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ such that $f|_{F_k}$ is finite and continuous on F_k for each $k \geq 1$, which will be denoted in abbreviation as $f \in C(\{F_k\})$.
(v) An increasing sequence $\{F_k\}$ of closed sets of E is *$\mathrm{Cap}_{h,\alpha}$-nest* if

$$\lim_{k\to\infty} \mathrm{Cap}_{h,\alpha}(E \setminus F_k) = 0.$$

(vi) A subset N of E is *$\mathrm{Cap}_{h,\alpha}$-polar* if $\mathrm{Cap}_{h,\alpha}(N) = 0$.

Obviously, if $\{F_n, n \geq 1\}$ of E is an $\mathcal{E}$-nest, then so is $\{K_n, n \geq 1\}$ where $K_n = \mathrm{supp}[\mathbf{1}_{F_n} \cdot m]$. Since $\mathcal{F}$ is a dense linear subspace in $L^2(E;m)$, every $\mathcal{E}$-polar set is m-null.

THEOREM 1.2.13. *Fix an arbitrary $\alpha > 0$ and let $h = G_\alpha\varphi$ for some strictly positive $\varphi \in L^2(E;m)$. Let $\{F_k, k \geq 1\}$ be an increasing sequence of closed subsets. Then*

(i) $\{F_k, k \ge 1\}$ *is an* $\mathcal{E}$*-nest if and only if it is a* $\mathrm{Cap}_{h,\alpha}$*-nest.*
(ii) *A set* $N \subset E$ *is* $\mathcal{E}$*-polar if and only if it is* $\mathrm{Cap}_{h,\alpha}$*-polar.*
(iii) *If* $\{F_k^1, k \ge 1\}$ *and* $\{F_k^2, k \ge 1\}$ *are two* $\mathcal{E}$*-nests, then* $\{F_k^1 \cap F_k^2, k \ge 1\}$ *is also an* $\mathcal{E}$*-nest.*

Proof. Let $h_k := h_{F_k^c}$. By Theorem 1.2.9(iv), h_k is decreasing to as well as $\mathcal{E}_\alpha$-convergent to some non-negative $h_\infty \in \mathcal{F}$ and

$$\lim_{k\to\infty} \mathrm{Cap}_{h,\alpha}(F_k^c) = \mathcal{E}_\alpha(h_\infty, h_\infty). \tag{1.2.7}$$

In particular, for every $v \in \cup_{k\ge1}\mathcal{F}_{F_k}$, by Theorem 1.2.5(iii),

$$\mathcal{E}_\alpha(h_\infty, v) = \lim_{k\to\infty} \mathcal{E}_\alpha(h_k, v) = 0. \tag{1.2.8}$$

Now suppose $\{F_k, k \ge 1\}$ is an $\mathcal{E}$-nest. Then by (1.2.8), $h_\infty = 0$ and so $\lim_{k\to\infty} \mathrm{Cap}_{h,\alpha}(F_k^c) = 0$ by (1.2.7).

Conversely, suppose that $\lim_{k\to\infty} \mathrm{Cap}_{h,\alpha}(F_k^c) = 0$. Then $h_\infty = 0$ by (1.2.7). For any α-excessive function $v \in \mathcal{F}$, denote $v_{F_k^c}$ by v_k so $v - v_k \in \mathcal{F}_{F_k}$. By the same reasoning as above for h, we see that v_k is decreasing to as well as $\mathcal{E}_\alpha$-convergent to some $v_\infty \in \mathcal{F}$. By Remark 1.2.6(i),

$$\begin{aligned}\int_E \varphi(x)v_\infty(x)m(dx) = \mathcal{E}_\alpha(h, v_\infty) &= \lim_{k\to\infty} \mathcal{E}_\alpha(h, v_k) \\ &= \lim_{k\to\infty} \mathcal{E}_\alpha(h_k, v) = \mathcal{E}_\alpha(h_\infty, v) = 0.\end{aligned}$$

This implies that $v_\infty = 0$ $[m]$ on E as $\varphi > 0$ $[m]$ on E. Therefore, $v = \lim_{k\to\infty}(v - v_k)$ is in the $\mathcal{E}_\alpha$-completion of $\cup_{k\ge1}\mathcal{F}_{F_k}$. Since $G_\alpha L^2(E; m)$ is $\mathcal{E}_\alpha$ dense in $\mathcal{F}$, we have that $\cup_{k\ge1}\mathcal{F}_{F_k}$ is $\mathcal{E}_\alpha$-dense (and hence $\mathcal{E}_1$-dense) in $\mathcal{F}$; that is, $\{F_k, k \ge 1\}$ is an $\mathcal{E}$-nest.

The second assertion of the theorem is immediate from the first. The third follows from the first and Theorem 1.2.9. □

Theorem 1.2.14. *Suppose that* $h > 0$ *is a function that satisfies one of the conditions in Definition 1.2.7 for* $\alpha = 1$. *Suppose there is an increasing sequence of open sets* $\{D_k, k \ge 1\}$ *of finite* $(h, 1)$*-capacity such that* $\overline{D}_k \subset D_{k+1}$, $k \ge 1$, *and* $\{\overline{D}_k, k \ge 1\}$ *constitutes an* $\mathcal{E}$*-nest.*
(i) *An increasing sequence* $\{F_k, k \ge 1\}$ *of closed subsets of* E *is an* $\mathcal{E}$*-nest if and only if* $\lim_{k\to\infty} \mathrm{Cap}_{h,1}(D_n \setminus F_k) = 0$ *for every* $n \ge 1$.
(ii) *A set* $N \subset E$ *is* $\mathcal{E}$*-polar if and only if it is* $\mathrm{Cap}_{h,1}$*-polar.*

Proof. (i) *Proof of the "only if" part:* Suppose $\{F_k, k \ge 1\}$ is an $\mathcal{E}$-nest. For a fixed n, let $g_k = h_{D_n\setminus F_k}$. By Theorem 1.2.9(iv), $\{g_k, k \ge 1\}$ is then decreasing to and $\mathcal{E}_1$-convergent to a function $g_\infty \in \mathcal{F}$. By Theorem 1.2.5(iii), g_k is $\mathcal{E}_1$-orthogonal to the space $\mathcal{F}_{\{D_n\setminus F_\ell\}^c} \supset \mathcal{F}_{F_\ell}$ for any $k \ge \ell$. Hence g_∞ is

$\mathcal{E}_1$-orthogonal to $\cup_\ell \mathcal{F}_{F_\ell}$, which is $\mathcal{E}_1$-dense in $\mathcal{F}$. Thus $g_\infty = 0$ and so for each fixed $n \geq 1$, $\mathrm{Cap}_{h,1}(D_n \setminus F_k) = \mathcal{E}_1(g_k, g_k) \to 0$ as $k \to \infty$.
Proof of the "if" part: By the assumption and Theorem 1.2.5(iv),

$$h_1 := \sum_{n=1}^{\infty} 2^{-n} \|h_{D_n}\|_2^{-1} h_{D_n}$$

is a 1-excessive function in $L^2(E;m)$ with $0 < h_1 \leq \|h_{D_1}\|_2^{-1} h$ $[m]$ on E. Let $h_2 := G_2 h_1$. Clearly $h_2 \leq h_1 \leq \|h_{D_1}\|_2^{-1} h$. As h_1 is 1-excessive, h_2 is 2-excessive, and $h_2 \leq h_1$, we see from Remark 1.2.8(iv)–(v) that, for any open set D,

$$\mathrm{Cap}_{h_2,2}(D) \leq 2\mathrm{Cap}_{h_1,1}(D) \leq 2\|h_{D_1}\|_2^{-2} \mathrm{Cap}_{h,1}(D). \tag{1.2.9}$$

Now suppose that $\lim_{k\to\infty} \mathrm{Cap}_{h,1}(D_n \setminus F_k) = 0$ for every $n \geq 1$. Since $F_k^c \subset \overline{D}_n^c \cup (D_{n+1} \setminus F_k)$, we see by Theorem 1.2.9(ii) that

$$\mathrm{Cap}_{h_2,2}(F_k^c) \leq \mathrm{Cap}_{h_2,2}(\overline{D}_n^c) + \mathrm{Cap}(D_{n+1} \setminus F_k).$$

By noting (1.2.9) and Theorem 1.2.13, we let $k \to \infty$ and then $n \to \infty$ to get $\lim_{k\to\infty} \mathrm{Cap}_{h_2,2}(F_k^c) = 0$, which means that $\{F_k, k \geq 1\}$ is an $\mathcal{E}$-nest by Theorem 1.2.13.
(ii) If N is $\mathcal{E}$-polar, then it is a subset of $\cap_k (E \setminus F_k)$ for some $\mathcal{E}$-nest $\{F_k\}$. By (i), $\mathrm{Cap}_{h,1}(D_n \cap N) = 0$ for each n, and by letting $n \to \infty$, we get $\mathrm{Cap}_{h,1}(N) = 0$ on account of Theorem 1.2.10(ii). Conversely, suppose $\mathrm{Cap}_{h,1}(N) = 0$. For the 2-excessive function $h_2 := G_2 h_1$ defined in the "if" part of the proof of (i), the inequality (1.2.9) holds for any set D and so $\mathrm{Cap}_{h_2,2}(N) = 0$, which implies that N is $\mathcal{E}$-polar in view of Theorem 1.2.13. □

Remark 1.2.15. Let $h > 0$ be a function that satisfies one of the conditions in Definition 1.2.7.
(i) Any $\mathrm{Cap}_{h,1}$-nest is an $\mathcal{E}$-nest.
(ii) If $h \in \mathcal{F}$, then one can take $D_n = E$, $n \geq 1$, in Theorem 1.2.14 and hence a $\mathrm{Cap}_{h,1}$-nest becomes a synonym of an $\mathcal{E}$-nest.
(iii) $h = 1$ is an important case for Theorem 1.2.14. □

1.3. QUASI-REGULAR DIRICHLET FORMS

We maintain the same assumptions on $(E, \mathcal{B}(E), m)$ as in the preceding section. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(E;m)$.

Lemma 1.3.1. *Let S be a countable family of $\mathcal{E}$-quasi-continuous functions on E. Then there is an $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ such that $S \subset C(\{F_k\})$.*

Proof. Fix some $\varphi \in L^2(E;m)$ with $0 < \varphi \leq 1$ and set $h = G_1\varphi$. Spell out $S = \{f_k, k \geq 1\}$. For each $n \geq 1$, there is an $\mathcal{E}$-nest $\{F_{n,k}, k \geq 1\}$ such that $f_n \in C(\{F_{n,k}\})$ and $\text{Cap}_{h,1}(F_{n,k}^c) \leq 2^{-nk}$. Define $F_k := \cap_{n\geq 1} F_{n,k}$, which is closed. By Theorems 1.2.9 and 1.2.10,

$$\text{Cap}_{h,1}(F_k^c) = \text{Cap}_{h,1}(\cup_{n\geq 1} F_{n,k}^c) \leq \sum_{n\geq 1} \text{Cap}_{h,1}(F_{n,k}^c) \leq 2^{-k} \quad \text{for } k \geq 1.$$

So $\{F_k, k \geq 1\}$ is an $\mathcal{E}$-nest by Theorem 1.2.13 and clearly $S \subset C(\{F_k\})$. □

THEOREM 1.3.2. *Let $h = G_1\varphi$ for some $\varphi \in L^2(E;m)$ with $0 < \varphi \leq 1$. Suppose that $u \in \mathcal{F}$ has an $\mathcal{E}$-quasi-continuous m-version $\widetilde{u}$. Then*

$$\text{Cap}_{h,1}(|\widetilde{u}| > \lambda) \leq \mathcal{E}_1(u,u)/\lambda^2 \quad \textit{for every } \lambda > 0.$$

Proof. Let $\{F_k, k \geq 1\}$ be an $\mathcal{E}$-nest such that $\widetilde{u} \in C(\{F_k\})$. For $\lambda > 0$, let $D_k := \{x \in F_k : |\widetilde{u}(x)| > \lambda\} \cup F_k^c$, which is an open subset of E. Let $u_k := \lambda^{-1}|\widetilde{u}| + h_{F_k^c} \in \mathcal{F}$. Since $0 < h \leq 1$ $[m]$ on E, $u_k \geq h$ on D_k. Thus

$$\begin{aligned}\text{Cap}_{h,1}(|\widetilde{u}| > \lambda) &\leq \text{Cap}_{h,1}(D_k) \leq \mathcal{E}_1(u_k, u_k)\\ &\leq \lambda^{-2}\mathcal{E}_1(|u|,|u|) + 2\lambda^{-1}\mathcal{E}_1(|u|, h_{F_k^c}) + \text{Cap}_{h,1}(F_k^c).\end{aligned}$$

It follows then $\text{Cap}_{h,1}(|\widetilde{u}| > \lambda) \leq \limsup_{k\to\infty} \text{Cap}_{h,1}(D_k) \leq \mathcal{E}_1(u,u)/\lambda^2$. □

THEOREM 1.3.3. *Suppose each $u_k \in \mathcal{F}$ has an $\mathcal{E}$-quasi-continuous m-version $\widetilde{u}_k$ and that u_k converges to u in $(\mathcal{F}, \mathcal{E}_1)$ as $k \to \infty$. Then there exists a subsequence $\{u_{n_k}, k \geq 1\}$ such that $\widetilde{u}_{n_k}$ converges to an $\mathcal{E}$-quasi-continuous m-version $\widetilde{u}$ of u quasi uniformly; that is, there is an $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ such that $\{\widetilde{u}, \widetilde{u}_{n_j}, j \geq 1\} \subset C(\{F_k\})$ and $\widetilde{u}_{n_k}$ converges to $\widetilde{u}$ uniformly on each F_k.*

Proof. Taking a subsequence if necessary, we may assume that $\mathcal{E}_1(u_{k+1} - u_k, u_{k+1} - u_k) < 2^{-3k}$ for every $k \geq 1$. Define

$$A_k := \left\{x \in E : |\widetilde{u}_{k+1}(x) - \widetilde{u}_k(x)| > 2^{-k}\right\}.$$

By Theorem 1.3.2, $\text{Cap}_{h,1}(A_k) \leq 2^{2k}\mathcal{E}_1(u_{k+1} - u_k, u_{k+1} - u_k) < 2^{-k}$. Let $\{E_\ell, \ell \geq 1\}$ be an $\mathcal{E}$-nest such that $\{\widetilde{u}_k, k \geq 1\} \subset C(\{E_k\})$. Define $F_k := E_k \cap \left(\cap_{l\geq k} A_l^c\right)$, which is closed. Since

$$\text{Cap}_{h,1}(F_k^c) \leq \text{Cap}_{h,1}(E_k^c) + \sum_{l\geq k} \text{Cap}_{h,1}(A_l) < \text{Cap}_{h,1}(E_k^c) + 2^{-k+1},$$

which tends to 0 as $k \to \infty$, $\{F_k, k \geq 1\}$ is an $\mathcal{E}$-nest and clearly $\widetilde{u}_n$ converges to some $\widetilde{u}$ uniformly on each F_k. Thus $\widetilde{u} \in C(\{F_k\})$ and therefore it is an $\mathcal{E}$-quasi-continuous m-version of u. □

Definition 1.3.4. An $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ is called *m-regular* if for each $k \geq 1$, $\text{supp}[\mathbf{1}_{F_k} m] = F_k$; that is, for every $x \in F_k$ and every neighborhood $U(x)$ of x, $m(U(x) \cap F_k) > 0$.

Definition 1.3.5. A Hausdorff topological space is called a *Lindelöf space* if every open covering of the space has a countable subcover.

Lindelöf theorem asserts that a topological space is Lindelöf if its topology has a countable base (see, e.g., [103, p. 49]). On a Lindelöf space, the topological support of any σ-finite measure μ is well defined to the smallest closed set F that μ does not charge on its complement F^c.

Lemma 1.3.6. *Let $\{F_k, k \geq 1\}$ be an $\mathcal{E}$-nest. Suppose that the relative topology on each F_k is Lindelöf. Let $\widehat{F}_k = \text{supp}[\mathbf{1}_{F_k} m]$. Then $\{\widehat{F}_k, k \geq 1\}$ is an m-regular $\mathcal{E}$-nest.*

Proof. Let $h := G_1\varphi$ for some $\varphi \in L^2(E; m)$ such that $0 < \varphi \leq 1$. Note that

$$F_k \setminus \widehat{F}_k = \{x \in F_k : \text{ there is an open neighborhood } U(x) \text{ of } x \text{ such that } m(U(x) \cap F_k) = 0\}$$

By the Lindelöf property, $m(F_k \setminus \widehat{F}_k) = 0$. Thus $\mathcal{L}_{F_k^c, h} = \mathcal{L}_{\widehat{F}_k^c, h}$ and therefore $\text{Cap}_{h,1}(\widehat{F}_k^c) = \text{Cap}_{h,1}(F_k^c)$. This proves that $\{\widehat{F}_k, k \geq 1\}$ is an m-regular $\mathcal{E}$-nest. □

Theorem 1.3.7. *Suppose $\{F_k, k \geq 1\}$ is an m-regular $\mathcal{E}$-nest and $f \in C(\{F_k\})$. If $f \geq 0$ $[m]$ on an open set D, then $f(x) \geq 0$ for every $x \in D \cap \left(\cup_{k \geq 1} F_k\right)$; i.e., $f \geq 0$ $\mathcal{E}$-q.e. on D.*

Proof. Since $f \geq 0$ $[m]$ on D and f is continuous on each F_k, $f \geq 0$ on $F_k \cap D$ due to the assumption $F_k = \text{supp}[\mathbf{1}_{F_k} m]$. Thus $f \geq 0$ on $\cup_{k \geq 1} (F_k \cap D)$. □

Definition 1.3.8. A Dirichlet form $(\mathcal{F}, \mathcal{E})$ on $L^2(E; m)$ is called *quasi-regular* if:

(i) there exists an $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ consisting of compact sets;

(ii) there exists an $\mathcal{E}_1$-dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-quasi-continuous m-versions;

(iii) there exists $\{f_k, k \geq 1\} \subset \mathcal{F}$ having $\mathcal{E}$-quasi-continuous m-versions $\{\widetilde{f}_k, k \geq 1\}$ and an $\mathcal{E}$-polar set $N \subset E$ such that $\{\widetilde{f}_k : k \geq 1\}$ separates the points of $E \setminus N$.

Remark 1.3.9. (i) Part (i) of Definition 1.3.8 implies that, for any α-excessive function h in $\mathcal{F}$ with $\alpha > 0$, $\mathrm{Cap}_{h,1}$ is tight; that is, there is an increasing sequence of compact sets $\{K_j, j \geq 1\}$ such that $\lim_{j\to\infty} \mathrm{Cap}_{h,1}(E \setminus K_j) = 0$.
(ii) Part (ii) of Definition 1.3.8 implies by Theorem 1.3.3 that every function in $\mathcal{F}$ has an $\mathcal{E}$-quasi-continuous m-version, which will be denoted by $\widetilde{f}$.
(iii) We may assume, by parts (i) and (iii) of Definition 1.3.8 together with Theorem 1.2.13 and Lemma 1.3.1, that there is an $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ consisting of compact sets so that $\{\widetilde{f}_k, k \geq 1\} \subset C(\{F_k\})$ and $\{\widetilde{f}_k, k \geq 1\}$ separates points of $\cup_k F_k$. Define

$$\rho(x,y) := \sum_{j=1}^{\infty} 2^{-j} \left(|\widetilde{f}_j(x) - \widetilde{f}_j(y)| \wedge 1 \right) \qquad \text{for } x, y \in \cup_{k\geq 1} F_k.$$

Then $\rho(x,y)$ is a (separating) metric on each F_k, which by the compactness of F_k is compatible with the original topology on F_k inherited from E. Since the topology induced by ρ on each F_k has countable base, F_k is a separable metric space. Hence $L^2(E;m) = L^2(\cup_{j\geq 1} F_j; m)$ is separable and therefore so is $(\mathcal{F}, \mathcal{E}_1)$. Thus Definition 1.3.8(ii) can be replaced by

(ii). There exists an $\mathcal{E}_1$-dense countable subset $\{u_k, k \geq 1\}$ of $\mathcal{F}$ whose elements have $\mathcal{E}$-quasi-continuous m-version.
(iv) By the Lindelöf theorem, each compact set F_k in (iii) is Lindelöf. Hence for a quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, if $f \geq 0$ $[m]$ on an open set D and if f is $\mathcal{E}$-quasi-continuous, then $f \geq 0$ $\mathcal{E}$-q.e. on D by Lemma 1.3.6 and Theorem 1.3.7.
(v) By Corollary 2 on p.12 of [136] $Y := \cup_{k\geq 1} F_k$ is a Lusin space (i.e., it is homeomorphic to a Borel subset of a compact metric space). Since $L^2(E;m)$ can be identified with $L^2(Y;m)$, when dealing with quasi-regular Dirichlet forms, we can assume that E is a topological Lusin space. □

For an m-measurable function f defined and finite m-a.e. on E, the *support* of f is defined to be the support of the measure $f \cdot m$. When f is continuous, the support of f is just the closure of the set $\{f \neq 0\}$. When E is a locally compact separable metric space, we shall denote by $C_c(E)$ the family of all continuous functions on E with compact support, and by $C_\infty(E)$ the family of all continuous functions f on E which vanishes at ∞, namely, there exists for any $\varepsilon > 0$ a compact set with $|f(x)| < \varepsilon$ for every $x \in E \setminus K$. $C_\infty(E)$ is a Banach space with respect to the uniform norm $\|f\|_\infty = \sup_{x\in E} |f(x)|$.

DEFINITION 1.3.10. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$ is said to be *regular* if
(i) E is a locally compact separable metric space and m is a Radon measure on E with full support;

(ii) $\mathcal{F} \cap C_c(E)$ is $\mathcal{E}_1$-dense in $\mathcal{F}$;
(iii) $\mathcal{F} \cap C_c(E)$ is uniformly dense in $C_c(E)$.

Remark 1.3.11. (i) By Stone-Weierstrass theorem (cf. [58, Theorem 4.45]), (iii) in Definition 1.3.10 is equivalent to
(iii′) $\mathcal{F} \cap C_c(E)$ separates the points of E.
(ii) Clearly, a regular Dirichlet form is quasi-regular. □

LEMMA 1.3.12. *Let E be a locally compact separable metric space and m a Radon measure on E with full support. Suppose that $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(E; m)$. If the space $\mathcal{F} \cap C_\infty(E)$ is dense both in $(\mathcal{F}, \|\cdot\|_{\mathcal{E}_1})$ and in $(C_\infty(E), \|\cdot\|_\infty)$, then $(\mathcal{E}, \mathcal{F})$ is regular.*

Proof. For a fixed $f \in \mathcal{F} \cap C_\infty(E)$, we consider its composition $f_\ell = \varphi_\ell \circ f$ with specific normal contractions defined by

$$\varphi_\ell(t) := t - ((-1/\ell) \vee t) \wedge (1/\ell), \qquad t \in \mathbb{R},\ \ell \geq 1, \tag{1.3.1}$$

then

$$f_\ell \in \mathcal{F} \cap C_c(E) \ \text{ and } \ \|f_\ell - f\|_\infty \leq \frac{1}{\ell} \ \text{ for } \ell \geq 1,$$

and we see that $\mathcal{F} \cap C_c(E)$ is dense in the space $(C_c(E), \|\cdot\|_\infty)$. On the other hand, Lemma 1.1.11 implies that f_ℓ is $\mathcal{E}_1$-convergent to f and hence $\mathcal{F} \cap C_c(E)$ is dense in $(\mathcal{F}, \mathcal{E}_1)$. □

Exercise 1.3.13. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(E; m)$. Show that, for any $f \in C_c(E)$, there exist $f_n \in \mathcal{F} \cap C_c(E)$ such that $\text{supp}[f_n] \subset \text{supp}[f]$ for every $n \geq 1$, and f_n converges to f uniformly on E as $n \to \infty$.

For a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$, it is customary to use 1-capacity denoted by Cap_1, that is, (h, α)-capacity with $h = 1$ and $\alpha = 1$. This is because in this case $\text{Cap}_1(D) < \infty$ for every relatively compact open subset $D \subset E$. Note that since E is a locally compact separable metric space, there is a sequence of relatively compact open subsets $\{D_k, k \geq 1\}$ with $\overline{D}_k \subset D_{k+1}$, $k \geq 1$, and $\cup_{k \geq 1} D_k = E$. Thus Theorem 1.2.14 is applicable with $h = 1$. In particular, we have the following, which gives the equivalence of $\mathcal{E}$-polar set, $\mathcal{E}$-nest, and $\mathcal{E}$-quasi-continuity with the notions of set of capacity zero, generalized nest, and quasi continuity, respectively, defined in the book [73].

THEOREM 1.3.14. *Suppose that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E; m)$. Then*
(i) *A subset set of E is $\mathcal{E}$-polar if and only if it is* Cap_1*-polar.*

(ii) *An increasing sequence of closed subsets* $\{F_j, j \geq 1\}$ *is an* $\mathcal{E}$*-nest if and only if* $\lim_{j\to\infty} \mathrm{Cap}_1(K \setminus F_j) = 0$ *for every compact set* $K \subset E$.
(iii) *A function* f *is* $\mathcal{E}$*-quasi-continuous if and only if for every* $\varepsilon > 0$*, there is an open set* $D \subset E$ *with* $\mathrm{Cap}_1(D) < \varepsilon$ *such that* $f\big|_{E\setminus D}$ *is finite and continuous or, equivalently, there exists a* Cap_1*-nest* $\{F_k\}$ *such that* $f \in C(\{F_k\})$.

Proof. (i) and (ii) follow immediately from Theorem 1.2.14.

For (iii), if f is $\mathcal{E}$-quasi-continuous, then, in view of Theorem 1.2.13, there is an $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ consisting of closed sets so that $f \in C(\{F_k\})$. Let $\{D_k, k \geq 1\}$ be an increasing sequence of relatively compact open subsets with $\cup_{k\geq 1} D_k = E$ and $\mathrm{Cap}(D_k) < \infty$, $k \geq 1$. By Theorem 1.2.14, for every $\varepsilon > 0$ and $n \geq 1$, there is an integer $k_n \geq 1$ so that $\mathrm{Cap}_1(D_n \setminus F_{k_n}) < 2^{-n-1}\varepsilon$. Define $D := \cup_{n\geq 1}(D_n \setminus F_{k_n})$, which is an open set with $\mathrm{Cap}_1(D) \leq \sum_{n\geq 1} \mathrm{Cap}_1(D_n \setminus F_{k_n}) < \varepsilon$.

It is easy to check that $f\big|_{E\setminus D}$ is continuous. Indeed, as for every $x_0 \in E \setminus D$ there is some $r_0 > 0$ and $n_0 \geq 1$ so that $B(x_0, r_0) \subset D_{n_0}$ and that $E \setminus D = \cap_{n\geq 1}(D_n^c \cup F_{k_n})$, we have $B(x_0, r_0) \cap (E \setminus D) \subset B(x_0, r_0) \cap F_{k_{n_0}}$. It follows that $f|_{E\setminus D}$ is continuous at x_0.

The sufficiency in (iii) is obvious because any Cap_1-nest is an $\mathcal{E}$-nest by Remark 1.2.15. □

The last assertion of the above theorem can be strengthened as follows.

Let $E_\partial = E \cup \{\partial\}$ be the one-point compactification of the locally compact metric space E. For a closed set $F \subset E$, we regard $F \cup \{\partial\}$ as a topological subspace of E_∂. For an increasing sequence $\{F_k\}$ of closed sets, we denote by $C_\infty(\{F_k\})$ the collection of functions f on E such that, if f is extended to E_∂ by setting $f(\partial) = 0$, then $f\big|_{F_k\cup\{\partial\}}$ is finite and continuous for each k. Obviously the space $C_\infty(E)$ is contained in $C_\infty(\{F_k\})$.

Suppose, for a function f on E, there exists an $\mathcal{E}$-nest $\{F_k\}$ such that $f \in C_\infty(\{F_k\})$. Then f is said to be *quasi continuous in the restricted sense* relative to the $\mathcal{E}$-nest $\{F_k\}$.

LEMMA 1.3.15. *If* $(\mathcal{E}, \mathcal{F})$ *is a regular Dirichlet form on* $L^2(E; m)$*, then each element* $f \in \mathcal{F}$ *admits an m-version* $\widetilde{f}$ *which is quasi continuous in the restricted sense relative to a* Cap_1*-nest.*

Proof. For $f \in \mathcal{F} \cap C_c(E)$ and $\lambda > 0$, the set $D_\lambda = \{x \in E : |f(x)| > \lambda\}$ is an open set with $f/\lambda \in \mathcal{L}_{D_\lambda,1}$ so that

$$\mathrm{Cap}_1(D_\lambda) \leq \mathcal{E}_1(f, f)/\lambda^2, \tag{1.3.2}$$

which yields the above assertion as in Theorem 1.3.3 because $\mathcal{F} \cap C_c(E)$ is $\mathcal{E}_1$-dense in $\mathcal{F}$. □

Exercise 1.3.16. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form. Show that the statement in Theorem 1.3.3 holds with an $\mathcal{E}$-nest $\{F_k\}$ and the space $C(\{F_k\})$ being replaced by a Cap_1-nest $\{F_k\}$ and $C_\infty(\{F_k\})$, respectively.

We give the following definition for future use.

Definition 1.3.17. Let E be a locally compact separable metric space, m be a Radon measure on E with $\text{supp}[m] = E$, and $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(E; m)$.
(i) $\mathcal{C} \subset \mathcal{F} \cap C_c(E)$ is said to be a *core* of $(\mathcal{E}, \mathcal{F})$ if $\mathcal{C}$ is dense both in $(\mathcal{F}, \|\cdot\|_{\mathcal{E}_1})$ and in $(C_c(E), \|\cdot\|_\infty)$. Clearly the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular if it has a core.
(ii) A core $\mathcal{C}$ is said to be *standard* if it is a dense linear subspace of $C_c(E)$, and for any $\varepsilon > 0$, there exists a normal contraction φ_ε of (1.1.7) such that $\varphi_\varepsilon(\mathcal{C}) \subset \mathcal{C}$.
(iii) A standard core $\mathcal{C}$ is said to be *special* if $\mathcal{C}$ is a dense subalgebra of $C_c(E)$, and for any compact set K and relatively compact open set G with $K \subset G$, there exists $f \in \mathcal{C}_+$ such that $f = 1$ on K and $f = 0$ on $E \setminus G$.
(iv) $(\mathcal{E}, \mathcal{F})$ is called *local* if $\mathcal{E}(f, g) = 0$ whenever f, $g \in \mathcal{F}$ have disjoint compact supports.
(v) $(\mathcal{E}, \mathcal{F})$ is called *strongly local* if $\mathcal{E}(f, g) = 0$ whenever $f \in \mathcal{F}$ has a compact support and $g \in \mathcal{F}$ is constant on a neighborhood of the support of f.

1.4. QUASI-HOMEOMORPHISM OF DIRICHLET SPACES

Suppose $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(E; m)$. Let $(\widehat{E}, \mathcal{B}(\widehat{E}))$ be a second measurable space and $j : (E, \mathcal{B}(E)) \to (\widehat{E}, \mathcal{B}(\widehat{E}))$ be a measurable map. Define $\widehat{m} := m \circ j^{-1}$, the push forward measure of m under map j; that is, for $A \in \mathcal{B}(\widehat{E})$, $\widehat{m}(A) = m(j^{-1}(A))$. Then $j^* : L^2(\widehat{E}; \widehat{m}) \to L^2(E; m)$ is an isometry, where $j^*\widehat{f} := \widehat{f} \circ j$ for $\widehat{f} \in L^2(\widehat{E}; \widehat{m})$. $j^*L^2(\widehat{E}; \widehat{m})$ is, in general, a closed subspace of $L^2(E; m)$. Define $\widehat{\mathcal{F}} := \{\widehat{f} \in L^2(\widehat{E}; \widehat{m}) : j^*\widehat{f} \in \mathcal{F}\}$ and

$$\widehat{\mathcal{E}}(\widehat{f}, \widehat{g}) := \mathcal{E}(j^*\widehat{f}, j^*\widehat{g}) \qquad \text{for } \widehat{f}, \widehat{g} \in \widehat{\mathcal{F}}.$$

Clearly $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ is a closed form on $L^2(\widehat{E}; \widehat{m})$. If j^* maps $L^2(\widehat{E}, \widehat{m})$ onto $L^2(E; m)$, then $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ is a Dirichlet form on $L^2(\widehat{E}; \widehat{m})$, which is called the *image Dirichlet form* of $(\mathcal{E}, \mathcal{F})$ under j. We denote in the sequel $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ as $j(\mathcal{E}, \mathcal{F})$.

Definition 1.4.1. Given two Dirichlet forms $(\mathcal{E}, \mathcal{F})$ and $(\widehat{\mathcal{F}}, \widehat{\mathcal{E}})$ on $L^2(E; m)$ and $L^2(\widehat{E}; \widehat{m})$, respectively, where E and $\widehat{E}$ are two Hausdorff topological spaces and m and $\widehat{m}$ are σ-finite measures on E and $\widehat{E}$ respectively with $\text{supp}[m] = E$ and $\text{supp}[\widehat{m}] = \widehat{E}$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is said to be *quasi-homeomorphic*

to $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ if there is an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ and an $\widehat{\mathcal{E}}$-nest $\{\widehat{F}_n\}_{n\geq 1}$ and a map $j\colon \cup_{k\geq 1} F_k \to \cup_{k\geq 1}\widehat{F}_k$ such that
(a) j is a topological homeomorphism from F_k onto $\widehat{F}_k$ for each $k \geq 1$.
(b) $\widehat{m} = m \circ j^{-1}$.
(c) $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}}) = j(\mathcal{E}, \mathcal{F})$; that is, $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ is the image Dirichlet form of $(\mathcal{E}, \mathcal{F})$ under map j.

For every function $\widehat{f}$ on $\widehat{E}$, $j^*\widehat{f}$ is uniquely defined on E modulo an m-null set and j^* is an isometry from $L^2(\widehat{E}; \widehat{m})$ onto $L^2(E; m)$.

Exercise 1.4.2. Suppose two Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ and $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ on $L^2(\widehat{E}; \widehat{m})$ are quasi-homeomorphic by a map j as in the above definition. Prove that j is quasi notion preserving in the following sense:
(i) An increasing sequence $\{E_k\}$ of closed subsets of E is an $\mathcal{E}$-nest if and only if $\{j(E_k \cap F_k)\}$ is an $\widehat{\mathcal{E}}$-nest.
(ii) $N \subset E$ is $\mathcal{E}$-polar if and only if $j\left((\cup_{k\geq 1} F_k) \cap N\right)$ is $\widehat{\mathcal{E}}$-polar.
(iii) A function f defined $\mathcal{E}$-q.e. on E is $\mathcal{E}$-quasi-continuous if and only if $f \circ j^{-1}$ is $\widehat{\mathcal{E}}$-quasi-continuous.

The following theorem gives an important connection between quasi-regular Dirichlet forms and regular Dirichlet forms, which enables us to transfer known results for regular Dirichlet forms to quasi-regular Dirichlet forms.

THEOREM 1.4.3. *A Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ is quasi-regular if and only if $(\mathcal{E}, \mathcal{F})$ is quasi-homeomorphic to a regular Dirichlet space on a locally compact separable metric space.*

Proof. The "if" part is trivial. We only need to show the "only if" part. Take a strictly positive bounded $\varphi \in L^1(E; m)$ and let $h = G_1\varphi$, which is strictly positive m-a.e. on E and is in $\mathcal{F}$. Since $(\mathcal{E}, \mathcal{F})$ is quasi-regular on $L^2(E; m)$, by Theorem 1.3.3, h has an $\mathcal{E}$-quasi-continuous m-version $\widetilde{h}$. We claim that there is an $\mathcal{E}$-nest $\{K_j, j \geq 1\}$ consisting of compact sets so that $\widetilde{h} \in C(\{K_j\})$ and $\widetilde{h} \geq 1/j$ on each K_j. By Theorems 1.2.13 and 1.3.3, there is an $\mathcal{E}$-nest $\{\widetilde{K}_j, j \geq 1\}$ consisting of compact sets so that $\widetilde{h} \in C(\{\widetilde{K}_j\})$. For each $j \geq 1$, define $K_j = \widetilde{K}_j \cap \{\widetilde{h} \geq 1/j\}$, which is compact. We show that $\{K_j, j \geq 1\}$ is an $\mathcal{E}$-nest. Note that $K_j^c = \widetilde{K}_j^c \cup \{x \in \widetilde{K}_j : \widetilde{h}(x) < 1/j\}$ and $v_j := h_{\widetilde{K}_j^c} + (1/j) \wedge h \in \mathcal{L}_{K_j^c, h}$. Thus

$$\begin{aligned}
\lim_{j\to\infty} \mathrm{Cap}_{h,1}(K_j^c) &\leq \lim_{j\to\infty} \mathcal{E}_1\left(h_{\widetilde{K}_j^c} + (1/j) \wedge h,\ h_{\widetilde{K}_j^c} + (1/j) \wedge h\right) \\
&\leq 2 \lim_{j\to\infty} \mathcal{E}_1\left(h_{\widetilde{K}_j^c}, h_{\widetilde{K}_j^c}\right) + 2 \lim_{j\to\infty} \mathcal{E}_1\left((1/j) \wedge h, (1/j) \wedge h\right) \\
&= 0,
\end{aligned}$$

where in the last equality we used Lemma 1.1.11(i) applied to normal contractions $\psi_j(t) := t - ((-1/j) \vee t) \wedge (1/j)$. This establishes that $\{K_j, j \geq\}$ is an $\mathcal{E}$-nest. Observe that for $f \in L^2(E; m)$, $\psi_j(f) \in L^1(E; m)$. So by Lemma 1.1.11 and Remark 1.3.9, there exists a countable $\mathcal{E}_1$-dense set $B_0 = \{f_n, n \geq 1\}$ of bounded $\mathcal{E}$-quasi-continuous functions in $\mathcal{F} \cap L^1(E; m)$ such that
(i) $\widetilde{h} \in B_0$ and B_0 is an algebra over the rational numbers,
(ii) There is an $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ consisting of compact sets such that $B_0 \subset C(\{F_k\})$ and B_0 separates points of $\cup_{k\geq 1} F_k$ and $\widetilde{h} \geq 1/k$ on F_k.

We make functions in B_0 to take value zero on $E \setminus \cup_{k\geq 1} F_k$. Define $B := \overline{B_0}^{\|\cdot\|_\infty}$, which is a commutative Banach algebra. We now construct a regular Dirichlet form $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ on a locally compact separable metric space $\widehat{E}$ via the Gelfand transform which will be quasi-homeomorphic to $(\mathcal{E}, \mathcal{F})$.

Step 1. Construct a locally compact separable metric space $\widehat{E}$.

Let $\widehat{E}$ be a collection of non-trivial real-valued functionals γ on B which satisfy for $f, g \in B$ and for rational numbers a and b,

(a) $|\gamma(f)| \leq \|f\|_\infty$,
(b) $\gamma(fg) = \gamma(f)\gamma(g)$,
(c) $\gamma(af + bg) = a\gamma(f) + b\gamma(g)$.

We equip $\widehat{E}$ with the weakest topology so that the function $\Phi_f : \gamma \mapsto \gamma(f)$ is continuous for every $f \in B$. It is well-known that $\widehat{E}$ is a separable locally compact Hausdorff space which is compact if and only if $1 \in B$, and $\{\Phi_f, f \in B\} \subset C_\infty(\widehat{E})$. The topological space $\widehat{E}$ is metrizable with metric δ defined by

$$\delta(\gamma, \eta) := \sum_{n\geq 1} 2^{-n} \left(|\gamma(f_n) - \eta(f_n)| \wedge 1 \right), \qquad \gamma, \eta \in \widehat{E}.$$

Let j be the unique map from $\cup_{k\geq 1} F_k$ into $\widehat{E}$ such that

$$(jx)(f) := f(x) \qquad \text{for } f \in B \text{ and } x \in \cup_{k\geq 1} F_k.$$

By (ii) above, j is a continuous one-to one map on each F_k. Hence $\widehat{F}_k := j(F_k)$ is compact in $\widehat{E}$ and j is a topological homeomorphism from F_k onto $\widehat{F}_k$ for every $k \geq 1$. Note that $j : \cup_{k\geq 1} F_k \to \widehat{E}$ is Borel measurable and $m(E \setminus \cup_{k\geq 1} F_k) = 0$. Define $\widehat{m} := m \circ j^{-1}$. Clearly $\widehat{m}(\widehat{E} \setminus \cup_{k\geq 1} \widehat{F}_k) = 0$. It follows from the m-integrability of functions in B_0 that $\widehat{m}$ is a Radon measure, and it is easy to check that $\text{supp}[\widehat{m}] = \widehat{E}$ (see [138, p. 23]). Since B_0 is dense in $L^2(E; m)$, j^* is a unitary map from $L^2(\widehat{E}; \widehat{m})$ onto $L^2(E; m)$.

Step 2. Φ maps B onto $C_\infty(\widehat{E})$.

For $f \in B$, $\Phi_f \in C_\infty(\widehat{E})$, where $\Phi_f(\gamma) = \gamma(f)$. Clearly, $\|\Phi_f\|_\infty = \|f\|_\infty$. So $\Phi(B)$ is closed under uniform norm. Since $h \in B$ and $\Phi(B)$ is an algebra of real-valued functions that vanish at infinity and separates points in $\widehat{E}$ with $\Phi(\widehat{h}) > 0$

on $\widehat{E}$, by Stone-Weierstrass theorem (cf. [58, Theorem 4.52]), $\Phi(B) = C_\infty(\widehat{E})$.

Step 3. The image Dirichlet form $j(\mathcal{E}, \mathcal{F})$ is regular on $L^2(\widehat{E}; \widehat{m})$.

Let $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}}) := j(\mathcal{E}, \mathcal{F})$. Then $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ is a Dirichlet form on $L^2(\widehat{E}; \widehat{m})$ as j^* is an isometry from $L^2(\widehat{E}; \widehat{m})$ onto $L^2(E; m)$. Since $\widehat{\mathcal{F}} \cap C_\infty(\widehat{E}) \supset \Phi(B_0)$ and the latter is uniformly dense in $C_\infty(\widehat{E})$ and $\widehat{\mathcal{E}}_1$-dense in $\widehat{\mathcal{F}}$, $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ is a regular Dirichlet form on $L^2(\widehat{E}; \widehat{m})$. Since $j^* \widehat{\mathcal{F}}_{\widehat{F}_k} = \mathcal{F}_{F_k}$ for every $k \geq 1$, $\{\widehat{F}_k, k \geq 1\}$ is an $\widehat{\mathcal{E}}$-nest and therefore j is a quasi-homeomorphism from $(\mathcal{E}, \mathcal{F})$ to $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$. ☐ This completes the proof of the theorem.

1.5. SYMMETRIC RIGHT PROCESSES AND QUASI-REGULAR DIRICHLET FORMS

THEOREM 1.5.1. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(E; m)$, where E is a locally compact separable metric space and m is a Radon measure on E with full support. There exists then a Hunt process X on E with an m-symmetric transition function so that $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form of the transition function of X.*

This theorem was proved by the second-named author [62] in 1971. A rather different proof from that of [62] is presented in [73]. Theorem A.1.37 of Appendix A on the Feller semigroup and this theorem constitute basic existence theorems of Hunt processes on locally compact spaces.

We can now combine Theorem 1.4.3 with Theorem 1.5.1 to show that there is a nice Markov process called an m-tight special Borel standard process associated with every quasi-regular Dirichlet form. See Section A.1.3 for the definition of a right process and a special Borel standard process.

Let $(E, \mathcal{B}^*(E))$ be a Radon space, m be a σ-finite measure on it, and X a right process on it. If the transition function $\{P_t; t \geq 0\}$ is m-symmetric, we say that X is *m-symmetric*. In this case, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ of $\{P_t; t \geq 0\}$ is called the *Dirichlet form* of the m-symmetric right process X. We say further that X is *properly associated with* $(\mathcal{E}, \mathcal{F})$ if $P_t f$ is an $\mathcal{E}$-quasi-continuous m-version of $T_t f$ for any $f \in \mathcal{B}(E) \cap L^2(E; m)$ and $t > 0$, where $\{T_t; t > 0\}$ is the $L^2(E; m)$-semigroup generated by $(\mathcal{E}, \mathcal{F})$.

A right process X is called *m-tight* if there is an increasing sequence of compact sets $\{K_j, j \geq 1\}$ so that $\mathbf{P}_m(\lim_{j\to\infty} \tau_{K_j} < \zeta) = 0$. Here $\tau_{K_j} := \inf\{t \geq 0 : X_t \notin K_j\}$ is the first exit time from K_j by X and ζ is the lifetime of X.

THEOREM 1.5.2. *Suppose that $(\mathcal{E}, \mathcal{F})$ is a quasi-regular Dirichlet form on $L^2(E; m)$, where E is a Hausdorff topological space such that the Borel σ-field $\mathcal{B}(E)$ is generated by the continuous functions on E. Then there is an $\mathcal{E}$-polar Borel set $N \subset E$ and an m-symmetric, m-tight, special Borel standard process X on $E \setminus N$ that is properly associated with $(\mathcal{E}, \mathcal{F})$.*

Proof. By Theorem 1.4.3, $(\mathcal{E},\mathcal{F})$ is quasi-homeomorphic to an $\widehat{m}$-symmetric regular Dirichlet form $(\widehat{\mathcal{E}},\widehat{\mathcal{F}})$ on a locally compact separable metric space $\widehat{E}$ through quasi-homeomorphism j. More precisely, $\widehat{m}=m\circ j^{-1}$, $(\widehat{\mathcal{E}},\widehat{\mathcal{F}})=j(\mathcal{E},\mathcal{F})$, and there are $\mathcal{E}$-nest $\{F_k, k\geq 1\}$ and $\widehat{\mathcal{E}}$-nest $\{\widehat{F}_k, k\geq 1\}$ so that j is a topological homeomorphism from F_k onto $\widehat{F}_k$ for every $k\geq 1$. j is a one-to-one map from $E_1=\cup_{k=1}^{\infty}F_k$ onto $\widehat{E}_1=\cup_{k=1}^{\infty}\widehat{F}_k$ and it can be extended to a one-to-one map from $E_1\cup\{\partial\}$ onto $\widehat{E}_1\cup\{\widehat{\partial}\}$, where ∂ is an extra point adjoined to E and $\widehat{\partial}$ is the point at infinity of $\widehat{E}$. On account of Theorem 1.2.13 and Theorem 1.3.14, we may and do assume that each $\widehat{F}_k$ is compact (consequently, each F_k is compact) by taking an intersection with another $\widehat{\mathcal{E}}$-nest if necessary.

By virtue of Theorem 1.5.1, there is an $\widehat{m}$-symmetric Hunt process

$$\widehat{X}=(\widehat{\Omega},\{\widehat{\mathcal{F}}_t\},\widehat{\zeta},\widehat{X}_t,\widehat{\mathbf{P}}_{\widehat{x}})$$

on $\widehat{E}$ such that $(\widehat{\mathcal{E}},\widehat{\mathcal{F}})$ is the Dirichlet form of $\widehat{X}$ on $L^2(\widehat{E},\widehat{m})$. We shall make use of some theorems in Section 3.1 concerning the relations between $\widehat{X}$ and $\widehat{\mathcal{E}}$. (This is the only proof in the book that uses forward references.) In view of Proposition 3.1.9, $\widehat{X}$ is automatically properly associated with $(\widehat{\mathcal{E}},\widehat{\mathcal{F}})$.

Denote by $\widehat{\tau}_{\widehat{F}_k}$ the first exit time from $\widehat{F}_k$ of $\widehat{X}$. By Theorem 3.1.4 and Theorem 3.1.5, there exists a Borel set $\widehat{N}$ containing $\widehat{E}\setminus\widehat{E}_1$ such that $\widehat{m}(\widehat{N})=0$ and $\widehat{\mathbf{P}}_{\widehat{x}}(\widehat{\Lambda})=1$ for every $\widehat{x}\in\widehat{E}\setminus\widehat{N}$, where

$$\widehat{\Lambda}=\left\{\widehat{\omega}\in\widehat{\Omega}:\ \lim_{k\to\infty}\widehat{\tau}_{\widehat{F}_k}=\widehat{\zeta},\ \widehat{X}_t,\ \widehat{X}_{t-}\in\widehat{E}_{\widehat{\partial}}\setminus\widehat{N}\text{ for every } t\geq 0\right\}.$$

The above set $\widehat{N}$ is called a *Borel properly exceptional set* for the Hunt process $\widehat{X}$.

We define an $\mathcal{E}$-polar Borel set $N\subset E$ by $E\setminus N=j^{-1}(\widehat{E}_1\setminus\widehat{N})$. We let $\Omega=\widehat{\Lambda}$, $\mathcal{F}_t=\widehat{\mathcal{F}}_t\cap\widehat{\Lambda}$, $t\in[0,\infty]$, and denote an element of Ω (resp. $\mathcal{F}_\infty$) by ω (resp. Γ). Finally we define $X=(\Omega,\{\mathcal{F}_t\},X_t,\zeta,\mathbf{P}_x)$ by

$$X_t(\omega):=j^{-1}(\widehat{X}_t(\omega))\quad\text{and}\quad\zeta(\omega):=\widehat{\zeta}(\omega)\qquad\text{for }\omega\in\Omega\text{ and }t\geq 0,$$

and

$$\mathbf{P}_x(\Gamma):=\widehat{\mathbf{P}}_{j(x)}(\Gamma)\quad\text{for }x\in E\setminus N\text{ and }\Gamma\in\mathcal{F}_\infty.$$

Observe that $\tau_{F_k}=\widehat{\tau}_{\widehat{F}_k}$ for every $k\geq 1$, where τ_{F_k} is the first exit time from F_k by X. It is straightforward to check that X is an m-symmetric, m-tight, special Borel standard process on $E\setminus N$ properly associated with $(\mathcal{E},\mathcal{F})$. □

As will be shown in Theorems 3.1.12 and 3.1.13, the Hunt process (respectively, m-symmetric right process) associated with a regular Dirichlet form (respectively, quasi-regular Dirichlet form) is unique in distribution. Moreover, it will be shown in Theorem 3.1.13 that for a quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, $\{F_k, k \geq 1\}$ is an $\mathcal{E}$-nest if and only if

$$\lim_{k\to\infty} \tau_{F_k} = \zeta \qquad \mathbf{P}_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in E,$$

where ζ is the lifetime of for the right process X associated with $(\mathcal{E}, \mathcal{F})$. Thus quasi-homeomorphism is not only an isometry at the Dirichlet form level but also a topological isometry at the process level up to its lifetime, as $j : F_k \mapsto \widehat{F}_k$ is a topological homeomorphism. In view of Theorem 1.5.2, we can assume without loss of generality, in most of the rest of the book, that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular, as corresponding results for quasi-regular Dirichlet forms can be easily deduced via quasi-homeomorphism.

In fact, the quasi-regularity of a Dirichlet form is not only sufficient but also necessary for the association of an m-tight special Borel standard process. More generally the following theorem holds:

Theorem 1.5.3. *Let E be a Radon space and m be a σ-finite measure on E with full support. Suppose that X is an m-symmetric and m-tight right process on E. Then the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ of X is quasi-regular and X is properly associated with $(\mathcal{E}, \mathcal{F})$.*

If in particular E is a Lusin space, then the Dirichlet form of an m-symmetric right process is quasi-regular and X is properly associated with $(\mathcal{E}, \mathcal{F})$.

The second statement follows from the first because an m-symmetric right process on a Lusin space E is necessarily m-tight (see [119, Theorem IV.1.15]).

When X is an m-tight, m-special Borel standard process on E, this result was proved by S. Albeverio and Z. M. Ma [2] in 1991 and its proof can be found in the book by Z. M. Ma and M. Röckner [119, Theorem IV.5.1] under a more general assumption on the state space E. The result under the current condition follows from the aforementioned result in [119] together with a result of P. J. Fitzsimmons [53, Theorem 3.22], who showed that the restriction of X on the complement of some m-inessential set is an m-special standard process and the Borel measurability assumption on the transition function can be weakened to the universal measurability. As a matter of fact, the stated results in [119] and [53] are formulated for a more general (not necessarily symmetric) sectorial Dirichlet form $(\mathcal{E}, \mathcal{F})$. Moreover, Theorem 1.4.3 also holds for more general sectorial Dirichlet forms; see [31].

It is important to consider a general right process in applications as it is invariant under variety of transformations (for example, time change, killing, h-transformations) while Borel measurability of the transition function is not. However, we shall prove in Theorem 3.1.13 of Section 3.1 that any m-symmetric right process properly associated with a quasi-regular Dirichlet form is, when restricted to the complement of an m-inessential set, a Borel special standard process properly associated with the form. This combined with Theorem 1.5.3 means that any m-tight m-symmetric right process on a Radon space or any m-symmetric right process on a Lusin space can always be modified to be a Borel special standard process (see Corollary 3.1.14 below).

We end this chapter by noting that any quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ admits the following expression in terms of an associated m-symmetric right process $X = (X_t, \mathbf{P}_x, \zeta)$ on E: for any $f \in \mathcal{F}_e$,

$$\mathcal{E}(f,f) = \lim_{t \to 0} \frac{1}{2t} \left(\mathbf{E}_m \left[(f(X_t) - f(X_0))^2; t < \zeta \right] + 2 \int_E f(x)^2 \mathbf{P}_x(\zeta \geq t) m(dx) \right), \tag{1.5.1}$$

where $(\mathcal{F}_e, \mathcal{E})$ is the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$. To see this, let $\{T_t; t > 0\}$ be the semigroup on $L^\infty(E; m)$ determined by the transition function of X. Then by (1.1.13), for $f \in L^\infty(E; m)$,

$$\mathcal{A}_{T_t}(f,f) = \frac{1}{2} \mathbf{E}_m \left[(f(X_t) - f(X_0))^2; t < \zeta \right] + \int_E f(x)^2 \mathbf{P}_x(\zeta \geq t) m(dx).$$

In view of (1.1.20), (1.1.22),

$$\mathcal{E}(f,f) = \lim_{t \to 0} \frac{1}{t} \mathcal{A}_{T_t}(f,f) = \lim_{t \to 0} \lim_{\ell \to \infty} \frac{1}{t} \mathcal{A}_{T_t}(\varphi^\ell \circ f, \varphi^\ell \circ f), \quad f \in \mathcal{F}_e,$$

for the normal contraction φ^ℓ defined by (1.1.19), and consequently, we get (1.5.1) by the monotone convergence theorem.