Chapter One

Introduction

Conformal geometry is the study of spaces in which one knows how to measure infinitesimal angles but not lengths. A conformal structure on a manifold is an equivalence class of Riemannian metrics, in which two metrics are identified if one is a positive smooth multiple of the other. The study of conformal geometry has a long and venerable history. From the beginning, conformal geometry has played an important role in physical theories.

A striking historical difference between conformal geometry compared with Riemannian geometry is the scarcity of local invariants in the conformal case. Classically known conformally invariant tensors include the Weyl conformal curvature tensor, which plays the role of the Riemann curvature tensor, its three-dimensional analogue the Cotton tensor, and the Bach tensor in dimension four. Further examples are not so easy to come by. By comparison, in the Riemannian case invariant tensors abound. They can be easily constructed by covariant differentiation of the curvature tensor and tensorial operations. The situation is similar for other types of invariant objects, for example for differential operators. Historically, there are scattered examples of conformally invariant operators such as the conformally invariant Laplacian and certain Dirac operators, whereas it is easy to write down Riemannian invariant differential operators, of arbitrary orders and between a wide variety of bundles.

In Riemannian geometry, not only is it easy to write down invariants, it can be shown using Weyl's classical invariant theory for the orthogonal group that all invariants arise by the covariant differentiation and tensorial operations mentioned above. In the case of scalar invariants, this characterization as "Weyl invariants" has had important application in the study of heat asymptotics: one can immediately write down the form of coefficients in the expansion of heat kernels up to the determination of numerical coefficients.

In [FG], we outlined a construction of a nondegenerate Lorentz metric in $n+2$ dimensions associated to an n-dimensional conformal manifold, which we called the ambient metric. This association enables one to construct conformal invariants in n dimensions from pseudo-Riemannian invariants in $n+2$ dimensions, and in particular shows that conformal invariants are plentiful. The construction of conformal invariants is easiest and most effective for scalar invariants: every scalar invariant of metrics in $n+2$ dimensions immediately determines a scalar conformal invariant in n dimensions (which may vanish, however). For other types of invariants, for example for differential operators, some effort is required to derive a conformal invariant from a pseudo-Riemannian invariant in two higher dimensions, but in many cases this can be carried out and has led to important new examples.

The ambient metric is homogeneous with respect to a family of dilations on the $n+2$-dimensional space. It is possible to mod out by these dilations and thereby obtain a metric in $n+1$ dimensions, also associated to the given conformal manifold in n dimensions. This gives the "Poincaré" metric associated to the conformal manifold. The Poincaré metric is complete and the conformal manifold forms its boundary at infinity.

The construction of the ambient and Poincaré metrics associated to a general conformal manifold is motivated by the conformal geometry of the flat model, the sphere S^n, which is naturally described in terms of $n+2$-dimensional Minkowski space. Let

$$Q(x) = \sum_{\alpha=1}^{n+1} (x^\alpha)^2 - \left(x^0\right)^2$$

be the standard Lorentz signature quadratic form on $\mathbb{R}^{n+2}$ and

$$\mathcal{N} = \left\{x \in \mathbb{R}^{n+2} \setminus \{0\} : Q(x) = 0\right\}$$

its null cone. The sphere S^n can be identified with the space of lines in $\mathcal{N}$, with projection $\pi : \mathcal{N} \to S^n$. Let

$$\widetilde{g} = \sum_{\alpha=1}^{n+1} (dx^\alpha)^2 - \left(dx^0\right)^2$$

be the associated Minkowski metric on $\mathbb{R}^{n+2}$. The conformal structure on S^n arises by restriction of $\widetilde{g}$ to $\mathcal{N}$. More specifically, for $x \in \mathcal{N}$ the restriction $\widetilde{g}|_{T_x\mathcal{N}}$ is a degenerate quadratic form because it annihilates the radial vector field $X = \sum_{I=0}^{n+1} x^I \partial_I \in T_x\mathcal{N}$. So $\widetilde{g}|_{T_x\mathcal{N}}$ determines an inner product on $T_x\mathcal{N}/\operatorname{span} X \cong T_{\pi(x)}S^n$. As x varies over a line in $\mathcal{N}$, the resulting inner products on $T_{\pi(x)}S^n$ vary only by scale and are the possible values at $\pi(x)$ for a metric in the conformal class on S^n. The Lorentz group $O(n+1,1)$ acts linearly on $\mathbb{R}^{n+2}$ by isometries of $\widetilde{g}$ preserving $\mathcal{N}$. The induced action on lines in $\mathcal{N}$ therefore preserves the conformal class of metrics and realizes the group of conformal motions of S^n.

If instead of restricting $\widetilde{g}$ to $\mathcal{N}$, we restrict it to the hyperboloid

$$\mathcal{H} = \{x \in \mathbb{R}^{n+2} : Q(x) = -1\},$$

then we obtain the Poincaré metric associated to S^n. Namely, $g_+ := \widetilde{g}|_{T\mathcal{H}}$ is the hyperbolic metric of constant sectional curvature -1. Under an appropriate identification of one sheet of $\mathcal{H}$ with the unit ball in $\mathbb{R}^{n+1}$, g_+ can be realized as the Poincaré metric

$$g_+ = 4\left(1 - |x|^2\right)^{-2} \sum_{\alpha=1}^{n+1} (dx^\alpha)^2,$$

and has the conformal structure on S^n as conformal infinity. The action of $O(n+1,1)$ on $\mathbb{R}^{n+2}$ preserves $\mathcal{H}$. The induced action on $\mathcal{H}$ is clearly by isometries of g_+ and realizes the isometry group of hyperbolic space.

The ambient and Poincaré metrics associated to a general conformal manifold are defined as solutions to certain systems of partial differential equations with initial data determined by the conformal structure. Consider the ambient metric. A conformal class of metrics on a manifold M determines and is determined by its metric bundle $\mathcal{G}$, an $\mathbb{R}_+$-bundle over M. This is the subbundle of symmetric 2-tensors whose sections are the metrics in the conformal class. In the case $M = S^n$, $\mathcal{G}$ can be identified with $\mathcal{N}$ (modulo ± 1). Regard $\mathcal{G}$ as a hypersurface in $\mathcal{G} \times \mathbb{R}$: $\mathcal{G} \cong \mathcal{G} \times \{0\} \subset \mathcal{G} \times \mathbb{R}$. The conditions defining the ambient metric $\widetilde{g}$ are that it be a Lorentz metric defined in a neighborhood of $\mathcal{G}$ in $\mathcal{G} \times \mathbb{R}$ which is homogeneous with respect to the natural dilations on this space, that it satisfy an initial condition on the initial hypersurface $\mathcal{G}$ determined by the conformal structure, and that it be Ricci-flat. The Ricci-flat condition is the system of equations intended to propagate the initial data off of the initial surface.

This initial value problem is singular because the pullback of $\widetilde{g}$ to the initial surface is degenerate. However, for the applications to the construction of conformal invariants, it is sufficient to have formal power series solutions along the initial surface rather than actual solutions in a neighborhood. So we concern ourselves with the formal theory and do not discuss the interesting but more difficult problem of solving the equations exactly.

It turns out that the behavior of solutions of this system depends decisively on the parity of the dimension. When the dimension n of the conformal manifold is odd, there exists a formal power series solution $\widetilde{g}$ which is Ricci-flat to infinite order, and it is unique up to diffeomorphism and up to terms vanishing to infinite order. When $n \geq 4$ is even, there is a solution which is Ricci-flat to order $n/2 - 1$, uniquely determined up to diffeomorphism and up to terms vanishing to higher order. But at this order, the existence of smooth solutions is obstructed by a conformally invariant natural trace-free symmetric 2-tensor, the ambient obstruction tensor. When $n = 4$, the obstruction tensor is the same as the classical Bach tensor. When $n = 2$, there is no obstruction, but uniqueness fails.

It may seem contradictory that for n odd, the solution of a second order initial value problem can be formally determined to infinite order by only one piece of Cauchy data: the initial condition determined by the conformal structure. In fact, there are indeed further formal solutions. These correspond to the freedom of a second piece of initial data at order $n/2$. When n is odd, $n/2$ is half-integral, and this freedom is removed by restricting to formal power series solutions. It is crucially important for the applications to conformal invariants that we are able to uniquely specify an infinite order solution in an invariant way. On the other hand, the additional formal solutions with nontrivial asymptotics at order $n/2$ are also important; they necessarily arise in the global formulation of the existence problem as a boundary value problem at infinity for the Poincaré metric. When n is even, the obstruction to the existence of formal power series solutions can be incorporated into log terms in the expansion, in which case there is again a formally undetermined term at order $n/2$.

The formal theory described above was outlined in [FG], but the details were not given. The first main goal of this monograph is to provide these details. We give the full infinite-order formal theory, including the freedom at order $n/2$ in all dimensions and the precise description of the log terms when $n \geq 4$ is even. This formal theory for the ambient metric forms the content of Chapters 2 and 3. The description of the solutions with freedom at order $n/2$ and log terms extends and sharpens results of Kichenassamy [K]. Convergence of the formal series determined by singular nonlinear initial value problems of this type has been considered by several authors; these results imply that the formal series converge if the data are real-analytic.

In Chapter 4, we define Poincaré metrics: they are formal solutions to the equation $\mathrm{Ric}(g_+) = -ng_+$, and we show how Poincaré metrics are equivalent to ambient metrics satisfying an extra condition which we call straight. Then we use this equivalence to derive the full formal theory for Poincaré metrics from that for ambient metrics. We discuss the "projectively compact" formulation of Poincaré metrics, modeled on the Klein model of hyperbolic space, as well as the usual conformally compact picture. As an application of the formal theory for Poincaré metrics, in Chapter 5 we present a formal power series proof of a result of LeBrun [LeB] asserting the existence and uniqueness of a real-analytic self-dual Einstein metric in dimension 4 defined near the boundary with prescribed real-analytic conformal infinity.

In Chapter 7, we analyze the ambient and Poincaré metrics for locally conformally flat manifolds and for conformal classes containing an Einstein metric. The obstruction tensor vanishes for even dimensional conformal structures of these types. We show that for these special conformal classes, there is a way to uniquely specify the formally undetermined term at order $n/2$ in an invariant way and thereby obtain a unique ambient metric up to terms vanishing to infinite order and up to diffeomorphism, just like in odd dimensions. We derive a formula of Skenderis and Solodukhin [SS] for the ambient or Poincaré metric in the locally conformally flat case which is in normal form relative to an arbitrary metric in the conformal class, and prove a related unique continuation result for hyperbolic metrics in terms of data at conformal infinity. The case $n = 2$ is special for all of these considerations. We also derive the form of the GJMS operators for an Einstein metric.

In [FG], we conjectured that when n is odd, all scalar conformal invariants arise as Weyl invariants constructed from the ambient metric. The second main goal of this monograph is to prove this together with an analogous result when n is even. These results are contained in Theorems 9.2, 9.3, and 9.4. When n is even, we restrict to invariants whose weight w satisfies $-w \leq n$ because of the finite order indeterminacy of the ambient metric: Weyl invariants of higher negative weight may involve derivatives of the ambient metric which are not determined. A particularly interesting phenomenon occurs in dimensions $n \equiv 0 \mod 4$. For all n even, it is the case that all even (i.e., unchanged under orientation reversal) scalar conformal invariants with $-w \leq n$ arise as Weyl invariants of the ambient metric. If $n \equiv 2 \mod 4$, this is also true for odd (i.e., changing sign under orientation reversal) scalar conformal invariants with $-w \leq n$ (in fact, these all vanish). But if $n \equiv 0 \mod 4$, there are odd invariants of weight $-n$ which are exceptional in the sense that they do not arise as Weyl invariants of the ambient metric. The set of such exceptional invariants of weight $-n$ consists precisely of the nonzero elements of the vector space spanned by the Pontrjagin invariants whose integrals give the Pontrjagin numbers of a compact oriented n-dimensional manifold (Theorem 9.3).

The parabolic invariant theory needed to prove these results was developed in [BEGr], including the observation of the existence of exceptional invariants. But substantial work is required to reduce the theorems in Chapter 9 to the results of [BEGr]. To understand this, we briefly review how Weyl's characterization of scalar Riemannian invariants is proved.

Recall that Weyl's theorem for even invariants states that every even scalar Riemannian invariant is a linear combination of complete contractions of the form $\operatorname{contr}\left(\nabla^{r_1} R \otimes \cdots \otimes \nabla^{r_L} R\right)$, where the r_i are nonnegative integers, $\nabla^r R$ denotes the r-th covariant derivative of the curvature tensor, and contr denotes a metric contraction with respect to some pairing of all the indices. There are two main steps in the proof of Weyl's theorem. The first is to show that any scalar Riemannian invariant can be written as a polynomial in the components of the covariant derivatives of the curvature tensor which is invariant under the orthogonal group $O(n)$. Since a Riemannian invariant by definition is a polynomial in the Taylor coefficients of the metric in local coordinates whose value is independent of the choice of coordinates, one must pass from Taylor coefficients of the metric to covariant derivatives of curvature. This passage is carried out using geodesic normal coordinates. We refer to the result stating that the map from Taylor coefficients of the metric in geodesic normal coordinates to covariant derivatives of curvature is an $O(n)$-equivariant isomorphism as the jet isomorphism theorem for Riemannian geometry. Once the jet isomorphism theorem has been established, one is left with the algebraic problem of identifying the $O(n)$-invariant polynomials in the covariant derivatives of curvature. This is solved by Weyl's classical invariant theory for the orthogonal group.

In the conformal case, the role of the covariant derivatives of the curvature tensor is played by the covariant derivatives of the curvature tensor of the ambient metric. These tensors are of course defined on the ambient space. But when evaluated on the initial surface, their components relative to a suitable frame determined by a choice of metric in the conformal class define tensors on the base conformal manifold, which we call conformal curvature tensors. For example, the conformal curvature tensors defined by the curvature tensor of the ambient metric itself (i.e., with no ambient covariant derivatives) are the classical Weyl, Cotton, and Bach tensors (except that in dimension 4, the Bach tensor does not arise as a conformal curvature tensor because of the indeterminacy of the ambient metric). The covariant derivatives of curvature of the ambient metric satisfy identities and relations beyond those satisfied for general metrics owing to its homogeneity and Ricci-flatness. We derive these identities in Chapter 6. We also derive the transformation laws for the conformal curvature tensors under conformal change. Of all the conformal curvature tensors, only the Weyl tensor (and Cotton tensor in dimension 3) are conformally invariant. The transformation law of any other conformal curvature tensor involves only first derivatives of the conformal factor and "earlier" conformal curvature tensors. These transformation laws may also be interpreted in terms of tractors. When n is even, the definitions of the conformal curvature tensors and the identities which they satisfy are restricted by the finite order indeterminacy of the ambient metric. The ambient obstruction tensor is not a conformal curvature tensor; it lies at the boundary of the range for which they are defined. But it may be regarded as the residue of an analytic continuation in the dimension of conformal curvature tensors in higher dimensions (Proposition 6.7).

Having understood the properties of the conformal curvature tensors, the next step in the reduction of the theorems in Chapter 9 to the results of [BEGr] is to formulate and prove a jet isomorphism theorem for conformal geometry, in order to know that a scalar conformal invariant can be written in terms of conformal curvature tensors. The Taylor expansion of the metric on the base manifold in geodesic normal coordinates can be further simplified since one now has the freedom to change the metric by a conformal factor as well as by a diffeomorphism. This leads to a "conformal normal form" in which part of the base curvature is normalized away to all orders. Then the conformal jet isomorphism theorem states that the map from the Taylor coefficients of a metric in conformal normal form to the space of all conformal curvature tensors, realized as covariant derivatives of ambient curvature, is an isomorphism. Again, the spaces must be truncated at finite order in even dimensions. The proof of the conformal jet isomorphism theorem is much more involved than in the Riemannian case; it is necessary to relate the normalization conditions in the conformal normal form to the precise identities and relations satisfied by the ambient covariant derivatives of curvature. We carry this out in Chapter 8 by making a direct algebraic study of these relations and of the map from jets of normalized metrics to conformal curvature tensors. A more conceptual proof of the conformal jet isomorphism theorem due to the second author and K. Hirachi uses an ambient lift of the conformal deformation complex and is outlined in [Gr3].

The orthogonal group plays a central role in Riemannian geometry because it is the isotropy group of a point in the group of isometries of the flat model $\mathbb{R}^n$. The analogous group for conformal geometry is the isotropy subgroup $P \subset O(n+1,1)$ of the conformal group fixing a point in S^n, i.e., a null line. Because of its algebraic structure, P is referred to as a parabolic subgroup of $O(n+1,1)$. Just as geodesic normal coordinates are determined up to the action of $O(n)$ in the Riemannian case, the equivalent conformal normal forms for a metric at a given point are determined up to an action of P. Since P is a matrix group in $n+2$ dimensions, there is a natural tensorial action of P on the space of covariant derivatives of ambient curvature, and the conformal transformation law for conformal curvature tensors established in Chapter 6 implies that the map from jets of metrics in conformal normal form to conformal curvature tensors is P-equivariant.

The jet isomorphism theorem reduces the study of conformal invariants to the purely algebraic matter of understanding the P-invariants of the space of covariant derivatives of ambient curvature. This space is nonlinear since the Ricci identity for commuting covariant derivatives is nonlinear in curvature and its derivatives. The results of [BEGr] identify the P-invariants of the linearization of this space. So the last steps, carried out in Chapter 9, are to formulate the results about scalar invariants, to use the jet isomorphism theorem to reduce these results to algebraic statements in invariant theory for P, and finally to reduce the invariant theory for the actual nonlinear space to that for its linearization. The treatment in Chapters 8 and 9 is inspired by, and to some degree follows, the treatment in [F] in the case of CR geometry.

Our work raises the obvious question of extending the theory to higher orders in even dimensions. This has recently been carried out by the second author and K. Hirachi. An extension to all orders of the ambient metric construction, jet isomorphism theorem, and invariant theory has been announced in [GrH2], [Gr3] inspired by the work of Hirachi [Hi] in the CR case. The log terms in the expansion of an ambient metric are modified by taking the log of a defining function homogeneous of degree 2 rather than homogeneous of degree 0. This makes it possible to define the smooth part of an ambient metric with log terms in an invariant way. The smooth part is smooth and homogeneous but no longer Ricci flat to infinite order. There is a family of such smooth parts corresponding to different choices of the ambiguity at order $n/2$. They can be used to formulate a jet isomorphism theorem and to construct invariants, and the main conclusion is that up to a linear combination of finitely many exceptional odd invariants in dimensions $n \equiv 0 \mod 4$ which can be explicitly identified, all scalar conformal invariants arise from the ambient metric. An alternate development of a conformal invariant theory based on tractor calculus is given in general dimensions in [Go1].

A sizeable literature concerning the ambient and Poincaré metrics has arisen since the publication of [FG]. The subject has been greatly stimulated by its relevance in the study of the AdS/CFT correspondence in physics. We have tried to indicate some of the most relevant references of which we are aware without attempting to be exhaustive. Juhl's recent book [J] has some overlap with our material and much more in the direction of Q-curvature and holography.

A construction equivalent to the ambient metric was derived by Haantjes and Schouten in [HS]. They obtained a version of the expansion for straight ambient metrics, to infinite order in odd dimensions and up to the obstruction in even dimensions. In particular, they showed that there is an obstruction in even dimensions $n \geq 4$ and calculated that it is the Bach tensor in dimension 4. They observed that the obstruction vanishes for conformally Einstein metrics and in this case derived the conformally invariant normalization uniquely specifying an infinite order ambient metric in even dimensions. They also obtained the infinite-order expansion in the case of dimension 2, including the precise description of the non-uniqueness of solutions. Haantjes and Schouten did not consider applications to conformal invariants and, unfortunately, it seems that their work was largely forgotten.

We are grateful to the National Science Foundation for support. In particular, the second author was partially supported by NSF grants # DMS 0505701 and 0906035 during the preparation of this manuscript.

Throughout, by smooth we will mean infinitely differentiable. Manifolds are assumed to be smooth and second countable; hence paracompact. Our setting is primarily algebraic, so we work with metrics of general signature. In tensorial expressions, we denote by parentheses (ijk) symmetrization and by brackets $[ijk]$ skew-symmetrization over the enclosed indices.