# Chapter One

## Introduction

The notion of a derivative is one of the main tools used in analyzing various types of functions. For vector-valued functions there are two main versions of derivatives: Gâteaux (or weak) derivatives and Fréchet (or strong) derivatives. For a function $f$ from a Banach space $X$ into a Banach space $Y$ the Gâteaux derivative at a point $x_0 \in X$ is by definition a bounded linear operator $T\colon X \longrightarrow Y$ such that for every $u \in X$,

$$\lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = Tu. \tag{1.1}$$

The operator $T$ is called the Fréchet derivative of $f$ at $x_0$ if it is a Gâteaux derivative of $f$ at $x_0$ and the limit in (1.1) holds uniformly in $u$ in the unit ball (or unit sphere) in $X$. An alternative way to state the definition is to require that

$$f(x_0 + u) = f(x_0) + Tu + o(\|u\|) \text{ as } \|u\| \to 0.$$

Thus $T$ defines the natural linear approximation of $f$ in a neighborhood of the point $x_0$. Sometimes $T$ is called the first variation of $f$ at the point $x_0$.

Clearly, for both notions of derivatives we have only to require that $f$ be defined in a neighborhood of $x_0$.

The existence of a derivative of a function $f$ at a point $x_0$ is not obvious. The question of existence of a derivative for functions from $\mathbb{R}$ to $\mathbb{R}$ was the subject of research and much discussion among mathematicians for a long time, mainly in the nineteenth century. If $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ has a derivative at $x_0$ then it must be continuous at $x_0$. While it is obvious how to construct a continuous function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ which fails to have a derivative at a given point, the problem of finding such a function which is nowhere differentiable is not easy. The first to do this was the Czech mathematician Bernard Bolzano in an unpublished manuscript about 1820. He did not supply a full proof that his function had indeed the desired properties. Later, around 1850, Bernhard Riemann mentioned in passing such an example. It was found out later that his example was not correct. The first one who published such an example with a valid proof was Karl Weierstrass in 1875. The first general result on existence of derivatives for functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ was found by Henri Lebesgue in his thesis (around 1900). He proved that a monotone function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable almost everywhere. As a consequence it follows that every Lipschitz function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, that is, a function which satisfies

$$|f(s) - f(t)| \leq C|s - t|$$

for some constant $C$ and every $s, t \in \mathbb{R}$, has a derivative a.e. This result is sharp in the sense that for every $A \subset \mathbb{R}$ of measure zero there is a Lipschitz (even monotone) function $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ which fails to have a derivative at any point of $A$.

Lebesgue's result was extended to Lipschitz functions $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ by Hans Rademacher, who showed that in this case $f$ is also differentiable a.e. However, this result is not as sharp as Lebesgue's: there are planar sets of measure zero that contain points of differentiability of all Lipschitz functions $f \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$. This can be seen by detailed inspection of our arguments in Chapter 12 (more details are in [39]). Questions related to sharpness of Rademacher's theorem have recently received considerable attention. See, for example, [2] or [15]. We do not cover this development here since its main interest and deepest results are finite dimensional, whereas our aim is to contribute to the understanding of the infinite dimensional situation.

The concept of a Lipschitz function makes sense for functions between metric spaces. Consequently, this gives rise to the study of derivatives of Lipschitz functions between Banach spaces $X$ and $Y$. It is easy to see that in view of the compactness of balls in finite dimensional Banach spaces both concepts of a derivative, defined above, coincide if $\dim X < \infty$ and $f$ is Lipschitz. However, if $\dim X = \infty$ easy examples show that there is a big difference between Gâteaux and Fréchet differentiability even for simple Lipschitz functions.

In the formulation of Lebesgue's theorem there appears the notion of a.e. (almost everywhere). If we consider infinite dimensional spaces and want to extend Lebesgue's theorem to functions on them, we have first to extend the notion of a.e. to such spaces. In other words, we have to define in a reasonable way a family of negligible sets on such spaces. (These sets are also often called exceptional or null.) The negligible sets should form a proper $\sigma$-ideal of subsets of the given space $X$, that is, be closed under subsets and countable unions, and should not contain all subsets of $X$. Since sets that are involved in differentiability problems are Borel, we can equivalently consider $\sigma$-ideals of Borel subsets of $X$, that is, families of Borel sets, closed under taking Borel subsets and countable unions. It turns out that this can be done in several nonequivalent ways (in our study below we were led to an infinite family of such $\sigma$-ideals). Thus the study of derivatives of functions defined on Banach spaces leads in a natural way to questions of descriptive set theory.

In the study of differentiation of Lipschitz functions on Banach spaces, one obstacle has been apparent from the outset. It was recognized already in 1930. The isometry $t \to \mathbf{1}_{[0,t]}$ (the indicator function of the interval $[0,t]$) from the unit interval to $L_1[0,1]$ does not have a Gâteaux derivative at a single point. The class of Banach spaces where this pathology does not appear was singled out already in the 1930s and characterized in various ways. The "good" Banach spaces (i.e., spaces $X$ so that Lipschitz maps from $\mathbb{R}$ to $X$ have a derivative a.e.) are now called spaces with the Radon-Nikodým property (or RNP spaces). The reason for this terminology is that one of their characterizations is that a version of the Radon-Nikodým theorem holds for measure with values in them. A detailed study of this class of spaces is presented in the books [4], [9], and [14]. All separable conjugate (in particular reflexive) spaces are RNP spaces. As we have just noted, this class does not include Banach spaces having $L_1[0,1]$ as a subspace. A similar easy argument shows that an RNP space cannot contain $c_0$ as a subspace. More sophisticated arguments are needed to show that there are separable Banach spaces with the RNP which are not subspaces of separable conjugate spaces or that there are spaces which fail to have the RNP but do not contain subspaces isomorphic to $L_1[0,1]$ or $c_0$. Such examples are presented in detail in [4].

The theorem of Lebesgue can be extended to Gâteaux differentiability of Lipschitz functions from an open subset of a separable Banach space into Banach spaces with the RNP. This was done by various authors independently in the 1970s by using different $\sigma$-ideals of negligible sets. The proofs are not difficult, and again all the details may be found in [4]. The situation concerning the existence of Gâteaux derivatives is at present quite satisfactory. On the other hand, the question of existence of Fréchet derivatives seems to be deep and our current knowledge concerning it is rather incomplete. This book is devoted to the study of this topic. Most of it consists of new material. We also recall the known results concerning this question and mention several of the problems which are still open. The proofs of most known results in this direction are at present quite difficult. It is not clear to us whether they can be considerably simplified. We present the proofs of the main results with all details and often accompany them with some words of motivation. Some examples we present seem to indicate that the fault in the difficulty lies mainly in the nature of things.

In dealing with Fréchet differentiability it turns out quite soon that we have to restrict the Banach spaces that can serve as domain spaces. The function $x \to \|x\|$ from $X$ to the reals is obviously continuous and convex and thus Lipschitz. If $X = \ell_1$ this function is easily seen not to be Fréchet differentiable at a single point. A similar situation may occur whenever $X$ is separable but $X^*$ is not. A separable Banach space $X$ is called an Asplund space if $X^*$ is again separable. The reason for this terminology is that Asplund was the first to prove that in such spaces real-valued convex and continuous functions have many points of Fréchet differentiability. "Many points" means here a set whose complement is a set of the first category (i.e., small in the sense of category and not in general in the sense of measure). This shows again why the study of Fréchet differentiability is strongly connected to descriptive set theory. Thus our real subject of study in this book is the existence of Fréchet derivatives of Lipschitz functions from $X$ to $Y$, where $X$ is an Asplund space and $Y$ has the RNP.

Perhaps the best known open question about differentiability of Lipschitz mappings is whether every countable collection of real-valued Lipschitz functions on an Asplund space has a common point of Fréchet differentiability.

Optimistic conjectures would assert Fréchet differentiability of Lipschitz functions almost everywhere with respect to a suitable proper $\sigma$-ideal of exceptional sets (or null sets). Based on what we currently know (including the results proved here), an optimistic differentiability conjecture may be stated in the following way.

**Conjecture.** In every Asplund space $X$ there is a nontrivial notion of exceptional sets such that, for every locally Lipschitz map $f$ of an open subset $G \subset X$ into a Banach space $Y$ having the RNP:

(C1) $f$ is Gâteaux differentiable almost everywhere in $G$.

(C2) If $S \subset G$ is a set with null complement such that $f$ is Gâteaux differentiable at every point of $S$, then $\mathrm{Lip}(f) = \sup_{x \in S} \|f'(x)\|$.

(C3) If the set of Gâteaux derivatives of $f$ attained on some $E \subset G$ is norm separable, then $f$ is Fréchet differentiable at almost every point of $E$.

There is very little evidence for validity of this Conjecture in the generality given above. On the positive side it holds if $Y = \mathbb{R}$ and, as we shall see in Chapter 6, it also holds for some infinite dimensional spaces $X$. On the negative side, it is unknown even whether every three real-valued Lipschitz functions on a Hilbert space have a common point of Fréchet differentiability. (The fact that every two real-valued Lipschitz functions on a Hilbert space have a common point of Fréchet differentiability is one of the new results we prove here.) Moreover, as far as we know, the Conjecture fails with any known nontrivial $\sigma$-ideal of subsets of infinite dimensional Hilbert spaces.

In any detailed study of Fréchet differentiability one immediately encounters the notion of porous sets (and of $\sigma$-porous sets that are their countable unions). We will give their usual definition later, since at this moment the only relevant fact is that a set $E \subset X$ is porous if an only if the function $x \to \mathrm{dist}(x, E)$ is Fréchet nondifferentiable at any point of $E$. It follows that porous sets have to belong to the $\sigma$-ideal hoped for in the above Conjecture. Of course, it also has to contain the sets of Gâteaux nondifferentiability of Lipschitz maps to RNP spaces. Denoting just for the purpose of this discussion by $\mathcal{I}$ the $\sigma$-ideal of subset of $X$ generated by the porous sets and sets of Gâteaux nondifferentiability of Lipschitz maps from $X$ into RNP spaces, one may hope that the Conjecture holds with exceptional sets being defined as elements of $\mathcal{I}$. Both subproblems of this variant of the Conjecture are open: it is unknown whether it holds with these exceptional sets, and it is unknown whether $\mathcal{I}$ is nontrivial.

A weaker variant of the Conjecture than the one at which we arrived above is in fact true. For the $\Gamma$-null sets (which will be defined in Chapter 5) we show that (C1) and (C2) hold, and that (C3) holds for the given space $X$ if and only if every porous subset in $X$ is $\Gamma$-null. This result coming from [28] was the first showing that the problem of smallness of porous sets is related to the problem of existence of Fréchet differentiability points of Lipschitz functions. One of the contributions of this text is to bring better understanding of this relation.

The statement (C2) is a weak form of the mean value estimate. Although it is stated for vector-valued functions, it can be equivalently asked only for real-valued functions (as the general case follows by considering $x^* \circ f$ for a suitable $x^* \in X^*$). One can argue that without the validity of this statement a differentiability result would not be very useful. For vector-valued functions there is, however, a stronger mean value estimate, the one that one would obtain by estimating in the Gauss-Green divergence theorem the integral of the divergence by its supremum. We will explain this concept, which we call a multidimensional mean value estimate, in detail in the last section of Chapter 2. The main results of this book give a fairly complete answer to the question under what conditions all Lipschitz mapping of $X$ to finite dimensional spaces not only possess points of Fréchet differentiability, but possess so many of them that even the multidimensional mean value estimate holds. It turns out that this property is much stronger that mere existence of points of Fréchet differentiability: for example, for mappings on Hilbert spaces it holds if the target is two-dimensional, but fails if it is three-dimensional.

We now describe some of the contents of this book in more detail. Every chapter starts with a brief information about its content and basic relation to results proved elsewhere. In most cases this is followed by an introductory section, which may also state the main results and prove their most important corollaries. However, some chapters contain rather diverse sets of results, in which case their statements are often deferred to the section in which they are proved. The key notions and notation are introduced at the end of this Introduction; more specialized notions and notation are given only when they are needed. The index and index of notation at the end should help the reader to find the definitions quickly.

The main point of the starting chapters is to revise some basic notions and results, although they also contain new results or concepts. Proofs that are well covered in the main reference [4] are not repeated here.

Chapter 2 recalls the notion of the Radon-Nikodým property and main results on Gâteaux differentiability of Lipschitz functions and related notions of null sets. Throughout the text, we will be interested not only in mere existence of points of Fréchet differentiability, but also, and often more important, in validity of the mean value estimates. We therefore explain this concept here in some detail. In particular, we spend some time on explaining the meaning of multidimensional mean value estimates, as this seems to be the concept behind nearly all positive results as well as the main counterexamples.

In Chapter 3 we meet some of the most basic concepts that we will use in the rest of the book. The existence of a Fréchet smooth equivalent norm on a Banach space with separable dual is of such fundamental importance that we prove it here even though it is treated in [4] as well. In separable Banach spaces with nonseparable dual we construct an equivalent norm which not only is rough in the usual sense, but has roughness directions inside every finite codimensional subspace (and even inside every subspace for which the quotient has separable dual). We also establish here the basic relations between porosity and differentiability, many of which will be further developed in the following text. Some of our finer results are deeply ingrained in descriptive theoretical complexity of sets related to differentiability. We therefore show here also that the set of points of Fréchet differentiability of maps between Banach spaces is of type $F_{\sigma\delta}$. In this book we normally assume that the Banach spaces we work with are separable (although we sometimes remark what happens in the nonseparable situation). The last section of this chapter justifies these assumptions: it describes how one can show that many differentiability type results hold in nonseparable spaces provided they hold in separable ones.

In Chapter 4 we treat results on $\varepsilon$-Fréchet differentiability of Lipschitz functions in asymptotically smooth spaces. In our context, these results are highly exceptional in the sense that they are the only differentiability results in which we do not prove that the multidimensional version of the mean value estimate holds; in fact we will show later that it may be false. The chapter, however, is not isolated from the others: some of the concepts and techniques introduced here will be used in what follows.

Chapter 5 introduces the notions of $\Gamma$- and $\Gamma_n$-null sets. The former is key for the strongest known general Fréchet differentiability results in Banach spaces; the latter presents a new, more refined concept. The reason for these notions comes from an (imprecise) observation that differentiability problems are governed by measure in finite dimension, but by Baire category when it comes to behavior at infinity. $\Gamma_n$-null sets are thus defined as those (Borel) sets that are null on residually many $n$-dimensional surfaces; to define $\Gamma$-null sets we use the same idea with $n = \infty$. The results connecting $\Gamma$- and $\Gamma_n$-null sets will later lead to finding a new class of spaces for which the strong Fréchet differentiability result holds.

The results of Chapter 6 are based on the already mentioned simple but important observation that for the above Conjecture to hold, porous sets have to be negligible. We show here that for $\Gamma$-null sets the converse is also true: the Conjecture holds (for the given space $X$) with $\Gamma$-null sets provided all sets porous in $X$ are $\Gamma$-null. In the proof we meet for the first time the concept of regular differentiability. It was observed a long time ago that if $f$ is Fréchet differentiable at $x$, then for any $c > 0$ and any direction $u$, $f(y + tu) - f(y)$ is well approximated by $tf'(x)(u)$ provided $t$ is small and $\|y-x\| \leq ct$. (This observation was crucial, for example, in Zahorski's deep study [47] of derivatives of real-valued functions of one real variable.) The validity of this observation or of its variants is called regular (Fréchet) differentiability. In general, it is weaker than Fréchet differentiability. From this moment on, proofs of Fréchet differentiability become two-stage: we show first that the function is, at a particular point, regularly differentiable, and only then proceed to the proof of differentiability. In this chapter the first stage is based on the relatively simple result that the set of points of irregular differentiability is $\sigma$-porous. In what follows, we will have to use considerably more sophisticated methods to handle this stage. Much of the material in this chapter follows ideas from [28], but the study of spaces in which porous sets are $\Gamma$-null is based on new results connecting $\Gamma$- and $\Gamma_n$-null sets that we will show later (in Chapter 10).

Chapters 7–14 contain our main new results. The first three contain new, sometimes technical, methods on which these results are based.

To explain the direction we have taken, we recall one of the facts that is in the background of the proofs of existing general Fréchet differentiability results in Banach spaces: for real-valued Lipschitz functions on spaces with smooth norms, attainment of the (local) Lipschitz constant by some directional derivative $f'(x; u)$, where $\|u\| = 1$, implies Fréchet differentiability of $f$ at $x$. In this form, this idea was used in [38] to prove that everywhere Gâteaux differentiable Lipschitz functions on Asplund spaces have points of Fréchet differentiability. Fitzpatrick [18] replaced attainment of the Lipschitz constant by attainment of what he called a modified version of the local Lipschitz constant, which is related to the notion of regularity mentioned above, and applied it to the problem of Fréchet differentiability of distance functions. The first Fréchet differentiability result for Lipschitz functions in [39] replaced the attainment of the Lipschitz constant by the requirement that the function $(x, u) \mapsto f'(x; u)$, perturbed in a suitable way, attains its maximum on a suitable set of pairs $(x, u)$. These perturbations are similar to those used in the proof of the Bishop-Phelps theorem. One of the contributions of the present work is the recognition that this similarity can be carried much farther, to the use of a variational principle of Ekeland's type (see [16]). We therefore devote Chapter 7 to the study in some detail of such principles, in particular, of smooth variational principles of the type found in [7]. We describe these principles as infinite two-player games, which allows some unusual applications later. The material of this chapter should be relevant to other areas of nonlinear analysis, and readers interested in such applications may find it useful that the chapter can be read independently of the others.

As we have already mentioned in description of Chapter 4, asymptotic smoothness of the norm can be successfully used to prove a weaker form of differentiability. We intend to extend this idea to showing that higher order asymptotic smoothness of the norm implies existence of common points of differentiability of more functions. In Chapter 8 we study general forms of such smoothness assumptions, which we make on bump functions rather than on norms. (This is more a matter of technical convenience than a serious generalization, and so in the following description of results we will speak only about asymptotic smoothness of norms.) Unlike ordinary smoothness, asymptotic smoothness allows arbitrary moduli, and one of the main contributions of our work is in isolating the “right” modulus of asymptotic uniform smoothness to guarantee existence of many common points of Fréchet differentiability of $n$ real-valued Lipschitz functions, namely, $o(t^n \log^{n-1}(1/t))$.

Chapter 9, as its title “Preliminaries to main results” indicates, gives a number of results and notions that will be needed in the sequel. In particular, it treats the important notion of regular differentiability in some detail and proves inequalities controlling the increment of functions by the integral of their derivatives. Another very important technical tool introduced here describes a particular deformation of $n$-dimensional surfaces. The idea is to deform a flat surface passing through a point $x$ (along which we imagine that a certain function $f$ is almost affine) to a surface passing through a point witnessing that $f$ is not Fréchet differentiable at $x$. This is done in such a way that certain “energy” associated to surfaces increases less than the “energy” of the function $f$ along the surface, which is precisely what enables the use of the variational principle to conclude that $f$ is in fact Fréchet differentiable at $x$.

The ideal goal of Chapter 10 would be to show that in spaces with modulus of asymptotic uniform smoothness $o(t^n \log^{n-1}(1/t))$ porous sets are $\Gamma_n$-null. We prove that this is the case for $n = 1, 2$, but for $n \geq 3$ we were unable to decide whether this holds in finite dimensional spaces. Nevertheless, we decompose every porous set in such a space into a union of its “finite dimensional” part (which is, for example, Haar null) and of a set porous “at infinity.” We show that the latter set is $\Gamma_n$-null. In fact, we use a condition weaker than asymptotic uniform smoothness $o(t^n \log^{n-1}(1/t))$. The main advantage of this is that in the case $n = 1$ our results imply that a separable space has separable dual if and only if all its porous subsets are $\Gamma_1$-null. Finally, we establish a (not completely straightforward) relation between $\Gamma_n$-nullness for infinitely many values of $n$ and $\Gamma$-nullness. This produces a new class of spaces for which the Conjecture holds, containing all spaces for which it was known before.

In Chapter 11 we investigate whether the result of Chapter 6 that $\Gamma$-nullness of porous sets implies the Conjecture has a meaningful analogy for $\Gamma_n$-null sets. One of the difficulties is that for $n \geq 3$ there are no (infinite dimensional) spaces in which we are able to verify the assumption that the porous sets are $\Gamma_n$-null. From the previous chapter we know only that in spaces admitting an appropriately asymptotically smooth norm every porous set can be covered by a union of a Haar null set and a $\Gamma_n$-null $G_\delta$ set. We therefore consider a space $X$ having this property and show that the set of Gâteaux derivatives of a Lipschitz function $f\colon X \longrightarrow Y$, where $\dim Y = n$, is, for every $\varepsilon > 0$, contained in the closed convex hull of the "$\varepsilon$-Fréchet derivatives," that is, of those $f'(x)$ such that

$$\|f(x+u) - f(x) - f'(x)(u)\| \le \varepsilon\|u\|$$

for all $u$ with $\|u\|$ small enough.

In Chapter 12 we use the ideas developed in the previous chapters to improve known results in the classical case of real-valued functions on spaces with separable dual. Here the main novelty is that the variational principle is used to organize the arguments in a way that leads to two new results. First, for Lipschitz functions we prove not only the known result that they have many points of Fréchet differentiability, but also that they have many points of Fréchet differentiability outside any given $\sigma$-porous set. Second, we show that even cone-monotone functions have points of Fréchet differentiability, and again such points may be found outside any given $\sigma$-porous set.

Chapter 13 contains perhaps the most surprising results of this text. We show that if a Banach space with a Fréchet smooth norm is asymptotically smooth with modulus $o(t^n \log^{n-1}(1/t))$ then every Lipschitz map of $X$ to a space of dimension not exceeding $n$ has many points of Fréchet differentiability. In particular, we get that two real-valued Lipschitz functions on a Hilbert space have a common point of Fréchet differentiability and, more generally, that every collection of $n$ real-valued Lipschitz functions on $\ell^p$ has a common point of Fréchet differentiability provided $1 < p < \infty$ and $n \le p$. The argument combines a significant extension of methods developed in Chapter 12 with understanding of the role of a particular porous set: the set of points of irregular differentiability. However, the results about porosity do not enter the proof directly.

In Chapter 14 we explain the need for the rather strong smoothness assumptions made in Chapters 10 and 13 to show $\Gamma_n$-nullness of sets porous "at infinity" and/or existence of many points of Fréchet differentiability of Lipschitz maps into $n$-dimensional spaces. Although it is probable that our assumptions are not optimal, we show that they are rather close to being so. As a corollary we obtain that Lipschitz maps of Hilbert spaces to $\mathbb{R}^3$ and, more generally, maps of $\ell^p$ to $\mathbb{R}^n$ for $n > p$ do not have many points of Fréchet differentiability. Here, of course, the word "many" (in the "mean value" sense explained above) is crucial; without it we would have disproved the Conjecture.

The two last chapters do not follow the main direction of the book in the sense that they return to the iterative procedure for finding points of Fréchet differentiability. Chapter 15 actually presents the current development of our first, unpublished proof of existence of points Fréchet differentiability of Lipschitz mappings to two-dimensional spaces. Unlike in Chapter 13, for functions into higher dimensional spaces the method does not lead to a point of Gâteaux differentiability but constructs points of so-called asymptotic Fréchet differentiability. Although some of these results follow from Chapter 13, in full generality they are new. The proof does not use the variational approach;

indeed, it uses perturbations that are not additive. It nevertheless still provides (asymptotic) Fréchet derivatives in every slice of Gâteaux derivatives, and so by results of Chapter 14 it also cannot be used to show existence of points of Fréchet differentiability of Lipschitz mappings of Hilbert spaces to three-dimensional spaces. Readers interested in attempts to find such a point by modifying the iterative (or variational) argument should probably start by checking in what way their approach differs from ours, and why the difference may allow for slices of Gâteaux derivatives without Fréchet derivatives.

The final Chapter 16 gives a separate, essentially self-contained, nonvariational proof of existence of points of Fréchet differentiability of $\mathbb{R}^2$-valued Lipschitz maps on Hilbert spaces. It is mainly directed toward readers whose main interest is in Hilbert spaces. The structure of the Hilbert space is heavily used, but even then some readers interested in the much stronger results of Chapter 13 may find useful that it covers some of the basic ideas in a substantially simpler way.

## 1.1 KEY NOTIONS AND NOTATION

We finish the Introduction by quickly recapitulating the main notions defined above and introducing some notation that will be used throughout or for which there was no other natural place. More specific notions and notation will be given only when needed (and can be found in the Index or Index of Notation at the end).

First, we should remind the reader that not much would be lost by assuming that the Banach spaces in this book are separable. We tend to give separability assumptions only in key statement and notions and only when there is a significant difference between the separable and nonseparable case. For example, in the definition of Haar null sets in Chapter 2 we assume that the space is separable, although there is a natural definition also in the nonseparable situation. The reason is that for separable spaces we may use in this definition Borel measures (which we call simply measures) while for nonseparable spaces we have to use Radon measures.

We will use the following notation for derivatives of functions between Banach spaces $X$ and $Y$. The *directional derivative of a function $f$ at $x \in X$ in the direction $u \in X$* is defined by

$$f'(x;u) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t},$$

provided the limit exists. To avoid any misunderstanding, we agree that by saying that "a limit exists" we understand that it is finite.

The Gâteaux derivative of $f$ at $x$ (defined at the beginning of the Introduction) will be denoted by $f'(x)$. So, provided $f$ is Gâteaux differentiable at $x$, we have two ways of writing the directional derivatives at $x$: $f'(x)(u)$ and $f'(x;u)$. The second expression may be meaningful even if the first is not. We will not use any particular notation for the Fréchet derivative (also defined at the beginning of the Introduction), since its value is the same as that of the Gâteaux derivative and so we may use $f'(x)$ if needed. (But this expression by itself does not say that $f$ is Fréchet differentiable at $x$.)

Although Fréchet differentiability is the main problem we study, we will also have to use various other notions of differentiability. Gâteaux or Fréchet differentiability of a function $f\colon X \longrightarrow Y$ in the direction of a subspace $Z$ of $X$ is defined in the obvious way as Gâteaux or Fréchet differentiability of the function $z \in Z \to f(x+z)$ at the point $z = 0$. We will use without any reference that the two notions of differentiability in the direction of $Z$ are equivalent when $f$ is Lipschitz and $Z$ is finite dimensional; in this case we will often speak just about differentiability or derivative. Other, more special notions such as regular differentiability and differentiability in directions of linear maps will be introduced only when needed, most of them in Definition 9.2.1.

Since we will work with a number of (pseudo)norms, we fix basic notation for them. (The only difference between a norm and a pseudonorm is that the latter may be zero for some nonzero vectors. Similarly for pseudometrics, although for them we will occasionally allow also infinite values.) The symbol $\|\cdot\|$ is used for most norms: for example, it may denote the norm on the Banach space $X$ we study or the usual operator norm on the space $L(X,Y)$ of bounded linear operators from $X$ to $Y$. However, since the Euclidean norm in $\mathbb{R}^n$ will often be used at the same time as other norms (on other spaces), we will denote it by $|\cdot|$. When needed, we add an index to give more information about a particular norm. By $\|\cdot\|_\infty$ we denote the supremum norm on the space of (Banach space–valued) functions. More special norms that we will use include $\|\gamma\|_{C^1} = \|\gamma\|_{C^1(S)} = \max\{\|\gamma\|_\infty, \|\gamma'\|_\infty\}$ on the space of (Banach space–valued) $C^1$ functions on a subset $S$ of $\mathbb{R}^n$ and the norms $\|\gamma\|_k$ and $\|\gamma\|_{\leq k}$ on the space of surfaces introduced in Chapter 5.

By $B(x,r)$ we will denote the open ball with center $x$ and radius $r$; in situations when the space or the norm is not clear, we invent a way of indicating it.

After derivatives, the notion of $\sigma$-porosity is perhaps the most often occurring notion in this book. The underlying notion of porosity comes in several variants, of which we will use the following three.

A set $E$ in a Banach space $X$ (or in a general metric space $X$) is called *porous* if there is $0 < c < 1$ such that for every $x \in E$ and every $\varepsilon > 0$ there is a $y \in X$ with $0 < \operatorname{dist}(x,y) < \varepsilon$ and

$$B\big(y, c \operatorname{dist}(x,y)\big) \cap E = \emptyset.$$

In this situation we also say that $E$ is *porous with constant c.*

If $Y$ is a subspace of $X$, then $E$ is called *porous in the direction of* $Y$ if there is $0 < c < 1$ such that for every $x \in E$ and $\varepsilon > 0$ there is $y \in Y$ so that $0 < \|y\| < \varepsilon$ and

$$B(x+y, c\|y\|) \cap E = \emptyset.$$

Instead of "porous in the direction of the linear span of a vector $u$" we say "porous in the direction $u$"; we also say that $E \subset X$ is *directionally porous* if it is porous in some direction.

The sets that will be really important for us are not the porous sets we have just defined, but their countable unions. A subset of $X$ is termed *$\sigma$-porous*, *$\sigma$-porous in the direction of* $Y$, or *$\sigma$-directionally porous* if it is a union of countably many porous, porous in the direction of $Y$, or directionally porous sets, respectively.

It remains to introduce some notation which is used more often but for which there is no reasonable place elsewhere. The symbol $\fint$ means the average integral: provided $0 < \mu A < \infty$,

$$\fint_A f\,d\mu = \frac{1}{\mu A}\int_A f\,d\mu.$$

In this book we will never encounter serious measurability problems, but to make some assumptions more compact we will identify measures with outer measures. So, for example, $\mathscr{L}^n$ denotes the outer Lebesgue measure in $\mathbb{R}^n$. The Lebesgue measure of the Euclidean unit ball in $\mathbb{R}^n$ is $\boldsymbol{\alpha}_n$.

As one of our main goals is finding points of Fréchet differentiability in arbitrary slices of the set of Gâteaux derivatives, we recall the relevant notions here. In general, a *slice* of a subset $M$ of a Banach space $X$ is any nonempty set of the form

$$S = \{x \in M \mid x^*(x) > c\},$$

where $x^* \in X^*$ and $c \in \mathbb{R}$. However, for us the more pertinent concept is that of $w^*$*-slice*, which is defined for a subset $M$ of the dual space $X^*$ as any nonempty set of the form

$$S = \{x^* \in M \mid x^*(x) > c\},$$

where $x \in X$ and $c \in \mathbb{R}$. More generally, we will use the concept of $w^*$-slices also in the space of operators $L(X, Y)$. A $w^*$-slice $S$ of a subset $M \subset L(X, Y)$ is any nonempty set of the form

$$S = \Big\{L \in M \;\Big|\; \sum_{i=1}^{m} y_i^* L x_i > c\Big\},$$

where $m \in \mathbb{N}$, $x_1, \dots, x_m \in X$, $y_1^*, \dots, y_m^* \in Y^*$ and $c \in \mathbb{R}$. Since sometimes the number $m$ of points and functionals determining the given slice plays a role, we will call the least such number the *rank of the slice* $S$.

Finally, we make agreements about the use of some standard terms. We will use the terms increasing and strictly increasing in the sense for which some authors use nondecreasing and increasing, respectively. Similarly, we use the terms decreasing and strictly decreasing, positive and strictly positive. However, we will not be pedantic in the use of these terms; in particular, we may say non-negative instead of positive and positive instead of strictly positive, the former since it may be clearer and the latter when the meaning is obvious from the context. In most situations it does not matter whether the set of natural numbers $\mathbb{N}$ starts from zero or one, but when it does, we assume it starts from one.