Chapter One

Complex Media

In recent years technology has replaced Hercules as far as the labours are concerned: the progress in theoretical studies, followed by impressive experimental work and achievements, is reaching the everyday lives of ordinary people and is rapidly changing our habits and lives.

A big part of this technological revolution, which emerged in the late twentieth century and is propagating with increasing speed and expanding front, is the result of complex media. Complex media are artificial materials exhibiting properties, based on their structure rather than their composition, superior to those in naturally existing materials. Nevertheless, there certainly do exist materials in nature displaying "exotic" properties.

A characteristic of a fast-growing research area, such as the one concerned with the study of complex media, is its interdisciplinary nature; scientists from a wide provenance spectrum, including electrical engineering, electromagnetics, solid state physics, microwave and antenna engineering, optoelectronics, classical optics, materials science, semiconductor engineering, and nanoscience, are engaged in this field. Of course, mathematics has its usual share, as well!

A discrimination between left and right has proved to be a fertile concept in the many branches of science that feed into electromagnetics: *handedness* is a term that is used extensively in the complex media[1] literature. There are actually three notions of handedness of interest in electromagnetics[2]:

Left-handedness: The term *left-handed* as a description of a certain class of metamaterials springs from the handedness of the vector triplet (E, H, K) (E being the electric field, H the magnetic field, and K the wave vector[3], respectively) of a linearly[4] polarised wave propagating in such media. This type of left-handedness refers to materials whose electric permittivity and magnetic permeability are both negative. The theoretical prediction of their existence was made by V. Veselago [420] in 1964.

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[1] The very fashionable term *metamaterials* refers to a wide class of complex media. A thorough discussion on the use and meaning of this term can be found in [383].

[2] For more details see [384], on which the following discussion of the notions of handedness is based.

[3] Recall that for a three-dimensional travelling plane wave $\Psi(t, x) = A\, cos(K \cdot x - \varpi\, t + \phi)$, with position vector $x \in \mathbb{R}^3$, at time $t > 0$, of angular frequency ϖ, amplitude A, and "phase offset" ϕ, the vector K is the "wave vector" and its magnitude is the angular wave number $|K| = 2\pi/\lambda$, λ being the wavelength.

[4] See Section 5.2.

Handedness of a circularly[5] *polarised wave:* In the electrical engineering community, handedness is manifested in relation to polarisation, which refers to the direction and behaviour of the electric field vector, which in the case of circular (or elliptical) polarisation exhibits a form of helicity (or handedness). The wave propagates in a certain direction, and (for isotropic media) the electric field is transverse. In the transverse plane, the temporal oscillations of the field vector are described by an ellipse or a circle (in the case of linear polarisation, the ellipse shrinks to a straight line). Along its direction of propagation, the wave may rotate to the left or to the right. Of course, these notions are meaningless unless one of them is properly defined: according to the U.S. Federal Standard 1037C (http://www.its.bldrdoc.gov/fs-1037/), the polarisation is defined as right-handed if the temporal rotation is clockwise when viewed from the transmitter (in the propagation direction) and left-handed if the rotation is counterclockwise. By contrast, astronomers look towards the source (transmitter), and therefore in the direction opposite that in which the wave propagates; hence the terms "clockwise" and "counter-clockwise" attribute meanings opposite to right- and left-handedness. Nevertheless, the handedness of a specific object remains invariant under orthogonal transformations.

Chirality and geometry: Handedness is a characteristic of material objects, such as corkscrews, doors, cookers, sinks, computer mice, keyboards, scissors, and a variety of construction tools. The mirror image of a right-handed object is the same as the original except that it is left-handed (the original image cannot be superimposed on its mirror image.) A nonhanded object remains the same within this mirror-image operation[6] since, after imaging, it can be brought into congruence with the original by simple translations and rotations. A handed object is called *chiral* (a term coined in 1888 by Kelvin[7], from the Greek word $\chi\varepsilon\iota\varrho$, meaning "hand"). Chiral media possess optical activity, or the ability to rotate the plane of polarisation of a beam of light passing through them. The relation between the chiral (micro)structure and the (macroscopic) optical rotation was discovered by Pasteur in the 1840s. The mirror-image operation is also called *parity transformation* (all spatial axes are reversed when parity is changed); it is a fundamental property of

[5] See Section 5.2.

[6] In a much more general setting, "mirror symmetry" is an example of a phenomenon known as *duality*, which occurs when two seemingly different physical systems are isomorphic in a nontrivial way. The nontriviality of this isomorphism makes quantum corrections necessary. In mathematics, an analogy is the Fourier transform: a local concept as the multiplication of two functions is equivalent to a convolution product, requiring integration over the whole space. Finding such dualities leads to solving complicated physical questions in terms of simple ones in the dual framework. A deep understanding of the inner mechanisms of duality symmetries is, in general, not yet feasible, with one exception: mirror symmetry. A mathematical framework to rigorise physical statements is already in an advanced stage of development. An excellent source elaborating aspects of this theory for physicists and mathematicians is [194].

[7] "I call any geometrical figure, or group of points, *chiral*, and say that it has chirality, if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself" ([234], p. 619).

physics that parity symmetry is broken in subatomic interactions. On several different scales and levels of nature, parity is not balanced. From amino acids through bacteria, winding plants and right-handed human beings to spiral galaxies, one of the handednesses dominates the other. The handedness of an optically active substance is called *dextrorotary* (resp., *levorotary*) if polarised light is rotated clockwise (resp., anticlockwise) as the observer faces the substance, with the substance between the observer and the light source. The handedness is indicated by prefixing "*d*-" (resp., "*l*-") to the substance's name.[8]

In geometry, an object is chiral if it does not coincide with its image under rotations and translations: in three dimensions any object with a plane of symmetry or a centre of symmetry is not chiral, but there are objects that, although they have neither a plane of symmetry nor a centre of symmetry, they are nonchiral. In two dimensions, any bounded nonchiral figure has an axis of symmetry. Typical chiral ones in two dimensions are rhomboids and spirals, while in three dimensions they are irregular tetrahedra[9] and Möbius strips.

A right-handed object and its corresponding left-handed object would be considered identical by usual symmetry. So, in what sense do the three above ways of looking at handedness differ, as far as the left-right classification is concerned? Obviously, the circular-polarisation-based handedness property is fully symmetric. Although the conventions differ and the definitions of left- and right-handedness are not alike in different scientific fields, the handedness of the polarisation in dipole antennas is only a matter of phase shift. Only metamaterials (which according to certain definitions cannot exist naturally) can display material parameters that are both simultaneously negative. As *left-handed* materials they belong to a class of media that by no means can be considered to be identified with that of the *right-handed* ones. As far as structural chirality is concerned, if all DNA molecules[10] were to twist in the right-handed sense, there would be no chance of the opposite handedness surviving. This is the reason that justifies the use of the term dyssymmetry[11] for this specific type of partial asymmetry. This phenomenon was discovered in 1811 by Arago [16], experimenting with quartz crystals (an anisotropic material), and one year later by Biot [63],

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[8]From the Latin words *dexter* meaning right and *laevus* meaning left.

[9]See *Chiral Polyhedra*, by E. W. Weisstein, in the framework of "The Wolfram Demonstrations Project" (http://demonstrations.wolfram.com/ChiralPolyhedra/).

[10]The DNA double helix is a spiral polymer of nucleic acids, held together by nucleotides that base pair together. In B-DNA, the most common double helical structure, the double helix is right-handed. Z-DNA is another of the many possible double helical structures of DNA. It is a left-handed double helical structure in which the double helix winds to the left in a zig-zag pattern. A-DNA is yet another of the possible double helical structures of DNA. A-DNA is thought to be one of three biologically active double helical structures, along with B- and Z-DNA. It is a right-handed double helix fairly similar to the B-DNA form, but with a shorter, more compact helical structure.

[11]From the Greek prefix $\delta\upsilon\sigma$- (dys-) meaning "difficult", "bad" or "ill" (it appears in many medical terms, e.g., dyspepsia, dysphagia, dyspnoea, etc.), and the word $\sigma\upsilon\mu\mu\varepsilon\tau\varrho\iota\alpha$, ("symmetry").

experimenting with turpentine vapour (an isotropic medium). Fresnel also examined optical activity in a chiral medium [154], as did in 1842 Cauchy [90]; this was the first mathematical study of chirality. The answer to the question of what is this strange property of media that makes them optically active was given by Pasteur in 1848 [345]: he noticed that two substances that were chemically identical in the classification scheme at the time but that had physical structures that were mirror images of each other exhibited different physical properties. Thus, Pasteur introduced geometry into chemistry and originated the branch of chemistry today called stereochemistry. Much more recently the studies of Prelog were extremely important; he shared[12] the 1975 Nobel Prize in Chemistry for his work in the field of natural compounds and stereochemistry. His lecture [351] at the Nobel Prize award ceremony regarding the rôle of chirality in chemistry, is very interesting.

Although they contain identical atoms in equal numbers, enantiomers[13] can, as mentioned above, have different properties. As Lakhtakia has written [272], "one enantiomer of the chiral compound thalidomide may be used to cure morning sickness, but its mirror image induces fetal malformation. Aspartame, a common artificial sweetener, is one of the four enantiomers of a dipeptive derivative. Of these four, one (i.e. aspartame) is sweet, another is bitter, while the remaining two are tasteless. Of the approximately 1850 natural, semisynthetic and synthetic drugs marketed these days, no less than 1045 can exist as two or more enantiomers; but only 570 were being marketed in the late 1980s as single enantiomers, 61 of which were totally synthetic. But since 1992, the U.S. Food and Drug Administration has insisted that only one enantiomer of a chiral drug be brought into market." Another example is mint flavored chewing gums containing chiral enantiomers; they create a different taste sensation to different people because the human taste sensors contain chiral molecules.

The great philosopher Kant was probably the first eminent scholar to point out the philosophical significance of mirror operations. The interested reader may refer to Section 13 of his 1783 "Prolegomena to Any Future Metaphysics" [222], where a most interesting discussion involving the notion of what is today called chirality is found.

Some of the history of the development of ideas about chirality may be found in the monographs [273], [268], [289] and the papers [272], [142], [213]. Also, the general audience oriented-books [158], [212] are very inspiring. See also [188].

The formalisation of the mathematical description of electrostatics took place around 1800, by giants such as J. L. Lagrange, P.-S. Laplace, S.-D. Poisson, G. Green and C. F. Gauss. However, there was no idea at the time of how electricity and magnetism were related. It was another giant, J. C.

[12] With Sir J. W. Cornforth.

[13] In chemistry, enantiomers (from the Greek words $\epsilon\nu\acute{\alpha}\nu\tau\iota o\varsigma$, meaning "opposite", and $\mu\acute{\epsilon}\varrho o\varsigma$, meaning "part" or "portion") are stereoisomers that are nonsuperimposable complete mirror images of each other.

Maxwell, who in the 1860s unified optics with electricity and magnetism in his monumental *A Treatise on Electricity and Magnetism*, first published in 1873 [306]. For a concise account of the history of electromagnetism, see, e.g., [83], [150], [348].

In the last part of the nineteenth century, after Maxwell's unification, it became possible to establish the connection between optical activity and the electromagnetic parameters of materials. In 1914, Lindman was the first to demonstrate the effect of a chiral medium on electromagnetic waves (his work in this field was about forty years ahead of that of other scientists); he devised a macroscopic model for the phenomenon of "optical" activity that used microwaves instead of light and wire spirals instead of chiral molecules. His related work was published in 1920 and 1922; for a very interesting account of Lindman's work, see [288].

At the macroscopic level, the Maxwell equations read

$$\begin{aligned} &\mathrm{curl} H = \partial_t D + J, && \text{Ampère's law,} \\ &\mathrm{curl} E = -\partial_t B, && \text{Faraday's law,} \\ &\mathrm{div} D = \rho,\ \mathrm{div} B = 0, && \text{Gauss's laws,} \end{aligned}$$

where E, H are the electric and the magnetic field, D, B are the electric and magnetic flux densities, J is the electric current density, and ρ is the density of the (externally impressed) electric charge.

This system contains eight equations (three from each of the first two "vector" laws and one from each of the "scalar" Gauss laws) but twelve unknowns (three components for each of the vector fields E, H, D, B). Constitutive relations, i.e., relations of the form

$$D = D(E, H)\,,\ B = B(E, H)\,,$$

must therefore be introduced. As is well known, constitutive relations are relations between physical quantities that are specific to a material or substance, and approximate the response of that material to external forces. Some constitutive equations are simply phenomenological; others are derived from first principles. This topic is discussed in Chapter 2.

Intensive research that has resulted in an impressively extensive bibliography on electromagnetic fields in complex (and in particular in chiral) media has appeared in the applied physics and engineering communities since the mid-1980s. By contrast, not so many rigorous mathematical contributions have appeared on the study of complex media. The large majority of these publications deal with time-harmonic electromagnetic fields in chiral media and appeared in the mid-1990s. The 1994 paper by Petri Ola [341] opened the way, followed initially by publications of the group at the Centre de Mathématiques Appliquées, École Polytechnique, Palaiseau, Paris, France, and the group at the Department of Mathematics of the National and Kapodistrian University of Athens, Greece. Of course, many other researchers gradually came onto the stage, so that the study of complex media in electromagnetics today forms an identifiable branch of applied mathematics. The rigorous mathematical analysis of time domain problems for complex media was the next step, and important progress in this field has been made. The vast majority of existing work deal with linear media, although recently advances have been made in nonlinear complex media. While most of the theory refers to deterministic complex media, its stochastic counterpart is not negligible.

Although there are books of different levels of mathematical rigour in the applied physics and engineering literature on the electromagnetics of complex media (e.g., [260], [266], [268], [271], [273], [289], [299], [378]), it seems that no books (apart from some parts of [91] and [145]) are devoted exclusively to the related mathematical theory. It is our intention to try to fill this gap by providing an introduction to the mathematical theory of complex media, linear and nonlinear, deterministic and stochastic. Of course, not all topics can be or are covered.