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"Sports and Pastimes, Done by Number": Mathematical Tricks, Mathematical Games

You've probably played a mathematical game at one time or another. From the counting games we learn as children and the calculator tricks we play in the schoolyard to classics like Sprouts or Lewis Carroll's Game of Logic, there's a whole world of game playing to be had in the realm of numbers. Mathematicians used to be accused of doing magic (some still are), and while conjuring spirits or divining the future may be far from what most of us think of when we think of mathematics, there is a timeless innocent pleasure in the wool-over-the-eyes mathematical tricks of the kind that this chapter showcases.

The selections in this chapter cover the whole period from the middle of the sixteenth century to the end of the twentieth, and if they show one thing it is that tastes have not changed all that much. Some of the very first mathematics books to be printed in English contained "guess my number" tricks and questions about what happens when you double a number again and again and again: kinds that are still popular.

At the same time, there are some areas in which innovations in mathematics have opened up new ground for mathematical games and puzzles. Leonhard Euler, for instance (whom we will meet again in Chapter 7), did important work on the ways of traversing a maze or a set of paths, and this made it much easier for mathematical writers after him to set "route problems" with confidence. Rouse Ball's 1892 "recreations" included reports on this and on a mathematical problem—that of coloring a map—which was an unsolved problem in his day, and, like Alan Parr's open-ended family of Femto games in the final extract of this chapter, it shows how mathematical games can also be an invitation to explore, discover, and create for yourself.

The Well Spring of Sciences

Humfrey Baker, 1564

Humfrey Baker was a teacher in sixteenth-century London, the translator of a book on almanacs, and the author of the very successful arithmetic primer, *The Welspring of Sciences*, embodying its author's infectious enthusiasm for its subject (he once compared arithmetic to good wine, which needed no "garlande" to persuade buyers of its merits).

First published in 1562, *The Welspring* went into many editions down to 1670: the later versions were simply called *Baker's Arithmetick*. The final section of the book gave a selection of mathematical amusements, some of the first pieces of recreational mathematics to be printed in England.

Baker's dense prose is presented here in a simplified paraphrase.

Humfrey Baker (fl. 1557–1574), *The Welspring of Sciences, Which teacheth the perfecte worke and practise of Arithmetic both in vvhole numbers & fractions, with such easie and compendious instruction into the saide art, as hath not heretofore been by any set out nor laboured. Beautified vvith most necessary Rules and Questions, not onely profitable for Marchauntes, but also for all Artificers, as in the Table doth plainely appere. Novv nevvely printed and corrected. Sette forth by Humfrey Baker Citizen of London.* (London, 1564), 158^{v}–162^{r}.

If you would know the number that any man doth think or imagine in his mind, as though you could divine . . .

Bid him triple the number. Then, if the result be even, let him take half of it; if it be odd, let him take the "greater half" (that is, the next whole number above half of it). Then bid him triple again the said half. Next, tell him to cast out, if he can, 36, 27, 18, or 9 from the result: that is, ask him to subtract 9 as many times as is possible, and keep the number of times in his mind. And when he cannot take away 9 any more, tell him to take away 3, 2, or 1, if he can, so as to find out if there is anything left besides the nines.

This done, ask how many times he subtracted 9. Multiply this by 2. And if he had any thing remaining beside the nines, add 1.

For example, suppose that he thought of 6. Being tripled it is 18, of which a half is 9. The triple of that is 27; now ask him to subtract 18, or 9, or 27, and again 9. But then he will say to you that he cannot; ask him to subtract 3, or 2, or 1. He will say also that he cannot; thus, considering that you have made him to subtract three times 9, you shall tell him that he thought of 6, for 3 times 2 makes 6.

If he had thought of 5, the triple of it is 15, of which the "greater half" is 8. The triple of that makes 24, which contains two nines. Two times two makes four, and since there is something remaining we add 1. This makes 5, which is the number that he thought of.

If someone in a group has a ring upon his finger, and you wish to know, as though by magic, who has it, and on which finger and which joint . . .

Ask the group to sit down in order, numbering themselves 1, 2, 3, etc. Then leave the room, and ask one of the onlookers to do the following. Double the number of the person that has the ring, and add 5. Then multiply by 5, and add the number of the finger on which the ring is. Then ask him to append to the result the figure (1, 2, or 3) signifying which joint the ring is on. (Suppose the result was 89 and the ring was on the third joint; then he will make 893.)

This done, ask him what number he has. From this, subtract 250, and you will have a number with at least three digits. The first will be the number of the person who has the ring. The second will be the number of the finger. And the last will be the number of the joint. So, if the number was 893, you subtract 250, and there will remain 643. Which shows you that the sixth person has the ring on the fourth finger, and on the third joint.

But note that when you have made your subtraction, if there is a zero in the tens—that is, in the second digit—you must take one from the hundreds digit. And that "one" will be worth ten tenths, signifying the tenth finger. So, if there remains 703, you must say that the sixth person has the ring on his tenth finger and on the third joint.

In the same way, if a man casts three dice, you may know the score of each of them.

Ask him to double the score of one die, add 5, and then multiply by 5. Next, add the score of one of the other dice, and append to the result the score of the last die. Then ask him what number he has. Subtract 250, and there will remain 3 digits, which tell you the points of the three dice.

Similarly, if three of your companions—say, Peter, James, and John—give themselves different names in your absence—for example, Peter would be called a king, James a duke, and John a knight—you can divine which of them is called a king, which a duke, and which a knight.

Take twenty-four stones (or any other tokens), and, first, give one to one of your friends. Next, give two to another of them, and finally give three to the last of them. Keep a note of the order in which you have given them the stones. Then, leaving the eighteen remaining stones before them, leave the room or turn your back, saying: "whoever calls himself a king, for every stone that I gave him let him take one of the remaining ones; he that calls himself a duke, for every stone that I gave him let him take two of them that remain; and he that calls himself a knight, for every stone that I gave him let him take four."

This being done, return to them, and count how many stones are left. There cannot remain any number except one of these: 1, 2, 3, 5, 6, 7. And for each of these we have chosen a special name, thus: *Angeli, Beati, Qualiter, Messias, Israel, Pietas*. Each name contains the three vowels *a*, *e*, *i*, and these show you the names in order. *A* shows which is the king, *E* which is the duke, and *I* shows which is the knight, in the same order in which you gave them the stones. Thus, if there remains only one stone, the first name, *Angeli*, shows by the vowels *a*, *e*, *i* that your first friend is the king, the second the duke, and the third the knight. If there remain two stones, the second name, *Beati*, shows you by the vowels *e*, *a*, *i* that your first friend is the duke, the second the King, and the third the knight. And so on for the other numbers and names.

Mathematical Recreations

Henry van Etten, 1633

Henry van Etten's *Mathematicall Recreations*, first published in French in 1624, collected together a wide variety of different material. Some of the "problems" were physical tricks or illusions, like "How a Millstone or other ponderosity may hang upon the point of a Needle without bowing, or any wise breaking of it." Others were numerical tricks like those in Baker's *Welspring of Sciences*, above, and still others were optical effects or illusions. The extracts given below thus show some of the diversity in what could plausibly be called mathematics at the time: a diversity which is emphasized by the book's splendidly encyclopedic title. They include a remarkable early report of what Galileo had seen through his telescope, together with the cheery assertion that making a good telescope was a matter of luck ("hazard") as much as skill.

Van Etten was apparently a pseudonym of the French Jesuit Jean Leurechon (c. 1591–1670). The translation has been ascribed to various different people, but its real author remains a mystery.

Henry van Etten (trans. anon.), *Mathematicall Recreations. Or a Collection of sundrie Problemes, extracted out of the Ancient and Moderne Philosophers, as secrets in nature, and experiments in Arithmetic, Geometrie, Cosmographie, Horologographie, Astronomie, Navigation, Musicke, Opticks, Architecture, Staticke, Machanicks, Chimestrie, Waterworkes, Fireworks, etc. Not vulgarly made manifest untill this time: Fit for Schollers, Students, and Gentlemen, that desire to knovv the Philosophicall cause of many admirable Conclusions. Vsefull for others, to acuate and stirre them up to the search of further knowledge; and serviceable to all for many excellent things, both for pleasure and Recreation. Most of which were written first in Greeke and Latine, lately compiled in French, by Henry Van Etten Gent. And now delivered in the English tongue, with the Examinations, Corrections and Augmentations.* (London, 1633), pp. 47–50, 98–102, 167, 208–209, 240.

How to describe a Circle that shall touch 3 Points placed howsoever upon a plane, if they be not in a ⌞straight⌟ line

Let the three points be A, B, C. Put one foot of the Compass upon A and describe an Arc of a Circle at pleasure; and placed at B, cross that Arc in the two points E and F; and placed in C, cross the Arc in G and H. Then lay a ruler upon GH and draw a line, and plac⌞ing⌟ a Ruler upon E and F, cut the other line in K. So K is the Center of the Circumference of a Circle, which will pass by the said three points A, B, C.

Or it may be inverted: having a Circle drawn, to find the Center of that Circle. Make 3 points in the circumference, and then use the same way: so shall you have the Center, a thing most facile to every practitioner in the principles of Geometry.

How to change a Circle into a square form

Make a Circle upon pasteboard or other material, and ⌞label the centre *A*⌟; then cut it into 4 quarters, and dispose them so that *A*, at the center of the Circle, may always be at the Angle of the square. And so the four quarters of the Circle being placed so, it will make a perfect square, whose side *AA* is equall to the diameter. Now here is to be noted that the square is greater than the Circle by the vacuity in the middle.

With one and the same compasses, and at one and the same extent, or opening, how to describe many Circles concentrical, that is, greater or lesser one than another

It is not without cause that many admire how this proposition is to be resolved; yea, in the judgement of some it is thought impossible, who consider not the industry of an ingenious Geometrician, who makes it possible: and that most facile, sundry ways. For in the first place, if you make a Circle upon a fine plane, and upon the Center of that Circle a small peg of wood be placed, to be raised up and put down at pleasure by help of a small hole made in the Center, then with the same opening of the Compasses you may describe Circles Concentrical: that is, one greater or lesser than another. For the higher the Center is lifted up, the lesser the Circle will be.

Secondly, the compass being at that extent upon a Gibbous body, a Circle may be described, which will be less than the former, upon a plane, and more artificially upon a Globe, or round bowl. And this again is most obvious upon a round Pyramid, placing the Compasses upon the top of it, which will be far less than any of the former; and this is demonstrated by the 20⌞th⌟ Pro⌞position⌟ of the first ⌞book⌟ of Euclid's ⌞*Elements*⌟.

Of spectacles of pleasure

⌞. . .⌟ Now I would not pass this Problem without saying something of Galileo's admirable Glass: for the common simple perspective Glasses give to aged men but the eyes or sight of young men, but this of Galileo gives a man an Eagle's eye, or an eye that pierceth the heavens. First it discovereth the spotty and shadowed opacous bodies that are found about the Sun, which darkeneth and diminisheth the splendour of that beautiful and shining Luminary; secondly, it shows the new planets that accompany Saturn and Jupiter; thirdly, in Venus is seen the new, full, and quartal increase, as in the Moon by her separation from the Sun; fourthly, the artificial structure of this instrument helpeth us to see an innumerable number of stars, which otherwise are obscured by reason of the natural weakness of our sight. Yea, the stars in ⌞the Milky Way⌟ are seen most apparently; where there seems no stars to be, this instrument makes apparently to be seen, and further delivers them to the eye in their true and lively colour, as they are in the heavens: in which the splendour of some is as the Sun in his most glorious beauty. This Glass hath also a most excellent use in observing the body of the Moon in time of Eclipses, for it augments it manifold, and most manifestly shows the true form of the cloudy substance in the Sun, and by it is seen when the shadow of the Earth begins to eclipse the Moon, and when totally she is overshadowed.

Besides the celestial uses which are made of this Glass, it hath another notable property: it far exceedeth the ordinary perspective Glasses which are used to see things remote upon the Earth, for as this Glass reacheth up to the heavens and excelleth them there in his performance, so on the Earth it claimeth preeminency. For the objects which are farthest remote, and most obscure, are seen plainer than those which are near at hand, scorning, as it were, all small and trivial services, as leaving them to an inferior help. Great use may be made of this Glass in discovering of Ships, Armies, etc.

Now the apparel or parts of this instrument or Glass is very mean or simple, which makes it the more admirable (seeing it performs such great service), having but a convex Glass, thickest in the middle, to unite and amass the rays, and make the object the greater ⌞. . . ,⌟ augmenting the visual Angle. As also a pipe or trunk to amass the Species, and hinder the greatness of the light which is about it (to see well, the object must be well enlightened, and the eye in obscurity). Then there is adjoined unto it a Glass of a short sight to distinguish the rays, which the other would make more confused if alone. As for the proportion of those Glasses to the Trunk, though there be certain rules to make them, yet it is often by hazard that there is made an excellent one, there being so many difficulties in the action, therefore many ought to be tried, seeing that exact proportion, in Geometrical calculation, cannot serve for diversity of sights in the observation.

Of the Dial upon the fingers and the hand

Is it not a commodity very agreeable, when one is in the field or in some village without any other Dial, to see only by the hand what of the clock it is? which gives it very near, and may be practised by the left hand in this manner.

Take a straw, or like thing, of the length of the Index or the second finger. Hold this straw very tight between the thumb and the right finger, then stretch forth the hand and turn your back and the palm of your hand towards the Sun, so that the shadow of the muscle which is under the thumb touch⌞es⌟ the line of life, which is between the middle of the two other great lines, which is seen in the palm of the hand. This done, the end of the shadow will show what of the clock it is: for at the end of the great finger it is 7 in the morning or 3 in the evening; at the end of the Ring finger it is 8 in the morning or 4 in the evening; at the end of the little finger or first joint, it is 9 in the morning or 3 in the afternoon; 10 and 2 at the second joint; 11 and 1 at the third joint; and midday in the line following, which comes from the end of the Index ⌞finger⌟.

Of sundry Questions of Arithmetic, and first of the number of sands

It may be said, incontinent⌞ly⌟, that to undertake this were impossible, either to number the sands of Libya, or the sands of the Sea. And it was this that the Poets sung, and that which the vulgar believes—nay, that which long ago certaine Philosophers to Gelon King of Sicily reported: that the grains of sand were innumerable. But I answer, with Archimedes,° that not only one may number those which are at the border and about the Sea, but those which are able to fill the whole world, if there were nothing else but sand, and the grains of sands admitted to be so small that 10 may make but one grain of Poppy. For at the end of the account there need⌞s⌟ not⌞hing⌟ to express them but this number: 30,840,979,456, and 35 Ciphers at the end of it. Clavius and Archimedes make it somewhat more, because they make a greater firmament than Tycho Brahe doth, and if they augment the Universe, it is easy for us to augment the number, and declare assuredly how many grains of sand there is requisite to fill another world, in comparison that our visible world were but as one grain of sand, an atom or a point. For there is nothing to do but to multiply the number by itself, which will amount to ninety places, whereof twenty are these: 95,143,798,134,910,955,936, and 70 Ciphers at the end of it. Which amounts to a most prodigious number ⌞. . .⌟.

To measure an inaccessible distance, as the breadth of a River, with the help of one's hat only

The way of this is easy: for having one's hat upon his head, come near to the bank of the River, and holding your head upright (which may be by putting a small stick to some one of your buttons to prop up the chin), pluck down the brim or edge of your hat until you may but see the other side of the water. Then turn about ⌞. . .⌟ in the same posture ⌞as⌟ before, towards some plain, and mark where the sight, by the brim of the hat, glanceth on the ground: for the distance from that place to your standing, is the breadth of the River required.

Note

Archimedes (c. 287–c. 212 BC) had attempted to calculate the number of grains of sand the universe could contain in his *Sand Reckoner*, reaching a result of 8×10^{63}.

"How Prodigiously Numbers Do Increase"

William Leybourne, 1667

William Leybourne wrote on a range of subjects including astronomy, geography, and surveying, reflecting a career which took in a period as a bookseller and printer, as well as his later roles of mathematician, teacher, and surveyor.

His book of "recreations" begins with parlor tricks and ends with a set of strategies to help with arithmetic; some of the impressive wealth of material was in fact taken from Henry van Etten (see the previous extract). One section, shown here, contains several variations on the "geometric progression" story that, today, is sometimes told of a grain of rice doubled for each of the squares on a chessboard. Leybourne evidently felt that his readers might have difficulty reading the very large numbers involved and took pains to write them out both in figures and in words.

William Leybourne (1626–1716), *Arithmetical Recreations: Or, Enchiridion of Arithmetical Questions: Both Delightful and Profitable. Whereunto are added Diverse Compendious Rules* *in Arithmetic, by which some seeming difficulties are removed, and the performance of them rendred familiar and easie to such as desire to be Proficients in the Science of Numbers. All performed without Algebra. By Will. Leybourne.* (London, 1667), pp. 122–140.

Concerning two Neighbours Changing of their Land

Two Neighbours had either of them a piece of Land: the one field was four-square, every side containing 120 perches,° so that it was round about 480 perches; the other was square also, but the sides longer than the other's field, and the ends shorter, for the sides of this field were 140 perches long apiece, and the ends thereof were 100 perches apiece, so that this field was 480 perches about as well as the other. Now, which of these two had the best bargain?

It is wonderful to see how Numbers will discover that to be erroneous and absurd, which to common sense and man's apprehension appears reasonable, as in this bargain. First, for the field which is 120 perches on either side: multiply 120 by 120, and the product is 14,400, and so many square perches doth that piece of Land contain, that is, 80 acres. Then, for the other field: multiply 140 (the length of one of the longer sides of the field) by 100 perches (one of the shorter sides) and the product will be 14,000 ⌞square⌟ perches, the content of the other field, which is less then the former by 400 ⌞square⌟ perches, that is, 2 acres and an half. And so much would he have lost that had the field of 120 perches on every side, though the other field were as much about.

And this error would still grow greater, the narrower the second field had been. As, suppose the ends or shorter sides thereof had been but 40 perches apiece, and the longer sides 200 apiece; this field would still have been 480 perches about, but let us see how much it contains. Multiply 200, the longer side, by 40, the shorter side, and it will produce 8000, and so many square perches will it contain, which is but 50 acres. So that if he had changed for this field, which is as much about as his, he would have then lost 30 acres by the bargain.

About the borrowing of Corn

A Country Farmer had in his house a vessel of Wood full of Wheat, which was 4 foot high, 4 foot broad both at top and bottom, and in all parts 4 foot, as the sides of a Die. One of his Neighbours desires him to lend him half his Wheat till Harvest, which he doth. Harvest coming, and his Neighbour is to repay, he makes a Vessel 2 foot every way, and fills him that twice, in lieu of what he borrowed. Was there gain or loss in this particular?

Examine first what either of these Vessels will hold, and by that you will discover the fallacy. First, the vessel 4 foot high contains 48 inches of a side, wherefore multiply 48 by 48, and the product will be 2304, which multiply again by 48, and the product will be 110,592; and so many cubical or square inches of Corn do his vessel hold, the half whereof, which is 55,296, he lent his Neighbour. Now, secondly, let us examine how much the second vessel will hold, it being 2 foot on every side, that is 24 inches. Multiply 24 by 24; the product is 574. Which multiply again by 24, and the product will be 13,824; and so many cubical or square inches did the lesser vessel contain. Which being filled twice, it made 27,648 cubical inches of corn or wheat, which was all he paid his Neighbour in lieu of the 55,296 inches which he borrowed, which is but the just half. And so allowing 2256 cubical inches of Wheat to make a Bushel (for so many there is in a Bushel) he paid his Neighbour less by 12 bushels, and about a peck, than he borrowed of him. And this, and the reason of it, is evident, as I will demonstrate to you by a familiar precedent. If you cause a Die to be made of one inch every side, and 8 other Dice to be made of half an inch every side, these 8 being laid close, one to another, in a square form, these 8 will be but of the same bigness with the other one Die, whose side is but an inch.

A Bargain between a Farmer and a Goldsmith

A rich Farmer being in a Fair, espies at a Goldsmith's Shop a Necklace of Pearl, upon which were 72 Pearls. The Farmer cheapening of it, the Goldsmith asked 30 shillings a Pearl, at which rate the Necklace would come to £103. The Farmer, looking upon it as dear, goes his way, offering nothing. Whereupon the Goldsmith calls him, and tells him, if he thought much to part with Money, he would deal with him for Corn. To which the Farmer hearkens, asks him how much Corn he would have for it at two shillings the Bushel. The Goldsmith told him he would be very reasonable, and would take for the first Pearl one Barley corn only, for the second two corns, for the third four corns, and so doubling the corns till the 72 Pearls were out. To this the Farmer agrees, and immediately strikes the Bargain. But see the event.

He that hath any skill in Numbers, will easily discern the vanity that there is in this kind of bargaining, so that no man can be bound to them; for Numbers increasing in a Geometrical Progression do so prodigiously increase, that (to those that are ignorant of the reason) it will seem impossible they should do so. But that it is so will appear evident by this bargain, if you enquire: first, the quantity; secondly, the worth of so much Barley in Money; and thirdly, the weight of it, and how it should be removed, or where stowed. Wherefore,

1. If we allow 10,000—ten thousand—corns to a Pint (which is more than enough) then 5,120,000 corns will make a Quarter, but yet (for the ease of them that will make tryal) we will allow 10,000,000 corns to make a Quarter. By which number, if you divide the whole number of corns that the 72 Pearls would have amounted unto ⌞. . . ,⌟ the Quotient of that Division would be

472,236,648,286,964.

And so many whole Quarters of Barley would the Necklace have amounted unto, and some odd Bushels, which we here omit as superfluous.

2. Now for the worth of this Barley, suppose it were sold at 13 pence the Bushel (which is a reasonable rate), that is, 10 shillings the Quarter. Wherefore, divide the foregoing number of Quarters by 2—that is, take half of it—and it will be 236,118,324,143,482 pounds sterling; which sum rendered in words, is, *Two hundred thirty-six millions of millions, one hundred and eighteen thousand, three hundred twenty-four millions, one hundred forty-three thousand, four hundred eighty-two* pounds. A vast sum of money for a Farmer. ⌞. . .⌟ So great vanity may be agreed and concluded upon by people ignorant of this Science, and for want of serious premeditation. But,

3. Let us consider the weight of so much Barley. If we allow 8 Bushels (or one Quarter) to weigh Two hundredweight (but doubtless it weighs more), then the whole number of Quarters, multiplied by 2, gives the weight of all the Barley to be 944,473,296,573,928 hundredweight. And if you divide this number by 20, the Quotient will be 47,223,664,828,696 Tons; that is, *Forty-seven millions of millions, two hundred twenty-three thousand six hundred sixty-four millions, eight hundred twenty-eight thousand, six hundred ninety-six* Tons. Which will require 47,223,664,828—that is, *Forty-seven thousand two hundred twenty-three millions, six hundred*

sixty-four thousand, eight hundred twenty-eight—Ships of a thousand Ton apiece to carry it. And to conclude, If there were four Millions of Nations in the World, and every one of those Nations had Ten thousand Sail of such Ships of a Thousand Ton apiece, yet all those Ships would not contain it. Thus, by this, you may see how prodigiously numbers do increase, being multiplied according to Geometrical Progression.

Concerning an Agreement that a Country-Fellow made with a Farmer

A Country-Fellow comes to a Farmer, and offers to serve him for 8 years, all which time he would require no other Wages than One grain of Corn, and one quarter of an inch of Land to sow it in the first year, and Land enough to sow that one Corn, and the increase of it, for his whole 8 years: to which the Farmer assents.

Their Bargain being thus made, let us consider what his eight years' service will be worth. For the first year he hath only one quarter of an inch of Ground, and one Corn, which Corn we will suppose had in the Ear at the year's end 40 Corns (for that is few enough). Then the second year he must have 40 square quarters of inches of ground to sow those 40 Corns in: that is, 10 square inches of ground. And the third year, supposing those 40 Corns to produce 40 Ears, and in each Ear 40 Corns, as before, they will be in the third year increased to 1600 corns, so that he must have 1600 square quarters of inches to sow that increase in, which is 40 square inches. And thus continuing till the 8 years be expired, the increase would be 6,553,600,000,000 corns—that is, *Six millions of millions, five hundred fifty-three thousand and six hundred millions*—of corns, and so many square quarters of inches of Land must he have to sow this increase in. Now know that 3,600,000,000—that is, *Three thousand and six hundred millions*—of square inches do make a mile, upon the ⌞surface⌟ or plane, and that ⌞a square⌟ Mile will be capable to receive 14,400,000,000—that is, *Fourteen thousand and four hundred millions*—of Corns. Wherefore divide 6,553,600,000,000 (the whole number of the Corn's increase) by 14,400,000,000 (the number of Corns that one Mile square of Land is capable to receive), and in the Quotient you shall have 455; and so many miles square of Land must there be, to contain the sowing of the increase of one Corn in 8 years. Which will be about 420,000—that is *Four hundred and twenty thousand*—Acres of Land, which being rated at half a Crown an Acre by the year, it will amount unto 50,000—that is, *Fifty thousand*—pound, which is 6,250—that is, *Six thousand two hundred and fifty*—pound a year: a very considerable Salary for Eight years' service.

By these, and the like Contract⌞s⌟, we may see what absurdities are and may easily be committed, which you see the subtlety of Numbers easily discovers. I will give you, by way of Discourse, some precedents concerning the increase of Creatures of several kinds, which you may make trial of for your Recreation. As,

1. What think you, if one should tell the Great Turk (or any other Potentate in the World, who having a great Army in the Field in continual pay) that all the Revenue appertaining to his Crown, will not for one year's time maintain all the Pigs that one Sow, with all the Pigs of her race, and the increase issuing of them, shall produce in 12 years, notwithstanding he can maintain so great an Army in the field? Doubtless, he would take it unkindly. But consider: imagine the Sow brings forth but 6 Pigs at a Litter, of which we will allow 2 to be Barrow (and this supposition is as little as may be), and then imagine that every of those 4 bring as many every year, and the increase of them the like, during the term of 12 years. They and their race at the end of the time will be increased to *Three and thirty millions* of Pigs. Now if we allow 5 shillings for to maintain one of these Pigs for a year, which is not full half a farthing a day, yet there must be *Three and thirty millions* of Crowns to maintain them one year: which will make a great hole in a large Revenue.

2. Would any man (that hath not skill in Numbers) imagine that 100 Sheep, and the increase thereof, being preserved for the space of 16 years, should be worth above *One million, six hundred and twenty thousand* pounds sterling? Yet if every Sheep do produce but one every year, at the expiration of 16 years, 100 sheep will increase unto 6,553,600, which is *Six millions, five hundred fifty-three thousand, six hundred* Sheep. Now supposing these to be worth but 5 shillings apiece, they would at that rate amount unto 1,638,400—that is, *One million, six hundred thirty-eight thousand four hundred*—pounds sterling.

Note

perches: measure of length, eventually standardized as $5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet, about 5.03 meters.

Profitable and Delightful Problems

Jacques Ozanam, 1708

Jacques Ozanam, "Professor of the mathematicks at Paris," published many mathematical works, including mathematical tables and a version of the *Arithmetic* of Diophantus, from the 3rd century AD. His *Recreations* first appeared in French in 1694; the mathematical parts of the book presented a range of mathematical puzzles and tricks, geometrical constructions, and other diversions broadly similar to those of his predecessors like van Etten and Leybourn. The "physical" recreations featured a memorable section on pyrotechnics which began with "Problem I. To make Gun-Powder" and ended with "Problem XXXVIII. To make an Ointment excellent for Curing all sorts of Burnings."

Jacques Ozanam (1640–1717), *Recreations mathematical and Physical; Laying down, and Solving Many Profitable and Delightful Problems of Arithmetic, Geometry, Opticks, Gnomonicks, Cosmography, Mechanics, Physicks, and Pyrotechny* (London, 1708), pp. 82–83, 192, 350–351.

After filling one Vessel with Eight Pints of any Liquor, to put just one half of that Quantity into another Vessel that holds Five Pints, by means of a third Vessel that will hold three Pints

This Question is commonly put after the following manner: A certain Person, having a Bottle filled with 8 Pints of excellent Wine, has a mind to make a Present of the Half of it, or 4 Pints, to one of his Friends. But he has nothing to measure it out with but two other Bottles, one of which contains 5, and the other 3 Pints. ⌞I ask⌟, how he shall do to accomplish it?

To answer this Question, let's call the Bottle of 8 Pints A, the 5-Pint Bottle B, and the 3-Pint Bottle C. We suppose there are 8 Pints of Wine in the Bottle A, and the other two, B and C, are empty, as you see in ⌞row⌟ D (see the table below). Having filled the Bottle B with Wine out of the Bottle A, in which there will then remain but 3 Pints, as you see at E, fill the Bottle C with Wine out of the Bottle B, in which, by consequence, there will then remain but 2 Pints, as you see at F. This done, pour the Wine of the Bottle

	Bottle		
	A (8)	*B (5)*	*C (3)*
D	8	0	0
E	3	5	0
F	3	2	3
G	6	2	0
H	6	0	2
I	1	5	2
K	1	4	3

C into the Bottle *A*, where there will then be 6 Pints, as you see in *G*, and pour the 2 Pints of the Bottle *B* into the Bottle *C*, which will then have 2 Pints, as you see at *H*; then fill the Bottle *B* with Wine out of the Bottle *A*, by which means there will remain but 1 Pint in it, as you see at I, and conclude the Operation by filling the Bottle *C* with Wine out of the Bottle *B*, in which there will then remain just 4 Pints, as you see at *K*, and so the Question is solved.

To measure an accessible Line upon the Ground by means of the Flash and the Report of a Cannon

With a Musket Ball, make a Pendulum 11 Inches and 4 Lines long, calculating the Length from the Center of the Motion to the Center of the Ball; and the very moment that you perceive the Flash of the Cannon (which must be at the very place, the Distance of which, from the place where you are, is enquired after) put the Pendulum in motion, so that the Arches of the Vibrations do not exceed 30 Degrees. Multiply by 200 the Number of the Vibrations from the moment you perceived the Flash to the moment in which you hear the Report, and reckon as many Paris *Toises*° for the Distance of the place where the Gun was fired, from the place where you stood.

Much after the same manner you may measure the Height of a Cloud, when 'tis near the Zenith, and at a time of Thunder and Lightning. But this way of measuring Distances is very uncertain, and I only mentioned it here as a Recreation.

To make a deceitful Balance, that shall appear just and even both when empty and when loaded with unequal Weights

Make a Balance, the Scales of which are of unequal Weight, and of which the two Arms are of unequal length, and in reciprocal proportion to these unequal Weights. That is, the first weight is to the second, as the second length is to the first: for thus the two scales will continue *in Æquilibrio* round the fixed point. And the same will be the Case if the two Arms are of equal length and of unequal thickness, so that the thickness of the first is to that of the second as the weight of the second scale is to that of the first. This supposed, if you put into the two scales unequal weights which have the same Ratio with the Gravities of the two scales, the heavier weight being in the heavier scale, and the lighter in the lighter scale, these two Weights and Scales will rest *in Æquilibrio*.

We'll suppose that the first Arm is three ⌞Inches⌟, and the second Arm two ⌞Inches⌟, and reciprocally the second scale weighs three Ounces, and the first scale two; in which case the balance will be even when they are empty. Then we put a weight of two pound into the first scale, and one of three into the second, or else one of four into the first, and one of six into the second, etc. And the balance continues still even, because the weights with the gravity of the Scales are reciprocally proportional to the length of the Arms of the Beam. Such a pair of Scales is discovered by shifting the weights from one side to another, for then the Balance will cast to one side.

Note

Toise: 6 feet.

Lotteries and Mountebanks

L. Despiau, 1801

Although it contains some of the usual card tricks, number games, and basic arithmetic (see van Etten and Leybourne, above), a fair part of Despiau's book is taken up with the attempt to make mathematics amusing by applying it to dice games. Despiau (I have failed to discover either his dates or his first name) had been a professor of mathematics and philosophy in Paris, and his book was commended by Charles Hutton, who taught at the Royal Military Academy at Woolwich. Yet it seems to have occurred to neither of them that this thematic angle, and the attached "large table of the chances or odds at play" might make the book less than perfectly suited for use in schools.

L. Despiau, *Select Amusements in Philosophy and Mathematics; Proper for agreeably exercising the Minds of Youth. Translated from the French ⌞. . .⌟ with several corrections and additions, particularly a large table of the chances or odds at play. The whole recommended as an useful Book for Schools, by Dr. Hutton, Professor of Mathematics, at Woolwich.* (London, 1801), pp. 75–77, 94–95.

Application of the doctrine of combinations to games of chance and to probabilities

Though nothing, on the first view, seems more foreign to the province of the mathematics, than games of chance, the powers of analysis have, as we may say, enchained this Proteus, and subjected it to calculation: it has found means to measure the different degrees of the probability of certain events, and this has given rise to a new branch of mathematics, the principles of which we will here explain.

When an event can take place in several different ways, the probability of its happening in a certain determinate manner is greater when, in the whole of the ways in which it can happen, the greater number of them determine it to happen in that manner. In a lottery, for example, everyone knows that the probability of obtaining a prize is greater, according as the number of the prizes is greater, and as the whole number of the tickets is less. The probability of an event, therefore, is ⌞proportional to⌟ the number of cases in which it can happen, taken directly, and ⌞to⌟ the total number of those in which it can be varied, taken inversely; consequently it may be expressed by a fraction, having for its numerator the number of the favourable chances, and for its denominator the whole of the chances.

Thus, in a lottery containing 1000 tickets, 25 of which only are prizes, the probability of obtaining a prize will be represented by $\frac{25}{1000}$ or $\frac{1}{40}$; if there were 50 prizes, the probability would be double, for in that case it would be equal to $\frac{1}{20}$; but if the number of tickets, instead of 1000, were 2000, the probability would be only one half of the former, or $\frac{1}{80}$; and if the whole number of the tickets were infinitely great, the prizes remaining the same, it would be infinitely small, or 0; while, on the other hand, it would become certainty, and be expressed by unity, if the number of the prizes were equal to that of the tickets.

Another principle of this theory, the truth of which may be readily perceived, and which it is necessary here to explain, is as follows:

A person plays an equal game when the money staked, or risked, is in the direct ratio of the probability of winning; for to play an equal game is nothing else than to deposit a sum so proportioned to the probability of winning, that, after a great many throws, the player may find himself nearly at par; but for this purpose, the stakes must be proportioned to the probability which each of the players has in his favour. Let us suppose, for example, that A bets against B on a throw of the dice, and that there are two chances in favour of the former, and one for the latter; the game will be equal if, after a great number of throws, they separate nearly without any loss. But as there are two chances in favour of A, and only one for B, after 300 throws A will have won nearly 200, and B 100. A, therefore, ought to deposit 2 and B only 1, for by these means A, in winning 200 throws, will get 200, and B, in winning 100, will get 200 also. In such cases, therefore, it is said that there is two to one in favour of A.

⌞The mountebank⌟

A mountebank, at a country fair, amused the populace with the following game: he had 6 dice, each of which was marked only on one face, the first with 1, the second with 2, and so on to the sixth, which was marked 6; the person who played gave him a certain sum of money, and he engaged to return it a hundredfold if, in throwing these six dice, the six marked faces should come up only once in 20 throws.

Though the proposal of the mountebank does not, on the first view, appear very disadvantageous to those who entrusted him with their money, it is certain that there were a great many chances against them.

It may indeed be seen that, of the 46,656 combinations of the faces of 6 dice, there is only one which gives the 6 marked faces uppermost; the probability therefore of throwing them, at one throw, is expressed by $\frac{1}{46656}$: and as the adventurer was allowed 20 throws, the probability of his succeeding was only $\frac{20}{46656}$, which is nearly equal to $\frac{1}{2332}$. To play an equal game, therefore, the mountebank would have engaged to return 2332 times the money deposited.

Dodging the Mastodon and the Plesiosaurus

Henry Ernest Dudeney, 1917

Henry Ernest Dudeney's well-loved classic, *Amusements in Mathematics*, takes quite a spare approach, giving no solutions and no mathematical analysis of its puzzles. The section on route puzzles presented here, for instance, could—and would, in the hands of many later popularizers—have been a springboard for a general discussion of unicursal problems and their mathematics, but Dudeney gives us the problems alone, often charmingly illustrated and whimsically described. If they "enable one to generalise," as he claims, it is up to the reader to work out how. In this respect, perhaps, the book recalls some of the very earliest presentations of mathematical recreations we have seen, like Baker's in the sixteenth century.

Henry Ernest Dudeney (1857–1930), *Amusements in Mathematics* (London, 1917), pp. 68–71.

Unicursal and Route Problems

It is reasonable to suppose that from the earliest ages one man has asked another such questions as these: "Which is the nearest way home?" "Which is the easiest or pleasantest way?" "How can we find a way that will enable us to dodge the mastodon and the plesiosaurus?" "How can we get there without ever crossing the track of the enemy?" All these are elementary route problems, and they can be turned into good puzzles by the introduction of some conditions that complicate matters. A variety of such complications will be found in the following examples. I have also included some enumerations of more or less difficulty. These afford excellent practice for the reasoning faculties, and enable one to generalize in the case of symmetrical forms in a manner that is most instructive.

A Juvenile Puzzle

For years I have been perpetually consulted by my juvenile friends about this little puzzle. Most children seem to know it, and yet, curiously enough,

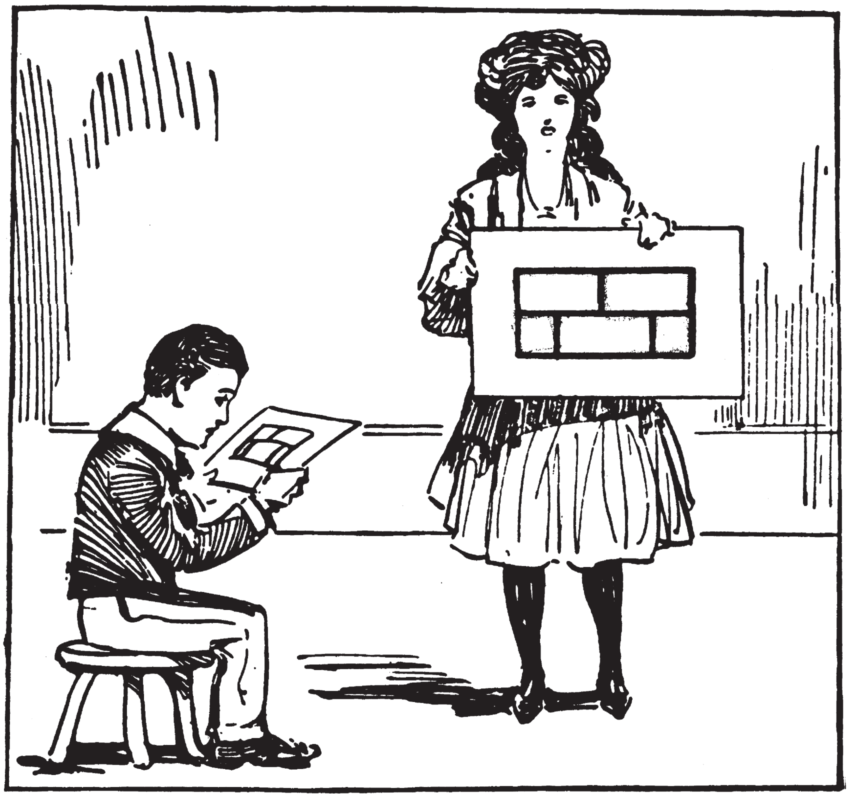

Figure 1.1. A juvenile puzzle, with two juveniles. (Dudeney, p. 69.)

they are invariably unacquainted with the answer. The question they always ask is, "Do, please, tell me whether it is really possible." I believe Houdini the conjurer used to be very fond of giving it to his child friends, but I cannot say whether he invented the little puzzle or not. No doubt a large number of my readers will be glad to have the mystery of the solution cleared up, so I make no apology for introducing this old "teaser."

The puzzle is to draw with three strokes of the pencil the diagram that the little girl is exhibiting in the illustration (Figure 1.1). Of course, you must not remove your pencil from the paper during a stroke or go over the same line a second time. You will find that you can get in a good deal of the figure with one continuous stroke, but it will always appear as if four strokes are necessary.

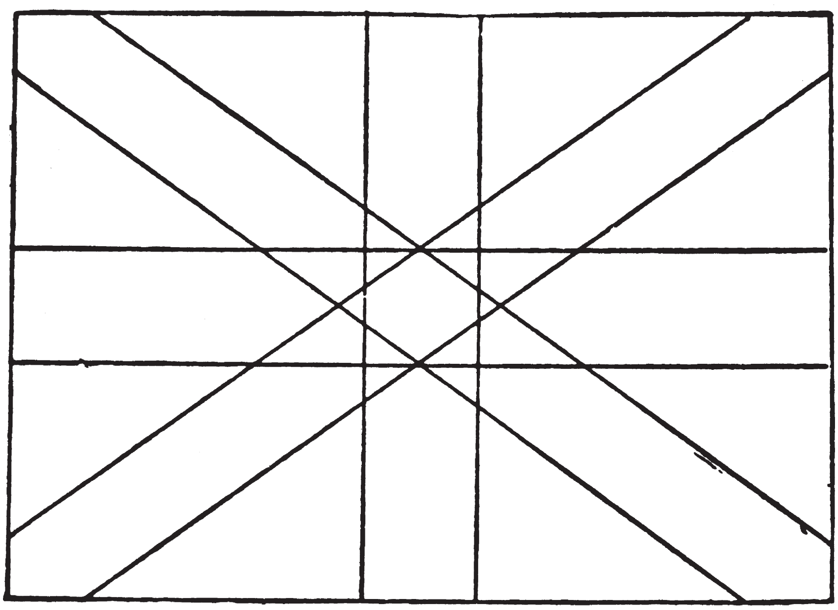

Figure 1.2. A rough sketch somewhat resembling the British flag. (Dudeney, p. 69.)

Another form of the puzzle is to draw the diagram on a slate and then rub it out in three rubs.

The Union Jack

The illustration (Figure 1.2) is a rough sketch somewhat resembling the British flag, the Union Jack. It is not possible to draw the whole of it without lifting the pencil from the paper or going over the same line twice. The puzzle is to find out just how much of the drawing it is possible to make without lifting your pencil or going twice over the same line. Take your pencil and see what is the best you can do.

The Dissected Circle

How many continuous strokes, without lifting your pencil from the paper, do you require to draw the design shown in our illustration (Figure 1.3)? Directly you change the direction of your pencil it begins a new stroke. You

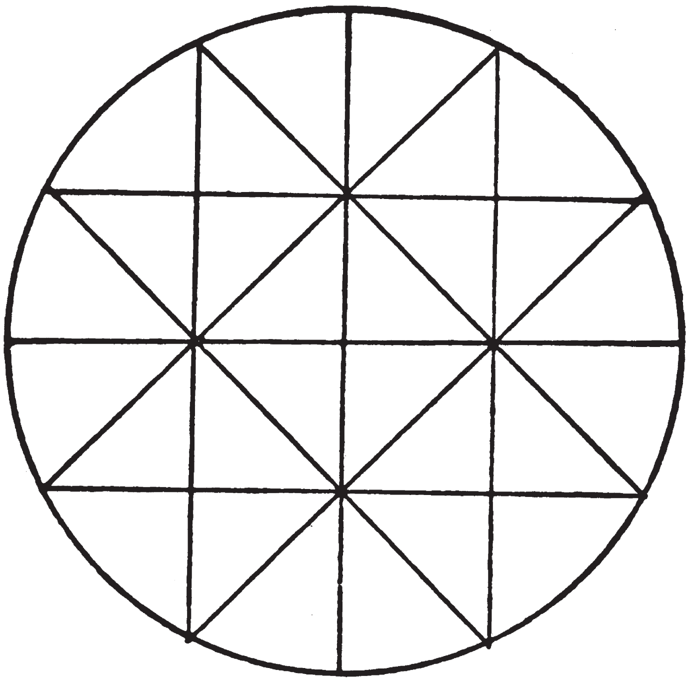

Figure 1.3. A dissected circle. (Dudeney, p. 69.)

may go over the same line more than once if you like. It requires just a little care, or you may find yourself beaten by one stroke.

The Tube Inspector's Puzzle

The man in our illustration (Figure 1.4) is in a little dilemma. He has just been appointed inspector of a certain system of tube railways, and it is his duty to inspect regularly, within a stated period, all the company's seventeen lines connecting twelve stations, as shown on the big poster plan that he is contemplating. Now he wants to arrange his route so that it shall take him over all the lines with as little travelling as possible. He may begin where he likes and end where he likes. What is his shortest route?

Could anything be simpler? But the reader will soon find that, however he decides to proceed, the inspector must go over some of the lines more than once. In other words, if we say that the stations are a mile apart, he will have to travel more than seventeen miles to inspect every line.

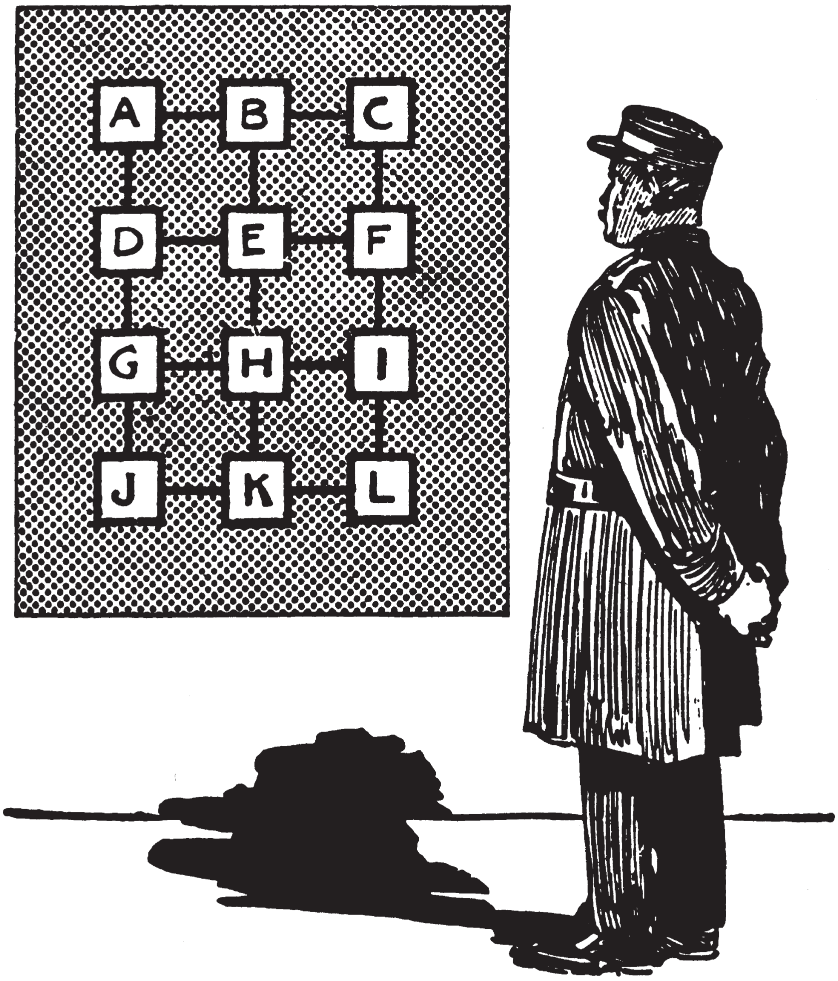

Figure 1.4. The tube inspector in a little dilemma. (Dudeney, p. 69.)

There is the little difficulty. How far is he compelled to travel, and which route do you recommend?

Visiting the Towns

A traveller, starting from town No. 1 (see Figure 1.5), wishes to visit every one of the towns once, and once only, going only by roads indicated by straight lines. How many different routes are there from which he can

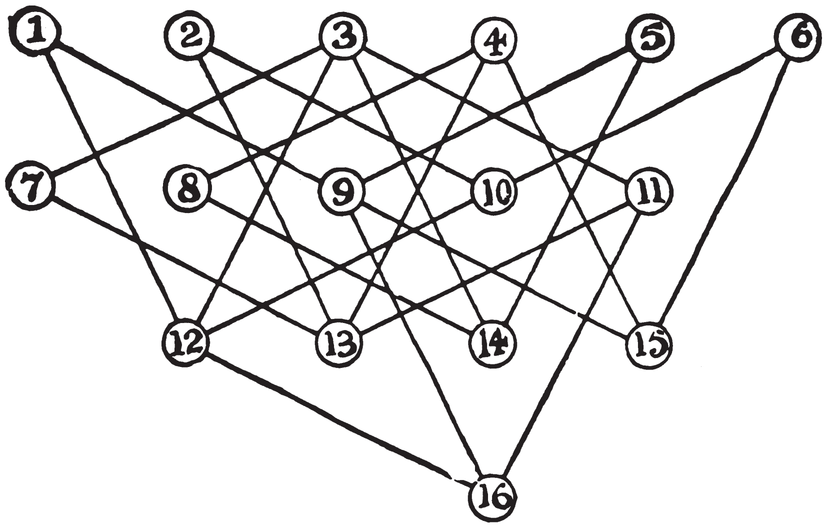

Figure 1.5. Visiting the towns, an absurdly easy puzzle. (Dudeney, p. 69.)

select? Of course, he must end his journey at No. 1, from which he started, and must take no notice of cross roads, but go straight from town to town. This is an absurdly easy puzzle, if you go the right way to work.

The Fifteen Turnings

Here is another queer travelling puzzle, the solution of which calls for ingenuity. In this case the traveller starts from the black town (see Figure 1.6) and wishes to go as far as possible while making only fifteen turnings and never going along the same road twice. The towns are supposed to be a mile apart. Supposing, for example, that he went straight to A, then straight to B, then to C, D, E, and F, you will then find that he has travelled thirty-seven miles in five turnings. Now, how far can he go in fifteen turnings?

The Fly on the Octahedron

"Look here," said the professor to his colleague, "I have been watching that fly on the octahedron, and it confines its walks entirely to the edges. What can be its reason for avoiding the sides?"

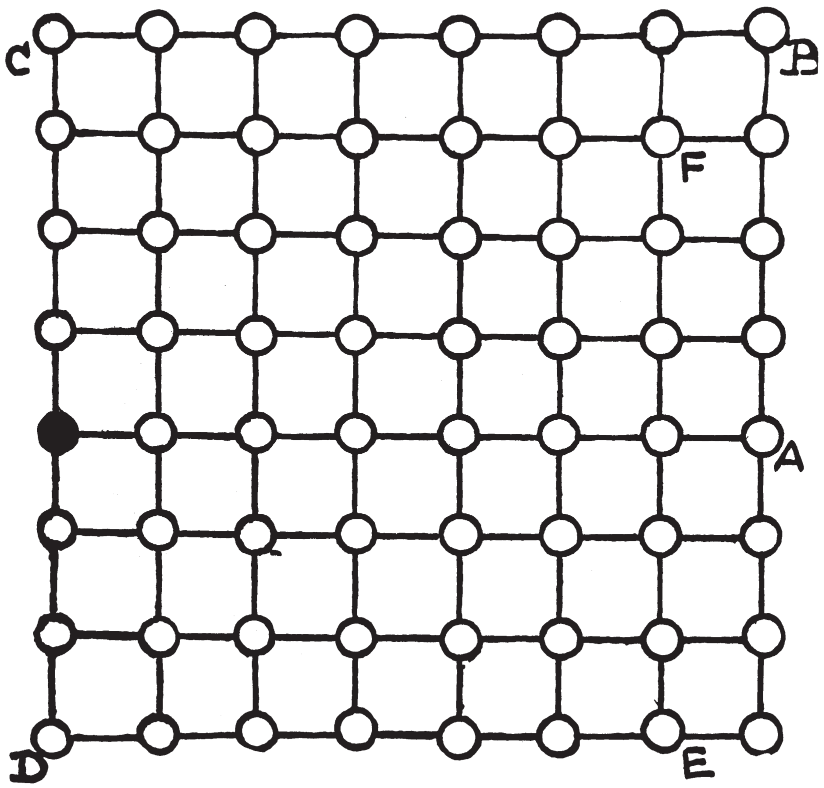

Figure 1.6. How far can he go in fifteen turnings? (Dudeney, p. 70.)

"Perhaps it is trying to solve some route problem," suggested the other. "Supposing it to start from the top point, how many different routes are there by which it may walk over all the edges, without ever going twice along the same edge in any route?"

The problem was a harder one than they expected, and after working at it during leisure moments for several days their results did not agree—in fact, they were both wrong. If the reader is surprised at their failure, let him attempt the little puzzle himself. I will just explain that the octahedron is one of the five regular, or Platonic, bodies, and is contained under eight equal and equilateral triangles. ⌞. . .⌟ In any route over all the edges it will be found that the fly must end at the point of departure at the top.

"Plenty of Interesting Things to Be Discovered"

NRICH, 1998–2004

The NRICH site at nrich.maths.org, founded in 1996, is one of the web's favorite resources for mathematical activities of every kind, and its collection of games and puzzles grows constantly. Alan Parr's remark about his game, Femto, could well apply to the site as a whole: there are always plenty of interesting things to be discovered.

These problems appear on the NRICH website http://nrich.maths.org and are used with permission.

Sprouts: nrich.maths.org/1208
Noughts and Crosses: nrich.maths.org/1199
Femto: nrich.maths.org/1179 (by Alan Parr)
Nim: nrich.maths.org/402

Sprouts

This is a game for two players. All you need is paper and a pencil. The game starts by drawing three dots. (See Figure 1.7.)

The first player has a turn by joining two of the dots and marking a new dot in the middle of the line. Or the line may start and end on the same dot. (See Figures 1.8 and 1.9.)

When drawing a line, it cannot cross another line. (This is important to remember!) A dot cannot have more than three lines branching to or from it. For example, in Figure 1.10, dots *A* and *B* cannot be used any more because they already have three lines.

The idea is to make it impossible for the other player to draw a line. So the last person to draw a line is the winner. What are the winning tactics? Does it matter who goes first? Why must the game end after a limited number of moves? How many? What happens when you start the game with four or five dots?

Endless Noughts and Crosses

This is a game for two players. You will need a sheet of grid paper (or rule lines down a sheet of writing paper).

Figure 1.7. The beginning of a game of Sprouts.

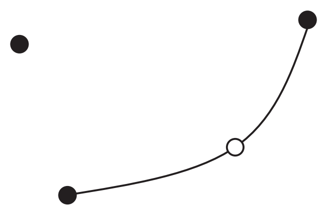

Figure 1.8. Sprouts: the first move.

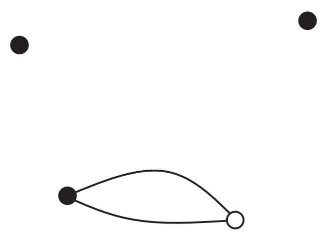

Figure 1.9. Sprouts: a different move.

The game is played like the ordinary game of noughts and crosses, with each player taking turns to mark a square with a nought or a cross, but it does not end with first string of three noughts or crosses. Keep going until either the grid is full or both players have had enough! The winner is the player who has the most strings-of-three. You might find it helpful to use different colour pens or to keep score as you play.

Playing on such a large grid means that the game is very unlikely to end in a draw and there is plenty of time to think about strategies for winning.

Once you have mastered strings-of-three, try a game with strings-of-four, then strings-of-five, maybe even strings-of-six!

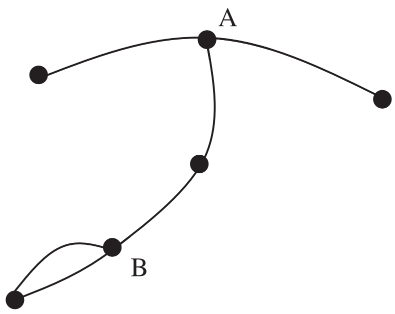

Figure 1.10. Sprouts: dots A and B are used up.

Femto

There is a book of mathematical games I'm rather fond of, even though it never made the best-seller lists. We called it "What's My Game?", and included in it was a two-player cardgame called Divide and Conquer, for which the entire equipment is a single suit of playing cards. This seemed to me to be just about the most minimal cardgame anyone could devise—until I discovered Pico, which is a German cardgame using just 11 cards.

In case anyone thinks Pico is still just a little too big I've put together ideas from both these games and come up with a new one, called Femto, which needs just eight cards.

Although the game is so small there are plenty of interesting things to be discovered in Femto ⌞. . .⌟.

Rules for FEMTO

1. Femto is a cardgame for two players; a Femto pack consists of eight cards numbered 2, 3, 4, 5, 6, 7, 8, 10.
2. The cards are shuffled and dealt, so each player gets four cards.
3. In each round of play each player puts out one card, face down. The two cards are then turned face up.
4. The round is won by the higher value card, unless the higher card is more than twice the value of the lower, in which case the lower card wins, e.g., 10 beats 8, 6 beats 5, 3 beats 10, 10 beats 5, . . .

5. Whoever played the winning card chooses one of the two cards and puts it, face up, on the table in front of him/her. The player of the losing card takes the remaining card and puts it back into his/her hand.
6. More rounds are played until one player has no cards left.
7. The winner is the player with the greater total value of cards in front of them at the end of the hand.

Well, for a tiny game there seem to be plenty of questions to explore. How long do games last? What is a typical winning score? How many different hands are possible? Which cards are the most powerful? Indeed, what is the most powerful hand (and how could it be beaten)?

Then there's another set of questions, and as a game developer these are the ones which particularly interest me. The first version of any game is often interesting enough, but needs a bit of tinkering to turn it into the finished version. Perhaps the most obvious question to ask is whether we've got the right set of cards. To take another set almost at random, would 5, 7, 9, 13, 16, 18, 20, 24 be any better? Or perhaps we should stick with the original set but add a 12: deal out four cards each so that one card is left unused. Or perhaps there's a role for a Joker card.

Other questions which I'd want to explore concern the game structure. I've suggested that the cards are dealt out, but what if the players take turns to choose cards to make up their starting hand? And instead of playing simultaneously, what if each round is played with one player leading a card face up, so that the second player can see its value before responding? I've also got a hankering to change rule (5), so that it's the losing player who chooses which card goes back into his/her hand. Or perhaps it should be not the winner or loser of the individual round who chooses, but whoever is currently winning (or perhaps losing), who makes the choice, . . .

Changing the overall winning criterion often has interesting consequences. The most usual way to do this is to make the winner the person with the lower score rather than the higher. Or in this case, the aim might be to score an odd total, or a total that is a multiple of 3, or a prime number, . . .

So it's not true after all that Femto is a game. It's more accurate to say that it may become any one of dozens of potential games. (If there's actually too much choice perhaps we'd be better off looking for a still smaller game, Atto?) I don't know how Femto will end up, but I am excited by the possibility of lots of people trying out different versions ⌞. . .⌟.

Nim

Challenge

Describe the strategy for winning the game of Nim. The rules are simple. Start with any number of counters in any number of piles. Two players take turns to remove any number of counters from a single pile. The loser is the player who takes the last counter.

Hint

Chance plays no part, and each game must end. The only advantage that either player can possibly have is to start or to play second. To work out how to win you need to start by analysing the "end game," and the losing position to be avoided, and then work back to earlier moves. What should you do if there are only two piles? If there are more piles, what happens if you reduce all the piles to one counter in each?

Solution

This is how you gain control of the game. Make a list of the binary numbers for the number of counters in each pile. Now to have control of the game you want to make the number of 1's in each column even. Whatever your opponent does, when you record the new binary number for the pile that has been changed, there will be an odd number of 1's in one or more of the columns. You will be able to make all the "column sums" even again.

For example, if the numbers of counters in the piles are 7, 6, 4 and 1 you get

1	1	1
1	1	0
1	0	1
1		

A good move now is to take 4 counters from the pile of 7 which makes the number of 1's in each column even.